Chapter 9
Forces and Constrained Motion

You have now learned to determine the motion of an object based on an analysis of the forces acting on the object and the application of Newton's second law. We use the structured problem-solving approach to first *identify* the moving object, *model* the forces acting to produce an equation of motion, *solve* the equations of motion to find the motion, and *analyze* the results to answer questions about the motion. This method is very robust—it works as long as we have good models for all the forces.

For example, if you drop a steel box onto a wooden table, you have learned to use a spring model to approximate the normal force from the table on the box while the two are in contact, as illustrated in Fig. 9.1. The normal force, N, depends on the position, x, of the bottom of the box:

$$N = \begin{cases} -kx & x < 0 \\ 0 & x \geq 0 \end{cases} \tag{9.1}$$

However, if the box is lying at rest on the table, we do not need a force model to find the normal force on the box. We can instead use our knowledge about the motion of the box—we know that it is not moving—and apply Newton's second law to determine the normal force, N, to be equal to the gravitational force, G, on the box:

$$\sum F = N - G = ma = 0 \;\Rightarrow\; N = G. \tag{9.2}$$

In this case the motion of the box is constrained: We know that it is not moving in the direction of gravity. For constrained motion we can therefore calculate some of the forces without a force model!

First, why is this interesting? If we already know that the box is not moving in the direction of gravity, why would we need to know the forces acting on the box in this direction? Was not the whole point of introducing quantitative models for the forces that we could use this to determine the motion? It turns out that some force

A. Malthe-Sørenssen, *Elementary Mechanics Using Matlab*,
Undergraduate Lecture Notes in Physics,
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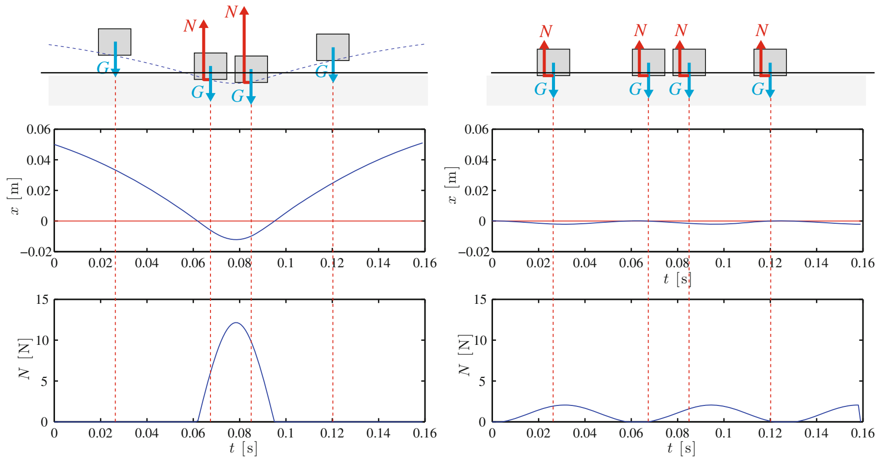

Fig. 9.1 Illustration of a box in contact with a surface. *Left figure* shows the gravitational, G, and normal force, N, on the box when the box is released from a small height above the floor. *Right figure* shows the gravitational and normal force when the box is lying (almost) still on the floor. The forces and vertical motion of the box has been exaggerated

models, such as the model for friction, depends on the normal force on the object. In this case, we need to know the normal force to find the force acting in the direction of the motion. Also, there are many cases where the motion is constrained only for a certain range of normal forces: For example, the attachment of a roller coaster cart may only sustain a certain maximum force before breaking, or the motion of a ball in a rope is only constrained by the rope as long as the rope is tight. In both these cases it is an advantage if we can find forces such as the normal force without introducing a force model.

Second, you may object that it is not really true that the box is not moving in the direction of gravity. The box is never completely at rest on the table, it still oscillates slightly up and down, and the normal force varies slightly. That is correct, but we always make many similar approximations. Generally, if the oscillations are very small, we ignore them, and the corresponding variation in forces, and assume that the motion is constrained.

In this chapter we will discuss the use of constrained motion to simplify problems. First, we demonstrate how we try to choose coordinate systems wisely so that motion only occurs along some of the axes. This technique is often called "decomposition of forces", and it simplifies the analysis of a problem, because only force components in the directions that the object can move can contribute to the acceleration of the object along its path.

We will also introduce a new force model, the friction force model, which allows us to model forces during the relative motion of two solids. This model has a long history and allows us to study many classical examples, even though the physical origin of friction forces is poorly understood.

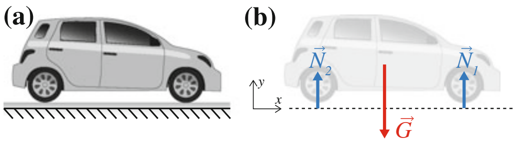

Fig. 9.2 Illustration of a car at rest on a flat surface. **a** Illustration of the car and the surface. **b** Free-body diagram for the car

We will also study more complicated constrained motions, such as the circular motion of an object attached to a rope, or the motion of a car driving along a curved road or a hilltop. In these cases the constraints are only valid for a limited range of the normal forces, and care must be taken to find when the limits are exceeded.

9.1 Linear Constraints

No Motion—Statics

Let us start with the simplest example of a constrained motion—an object that is not moving at all. In this case, we are not interested in determining the motion of the object, but instead we want to find the forces acting on the object from the constraint that the object is not moving, and that the net force on the object therefore is zero. Such problems are typically called equilibrium problems, or *static* problems.

For example, let us analyze a car standing on the ground, as illustrated in Fig. 9.2a. We follow the first steps in the structured problem-solving approach. First, we *identify* the object, which in this case is the car.

Second, we *model* the interactions between the object and the environment by identifying the forces acting on the object and by introducing quantitative models for the forces. The free-body diagram for the car is illustrated in Fig. 9.2b. The only contact forces are acting on the wheels of the car from the ground, and we call these contact forces $\mathbf{N}_1$ and $\mathbf{N}_2$.[1] In addition, there is a gravitational force, $\mathbf{G}$. Both the normal forces and the gravitational force act in the y-direction only.

We have a quantitative model for the gravitational force, $\mathbf{G} = -mg\,\mathbf{j}$, where m is the mass of the car. However, we do not know the normal forces. Instead, we will use Newton's second law to determine the forces given the constraint we have on the motion—the car is not moving.

Because the car is not moving, the acceleration of the car is zero, $\mathbf{a} = \mathbf{0}$. Newton's second law therefore gives:

$$\sum \mathbf{F} = \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{G} = m\mathbf{a} = \mathbf{0}. \tag{9.3}$$

$$\mathbf{N}_1 + \mathbf{N}_2 = -\mathbf{G} = mg\,\mathbf{j}. \tag{9.4}$$

[1] On a real car there are of course four contact forces acting on the four wheels of the car.

We have found that the *sum* of the normal forces equals the opposite of the gravitational force.

Generally, we do not know how the total normal force is divided between the normal forces.[2] For example, if the car has a very large load in the luggage room in the back, we expect the back suspensions to be more compressed than the front suspensions, which suggest that the normal force on the back of the car is larger than on the front of the car. On the other hand, if the engine in the front of the car is very heavy, we would expect the front wheels to experience a larger load than the back wheels.

It is therefore common to only draw a single contact point for the car, where the sum, $\mathbf{N}$, of the normal forces are acting:

$$\mathbf{N} = \mathbf{N}_1 + \mathbf{N}_2 . \tag{9.5}$$

Equilibrium Along One Axis, Motion Along the Other

Let us now look at a slightly more complicated problem, a car moving in a straight line along a flat surface, as illustrated in Fig. 9.3. In this case the motion of the car is constrained to a line, and the forces acting on the car are either directed along the line or are normal to the line. Notice that since the car is constrained to move along a line, the acceleration normal to the line is zero.

Let us analyze the motion of the car when a horizontal force $\mathbf{F}$ is applied, as show in Fig. 9.3. The forces acting on the car are illustrated in Fig. 9.3. The contact forces are the two normal force, $\mathbf{N}_1$ and $\mathbf{N}_2$, as we introduced above, the externally applied force, $\mathbf{F}$, acting horizontally, and the gravitational force $\mathbf{G}$. We assume that the effects of friction and air resistance are negligible.

We have a force model for the gravitational force, $\mathbf{G} = -mg\,\mathbf{j}$, and we assume that we know the value of the externally applied force, $\mathbf{F}$.

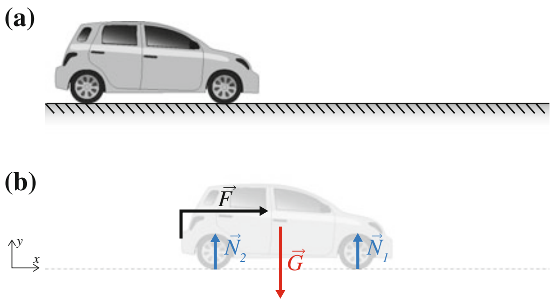

Fig. 9.3 **a** Illustration of a car pushed along a horizontal surface. **b** Free-body diagram for the car

[2]As we will see later, we can determine each of the normal forces by making a similar assumption about the rotation of the car: the car is not rotating.

We can determine both the motion of the object and the normal forces by applying Newton's second law. We recall that Newton's law is a vector equation, and that it can be applied along each unit vector independently:

$$\sum \mathbf{F} = \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{G} + \mathbf{F} = m\mathbf{a}. \tag{9.6}$$

$$N_1\mathbf{j} + N_2\mathbf{j} - G\mathbf{j} + F\mathbf{i} = m\mathbf{a} = ma_x\,\mathbf{i} + ma_y\,\mathbf{j}. \tag{9.7}$$

This can be written as two sets of equations, one for motion along the x-axis, and one for motion along the y-axis:

$$F = ma_x$$
$$N_1 + N_2 + G = ma_y \tag{9.8}$$

Since the car is moving along the x-axis, the acceleration along the y-axis is zero:

$$N_1 + N_2 + G = ma_y = 0, \tag{9.9}$$

and

$$N_1 + N_2 = -G = mg. \tag{9.10}$$

The forces in the y-direction are therefore the same as we found above.

The motion in the x-direction is given by the forces acting in the x-direction, as we have found previously.

This analysis demonstrates how we *decompose* the motion into

- motion *along* the constraint, and
- motion *normal* to the constraint.

This is done through the analysis of force, where we similarly *decompose* the forces into

- forces *along* the constraint, and
- forces *normal* to the constraint.

In this particular case all forces are acting either along or normal to the line of constraint. Consequently, the analysis is particularly simple.

Decomposition of Forces

We are now ready to add one more complication, and address motion where the external forces are not simply parallel to or normal to the motion, but where there are force components both parallel to and normal to the motion. In this case, the use of *decomposition* of forces becomes more important, and we will demonstrate how we use this technique combined with the application of Newton's second law through an example.

Let us study the motion of a car rolling down an inclined slope, as illustrated in Fig. 9.4. We realize that also in this case the car is constrained to move along the surface of the slope. We determine the motion of the car by identifying and modeling the forces acting on the car. The contact forces on the car are the normal force, $\mathbf{N}_1$ and $\mathbf{N}_2$. These act normal to the surface. We have chosen the coordinate system to be oriented along the inclined slope so that the x-axis points in the direction of motion—what we called along the constraint—and the y-axis points in a direction normal to the constraint. We assume that we can neglect the effects of friction and air resistance.

In addition, the car is subject to a gravitational force, $\mathbf{G}$. The gravitational force is directed along the vertical axis, which we have called the y'-axis in Fig. 9.4.

We decompose the forces in the x- and y-directions, parallel and normal to the direction of motion for the car. We notice that the normal forces are directed along the y-axis:

$$\mathbf{N}_1 = N_1\,\mathbf{j}, \;\; \mathbf{N}_2 = N_2\,\mathbf{j}. \tag{9.11}$$

The gravitational force, $\mathbf{G}$, is not parallel to the x- or the y-axis, but we can decompose it into two components along these directions. From Fig. 9.4 we see that the x- and y-components of gravity can be written as:

$$G_x = G\sin\alpha, \;\; G_y = -G\cos\alpha. \tag{9.12}$$

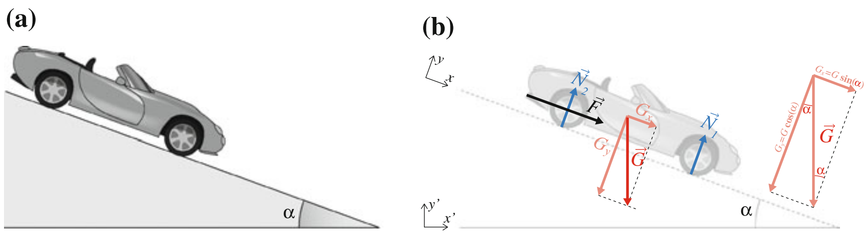

Fig. 9.4 **a** Illustration of a car rolling down an inclined slope. **b** Free-body diagram for the car

The gravitational force is therefore:

$$\mathbf{G} = G_x\,\mathbf{i} + G_y\,\mathbf{j}. \tag{9.13}$$

where the gravitational force model gives the value, $G = mg$, for the gravitational force on the car.

Notice that we have decomposed the forces in the direction of motion, which is given by the constraint, and in the direction normal to motion. This allows us to apply Newton's laws of motion in each direction, separating the motion along the surface from motion normal to the surface.

We apply Newton's second law of motion to the car:

$$\sum \mathbf{F} = \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{G} = m\,\mathbf{a}. \tag{9.14}$$

We write the vectors on component form:

$$N_1\,\mathbf{j} + N_2\,\mathbf{j} + G_x\,\mathbf{i} + G_y\,\mathbf{j} = ma_x\,\mathbf{i} + ma_y\,\mathbf{j}. \tag{9.15}$$

We can separate this into two equations, one equation for the x-axis and one equation for the y-axis:

$$G_x = ma_x \tag{9.16}$$

$$N_1 + N_2 + G_y = ma_y \tag{9.17}$$

Because we know that the car is only moving along the surface, the acceleration in the y-direction is zero, giving:

$$N_1 + N_2 = -G_y = G\cos\alpha = mg\cos\alpha. \tag{9.18}$$

We have therefore found that sum of normal forces on the car balances the component of the gravitational force that is normal to the surface.

We find that motion along the x-axis is determined by:

$$a_x = \frac{G_x}{m} = \frac{mg\sin\alpha}{m} = g\sin\alpha. \tag{9.19}$$

We can therefore use our tools to solve motion with constant acceleration to find the motion along the x-axis.

Notice the use of decomposition: We *choose a coordinate system so that one or more axes are directed in a direction where we know that there is no motion*. Above we chose the y-axis to be directed normal to the surface and the x-axis to be along the direction of motion. We apply Newton's second law in the directions where there are no motion to determine the other forces in this direction. Then, we find force components in the direction of motion, which determine the acceleration and hence the motion of the object along the surface. This approach is general, and can be applied to a wide range of problems where motion is constrained to a line.

9.1.1 Example: A Bead in the Wind

Problem: A bead is sliding without friction along a vertical wire. A wind with a constant wind speed of w is blowing horizontally. Find the force from the wire on the bead and the acceleration of the bead. You can use a linear air resistance law for the bead.

Approach: We find the net force along the bead, including the effect of air resistance, and use this to find the acceleration along the wire and the forces normal to the wire from Newton's second law.

Identify: The center of the bead, given by $\mathbf{r}(t)$, must move along the wire. We therefore choose the y-axis to be along the wire in the vertical direction, and the x-axis to be directed in the direction the wind is blowing (see Fig. 9.5a).

Model: The contact forces on the bead are the force from the wire on the bead, $\mathbf{N}$; the air drag force, $\mathbf{F}_D$; and gravity $\mathbf{G} = -mg\,\mathbf{j}$. The air drag force depends on the velocity $\mathbf{v}$ of the bead *relative to* the velocity $\mathbf{w}$ of the wind:

$$\mathbf{F}_D = -k_v\,(\mathbf{v} - \mathbf{w}), \tag{9.20}$$

These forces are illustrated in the free-body diagram for the bead in Fig. 9.5b.

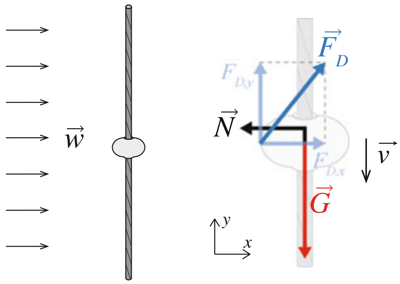

Fig. 9.5 **a** Illustration of bead on a vertical wire. **b** Free-body diagram of the bead

We decompose in the x- and y-direction. Gravity acts vertically, and therefore along the y-axis. The air drag depends on the velocity, $\mathbf{v}$, of the bead. Since the bead moves only in the vertical direction, along the wire, the velocity only has a vertical component:

$$\mathbf{v} = v_y\,\mathbf{j}. \tag{9.21}$$

The velocity of the air is in the positive x-direction:

$$\mathbf{w} = w\,\mathbf{i}. \tag{9.22}$$

The air drag force is therefore:

$$\mathbf{F}_D = F_{D,x}\,\mathbf{i} + F_{D,y}\,\mathbf{j}, \tag{9.23}$$

where we find F_x from:

$$F_{D,x} = \mathbf{F}_D \cdot \mathbf{i} = -k_v\,(\mathbf{v} - \mathbf{w}) \cdot \mathbf{i} = -k_v\,(\mathbf{v} \cdot \mathbf{i} - \mathbf{w} \cdot \mathbf{i}) = -k_v\,(0 - w) = k_v w \tag{9.24}$$

And, similarly, we find the y-component:

$$F_{D,y} = \mathbf{F}_D \cdot \mathbf{j} = -k_v v_y, \tag{9.25}$$

We also notice that the normal force, $\mathbf{N}$, only has a component along the x-axis. We have now decomposed the forces, and can apply Newton's second law along the x- and y-directions.

Since the bead is constrained not to move in the x-direction, we know that the acceleration in the x-direction is zero. Newton's law along the x-axis therefore gives:

$$\sum F_x = F_{D,x} - N = ma_x = 0 \;\Rightarrow\; N = F_{D,x} = k_v w. \tag{9.26}$$

The normal force is therefore constant for this particular model of the motion!

We find the acceleration of the bead by applying Newton's law in the y-direction:

$$\sum F_y = F_{D,y} - G = -k_v v_y - mg = ma_y \;\Rightarrow\; a_y = -\frac{k_v}{m} v_y - g. \tag{9.27}$$

From these equations you know how to find the motion of the bead. For this particular formulation of the problem, it is decoupled—the motion of the bead in the y-direction does not depend on the wind velocity.

Test your understanding: How would this problem change if you used a square-law for the air drag?

9.2 Force Model—Friction

We have now addressed constrained motion such as the motion of a block sliding along a surface as illustrated in Fig. 9.6. We have shown that we can use Newton's second law in combination with the constraint to determine the normal force on the block without including a force model for the contact. For the case illustrated in Fig. 9.6, we can apply Newton's second law in the vertical direction to find:

$$\sum F_y = N - G = N - mg = ma_y = 0, \tag{9.28}$$

$$N = mg, \tag{9.29}$$

But so far there has been no coupling between the forces normal to the surface and the forces along the surface—the tangential forces. The magnitude of the normal force has no impact on the motion of the block.

In real systems there are several mechanisms that introduce a coupling between the normal and tangential forces. When we discuss macroscopic systems it is common to group these mechanisms into a common term: friction. Friction is a force acting on the contact between two objects, it acts tangential to the contact, and it depends on the magnitude of the normal force at this contact. The introduction of a force model for friction forces is therefore essential in order to get a realistic description of contact forces. Unfortunately, the concept of friction is not that well defined. The mechanisms of friction are currently not well understood, and are subjects of current research. There is also not only one, single mechanism responsible for the tangential forces acting on a contact—there are many different mechanisms acting all at the same time, including effects such as surface geometry and roughness, the effect of thin fluid films, surfactants, and dirt on the surfaces, the effect of adhesion forces and plastic deformation of small asperities on the contact surface, the effect of the external boundaries, and many more. In addition, it is also clear that the scale you are studying is important: For example on very small scales inter-atomic forces and the presence of individual atoms or molecules along the surface can affect the tangential forces acting.

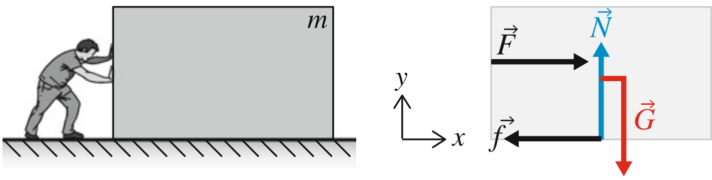

Fig. 9.6 *Left* Illustration of a crate on the floor. *Right* A free-body diagram of a crate on the floor. You are trying to push the crate with the force **F** directed along the floor

But wait, you interrupt, I already know something about friction, and it is a simple law! Yes. Already in 1699 Amontons reported the experimental observations that we today also call the "laws of friction":

Amonton's laws of dynamic friction:

- The friction force is proportional to the normal force: $F = \mu N$.
- The friction force does not depend on the apparent contact area.

This is an experimental observation that turns out to be surprisingly robust, in particular considering that it includes so many different effects! In this section we will motivate and discuss the force model for friction as a contact force acting in the tangential direction along any contact between two solids. We will introduce a model for the static friction force for the case when the two surfaces are not moving relative to each other, and a model for the dynamic friction force when the surfaces are moving relative to each other.

Static Friction

We all have a good intuition for solid friction because it is an important force in our macroscopic world. If you want to push a heavy crate along the floor, you know that you need to apply a certain force to "get it going". If you do not push hard enough, it will not start moving. You may also know that if you want to keep it moving, you must constantly apply a force to it, otherwise it will stop moving. If you give the crate a hard push, it will start moving, but you know that it will decelerate and stop after some distance. These simple insights provides us with some of the basic properties of a the friction force.

Sliding box model system: Let us analyze the motion of a crate on a flat floor in detail (see Fig. 9.6 for an illustration). You push on the crate with a horizontal force, **F**. The motion of the crate is determined by the forces acting on it: The applied force, **F**, the normal force **N** from the floor, and the gravitational force, **G**. The crate is cannot move down through the floor—it is constrained not to move in the y-direction. We can therefore use Newton's second law in the y-direction to determine the normal force:

$$\sum F_y = N - G = N - mg = ma_y = 0 \;\Rightarrow\; N = mg. \tag{9.30}$$

We also know, from our experience or from staging a simple experiment, that when the applied force, **F**, is small, the crate is not moving in the horizontal direction. However, according to Newton's second law of motion the horizontal acceleration is given by the net horizontal force:

$$\sum F_x = ma_x. \tag{9.31}$$

Because the crate is not moving, the sum of the forces must be zero. Consequently, the crate cannot only be affected by the applied force **F** in the horizontal direction. There must be an additional force, counteracting the applied force. This force must come from the contact with the floor. And the force has the apparently "magical" property that it is exactly equal to the applied, horizontal force! If you push with a very small force, $\mathbf{F}_0$, then the force from the floor will be exactly the same but in the opposite direction. If you push with a slightly larger force, $\mathbf{F}_1$, then the force from the floor will again be exactly the same, but in the opposite direction. In this respect the friction force is similar to the normal force from the floor on the crate, which in the static situation matches the force from gravity. We call the force from the floor on the crate the *static friction* force.

Microscopic model for static friction: The word static indicates that it is the friction along a surface when the object does not move relative to the surface. We can develop a simple model for the origin of the static friction force from a microscopic picture of the contact. On the microscopic scale, the contact between the crate and the floor consists of many, small irregularities: small bumps on the surface of the floor and the surface of the crate. When the crate is placed on the surface, these irregularities are pressed together. One possible explanation is that in this process, the irregularities are "glued" together due to adhesive forces between the two materials. This glue acts like small springs, so that when we try to push at the crate, these small springs are extended, exerting forces that counteract the applied force. However, if the applied force on the crate becomes too large, the adhesive forces become so large that the contacts break, and the crate starts to slip.

It is reasonable to assume that the number of irregularities that are in contact, and possibly the contact area of each irregularity, increase with the normal force. This means that we should discern between the apparent area of contact, which is simply the size of the side of crate facing the floor, and the actual area of contact, which is given by the contact areas of all the small irregularities. If the actual contact area is proportional to the normal force, we expect the static friction force also to be proportional to the normal force.

The simple model we have developed now, based only on our intuition, is surprisingly close to the most usual description of static friction. By the word "law" here, we mean a model for the force —it is not a fundamental law of nature, but rather a model that can often, but not always, be applied to contact problems.

Coefficient of static friction: The upper limit of the static friction force is $F_m = \mu_s N$, where N is the normal force for the contact, and μ_s is called the coefficient of static friction. If the friction force exceeds this upper limit, the object will start moving relative to the surface, and our model for static friction is no longer valid.

We see that the concept of static friction is closely related to a constraint on the motion of an object:

The **static friction force** is a tangential force acting on the interface between two solids in contact that are not moving relative to each other. The magnitude and direction of the force is so that the two object do not accelerate relative to each other. However, there is an upper limit on the magnitude of the static friction force

$$F_s < F_{s,max} = \mu_s N, \tag{9.32}$$

where N is the normal force at the contact.

In order to determine the static friction force we need to assume that the objects are not accelerating, that is, we need to assume a constraint on the motion, and use Newton's second law of motion to determine the static friction force. Then, we need to check that the force does not exceed the threshold. If it exceeds the threshold, the objects will start moving relative to each other, and we need another force model—the dynamic friction model.

This law is experimentally justified and it is only an approximation of the real behavior of materials, where many different effects contribute to bring about the effect of a static friction force.

Dynamic Friction

You are gradually increasing how hard you are pushing a crate on the floor. What happens when the crate slips? Our model so far has been for *static* friction. However, you know from your own experience that if you give the crate an initial velocity and release it along the floor, it slows downs and stops (see Fig. 9.7). That means that the crate decelerates. We can therefore use Newton's second law and conclude that because the crate decelerates, there must be a force acting along the floor in the direction opposite the movement, which is the cause of the deceleration. We call this force the dynamic friction force.

There is a surprisingly simple law for the dynamic friction force, $\mathbf{F}_d$, acting on an object in contact with another object:

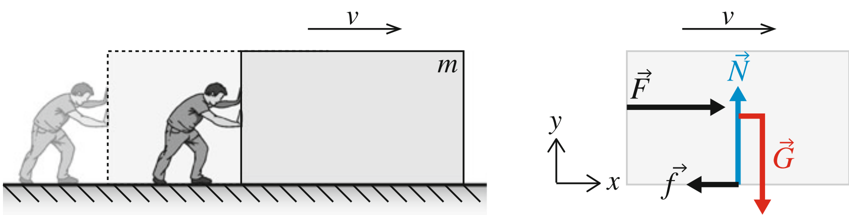

Fig. 9.7 *Left* Illustration of a crate moving along the floor. *Right* A free-body diagram of a crate on the floor. You are pushing the crate with the force **F** directed along the floor

The **dynamic friction force**, F_d, is tangential to the surface of contact. The magnitude of the force is $F_d = \mu_d N$ where N is the normal force across the contact. The direction of the force is opposite the relative motion of the two objects.

This law is empirically based. This means that it is the result of measurements. The law states that the dynamic friction force depends only on the normal force and on the coefficient of friction, which is a property of the two materials in contact. The dynamic friction force does not depend on the area of contact. Neither does it depend on the velocity of the object relative to the floor. This is surprising—we would expect the force to be higher for higher velocities. For high velocities, there will be a velocity dependence, but for most problems related to objects you typically find around you—macroscopic objects—the law holds reasonably well.

9.2.1 Example: Static Friction Forces

Problem: A book is lying on a table tilted an angle $\alpha = 30°$ with the horizontal. The coefficient of static friction between the book and the table is $\mu_s = 2/3$. What is the friction force on the book? At what tilting angle will the book start to slide?

Approach: Does the book move? We start from the assumption that the book is not moving—a special case of constrained motion—and check if the conditions for static friction are fulfilled. If they are not fulfilled, the book starts to slide.

Identify: As shown in Fig. 9.8 we choose the x-axis to be directed along the table so that forces tangential to the surface are along the x-axis and forces normal to the surface are along the y-axis.

Model: The contact forces from the table on the book are the normal force N and the friction force F. In addition, the book is affected by gravity, $G = mg$, as illustrated in the free-body diagram in Fig. 9.8.

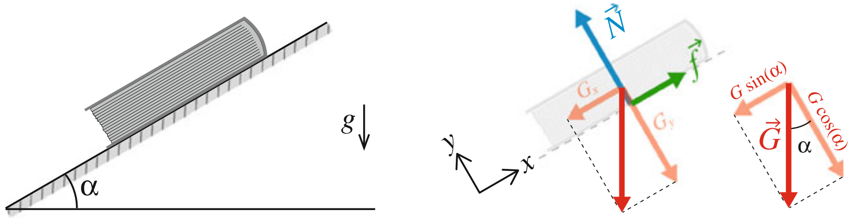

Fig. 9.8 *Left* Illustration of a book lying on an inclined table. *Right* Free-body diagram for the book

We apply Newton's second law in the x- and y-direction, decomposing W along each direction as illustrated in Fig. 9.8. We have assumed no motion in the x-direction:

$$\sum F_x = f - G_x = ma_x = 0 \;\Rightarrow\; f = G_x = G\sin\alpha = mg\sin\alpha. \tag{9.33}$$

And, similarly no motion in the y-direction:

$$\sum F_y = N - G_y = ma_y = 0 \;\Rightarrow\; N = G_y = mg\cos\alpha. \tag{9.34}$$

As long as the book is not moving, F is a result of static friction. But there is a limit on the static friction force: $f < f_{\text{max}} = \mu_s N$. If the static friction force exceeds this limit, the book starts to slide.

First, let us check if the static friction force is exceeded when $\alpha = 30°$ and $\mu_s = 2/3$. In this case $\cos\alpha = \sqrt{3}/2$,

$$N = mg\cos\alpha = \frac{\sqrt{3}}{2}mg, \tag{9.35}$$

and the maximum static friction force is:

$$f_{\text{max}} = \mu_s N = \frac{2}{3}\frac{\sqrt{3}}{2}mg = \frac{\sqrt{3}}{3}mg. \tag{9.36}$$

We found above that when the book is not moving the friction force is:

$$f = mg\sin\alpha = \frac{1}{2}mg, \tag{9.37}$$

which is smaller than the maximum friction force. The book does not slide.

Let us find at what angle the book starts to slide. Sliding occurs when the friction force becomes equal to the maximum friction force. For smaller angles than this, the book does not slide, but for an angle large than this, the book starts sliding. This angle is called the angle of marginal stability, α_m. It is found from

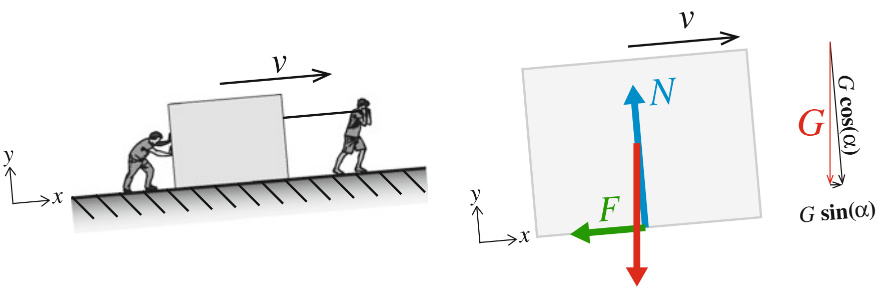

Fig. 9.9 *Left* Illustration of a rock block being pushed up along a hill. *Right* Free-body diagram for the block

$$f = f_{\rm m} \tag{9.38}$$

$$\sin(\alpha_m)\, mg = \mu_s \cos(\alpha_m)\, mg \tag{9.39}$$

$$\frac{\sin(\alpha_m)}{\cos(\alpha_m)} = \mu_s \tag{9.40}$$

$$\alpha_m = \arctan(\mu_s). \tag{9.41}$$

We should check if this result is reasonable by testing the behavior for large and small values of μ_s. When μ_s is small, α_m is also small, which is the correct behavior, When μ_s becomes very large, α_m approaches $\pi/2$, which is also the correct result.

9.2.2 Example: Dynamic Friction of a Block Sliding up a Hill

Problem: In order to build a pyramid, in case you are planning for greatness, you need to push large blocks of rock up an inclined hill. (Unless, of course, you have access to a lift). It has been suggested that the Egyptians made ramps of sand in order to push and pull the rocks up to their place. Let us assume they were able to make a ramp with an inclination of $\alpha = 5°$. What force is needed to push a rock of mass 6000 kg up this hill? A typical coefficient of dynamic friction between rock (sandstone) and sand is $\mu_d = 0.6$.

Sketch and Identify: The problem is illustrated in Fig. 9.9. We have chosen the coordinate system so that x is along the inclined hill.

Model: The rock is in contact with a pushing and/or pulling mechanism: The rock may be pulled by ropes and pushed at the bottom. We call the sum of both of these forces, F, which acts along the slope, in the x-direction. The rock is also in contact with the sand slope. There is a normal force, N, from the slope on the rock, and there is friction force acting along the slope from the slope on the rock. Because the rock is moving relative to the slope, we use the dynamic friction model for the friction force, $f = \mu_d N$, acting in the negative x-direction.

We apply Newton's second law in each direction. The block does not move in the y-direction:

$$\sum F_y = N - G_y = ma_y = 0, \tag{9.42}$$

where we find G_y from the inset in Fig. 9.9, $G_y = G\cos(\alpha)$, which gives

$$N = G_y = G\cos(\theta) = mg\cos(\theta). \tag{9.43}$$

Newton's law of motion in the x-direction gives:

$$\sum F_x = F - f - G_x = ma_x. \tag{9.44}$$

Now, the smallest force exerted in order to move the block up the hill, is the force required to keep the block at constant velocity, that is, the force that makes the acceleration $a_x = 0$:

$$F = f + G_x. \tag{9.45}$$

From the inset in Fig. 9.9 we find that $G_x = G\sin(\alpha) = mg\sin(\alpha)$. We solve by inserting $f = \mu_d N$ into (9.44).

$$\begin{aligned} F &= f + G_x \\ &= \mu_d N + G_x = \mu_d mg\cos(\theta) + mg\sin(\theta) \\ &= mg\left(\mu_d \cos(\theta) + \sin(\theta)\right) \\ &= 6000\,\text{kg} \cdot 9.81\,\text{m/s}^2 \left(0.6 \cdot \cos(5^\circ) + \sin(5^\circ)\right) \\ &\simeq 40.3\,\text{kN}. \end{aligned} \tag{9.46}$$

Test your understanding: The Egyptians knew that sliding a big block of rock on sand required too large forces, because the friction was too large. Therefore, they invented an improved method: They laid down wooden plates in front of the block, and poured water onto the plates. This reduced the effective dynamic coefficient of friction to $\mu_d = 0.2$. What is the force needed in this case?

9.2.3 Example: Oscillations During an Earthquake

The numerical implementation of the models for friction forces may seem straightforward, but sometimes poses unexpected problems. In this exercise you will learn to implement both dynamic and static friction in a numerical model of motion under friction.

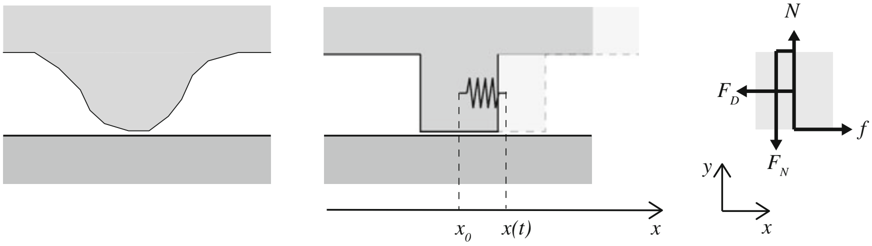

Fig. 9.10 Illustration of a single surface protrusion and its simplified model

Problem: A real, rough surface typically consists of many individual asperities—small protrusions that make up the real contacts along the surface. Here, we will address the motion of one such asperity along a contact surface during an earthquake. The system is sketched in Fig. 9.10. How can we make a simplified model of this system? The asperity will deform and slide as the two surfaces are moving relatively to each other during the earthquake. We propose a very simple model for this interaction: We model the asperity as a block of mass m, attached to the top surface with a spring of stiffness k. The block is pressed down onto the underlying surface with a force F_N. The attachment point x_0 of the spring follows the motion of the top surface. We will assume that the top surface, and therefore x_0, oscillates as $x_0(t) = A \sin \omega t$ during the earthquake. Find the motion of the asperity.

Identify and Sketch: We describe the position of the block by $x(t)$, its position relative to the bottom surface. We assume that the block starts at rest from $x(t) = 0$ at $t = 0$.

Model: The block is affected by contact forces from the top surface: a force F_N and the spring force, F_D, due to the spring attached at point $x_0(t)$. The block is also affected by contact forces from the bottom surface: A normal contact force N and a friction force f acting along the surface. The free-body diagram is shown in Fig. 9.10.

We apply Newton's second law in the direction normal to the surface

$$\sum F_y = N - F_N = 0 \;\Rightarrow\; N = F_N. \tag{9.47}$$

Newton's law of motion along the surface gives:

$$\sum F_x = F_D - f = -k\,(x - x_0(t)) - f = ma. \tag{9.48}$$

However, we now also need a model for the friction force f, but that depends on whether the block is moving. If it is moving, the force model is

$$f = -\text{sign}(v)\,\mu_d F_N, \tag{9.49}$$

where μ_d is the coefficient of dynamic friction. However, if the block is not moving, the force f will correspond to the force F_D, unless it then exceeds the maximum static friction force $\mu_s F_N$.

Initiation of sliding: The block starts from rest and will start moving if the force f exceeds the maximum static friction force, that is, if:

$$f = F_D = -k\,(x - x_0(t)) > \mu_s F_N, \tag{9.50}$$

where $x = 0$ and $x_0(t) = A \sin \omega t$. The maximum value of f occurs when $x_0 = A$, which gives

$$kA > \mu_s F_N. \tag{9.51}$$

The block will therefore only start moving if $A > \mu_s F_N / k$.

Block dynamics: We find the motion of the block by integrating Newton's second law for the horizontal motion:

$$ma = F_D - f = -k\,(x - x_0(t)) - f. \tag{9.52}$$

We solve this equation numerically, using an Euler-Cromer scheme for integration:

$$v(t + \Delta t) = v(t) + a(t)\,\Delta t \tag{9.53}$$

$$x(t + \Delta t) = x(t) + v(t + \Delta t)\,\Delta t, \tag{9.54}$$

However, to implement the effect of the friction force f, we need to consider the *state* of the block. The block may be sliding; still; or transitioning from sliding to still or from still to sliding. If the block is sliding, we use the dynamic friction model. If the block is still, we use the static friction model. The force calculation is implemented as follows:

```
x0 = A*sin(omega*t(i));
FD = -k*(x(i)-x0);
if v(i)==0.0 % Static friction
    f = -FD;
    if abs(f)>mus*N % Slips
        f = -sign(FD)*mud*N;
    end
else         % Dynamic friction
    f = -sign(v(i))*mud*N;
end
Fnet = FD + f;
```

First, we calculate the other forces, F_D, acting on the block. Then we check if the block is sticking, which occurs when the velocity is zero. In this case, the force will be the static friction force, F_s, which is equal to the other forces, $F_s = -F_D$, so that the net force is zero. However, this is only true if the static friction force is less than the friction threshold. If the force exceeds the friction threshold, the block will start moving in a direction given by the force F_D, and the block will experience a dynamic force acting in a direction opposite F_D and with a magnitude given by the dynamic friction force, $f = -\text{sign}(F_D)\mu_d N$.

Changing the frictional state: Notice that this method checks if the velocity is exactly equal to zero, but this will never occur during a simulation. During an integration step the velocity will move from being positive to being negative without ever being precisely zero. However, when the velocity of the block changes sign, the block will actually stop and start sticking to the surface with a static friction force. We therefore set the velocity to be exactly zero when the velocity changes sign. (Alternatively, you could have introduced a state variable, telling if the block is in a sliding or a sticking state). This check needs to be done during integration:

```
v(i+1) = v(i) + a*dt;
if (v(i)~=0.0)&&(sign(v(i+1))~=sign(v(i)))
    v(i+1) = 0.0; % The block has stopped
end
```

We now have all the components needed for a program to solve the motion of the block:

```
m = 2e-12;      % kg
A = 1e-5;    % m
k = 10.0;  % N/m
N = 1e-4;  % N
T = 0.01; % s
omega = 2*pi/T;
time = 2*T;
dt = time/100000;
n = ceil(time/dt);
mud = 0.2;
mus = 0.4;
t = zeros(n,1);
x = zeros(n,1);
v = zeros(n,1);
f  = zeros(n,1);
x(1) = 0.0;
v(1) = 0.0;
for i=1:n-1
    x0 = A*sin(omega*t(i));
    FD = -k*(x(i)-x0);
    if v(i)==0.0 % Static friction
        f(i) = -FD;
        if abs(f(i))>mus*N % Slips
            f(i) = -sign(FD)*mud*N;
        end
    else         % Dynamic friction
        f(i) = -sign(v(i))*mud*N;
    end
    Fnet = FD + f(i);
    a = Fnet/m;
    v(i+1) = v(i) + a*dt;
    if (v(i)~=0.0)&&(sign(v(i+1))~=sign(v(i)))
        v(i+1) = 0.0; % The block has stopped
    end
    x(i+1) = x(i) + v(i+1)*dt;
    t(i+1) = t(i) + dt;
end
```

(Here, we have introduced parameters corresponding to a cubic asperity of of size 10 μm and density 2.0 g/cm^3.) The resulting dynamics is shown in Fig. 9.11.

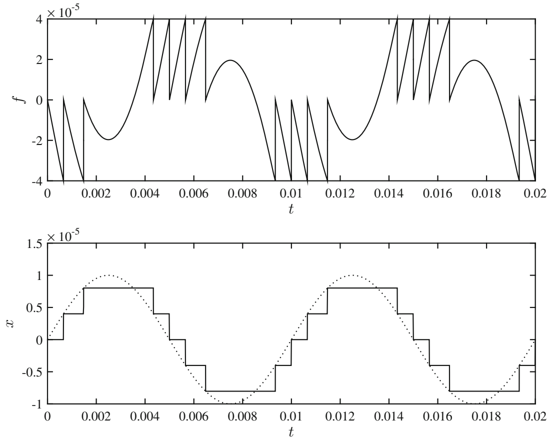

Fig. 9.11 **a** Plot of the friction force f. **b** Plot of the position x of the asperity (*solid*) and of the top surface, x_0 (*dashed*)

Notice that the block moves in slips: It follows the motion of the driving surface, but sticks and slips as it moves along. We call this stick-slip motion. You should try to change the parameters in the program to see if you can find other interesting regimes of behavior.

9.3 Circular Motion

How large forces acts on a roller coaster cart as it passes through a vertical loop, or on a car as it drives through a shape turn on the road? These are questions we can answer without having a specific model for the forces acting, because the motion is constrained: A roller coaster cart driving around a loop is constrained to follow the loop, since the cart is attached to the rail.

We have already seen how we can use constraints to find forces for linear motion. Fortunately, we can use exactly the same technique to determine the forces acting on an object following a curve. However, for curved motion we know that the acceleration also has a component normal to the curve: The centripetal acceleration. For curved motion we must therefore remember that even though the object follows a particular path, there must be a net force acting on the object in order for it to follow the path.

Linear and Curved Motion

We can illustrate the difference between linear and curved motion by looking at a bead moving along a wire, as illustrated in Fig. 9.12.

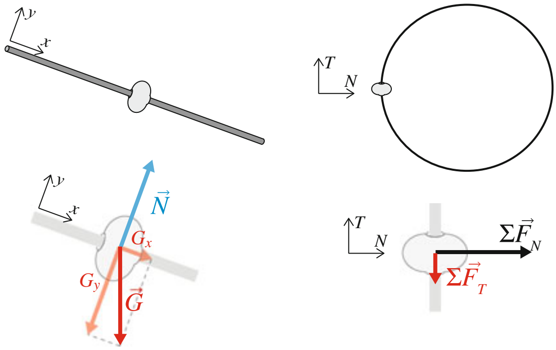

Fig. 9.12 *Left* Illustration of a bead moving along a straight (*linear*) wire (*left*) and a circular (*curved*) wire (*right*)

If the wire is shaped as a line, the bead can be accelerated along the wire but not normal to the wire. Hence, if we analyze the forces acting on the bead and apply Newton's second law, we recognize that the net force in the x-direction along the wire corresponds to the acceleration along the wire:

$$\sum F_x = ma_x. \tag{9.55}$$

But in the direction normal to the wire (the y-direction), there is no motion and no acceleration. Therefore the net force is zero:

$$\sum F_y = ma_y = 0. \tag{9.56}$$

If the wire is shaped as a circle, the analysis is different. Let us describe the motion of the bead using a coordinate system with one axis along the curve, the tangential axis, $\hat{u}_T$; and one axis normal to the curve pointing in towards the center of the circle, the normal axis, $\hat{u}_N$. Again, we analyze the forces acting on the bead and apply Newton's second law. The net force in the tangential direction is causing the tangential acceleration:

$$\sum F_T = ma_T, \tag{9.57}$$

But in this case, the motion is not linear. Therefore, even though the bead follows the wire, it is accelerated also in the direction normal to the wire because the wire is curved. The net force in the normal direction is therefore related to the centripetal acceleration:

$$\sum F_N = ma_N = m\frac{v^2}{R}. \tag{9.58}$$

This important difference between linear and curved motion is at the center of our analysis of curved constrained motion.

Structured Problem-Solving Approach

The method we apply when analyzing motion along a curved track consists of the following modification to the **Model** step in the structured problem-solving method:

- Find the forces acting on the object.
- Introduce models for the forces.
- In the normal direction the net force must produce the centripetal acceleration: $a_N = v^2/R$
- In the tangential direction the tangential acceleration is determined from the net tangential force.

You may then **Solve** to find the motion of the object and **Analyze** the solution to answer questions about the motion.

Types of Constraints

We can classify the constraints into two main types.

Non-conditional constraint: The constraint experienced by a bead moving along a wire. The bead follows the wire independently of the speed of the motion or any other property of the motion. Other similar cases are the motion of a weight attached to a rigid staff; the motion of a roller coaster car attached to a track; the motion of a person attached to a seat; or the motion of a train along a railway track. In the cases of a non-conditional constraint we can determine the forces and motion using the approach presented above.

Conditional constraint: The constraint experienced by a ball swung in a rope: As long as the rope is tight, the ball follows a circular path, because the rope pulls on the ball when stretched. But the rope does not exert any force when pushed. This means that the constraint in conditional: The ball is constrained to follow a circle only as long as the tensile force needed to make it follow a circle is positive. If the tensile force needed is negative, the rope cannot push the ball, and the constraint is no longer present. In this case the ball is only affected by gravity and air resistance until the rope is again tight. If you want to determine the motion of the ball when the rope is not tight, you apply Newton's laws of motion to find the acceleration and solve to find the position and velocity of the ball using methods you are now proficient in.

Examples of Constrained Systems

A car on a hill-top: Usually we assume that a car driving along a bumpy road follows the vertical motion of the road. This is really a constraint on the motion: We assume that the car follows the path given by the road. If the car drives along a flat part of the road, this gives us a way to estimate the normal force. However, the normal force can only be positive: The car is not glued to the surface. The constraint on the motion of the car is therefore conditional: It can only follow the shape of the surface if this requires a positive normal force. If a negative normal force is required in order to follow the surface, the car looses contact with the surface.

A car driving through a curve: Usually, we assume that a car driving around a turn does not slide. The car therefore follows a specific track—the track given by the road. We can therefore analyze the motion of the car around a turn as if it was constrained—following a given curved track. This requires a net force on the car to give it the necessary centripetal acceleration. An important component in the net force is the friction force from the ground on the tires. But the friction force is limited by the maximum static friction force. Hence, the motion of the car is constrained to follow the curve, but only as long as the friction force is not exceeding its maximum.

9.3.1 Example: A Car Driving Through a Curve

Problem: A car is driving through a circular curve at constant speed. The coefficient of static friction between the tires of the car and the ground is μ_s. The speed of the car is v and the radius of the circle is R. How fast can the car drive before slipping?

Approach: This is an example of a conditional constrained problem: The car is constrained to follow a circular path, but only as long as the static friction force needed to give the car the required centripetal acceleration does not exceed the maximum frictional force.

Identify: In this problem we address the motion of a car driving through a circular curve. We address the motion of the car as it passes the x-axis, as illustrated in Fig. 9.13. The motion of the car is in the xy-plane.

Model: The car is in contact with the road, giving rise to several contact forces: a normal force in the vertical direction, N, and a frictional force, f, from the ground on the car. The friction force has a component f_y along the road and a component f_x normal to the road (in the x-direction). In addition, the car is affected by the gravity, $G = mg$ in the z-direction. The free-body diagram of the car is shown in Fig. 9.13b.

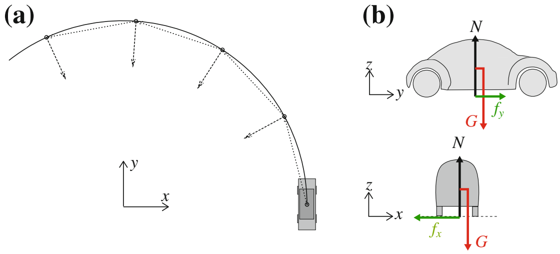

Fig. 9.13 **a** A car driving around a circular track. **b** Free-body diagram of the car

We relate the forces acting on the car to its acceleration by applying Newton's second law. The car is not moving in the vertical (z) direction, hence the acceleration a_z is zero:

$$\sum F_z = N - G = N - mg = ma_z = 0, \tag{9.59}$$

The normal force from the ground on the car is therefore $N = mg$.

Because the car drives at constant speed, the acceleration in the tangential direction along the road (the y-direction) is zero:

$$\sum F_y = f_y = ma_y = 0, \tag{9.60}$$

Therefore the y-component of the friction force is zero.

Since the car follows a circular path, the car is accelerated in the normal direction, in toward the center of the circle. This corresponds to the $-x$ direction in Fig. 9.13. Since the car is driving with a speed v, the normal acceleration corresponds to the centripetal acceleration of an object moving in a circle of radius R:

$$\sum F_x = f_x = ma_x = m\left(-\frac{v^2}{R}\right), \tag{9.61}$$

The friction force from the road on the car is therefore:

$$f = -m\frac{v^2}{R}. \tag{9.62}$$

However, we know that the maximum absolute value of the static friction force is

$$f_{\max} = \mu_s N, \tag{9.63}$$

(Notice that we use the static friction force model here because the car is not slipping). If the friction force f in (9.62) required to make the car follow the circular path exceeds the maximum static friction force, it means that our initial assumption that the motion is constrained fails and the car cannot follow a circular path. The car starts sliding at the velocity v_m when the static friction force f in (9.62) is equal to the maximum static friction force:

$$\left|-m\frac{v_m^2}{R}\right| = \mu_s N \;\Rightarrow\; v_m^2 = \mu_s gR \;\Rightarrow\; v_m = \sqrt{\mu_s gR}. \tag{9.64}$$

This means that as long as the friction force is smaller than $f_{\max}$ the car is able to follow the circular path. But if the friction force exceeds the static friction force, the car can no longer follow the circular path. Instead, it starts sliding and follows a non-circular path, which can also be found from Newton's second law. Sliding starts at a particular speed, v_m. For speeds below this, the car is able to follow the curve.

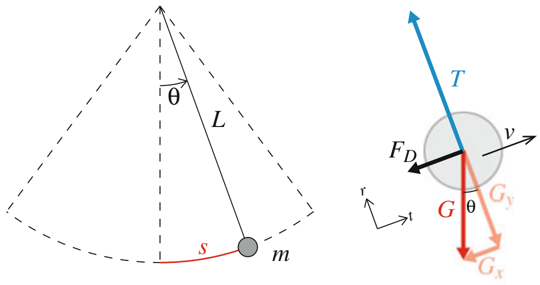

Fig. 9.14 *Left* Illustration of a pendulum. *Right* Free-body diagram of the pendulum ball

9.3.2 Example: Pendulum with Air Resistance

In this example we will address the motion of a pendulum with air resistance. The pendulum is a classic problem in mechanics, which can be solved analytically, but with the added complication of air resistance, we need to resort to numerical methods.

Problem: The pendulum consists of a ball of mass m attached to a massless rope of length L. The other end of the rope is attached to the ceiling as illustrated in Fig. 9.14. The ball starts at the bottom-most position with an initial velocity v_0. Find the motion of the pendulum with and without air resistance and discuss the differences. What initial velocity is needed for the ball to make a complete circle?

Approach: First, we will find the motion of the ball, assuming that the ball follows a circular pat, using Newton's second law. We will find the motion without air resistance analytically, and use this analytical solution to test a more general numerical method. The numerical model can be used to find the initial velocity needed to make a complete circle.

Identify: Our basic assumption is that the ball is constrained to follow a circular path. The position of the ball is measured by the distance $s(t)$ measured along the circle, starting at zero at the bottom of the circle. However, we will need to check the validity of this constraint to check that the conditions for the ball following the circular path are satisfied at all times. We can also describe $s(t)$ by the angle, $\theta(t)$, formed with the vertical as illustrated in Fig. 9.14. The arc length, $s(t)$, measured along the circle is relate to the angle, $\theta(t)$ measured in radians, by $s(t) = L\theta(t)$.

Model: The ball is in contact with the rope and the surrounding air, giving rise to the force from the rope on the ball, the rope tension, $\mathbf{T}$, and the air drag, $\mathbf{F}_D$. In addition, the ball is subject to the force from gravity, $\mathbf{G}$.

The forces are illustrated in Fig. 9.14 when the ball is at a position $s(t)$ corresponding to an angle $\theta(t)$. We decompose the forces using a local coordinate system with unit vectors $\hat{u}_T$ and $\hat{u}_N$ tangential and normal to the path respectively.

Trigonometrical decomposition: We may decompose the forces in two ways. Either we may examine the geometry in Fig. 9.14 and see that $G_x = G \sin\theta$ and $G_y = G \cos\theta$, and therefore

$$\mathbf{G} = -mg\sin\theta\,\hat{u}_T - mg\cos\theta\,\hat{u}_N. \tag{9.65}$$

Unit vector decomposition: Alternatively, we can find $\hat{u}_T$ and $\hat{u}_N$ expressed using θ, and use these to decompose the vectors using the dot product. From Fig. 9.14 we see $\hat{u}_N$, is directed towards the center of the circle, and that the tangential vector is normal to the normal vector:

$$\hat{u}_N = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j},\ \ \hat{u}_T = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}. \tag{9.66}$$

From these expressions and $\mathbf{G} = -mg\,\mathbf{j}$ we find

$$G_T = \mathbf{G}\cdot\hat{u}_T = -mg\sin\theta,\ \ G_N = \mathbf{G}\cdot\hat{u}_N = -mg\cos\theta, \tag{9.67}$$

just as we found above. This method is more mathematical, less intuitive, but more robust—it always works if you are able to write down the unit vectors.

Air resistance model: We assume that we may use a quadratic law for the air resistance:

$$\mathbf{F}_D = -Dv\,\mathbf{v} = -D|v(t)|v(t)\hat{u}_T, \tag{9.68}$$

where we have used that $\mathbf{v} = v(t)\hat{u}_T$. Notice that $\hat{u}_T$ points in the positive rotational direction and that $v(t)$ may be positive and negative depending on the direction of motion.

String tension: The string tension acts in the normal direction: $\mathbf{T} = T\hat{u}_N$.

Newton's second law: Newton's second law for the ball gives:

$$\sum\mathbf{F} = \mathbf{T} + \mathbf{F}_D + \mathbf{G} = m\,\mathbf{a}. \tag{9.69}$$

We decompose the acceleration in a tangential and a normal component (from Chap. 8):

$$\mathbf{a} = \frac{dv}{dt}\hat{u}_T + \frac{v^2}{\rho}\hat{u}_N, \tag{9.70}$$

where $\rho = L$ is the radius of the circle. The net force on component form is:

$$\sum\mathbf{F} = T\hat{u}_N - Dv|v|\hat{u}_T - mg\sin\theta\,\hat{u}_T - mg\cos\theta\,\hat{u}_N = \frac{dv}{dt}\hat{u}_T + \frac{v^2}{\rho}\hat{u}_N. \tag{9.71}$$

The normal component of this equation is

$$T - mg\cos\theta = m\left(v^2/L\right) \ \Rightarrow\ T = mg\cos\theta + m\left(v^2/L\right). \tag{9.72}$$

This allows us to calculate T given the velocity $v(t)$ and the angle $\theta(t)$. The condition for circular motion is that T is positive, since the rope can only pull and not push at the ball. If $T < 0$ our assumptions break down and the ball no longer follows a circular path.
In the tangential direction we get:

$$m\frac{dv}{dt} = -Dv|v| - mg\sin\theta = -Dv|v| - mg\sin\frac{s(t)}{L}, \tag{9.73}$$

where we have inserted $\theta = s(t)/L$. This gives us an equation of motion, just like we have found before, that we must now solve.

Solve: We want to find the position $s(t)$ as a function of time from (9.73). The initial conditions for the ball is that at the time $t_0 = 0\,\mathrm{s}$ the ball is at $s(t_0) = 0\,\mathrm{m}$, and the initial velocity is $v(t_0) = v_0$.

Analytical solution: This problem can be solved when there is no air drag ($D = 0$) and $s(t) \ll L$. We can then approximate $\sin(s/L) \simeq (s/L)$,[3] giving the equation of motion

$$\frac{d^2s}{dt^2} = -g\sin\frac{s(t)}{L} \simeq -\frac{g}{L}s, \tag{9.74}$$

which we recognize as the equation for harmonic motion, which we solved in Sect. 5.7. The general solution is

$$s(t) = A\cos\omega t + B\sin\omega t, \tag{9.75}$$

which is confined by the initial condition $s(0) = 0\,\mathrm{m/s}$, which gives that $A = 0$, and $v(0) = B\omega = v_0$, which gives $B = v_0/\omega$ The solution is therefore

$$s(t) = \frac{v_0}{\omega}\sin\omega t, \tag{9.76}$$

where we find ω by inserting $s(t)$ from (9.76) into (9.74), giving $\omega^2 = g/L$. The solution describes a periodic motion with period $T = 2\pi/\omega = 2\pi(L/g)^{1/2}$.

For small deviations and without air resistance, the pendulum will therefore oscillate back-and-forth with a period given by gravity and the length of the pendulum: A well know result from mechanics. We can use this simplified result to test our more general solution to the problem found below.

Numerical solution: The solution for small values of s is not relevant if we want to understand how far the ball can be swung. Instead, we must solve the equation of motion numerically. The differential equation of motion is:

[3] The result $\sin u \simeq u$ when $u \ll 1$ is a result of a first order Taylor expansion.

$$\frac{d^2 s}{dt^2} = -\frac{D}{m} v(t)|v(t)| - g \sin \frac{s(t)}{L}. \tag{9.77}$$

with initial conditions $s(0) = 0$ and $v(0) = v_0$. In addition, we need to check that the rope tension remains positive, since our solution breaks down if this is not true. Using T from (9.72) the condition for rope tension is $T(t) = mv^2/L + mg\cos(s/L) \geq 0$.

Let us now address a specific pendulum with a ball of mass 0.5 kg, the rope has a length of $L = 1$ m, and the drag coefficient is $C_D = 0.04$ kg/m, and $v_0 = 2$ m/s. We solve the equations of motion numerically using Euler-Cromer's method in the following program:

```
g = 9.8;      % m/s^2
dt = 0.01;    % s
time = 10.0;  % s
v0 = 2.0;     % m/s
D = 0.05;     %
L = 1.0;      % m
m = 0.5;      % kg
n = ceil(time/dt);
t = zeros(n,1);
s = zeros(n,1);
v = zeros(n,1);
T = zeros(n,1);
v(1) = v0;
s(1) = 0.0;
i = 1;
while (i<n&&T(i)>=0.0)
    t(i+1) = t(i) + dt;
    a = -D/m*v(i)*abs(v(i)) - g*sin(s(i)/L);
    v(i+1) = v(i) + a*dt;
    s(i+1) = s(i) + v(i+1)*dt;
    T(i+1) = m*v(i+1)^2/L + m*g*cos(s(i+1)/L);
    i = i + 1;
end
```

Notice the simple visualization trick using plot to draw the pendulum as a line with the following program:

```
for j = 1:i
    x = [0.0 sin(s(j)/L)];
    y = [0.0 -cos(s(j)/L)];
    plot(x,y,'-o'); axis equal;
    axis([-1 1 -1 1]); drawnow;
end
```

Rope tension T is calculated at every time step, and the simulation is stopped if the tension becomes negative. This does not mean that the motion of the pendulum stops if tension becomes negative, but the motion is no longer described by the equation of motion in (9.77). Instead the ball will fall until the rope again becomes tight, and the ball bounces about in the rope until it comes to rest or starts swinging back and forth.

Results: The result of the simulation is shown in Fig. 9.15. The pendulum oscillates with a relatively small deflection, corresponding to small values of s. The damping due to air resistance is clearly evident. However, the damping becomes less dominating for smaller velocities, because the damping term depends on square of the

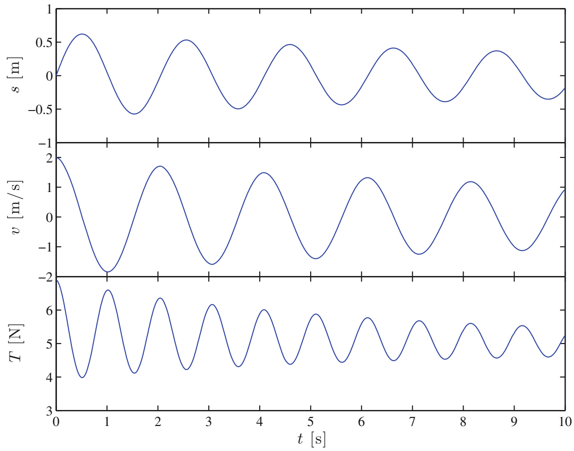

Fig. 9.15 Plot of the position, s, velocity, v, and rope tension, T for the damped pendulum with $v_0 = 2.0\,\text{m/s}$

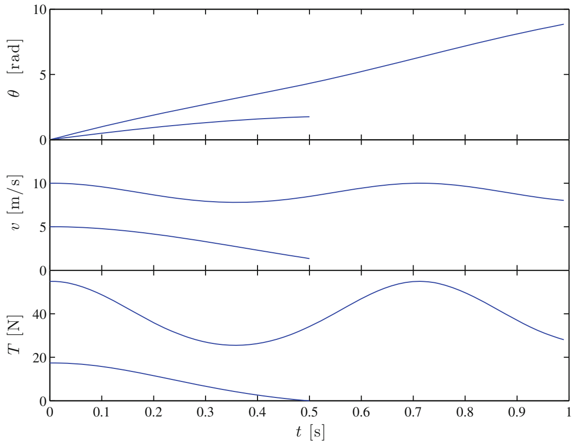

Fig. 9.16 Plot of the angle, θ, velocity, v, and rope tension, T for the undamped pendulum with $v_0 = 5.0\,\text{m/s}$ and for $v_0 = 10.0\,\text{m/s}$

velocity, $F_D \propto v^2$. If we turn air drag off completely, we can compare the numerical solution to the analytical solution in order to test our numerical model.

Discussion: What happens if we increase the initial velocity, v_0? We expect the pendulum to continue all the way around the circle. However, we need to ensure that the rope tension, T, remains positive.

First, we turn air resistance off. We give the pendulum an initial angular velocity, and see if it goes all the way round. We start from $v_0 = 5\,\text{m/s}$. Here, the pendulum ball falls down at $s = 1.76\,\text{m}$, which corresponds to $\theta = 1.76\,\text{m}/1.0\,\text{m} \simeq 0.54\pi$. Increasing v_0 to $10\,\text{m/s}$ allows the pendulum to go all the way round without falling down, and the pendulum continues to rotate in the vertical plane. The two paths are illustrated in Fig. 9.16.

We may now use our program to find the maximum angle, θ^*, reached by the pendulum—that is the angle at which the pendulum falls down—as a function of the initial velocity: $\theta^*(v_0)$. When there is no air drag, we can solve this problem analytically using energy conservation methods, but with air drag, there is no simple analytical solution, and our numerical study would be the only practical solution.

Test your understanding: What happens when $\theta > 2\pi$ in Fig. 9.16?

Summary

Linearly constrained motion:

- An object is **linearly constrained** if it is forced to follow a straight line.
- Choose the T-axis along the motion, and N-axis normal to the motion.
- Decompose forces in the T and N directions.
- Set $a_N = 0$. Use Newton's second law to find the forces in the N-direction.
- Apply Newton's second law to find acceleration in the T-direction.

Friction force model:

- Friction forces are tangential forces at the contact between two objects.
- The friction force may depend on the normal forces at the contact.
- If the two objects in contact **are not moving** relative to each other, the friction force f is limited by $f_m = \mu_s N$, where μ_s is the coefficient of static friction.
- If the two objects in contact **are moving** relative to each other, the friction force f is $f = \mu_d N$, where μ_d is the coefficient of dynamic friction. The friction force acts against the relative movement of the objects.

Curved motion:

- Objects constrained to follow a curved path has an acceleration normal to the path given by the centripetal acceleration $a_N = v^2/R$.
- The **net force in the direction normal to the curve** is therefore always non-zero for curved motion!

Exercises

Discussion Questions

9.1 Walking on ice. Walking on ice is usually more tiring than walking on a dry road. Why?

9.2 High g. You want to perform experiments under higher g than at the surface of the Earth. How could you design a setup where you may vary g?

9.3 Spaceship loop. You want your spaceship to make a loop. How would you direct your thrusters?

9.4 Spaceship helix. You want your spaceship to make a helix-like motion (spinning around). How would you direct your thrusters?

9.5 Driving a pendulum. You hold an improvised pendulum in your hand: a rope tied to a small weight. How do you need to move your hand in order to keep the pendulum moving with an approximately constant maximum angle?

Problems

9.6 Rope with finite mass. A homogeneous rope with mass m hangs between two equally high poles. The angle between the rope and the horizontal at each of the attachment points is α.
(a) Find the tension at each end of the rope.
(b) Find the tension at the lowest point of the rope.
(c) Is it possible to tighten the rope so much that $\alpha = 0$?

9.7 Fireman on pole. A fireman of mass m is in hurry, and jumps onto a vertical pole leading into the garage. He holds the pole so tight that he slides downwards with constant velocity.
(a) Draw a free-body diagram of the fireman on the pole.
(b) Determine the friction force on the fireman.
(c) The dynamic coefficient of friction between the fireman and the pole is μ_d. Find the normal force from the fireman on the pole.

9.8 Pulling a box. You pull a box of mass m along the floor using a rope attached to a top corner of the box. The dynamic coefficient of friction between the box and the floor is μ. The rope makes an angle α with the horizontal. You pull at the rope with a force of magnitude T.
(a) Draw a free-body diagram of the box.
(b) Find the normal force from the floor on the box as a function of α and T.
(c) Find the acceleration of the box as a function of α and T.
(d) For what value of α is the acceleration of the box the maximum?

9.9 Hanging rope. A homogeneous rope of length L and mass m is lying on top of a table. A length x of the rope is hanging over the edge. The coefficient of static friction between the rope and the table is μ. How large part of the rope can hang over the edge before the rope starts to slide? (You may use the following questions as hints.)
(a) Divide the system into two parts: The part of the rope on the table and the part of the rope hanging over the table. Draw a free-body diagram for each of the parts.
(b) Find the rope tension, T, acting from one part of the rope on the other.
(c) Find the normal force on the part of the rope on the table.
(d) Find the maximum friction force, and find how large part of the rope hangs over the edge when the friction force is equal to the maximum friction force.

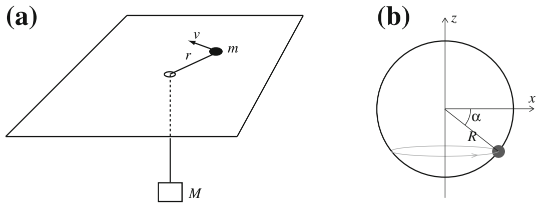

Fig. 9.17 **a** Rope through a hole. **b** Bead on a wire

9.10 Pulling out a book. Two books are lying on top of each other on a table. The upper book has a mass m_1, and the lower book has a mass m_2. The coefficient of static friction between the books is μ_1. The coefficient of static friction between the book and the table is μ_2 and the coefficient of dynamic friction between the book and the table is μ_d. You pull on the lower book with a horizontal force F.
(a) How large must F be for you to start pulling both books along the table.
(b) How large must F be for you to pull out only the lower book?

9.11 Forces on a 200 m runner. A 200 m runner with a mass of 70 kg has an approximately constant speed of 10 m/s through the first curve. The radius of curvature is $R = 25$ m.
(a) Find the friction force f on the sprinter through the curve.
(b) If we assume that the sprinter is not slipping, how large must the coefficient of static friction be for the sprinter to make the turn?

9.12 Rope through a hole. A weight of mass M is hanging from one end of a massless rope. The rope passes through a small hole in the flat table. A block is attached to the other end of the rope. The block is sliding without friction on the table. The length of the rope from the hole to the block is R, and the mass of the block is m, as illustrated in Fig. 9.17a. For a particular velocity v the weight remains at a constant height. Find this velocity.

9.13 Bead on a wire. A circular bead of mass m is moving freely (without friction) along a circular wire of radius R. The wire is in an vertical plane, and the vertical plane is rotating around a vertical axis through the center of the circle with a constant period T, as illustrated in Fig. 9.17b. Find the angular position, α, of the bead on the wire.

9.14 Man in a wheel. You are trying out a new theme park attraction. You walk into a vertical cylinder of radius $R = 3$ m. You are all told to lean your backs towards the round inner walls of the cylinder. The cylinder starts spinning, slowly picking up speed. Suddenly, the floor drops down, but you do not fall down!

The coefficient of static friction between you and the wall is $\mu_s = 0.2$. How fast must the cylinder spin to ensure that you do not fall down?

9.15 Motorcycle in a loop. As a stunt motorcyclist you are trying to run through a vertical loop with radius 3 m. What speed do you need to have at the top of the loop in order not to fall down?

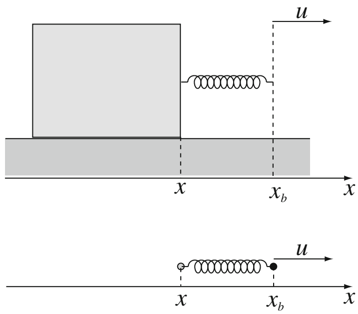

Fig. 9.18 Illustration of a block pulled by a spring

Projects

9.16 Stick-slip friction. In this project we will study a phenomenon called stick-slip friction. If you pull a block along a flat table with a soft spring, you will find that the block does not move continuously with a constant velocity, instead it moves in small jumps. This intermittent motion is called stick-slip friction, and it is the origin of the high-frequency vibrating tone you often hear from wheels that are not well lubricated. It is also one of the basic mechanisms leading to the wide distribution of earthquake sizes.

Here, we will introduce and study a model for stick-slip friction for a block pulled by a spring sliding over a flat, horizontal surface, as illustrated in Fig. 9.18.

The block has mass m. A massless spring (with spring constant k and equilibrium length b) is attached to the block at the point x. The free end (the right-hand end in Fig. 9.18) of the spring is at the point x_b. We move x_b, the free end of the spring, with a constant velocity u. The static and dynamic coefficients of friction for the contact between the block and the bottom surface are μ_s and μ_d respectively. The acceleration of gravity is $g = 9.8\,\text{m/s}^2$.

The block starts at the position $x(t_0) = 0$ at the time $t_0 = 0$. The position x_b of the free end of the spring is $x_b(t_0) = x(t_0) + b$ at t_0.

(a) Draw a free-body diagram for the block.
(b) Find the position of the spring attachment point $x_b(t)$ as a function of time.
(c) Show that the force, **F** on the block from the spring is $\mathbf{F} = k(x_b - x - b)\mathbf{i}$.
Stationary state: First, let us characterize the stationary state, where the block is moving at a *constant* velocity.
(d) Identify the forces acting on the block and draw a free-body diagram for the block in the stationary state.
(e) Introduce force models for all the forces acting on the block. Find the normal force, N, on the block.
(f) Find the acceleration of the block in the stationary state.
(g) Find the elongation ΔL of the spring in the stationary state.
(h) Find the position $x(t)$ of the block as a function of time in the stationary state.

Starting from rest: Let us now address the situation where the block starts from rest. That is, we assume that the block starts at $x(t_0) = 0$ m with $v(t_0) = 0$ m/s at the time $t_0 = 0$ s.
(i) Identify the force acting on the block and draw a free-body diagram of the block before the block starts moving. Introduce force models for all the forces.
(j) Find the elongation ΔL of the spring at the instant the block starts moving.
(k) Assume that the block starts at rest. Find the friction force on the block as a function of time in the period before the block starts moving. Sketch the friction force as a function of time until some time after the block has started moving.
(l) Show that the acceleration of the block immediately after it starts moving is $a = (k/m)(x_b - x - b) - \mu_d g$. Explain why you cannot use this relation for the acceleration to determine the subsequent motion of the block.
General motion: Now, we will develop a *general method* to find the motion, $x(t)$, of the block. First, we study the case when $u = 0$ m/s and the coefficients of friction are zero, $\mu_s = \mu_d = 0$.
(m) Find an expression for the horizontal acceleration of the block. Show that $x(t) = (v_0/\omega)\sin \omega t$, where $\omega = (k/m)^{1/2}$, when $v(0) = v_0$.
(n) Write a numerical algorithm to find the position and velocity of the block at a time $t_i + \Delta t$, $x(t_i + \Delta t)$ and $v(t_i + \Delta t)$, given the position and velocity of the block at a time t_i, $x(t_i)$ and $v(t_i)$.
(o) Implement the numerical algorithm in a program to find the position of the block as a function of time for $m = 0.1$ kg, $k = 100$ N/m, $b = 0.1$ m and $v_0 = 0.1$ m/s. Plot the behavior for a simulation of 2 s, and compare the result of your program with exact solution. Ensure that you choose a time-step Δt the reproduces the exact solution with sufficient accuracy. What happens if you choose a too large time-step Δt?

Let us now address the situation when the block is pulled at a finite velocity, u.
(p) Modify your program to find the position of the block when $u = 0.1$ m/s and the block starts at rest. In this case, the exact solution is:

$$x(t) = ut - \frac{u}{\omega}\sin \omega t. \tag{9.78}$$

Compare your result with the exact solution by plotting both the simulated x and the exact x in the same plot.
General motion with friction: Finally, we address the full complexity of the situation, and introduce non-zero friction forces.
(q) Modify your program to include friction using $\mu_s = 0.6$, $\mu_d = 0.3$. Show a plot of $x(t)$ for $m = 0.1$ kg and for $m = 1.0$ kg.
(r) Plot the spring force F on the block as a function of time for both values of m and explain the differences.
(s) What happens if you instead decrease k to $k = 10$ N for $m = 0.1$ kg. Can you explain the behavior?

9.17 Feather in tornado. In this project you will learn to use Newton's laws and the force model for air resistance in a wind field to address the motion of a light object in strong winds. We start from a simple model without wind and gradually add complexity to the model, until we finally address the motion in a tornado.

Motion without wind: First, we address the motion of the feather without wind.

(a) Identify the forces acting on a feather while it is falling and draw a free-body diagram for the feather.

(b) Introduce quantitative force models for the forces, and find an expression for the acceleration of the feather. You may assume a quadratic law for air resistance.

(c) If you release the feather from rest, its velocity will tend asymptotically toward the terminal velocity, v_T. Show that the terminal velocity is $v_T = -(mg/D)^{1/2}$, where D is the constant in the air resistance model.

(d) We release the feather from a distance h above the floor and measure the time t until the feather hits the floor. You may assume that the feather falls with a constant velocity equal to the terminal velocity. Show how you can determine D/mg by measuring the time t. Estimate D/mg when you release the feather from a height of 2.4 m above the floor and it takes 4.8 s until it hits the floor.

(e) We will now develop a more precise model where we do not assume that the velocity is constant. You release the feather from the height h at the time $t = 0$ s. Find the equation you have to solve to find the position of the feather as a function of time. What are the initial conditions?

(f) Write a program that solves this equation to find the velocity and position as a function of time t. Use the parameters you determined above, and test the program by ensuring that it produces the correct terminal velocity.

(g) Figure 9.19 shows the position and velocity calculated with the program using the parameters found above. Was the approximation in part (d) reasonable? Explain your answer.

Model with wind: We have now found a model that can be used to find the motion of the feather. We will now find the motion of the feather in three dimensions while it is blowing. The velocity of the wind varies in space, so that the wind velocity $\mathbf{w}$ is a function of the position $\mathbf{r}$. We write this as $\mathbf{w} = \mathbf{w}(\mathbf{r})$.

(h) Find an expression for the acceleration of the feather. The expression may include the wind velocity $\mathbf{w}(\mathbf{r})$. Let the z-axis correspond to the vertical direction.

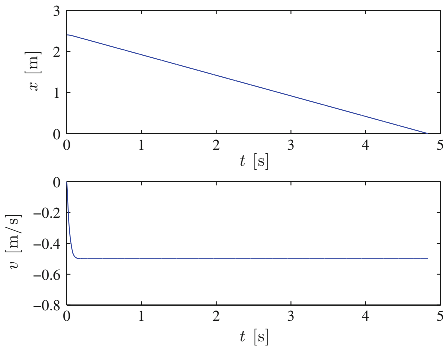

Fig. 9.19 Result of a simulation

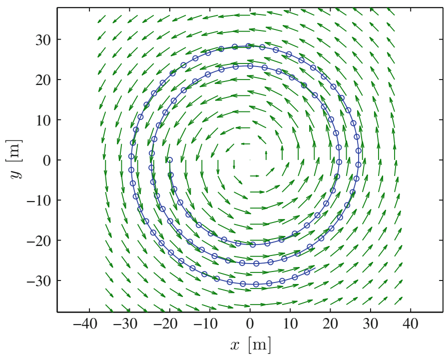

Fig. 9.20 Illustration of the velocity field of the tornado and a path for a feather released in a tornado

(i) Assume that the feather is moving in an approximately horizontal plane—that is you may assume that the vertical acceleration is negligible. How does the wind have to blow in order for the feather to move in a circular orbit of radius r_0 with a constant speed v_0?
Motion in a tornado: For a tornado with a center at the origin, the wind velocity is expected to be approximately given by the model:

$$\mathbf{w}(\mathbf{r}) = u_0 r e^{-r/R} \hat{u}_\theta = u_0 \left(-y, x, 0\right) e^{-r/R}, \tag{9.79}$$

where u_0 is a characteristic velocity for the wind, R is the radius of the tornado, and $\hat{u}_\theta$ is a tangential unit vector in the horizontal plane ($\hat{u}_\theta$ is normal to $\mathbf{r}$). Here, $\mathbf{r} = (x, y, z)$, and $r = |\mathbf{r}|$. The velocity field is illustrated in Fig. 9.20.
(j) Is is possible to choose an appropriate set of initial conditions so that the feather moves in a circular path in the tornado? Explain your answer.
(k) Rewrite your program to find the velocity and position of the feather as a function of time. For the tornado you may use the values $u_0 = 100$ m/s and $R = 20$ m.
(l) Find the trajectory for the feather if it is released from rest from a height of 2.4 m, and in a position corresponding to $\mathbf{r} = -R\,\mathbf{i} + h\mathbf{k}$.
(m) The trajectory of the feather is shown in Figs. 9.20 and 9.21. Compare the results with what happened when you dropped the feather without wind. Why does the feather now take longer to reach the ground?

9.18 Modelling Atomic Interactions. In this project we will create a model for an atom with mass m in the vicinity of two large molecules. We will assume that the molecules are massive and do not move. We will first look at a one-dimensional movement along the x-axis, and later expand to higher dimensions.

The interactions between the atom and the molecules can be described by the potential

$$U(x) = U_0 \left(\left[\left(\frac{x}{d}\right)^2 - 1 \right]^2 \right) = \frac{U_0}{d^4}(x^2 - d^2)^2 = \frac{U_0}{d^4}(x^4 - 2x^2 d^2 + d^4) \tag{9.80}$$

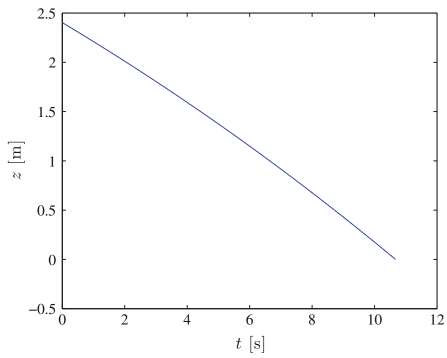

Fig. 9.21 Illustration of the (*vertical*) trajectory of a feather released from rest at $\mathbf{r} = -R\mathbf{i} + h\mathbf{k}$ at $t = 0\,\text{s}$

where U_0 is a known constant measured in Joules, x is the position of the atom, and d is a known length. We will assume all other forces on the atom are small in comparison, and neglect them in our model.
(a) Make an energy diagram. Find the equilibrium points, mark these on the diagram and characterize their stability.
(b) Choose two different energies that give two distinct types of motions, draw them into the diagram, and describe the motion in each case.
(c) If the atom starts at rest at $x = 2d$, what is the velocity of the atom at the point $x = d$?
(d) If the atom starts at rest at $x = d$ with the velocity v_0, how large must v_0 be for the atom to reach the point $x = -d$?
(e) Show that the acceleration of the atom is

$$a = -\frac{4U_0}{md^4}(x^3 - xd^2) \tag{9.81}$$

(f) Write a numerical algorithm to find the position and velocity of the atom at a time $t + \Delta t$, given the position and velocity of the atom at a time t.
(g) Implement the numerical algorithm in a program to find the position of the atom as a function of time from $t = 0\,\text{ns}$ to $t = 10\,\text{ns}$ with a timestep of $\Delta t = 0.01\,\text{ns}$ for $m = 1$, $d = 0.1\,\text{nm}$ and $U_0 = 1\,\text{nJ}$.
(h) Make a plot of two distinct x(t) from $t = 0\,\text{ns}$ to $t = 10\,\text{ns}$, the first from a running of the program with the initial conditions $x_0 = d$ and $v_0 = 0.5\,\text{m/s}$, and the other from a running of the program with the initial conditions $x_0 = d$ and $v_0 = 1.5\,\text{m/s}$. Make another plot of v(t) for the same initial conditions.
(i) Describe the behavior of the atom in both simulations and sketch the motion in an energy-diagram.

We will now look at the same system, but in two dimensions. The Atom interacts with a surface in such a way that the potential of the atom is given as

$$U(r) = U_0\left(\left[\left(\frac{r}{d}\right)^2 - 1\right]^2\right) = \frac{U_0}{d^4}(r^2 - d^2)^2 = \frac{U_0}{d^4}(r^4 - 2r^2d^2 + d^4) \tag{9.82}$$

where $r = \sqrt{x^2 + y^2}$ is the distance to the origin. (We can no longer interpret this interaction as an effect from two molecules. Instead we interpret it as an approximation of a more complicated interaction from many surrounding atoms.)

(j) Show that the acceleration on the atom can be written

$$\mathbf{a} = -\frac{4U_0}{md^4}(r^3 - rd^2)\frac{\mathbf{r}}{r} \tag{9.83}$$

(k) Rewrite your program to find the velocity and position of the atom using the new expression for the force F. Use vectorized expressions in your code.

(l) Plot the motion of an atom starting in $\mathbf{r_0} = (d, 0)$ from $t = 0$ ns to $t = 20$ ns for the initial velocities $\mathbf{v} = (0, 0.5\,\text{m/s})$, $\mathbf{v} = (0, 1\,\text{m/s})$ and $\mathbf{v} = (0, 1.5\,\text{m/s})$.

(m) Can you choose initial conditions $\mathbf{r_0}$ and $\mathbf{v_0}$ in such a manner that the atom moves in a circular orbit with a constant radius? If so, what initial conditions are those? Plot the motion for these conditions.