Chapter 5
Forces in One Dimension

What determines how far a bungee-jumper falls before he starts moving upward? In this chapter you acquire the tools to answer this, sometimes critical, question.

We have introduced a structured approach to find the motion of a object from its acceleration and the initial conditions (see Fig. 5.1). But how do we find the acceleration? We could measure it directly, as we did with an accelerometer, but this is not satisfactory. Physics is not only about describing what is happening, but rather about explaining and predicting motion. In order to determine the motion, we need to be able to *predict* the acceleration of an object.

In this chapter we will show you that the acceleration of an object is related to the forces acting on the object. In order to predict the motion, we need to:

- Find what forces are acting on an object.
- Introduce quantitative models for the forces—we need numbers for the forces in order to have numbers for the acceleration.
- Determine the acceleration from the forces using Newton's second law of motion.
- "Solve" the motion from the differential equations of motion and the initial conditions.

We will address these points in detail: First we show how to identify the forces acting on an object. Then we introduce Newton's second law that relates forces to acceleration. Finally, we introduce models for some of the most common forces in the macroscopic world.

5.1 What Is a Force?

We all have an intuitive notion of a force. Imagine you give one end of a soft rubber band to a friend (see Fig. 5.2). As you pull the rubber band, your friend will experience a pull in the rubber band. She feels the force acting on her. As you pull harder, she will feel that the pull becomes stronger—the force acting on her becomes larger.

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A. Malthe-Sørenssen, *Elementary Mechanics Using Matlab*,
Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-19587-2_5

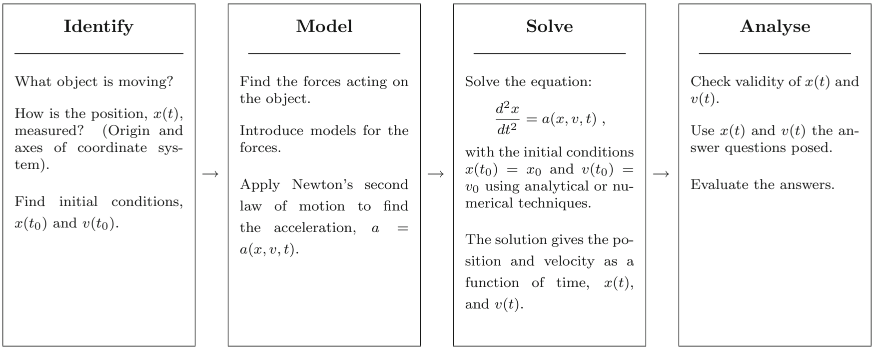


Fig. 5.1 The structured problem solving approach

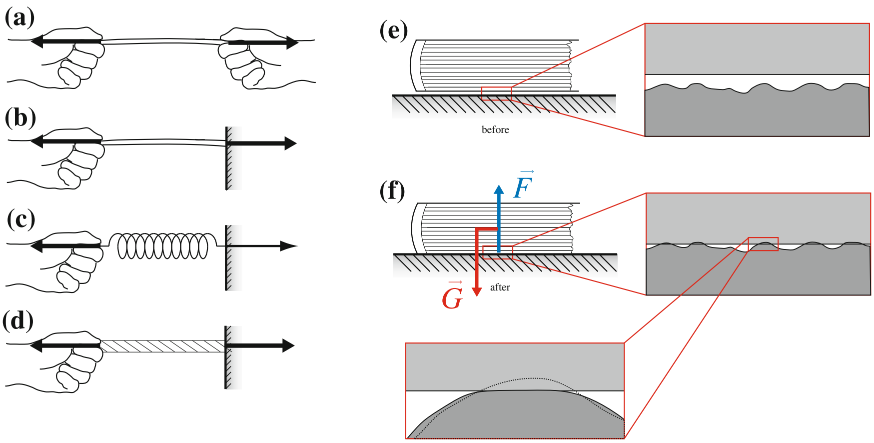


Fig. 5.2 Illustration of **a** two hands pulling on a rubber band, **b** a rubber band attached to a wall, **c** a spring attached to a wall, **d** a rope attached to a wall, **e** a book above a table, **f** a book on a table, **g** deformation of surface bump, **h** magnification of bump

In addition, the rubber band stretches. The harder you pull, the longer the rubber band becomes.

If you instead tie the rubber band to the wall, the rubber band will again elongate as you pull. But now it is not a person experiencing the pull, it is the wall. A force may indeed act on the wall as well as on a person. If we pull harder, the rubber band elongates further, and we expect the force on the wall to become larger. This suggests that the elongation of the band is a reasonable way to measure the magnitude of the force, and this is indeed the usual way to define a force: by prescribing how we can measure it. We can measure forces by how they deform rubber bands.

Now, there is nothing special about a rubber band. We could replace the rubber band by a spring or any other material. As you pull on the spring, the spring elongates. If the spring is stiff, it elongates less than the rubber band, but it still elongates somewhat. A rope may be even stiffer, and would deform even less, but a careful measurement would show that also a rope elongates when pulled.

We are nearing a definition of a force. We could define a force as an interaction—a pull or a push on an object—that can be measured by the deformation of a spring. In this case the magnitude of the force increases with the deformation of the spring. This definition is not altogether satisfactory, but it illustrates a particular type of force—what we call a *contact force*. Contact forces occur where an object is in contact with other objects.

What about a book lying on a table, are there any forces acting on the book? The book is not moving, so we may be tempted to say no. Unfortunately, this would be wrong. When we pulled on the wall with the spring, the wall was not moving, but there was still a force acting on it. What about the book—where are the forces acting on the book? First, there is one force we have not discussed so far, the force of gravity. This is one of the fundamental forces in nature: There are gravitation forces between any two objects pulling the objects toward each other. There is a gravitational force from the Earth on the book, which pulls the book downward.

What is stopping the book from moving? The table! But how? We cannot see any deformation as we could for the rubber band. This is only because you do not look carefully enough. If you zoomed in on the contact between the book and the table using a microscope, you would see that the surface of the table and the surface of the book are not flat, but rough. Small surface irregularities can be seen along the surfaces. When the book is placed on the table, these small irregularities deform (see Fig. 5.2). Each irregularity acts as a small spring, and when the irregularities are deformed, that deformation is related to the contact force between the two objects. The sum of the forces from all of these small springs is the force from the table on the book.

If we zoom further in on the contact between one surface irregularity and the table, we realize that the contact force really is a sum of electromagnetic forces between the atoms on the surface of the book and the atoms on the surface of the table. The atoms are never in actual contact, but as the book and the table are pressed toward each other, electromagnetic forces will act from the table on the book. The electromagnetic force has been shown to be part of the electromagnetic and the weak nuclear force, which is one of three fundamental forces. The other two are gravity and the strong nuclear force, which is responsible for the interactions between subatomic particles and for the interactions in the nucleus. These are the three main forces in nature, and all forces are reducible to these forces.

In most cases, we will study objects that consist of many atoms. In practice, we cannot find the sum of the forces from all the individual atoms to find the magnitude of the force, but we will instead develop simplified models for the macroscopic forces we encounter. We will call such models **force models**, and they will be our main tools for determining forces on macroscopic objects.

5.2 Identifying Forces

The first step in the "Model" box in our structured problem solving approach is to find the forces acting on the object. We therefore need a systematic way to find and identify the forces acting on an object, and we will do this by studying a specific example: A ball bouncing on the ground, as illustrated in Fig. 5.3. In the process of studying this example, we develop a general procedure for analyzing forces.

First, we need to discern between the *object*, also called the *system*, and the *environment*, which is everything else. In this case, the system is the ball, and the environment is everything else, such as the floor, the air surrounding the ball, and the Earth.[1] We have now found the first step in our procedure:

1. Divide the problem into *system* and *environment*.

In order to find the forces acting, we must realize a fundamental characteristic of a force:

> All forces acting on the *system* must have a source—an identifiable cause in the *environment*.

A force acting on the ball must be related to an interaction with something in the environment. This also means that we do not consider internal forces—forces between one part of the object and another part—we only consider external forces.

We have claimed that there are only three types of forces: gravity, the electromagnetic and weak nuclear force, and the strong nuclear force. However, this is not very helpful for our analysis of a macroscopic object such as the ball. Instead, we will divide forces into two main types:

> Forces are either *contact forces* or *long-range forces*.

Contact forces, as evident from the name, are forces that occur at the contact between the system and the environment. We find contact forces by examining our drawing of the ball in Fig. 5.3. From the drawing, we see that the ball is in contact with the floor.

We generalize the procedure to find the contact points in the following steps:

2. Draw a figure of the object and everything in contact with the object.
3. Draw a closed curve around the system.
4. Find contact points—these are the points where contact forces may act.

[1] You will see that we need to include the Earth, since the gravitational force on the ball comes from the interaction with the whole Earth, and not just with the floor.

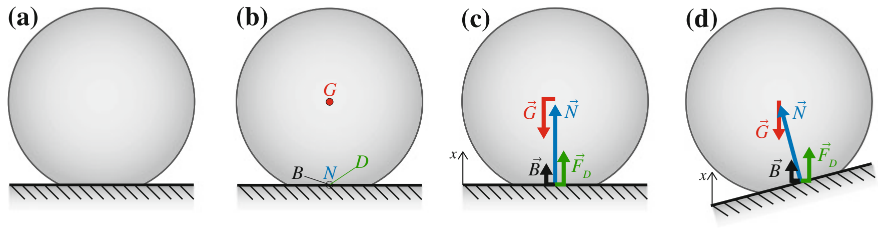


Fig. 5.3 Illustration of a ball bouncing off the floor

What is the contact force at this contact? The force on the ball is from the floor, and we call this force the normal force. It is similar to the force acting on the book lying on the table, and may be considered as a sum of many forces acting along the interface between the ball and the floor. We introduce the symbol N for this force. The symbol N represents a number (with a notation), giving the strength of the force. (As we will see later, forces are measured in Newton, and N is therefore measured in Newtons.) This step in the general procedure can be summarized as:

5. Give names and symbols to all the contact forces.

The direction of the normal force from the floor on the ball depends on the direction of the floor as illustrated in Fig. 5.3d. In order to show both the direction and the magnitude of the force, we realize that a force must be a vector, and we introduce the symbol $\mathbf{N}$ for the normal force. To illustrate the normal force acting on the ball, we draw a vector starting in the contact point, acting in the direction of the normal force, and with a length related to its magnitude, as illustrated in Fig. 5.3c.

So far we have only discussed *one* of the contact forces between the ball and the environment. What other contact forces are there? The ball is also in contact with the air. The contact with the air results in several forces. Everywhere along the surface of the ball there will be small drag forces because of the difference in velocity between the surface of the ball and the air. Again, we simplify by assuming that all these small forces sum to a single force, the air resistance, $\mathbf{F}_D$, which is drawn as acting in a single point on the surface of the ball, as illustrated in Fig. 5.3c.

Similarly, there are differences in the pressure in the air, which would give rise to a buoyancy force, $\mathbf{B}$, which we again assume to be acting on the surface of the ball.

Finally, we must also look for the *long-range forces* affecting the ball. The only long-range force is the gravitational force acting from the Earth on the ball. We call this force, $\mathbf{G}$, and draw it as acting in the center of the ball, in the direction toward the center of the Earth.

This sums up the final step of our procedure:

6. Identify the long-range forces.

Free-Body Diagram

We have now defined a general procedure to find what we call the *free-body diagram* for the system. This is a diagram that identifies all the forces acting on the object, where each force is drawn as a vector starting from the point where the force is acting. The construction of the free-body diagram is central to mechanics—and it will typically be one of the first tasks you will do whenever you are solving a mechanics problem. We will therefore provide you with a detailed prescription for how to draw the free-body diagram.

Drawing a free-body diagram:
Follow these steps to find and identify all the forces acting on an object and then to draw the free-body diagram for the system.

- Divide the problem into *system* and *environment*.
- Draw a figure of the object and everything in contact with the object.
- Draw a closed curve around the system.
- Find contact points—these are the points where contact forces may act.
- Give names and symbols to all the contact forces.
- Identify the long-range forces.
- Make a drawing of the *object*. Draw the forces as arrows, vectors, starting from where the force is acting. The direction of the vector indicates the (positive) direction of the force. Try to make the length of the arrow indicate the relative magnitude of the forces.
- Draw in the axes of the coordinate system. It is often convenient to make one axis parallel to the direction of motion. When you choose direction of the axis you also choose the positive direction for the axis.

Note that when you draw a force, you indicate the positive direction for this force. If you later calculate the force and find that it is negative, it simply means that the force is acting in the opposite direction of what you thought or defined as the positive direction when you made the drawing.[2]

5.3 Newton's Second Law of Motion

We are now able to find and identify the forces acting on an object. However, we still need a connection between the forces and the motion of an object. This connection can be found through Newton's second law of motion, which relates the acceleration of an object to the forces acting on the object:

[2]You should, however, be aware that in some cases, this may mean that you have made an error in your assumptions, because some forces, such as the normal force due to a contact, cannot be negative unless the objects are glued together.

Newton's second law of motion: The force **F** on an object of inertial mass m is related to the acceleration **a** of the object through $\mathbf{F} = m\mathbf{a}$.

Newton's laws of motion are laws of nature that have been found by experimental investigations and have been shown to hold up to continued experimental investigations. Newton's laws are valid over a wide range of length- and time-scales. We use Newton's laws of motion to describe everything from the motion of atoms to the motion of galaxies.

Aspects of Newton's Second Law

Vector equation: Newton's second law is a vector equation: The acceleration is in the direction of the force, and the acceleration is proportional to the force. In this chapter, we will only study forces and motion in one dimension. We will therefore write $\mathbf{F} = F_x\,\mathbf{i}$, where **i** is the unit vector along the x-axis. The one-dimensional version of Newton's second law is then:

$$F_x = ma_x = m\frac{d^2x}{dt^2}. \tag{5.1}$$

Intertial mass: Newton's second law introduces a new property of an object—*the inertial mass*, m. We determine the inertial mass of an object by measuring the acceleration for a given applied force. The inertial mass is measured in Grams, with the notation g. Experimental studies show that inertial masses are additive: If we add two objects of masses m_A and m_B together, their total inertial mass is:

$$m = m_A + m_B. \tag{5.2}$$

Unit: Forces are measured in *Newton*, with the notation N. The definition of one Newton is that it is the force that gives an object with (inertial) mass of 1 kg an acceleration of 1 m/s^2. That is:

$$1\,\mathrm{N} = 1\,\mathrm{kg\,m/s^2}. \tag{5.3}$$

Net external force: The force, **F**, in Newton's second law is the *net external force* acting on the object. By *external* we mean that the force has a cause outside the system, as we insisted when we drew a free-body diagram of an object. By *net force* we mean that if there are several forces acting on an object, it is the sum of all the external forces that causes the acceleration. We call this sum the net force:

$$\mathbf{F}_{\text{net}} = \sum_j \mathbf{F}_j = m\mathbf{a}. \tag{5.4}$$

Here we have written a sum over various forces, where each force is identified by a subindex j. This is a typical way of writing the net force in shorthand. In practice, we replace the sum, $\sum_j$, by a sum of each of the external forces found in the free-body diagram.

For example, for the ball bouncing off the floor studied above, the net force on the ball is:

$$\mathbf{F}_{\text{net}} = \sum_j \mathbf{F}_j = \mathbf{G} + \mathbf{N} + \mathbf{F}_D + \mathbf{B}. \tag{5.5}$$

Superposition: Forces are additive. We say that they obey the *superposition principle*. The acceleration due to many forces, $\mathbf{F}_i$, is the same as the acceleration of one force equal to the sum of all the small forces.

Point particles: Newton's second law applies to a *point particle*, an object located in a single point. The acceleration $\mathbf{a}$ is the acceleration of this point. While the concept of a point particle is mathematically useful, it is not that useful in a world of macroscopic objects that have spatial extent, such as any object we typically describe in mechanics.

Can we still use Newton's second law for extended objects? Yes! Newton's second law is valid for the motion of any macroscopic object—even objects that deform or change during the motions, such as bouncing football. But for a macroscopic object we need to be very precise in how we describe the position of the object, because the position is a single point whereas the object is located in many points. What point to choose? It turns out that Newton's second law is valid if we choose a particular point called the center of mass of the object (or any point on the object that does not move relative to the center of mass). We will introduce the center of mass formally later, but for now we can use the (geometric) center of the object or any other point that does not move relative to the center.

Because we can use Newton's second law for both point particles and extended objects, we do not need to discern between point particles and extended objects for now. Usually, we find it most convenient to work with real, extended physical objects, and we draw forces onto the extended objects, as shown in Fig. 5.3.

5.3.1 Example: Acceleration and Forces on a Lunar Lander

This example demonstrates how we can find forces from the acceleration, both in the case where the net force is zero and where the acceleration is measured.

You are leading a team that is building the return module for the next lunar expedition. You have designed a module that breaches if exposed to air resistance forces above 10^6 N for more than 5 s. The mass of the module is 5000 kg. To test your design you are using a numerical model that models the entry of the lander

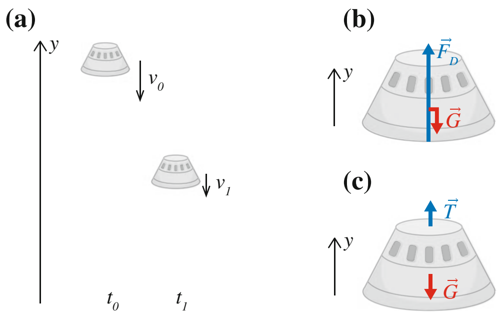


Fig. 5.4 **a** Sketch of descent of the reentry module, **b** free-body diagram of the module during reentry, and **c** during weighing

into the Earth's atmosphere. The result of such a simulation is in the form of the accelerations, $a(t_i)$, at a sequence of discrete times, t_i, in the file reentry.d.[3] Can the hull sustain this entry?

Free-body diagram: We illustrate the descent of the module in a sketch, as shown in Fig. 5.4. Our system is the module, and we describe its vertical position by $y(t)$.

We draw the system alone in a separate figure in order to draw the free-body diagram. The module is in contact with the surrounding air, giving rise to the air drag force, $\mathbf{F}_D$, which acts upward when the module is moving downward. This is the only contact force. In addition, it is affected by a long-range force—the gravitational force from the Earth, $\mathbf{G}$, which acts downward toward the Earth.

Newton's second law: The net force on the module is:

$$\mathbf{F}_{\text{net}} = \mathbf{F}_D + \mathbf{G}, \tag{5.6}$$

where $\mathbf{F}_D$ acts upward when the module is moving downward, hence $\mathbf{F}_D = F_D\mathbf{j}$, and $\mathbf{G}$ acts downward, that is, in the negative y-direction, $\mathbf{G} = -G\mathbf{j}$:

$$\mathbf{F}_{\text{net}} = (F_D - G)\,\mathbf{i}. \tag{5.7}$$

We remove the vector notation since we are looking at motion along the y-axis, getting the net force along the y-axis:

$$F_{\text{net}} = F_D - G, \tag{5.8}$$

which is the equation you will usually start from—we do not usually include the whole vector notation derivation when discussing a one-dimensional problem.

Newton's second law for motion along the y-axis gives:

$$F_{\text{net}} = F_D - G = ma_y, \tag{5.9}$$

[3] http://folk.uio.no/malthe/mechbook.

since we know $a_y(t_i)$ at given times t_i, we may use this relation to find $F_D(t_i)$ at the same time intervals, if we only knew $G(t_i)$.

Measuring the gravitational force: To determine $G(t_i)$, we need a force model—a model that gives us a number value (and unit) for the gravitational force. You probably already know this law, $G = mg$, and we will introduce it later on - but what could you do if you did not know it? We could device a experiment to measure the gravitational force G on the module. We hang the module in a wire, and measure the force in the wire when the module is at rest. This is the principle behind a weight. Figure 5.4c illustrates the free-body diagram for this experiment. Since the module is not moving the only contact force is **T**, the force from the wire on the module. Newton's second law in the y-direction becomes:

$$F_{\text{net}} = T - G = ma_y. \tag{5.10}$$

However, we have designed this experiment so that $a_y = 0\,\text{m/s}^2$, since the module is not moving, therefore

$$T - G = 0 \;\Rightarrow\; T = G. \tag{5.11}$$

We have found that we can measure G by measuring T, which gives $T = G = 49{,}000\,\text{N}$.

You will find that this use of Newton's second law is very common, and we will return to it many times. Problems where there is no motion—or motion with constant velocity—are often called static problems.

Calculating air resistance force: Since we now know that $G = 49{,}000\,\text{N}$ (and we assume this is a constant throughout the motion), we can now use the data we have for $a_y(t_i)$ for the module to find the air resistance force on the module. From Newton's second law we have:

$$F_D - G = ma_y \;\Rightarrow\; F_D = G + ma_y, \tag{5.12}$$

Since the acceleration is changing throughout the motion, the air resistance force F_D is also a function of time. We read the accelerations from the file reentry.d[4] using the following program.

```
load -ascii reentry.d
t = reentry(:,1);
a = reentry(:,2);
G = 49000.0; % N
m = 5000.0; % kg
FD = G + m*a;
plot(t,FD)
xlabel('t [s]')
ylabel('F_D [N]')
```

Analysis: We see from the plot of $F_D(t)$ in Fig. 5.5 that while F_D is decreasing, it is larger than the limit for more than 5 s. With this entry, the hull will breach!

[4] http://folk.uio.no/malthe/mechbook/reentry.d.

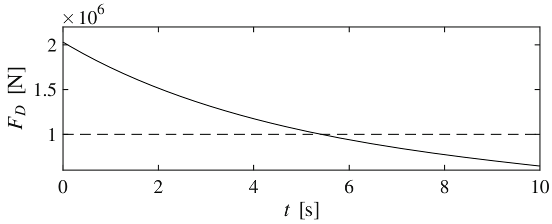


Fig. 5.5 Plot of the air resistance force, F_D, as a function of time

Additional material: We can find the time when the air resistance force becomes less than $F_D^C = 10^6$ N, by first finding the smallest i where $F_D(t_i)$ is less than F_D^C, and then finding the corresponding t_i. This is done by:

```
>> i = min(find(FD<1e6));
>> ti = t(i)

ti =      5.4246
```

This shows that the air resistance force falls to F_D^C after 5.42 s. The module needs to be redesigned. You may get ideas as to how when you learn about air resistance later in this chapter.

5.4 Force Models

In order to use Newton's second law to determine the acceleration of an object, we need to find out how large a force is—we need to determine its magnitude and direction. For this, we need theories that provide numerical values for the forces. We call such models "force models". The force models may be based on direct, experimental measurements. We often call such models phenomenological or experimental force models. The force models can also be based on a more fundamental model or a model based on a microscopic view of the interactions.

In the following we introduce models for some of the most common types of forces acting between macro- and microscopic objects. These models will be your toolbox for addressing physical processes—you need to continually build on this toolbox, as this will be your reservoir of physical knowledge. If you want to describe a ball falling through air, you need mathematical expressions for the forces on the ball: both the force due to gravity and the force due to air resistance. If you want to describe the motion of a nanometer sized particle in water close to a charged surface you need to introduce (probably sophisticated) models for the forces between the particle and the individual water molecules and between the particle and the surface. The range of problems you can solve depends on your knowledge of interactions—forces—and on your ability to simplify a complicated situation to a model that only contains forces you know how to address.

5.5 Force Model: Gravitational Force

Another of Newton's great accomplishments is his discovery of the law of gravity.

According to **Newton's law of gravity**, there are attractive, gravitational forces between all objects. The gravitational force on object A from object B is:

$$\mathbf{F}_{\text{from B on A}} = \gamma \frac{m \cdot M}{r_{AB}^3} \mathbf{r}_{AB}, \tag{5.13}$$

Here $\mathbf{r}_{AB}$ is a vector pointing from the center of object A to the center of object B, and r_{AB} is the length of this vector, corresponding to the distance between the centers of objects A and B (see Fig. 5.6). The quantities m and M are the gravitational masses of objects A and B respectively, and γ is the gravitational constant.

All experimental evidence shows that the gravitational masses are the same as the inertial masses. We will therefore use the same symbol, m for both the inertial mass and the gravitational mass of an object.

Constant Gravity

The general gravitational law becomes simpler for an object near the surface of the Earth or another planet. In this case, the distance r_{AB} from the object to the center of the Earth is approximately constant and equal to the radius R of the Earth. The gravitational force can therefore be approximated by:

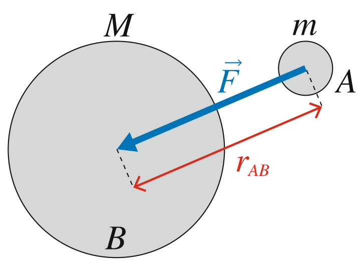


Fig. 5.6 The gravitational force **F** from object B on object A

The gravitational force near the surface of a planet is approximately constant, and equal to:

$$\mathbf{G} = -m \underbrace{\frac{\gamma M}{R^2}}_{=g} \mathbf{j} = -mg\,\mathbf{j}, \tag{5.14}$$

where the unit vector $\mathbf{j}$ points in the upward direction, and g is called the acceleration of gravity.

The constant g only depends on the radius and mass of the planet.

The Acceleration of Gravity

Why do we call g the acceleration is gravity? Because this is the acceleration of an object that is subject only to a gravitational force, which is easily seen from Newton's second law applied to an object only affected by gravity:

$$m\mathbf{a} = \mathbf{G} = -mg\,\mathbf{j}, \tag{5.15}$$

$$\mathbf{a} = -g\,\mathbf{j}, \tag{5.16}$$

For an object on the surface of the Earth, the acceleration of gravity is approximately $g = 9.81\ \text{m/s}^2$, whereas for an object on the surface of the Moon, the acceleration of gravity is $g_m = 0.17\,g$. You can find a table of the acceleration of gravity on the surface of various objects in the solar system in Table 5.1.

Table 5.1 The acceleration of gravity on the surface of various objects in the Solar system

Body	Mass (kg)	Radius (km)	g (m/s^2)	g/g_e
Sun	1.99×10^{30}	6.96×10^{5}	274.13	27.95
Mercury	3.18×10^{23}	2.43×10^{3}	3.59	0.37
Venus	4.88×10^{24}	6.06×10^{3}	8.87	0.90
Earth	5.98×10^{24}	6.38×10^{3}	9.81	1.00
Moon	7.36×10^{22}	1.74×10^{3}	1.62	0.17
Mars	6.42×10^{23}	3.37×10^{3}	3.77	0.38
Jupiter	1.90×10^{27}	6.99×10^{4}	25.95	2.65
Saturn	5.68×10^{26}	5.85×10^{4}	11.08	1.13
Uranus	8.68×10^{25}	2.33×10^{4}	10.67	1.09
Neptune	1.03×10^{26}	2.21×10^{4}	14.07	1.43
Pluto	1.40×10^{22}	1.50×10^{3}	0.42	0.04

There are local variations of g along the Earth's surface due to deviations of the spherical shape of the Earth, due to topographical variations, and due to differences in density in the Earth's crust. In addition, there are variations in the effective g due to the rotation of the Earth. Surveys of variations in g due to density differences in the crust are used as a remote sensing technique that gives important information about the properties of rocks present in the Earth's crust. This technique is routinely used for example for petroleum exploration.

5.6 Force Model: Viscous Force

You will often encounter objects that are in contact with a surrounding fluid such as a air or water. We therefore need a force model for the interaction between fluids and solid objects. Unfortunately, there is no fundamental law of nature for such an interaction. Instead, we must determine the force model from experiments or calculations based on an underlying model for fluid flow, and use this result as our model. Fortunately, experiments and calculations show that the force from the fluid has a simple form—it depends on the velocity of the object.

Drag Force at Low Velocities

If you pull a sphere through water, you expect a contact force F_D from the water on the sphere counteracting the motion of the sphere relative to water because water is forced to flow around the sphere. Both experiments and theoretical models show that for low velocities the fluid flows smoothly around the object (see Fig. 5.7), and the force, F_D, from the fluid is proportional to the velocity of the object relative to the fluid.

The **viscous force** on an object moving at a velocity v relative to a fluid is:

$$F_D = -k_v v, \tag{5.17}$$

where k_v is a constant that depends on the objects size, shape and surface, as well as on the (dynamic) viscosity of the fluid. Stokes showed that at low velocities

$$k_v = 6\pi \eta R, \tag{5.18}$$

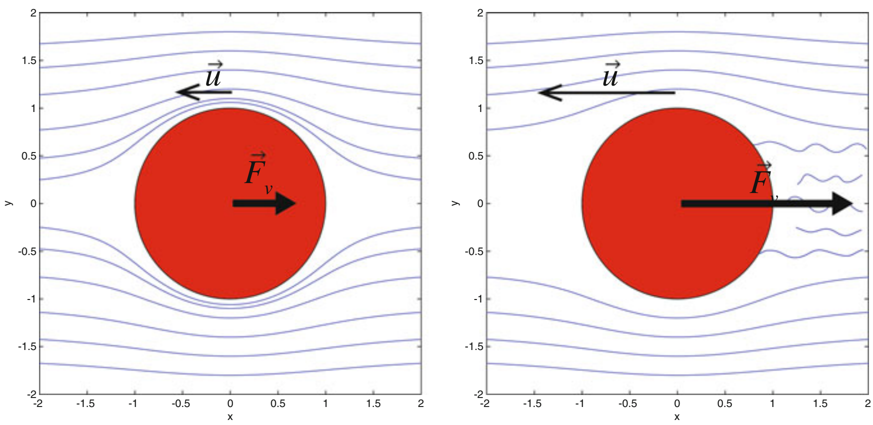


Fig. 5.7 Illustration of a drag force on an object due to the motion of the object relative to the surrounding fluid. At low velocities the fluid flow around the object is smooth, and the force is approximately proportional to the velocity. At higher velocities, the fluid flow becomes turbulent near the surface, the flow becomes irregular, and the force is approximately proportional to the square of the velocity

where R is the radius of the sphere and η is the viscosity of the fluid. The viscosity of air is $\eta = 1.82\,10^{-5}\,\mathrm{Nsm}^{-2}$ (at room temperature), and the viscosity of water is $\eta = 1.00\,10^{-3}\,\mathrm{Nsm}^{-2}$ (at room temperature).

We call a force that is proportional to the velocity (of the object) a *viscous force*. If the force is a contact force from a fluid we also often use the term *drag force* to describe the interaction.

Drag Force at High Velocities

At larger velocities the fluid flow around the object becomes more irregular (see Fig. 5.7), and the drag force is not mainly related to the forces required to drag the fluid along the surface of the object, but instead depends on the under-pressure generated behind the object. In this case, the force is proportional to the square of the velocity, and we call this law the square law of air resistance.

The **drag force** on an object moving at a velocity v relative to a fluid is:

$$F_D = -Dv^2, \tag{5.19}$$

acting in the direction opposite the velocity.

The minus sign shows that the force acts in the direction opposite the velocity. The prefactor D is a constant that depends on the objects size, shape and surface, and the density of the fluid. Experimental data gives an approximative value for D for a spherical object:

$$D \simeq 12.0\,\rho\,R^2. \tag{5.20}$$

where ρ is the density of the fluid, and R is the radius of the sphere.

General Model for Fluid Drag

The behaviors for low and high velocities are special cases of a more general model for the drag force. Experiments show that the drag force can be written in the general form:

$$F_D = \frac{1}{2}\rho\,S\,C_D(v, \eta, \rho, d)v^2, \tag{5.21}$$

where ρ is the density of the fluid. For air $\rho = 1.293\,\text{kg/m}^3$ at normal pressures and temperatures. S is the cross-sectional area of the object—that means the area of the object's projection on a plane normal to the direction of motion. For a sphere with radius r the cross-sectional area is $S = \pi r^2$. The speed of the object relative to the fluid is v. The viscosity of the fluid is η, and d is a characteristic length scale, such as the diameter of a sphere. The coefficient $C_D(\ldots)$ is called the drag coefficient and describes the details of the air resistance—the physics of drag is hidden in this function. For a given object type—such as a sphere of a given material—experiments show a remarkable feature: The drag coefficient only depends on one number, *Re*, called the Reynold's number:

$$C_D(v, \eta, \rho, d) = C_D(Re),\;\; Re = \frac{\rho d}{\eta}v. \tag{5.22}$$

This is a surprisingly compact description with many interesting implications. For example, if you increase the velocity by a factor 10 you get the same drag coefficient if you either reduce the radius by a factor 10 or increase the viscosity by a factor 10. The general function for the drag coefficient, $C_D(Re)$, for a smooth sphere is shown in Fig. 5.8. (You can find the data-set for part (a) in cdreynolds.d[5]).

For small velocities, we expect to recover Stokes' general result: $F_D = 6\pi\eta R v$, which implies that $C_D(Re) \propto Re^{-1}$ for small v (and Re). This is indeed what we observe in Fig. 5.8, where Re^{-1} is shown as a straight line. For large velocities, we see that the drag coefficient becomes approximately constant, and the force is therefore proportional to the square of the velocity. Our simplified models for small and large velocities are therefore consistent with the experimental results, and we

[5] http://folk.uio.no/malthe/mechbook/cdreynolds.

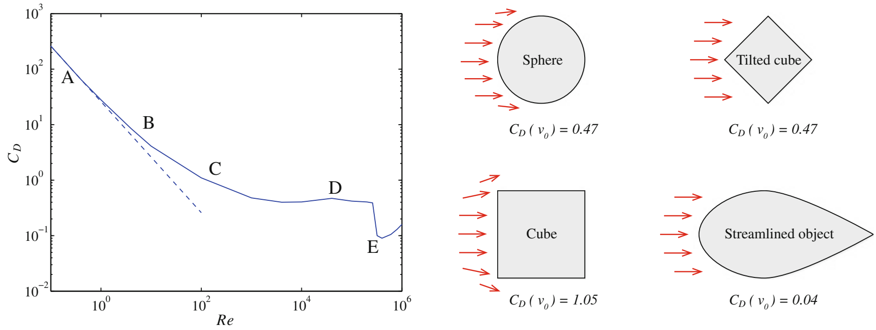


Fig. 5.8 **a** The drag coefficient C_D as a function of Reynold's number, Re $= \rho dv/\eta$ based on experimental data. **b** The drag coefficient C_D at the same velocity for object with the same cross-sectional area, but with different shapes

now also have a better understanding of what small and large means: Small means $Re \ll 1$ and large means $Re \gg 1000$.

Something strange happens when the Reynold's number reaches Re $= 3.2\,10^5$: The drag coefficient drops significantly! A careful calculation (which you can program yourself) shows that not only the drag coefficient but also the fluid drag force falls. How can the drag force decrease when the velocity increases? This effect is due to boundary-layer turbulence (which we will not explain here). The transition point where this effect kicks in depends on surface properties of the object. For a rough surface, such as that of a golf ball, the transition occurs for a lower Reynolds number than for a smooth ball. This is the reason why golf balls have a rough surface: The air drag force for large velocities is reduced by this design.

What happened to aerodynamic design? This is also hidden in the drag coefficient. The value of the drag coefficient depends on the shape of the object, and more aerodynamic designs have lower drag coefficients for approximately the same cross-sectional area, as illustrated in Fig. 5.8.

5.6.1 Example: Falling Raindrops

This example demonstrates how we can find the motion of an object subject to a constant force (gravity), and to a velocity-dependent force.

Raindrops are often as small as $d = 1$ mm in diameter as they start falling. Here, we apply the structured problem-solving approach to find their velocity, first without air resistance, and then with a model for viscous drag.

Sketch and Identify: We always start addressing a problem through a sketch which should include *the system*, a raindrop, *the environment*, and *a coordinate system* (see Fig. 5.9). We describe the position of the raindrop by its vertical position $y(t)$ as a function of time t.

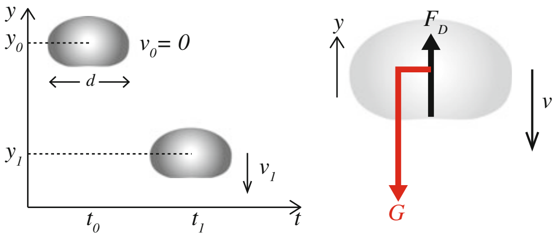


Fig. 5.9 (*Left*) Illustration of a raindrop falling down. (*Right*) Free-body diagram for the drop. (Notice that a real raindrop is neither perfectly spherical nor "drop-shaped", it is pushed flat by the air resistance force)

Model: The motion of the raindrop is determined by the forces acting on it. We draw the forces in a free-body diagram, as illustrated in Fig. 5.9. The sketch shows that the raindrop is only in contact with the surrounding air, which gives rise to an air resistance force F_D. In addition, the raindrop is affected by gravity, G, from the Earth.

We have force models for each of these forces. We know that gravity is $\mathbf{G} = -mg\mathbf{j}$, where m is the mass of the raindrop. For the air resistance F_D we assume that we can use the viscous law:

$$\mathbf{F}_D = -k_v v(t)\,\mathbf{j}, \tag{5.23}$$

where $v(t)$ is the velocity of the drop. You should check with yourself that this force indeed has the correct sign. Remember that when the drop falls downward its velocity is negative.

Newton's second law: We apply Newton's second law to find the acceleration of the drop:

$$\mathbf{F}^{\text{net}} = \mathbf{F}_D + \mathbf{G} = -mg\,\mathbf{j} - k_v v(t)\,\mathbf{j} = ma\,\mathbf{j}, \tag{5.24}$$

which corresponds to

$$-\,mg - k_v v(t) = ma. \tag{5.25}$$

We lack two of the numbers in this equation: m and k_v. We can find the mass of the drop by assuming that it is spherical and made of water. The volume of a sphere is $V = (4\pi/3)r^3$, where $r = d/2$ is the radius of the sphere, and the mass density of water is $\rho = 1000.0\,\text{kg/m}^3$. The mass of the drop is therefore

$$m = \rho\,V = \rho\,\frac{4\pi}{3}r^3 = \rho\,\frac{4\pi}{3}\left(\frac{d}{2}\right)^3 = 5.24\;10^{-7}\,\text{kg}. \tag{5.26}$$

We find k_v from Stokes' formula in (5.18): $k_v = 6\pi R\eta$. The radius of the raindrop is $r = 0.5\;10^{-3}$ m, and the viscosity of the air is $\eta = 1.82\;10^{-5}\,\text{Nsm}^{-2}$. This gives $k_v = 1.85\;10^{-7}\,\text{Nsm}^{-1}$.

Finally, to calculate the motion of the drop, we need to know its initial conditions: It starts from $y(0) = h$ at $t = 0$ s at rest, that is, with $v(0) = 0$ m/s.

Simplified model: No air resistance: First, what happens in the simplified case when we have no air resistance? In that case, $k_v = 0$, and the acceleration is a constant

$$a = -g. \tag{5.27}$$

We find the velocity as function of time by direct integration:

$$v(t) - v(0) = \int_0^t a\,dt = -gt. \tag{5.28}$$

This corresponds to a free fall, as we have seen previously. We expect this to only be a good approximation as long as the air resistance term is small, that is as long as $k_v v(t)$ is much smaller than mg, that is when

$$k_v v \ll mg \;\Rightarrow\; v \ll \frac{mg}{k_v} = \frac{5.24\,10^{-7}\,\text{kg}\,9.8\,\text{m/s}^2}{1.85\,10^{-7}\,\text{Nsm}^{-1}} \simeq 27.8\,\text{m/s}. \tag{5.29}$$

We will check how good this approximation is further on.

Simplified model: Constant velocity: What will happen as the drop starts to fall? It starts from zero velocity, hence the initial acceleration will be $a = -g - (k_v/m)v = -g$. As the drop falls, the velocity becomes a negative number, but with increasing magnitude. The acceleration, $a = -g - (k_v/m)v$ will therefore approach zero. However, if the acceleration becomes zero, the velocity will no longer change, and the drop will have reached a stationary velocity—a velocity that does not change with time. This occurs when

$$a = -g - \frac{k_v}{m}v = 0 \;\Rightarrow\; -g = \frac{k_v}{m}v \;\Rightarrow\; v = -\frac{mg}{k_v}. \tag{5.30}$$

We call this velocity the *terminal velocity*, v_T:

$$v_T = \frac{mg}{k_v}. \tag{5.31}$$

We therefore expect the drop to approach the velocity $v = -v_T$ as time increases.

Full model: Numerical solution: We now know both the initial behavior, $a = -g$, and the asymptotic behavior, $a \to 0\text{m/s}^2$, $v \to -v_T$. We can find the velocity by solving

$$a = \frac{dv}{dt} = -g - \frac{k_v}{m}v, \tag{5.32}$$

with initial conditions $v(t_0) = 0\,\text{m/s}$ using Euler's method:

$$v(t_i + \Delta t) = v(t_i) + \Delta t \cdot a(t_i, v(t_i)). \tag{5.33}$$

This method is implemented in the following program:

```
clear all; clf; % Physical constants
g = 9.81;
kv = 1.85e-7; % Nsm^-2
```

```
m = 5.2e-7; % kg
time = 20.0;
dt = 0.001;
v0 = 0.0;
y0 = 0.0;
n = time/dt;
v = zeros(n,1);
a = zeros(n,1);
t = zeros(n,1);
v(1) = v0;
for i = 1:n-1
  a(i) = -g - (kv/m)*v(i);
  v(i+1) = v(i) + a(i)*dt;
  t(i+1) = t(i) + dt;
end
```

Analysis of numerical solution: The resulting plot of $v(t)$ and $a(t)$ is shown in Fig. 5.10. We see that both $v(t)$ and $a(t)$ behaves as we expected. The drop starts with zero velocity and an initial acceleration of $-g$. The acceleration reduces as the velocity increases and the drop reaches a stationary state where it moves with constant velocity. We compare with the two simplified models. The behavior without air resistance is plotted as a dashed line, and it is indeed a reasonable approximation for small velocities, that is when $|v| \ll v_T = 27.5$ m/s. For long times the velocity approaches $v \rightarrow -v_T$, which is illustrated by the dotted line in the plot. The simplified solutions are therefore useful to check if our numerical solution is correct.

Full model: Analytical solution: Now that we have both found the simplified and the numerical solution, we are ready to attempt an analytical soultion. In this case we are fortunate, since the particular equation in (5.32) can be solved analytically using separation of variables. We simplify the equation by writing:

$$\frac{dv}{dt} = -g - \frac{k_v}{m} v = -g - \frac{g}{v_T} v = -g \left(1 + \frac{v}{v_T} \right), \tag{5.34}$$

with initial condition $v(0) = 0$ m/s. We separate v and t on each side of the equation:

$$\frac{dv}{1 + v/v_T} = -g dt, \tag{5.35}$$

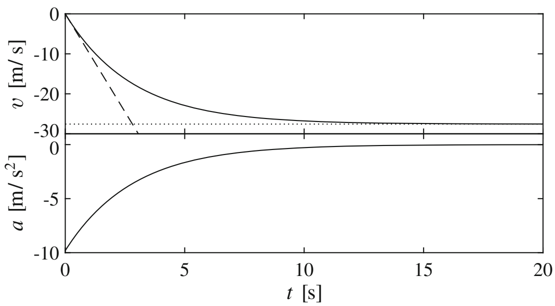


Fig. 5.10 Plot of $v(t)$ and $a(t)$ for the drop

and integrate each side from $t_0 = 0$ to t:

$$\int_{v_0}^{v(t)} \frac{dv}{1 + v/v_T} = -\int_0^t g dt. \tag{5.36}$$

We introduce $u = 1 + v/v_T$, $du = dv/v_T$, $dv = v_T du$:

$$\int_1^{1+v(t)/v_T} v_T \frac{du}{u} = -g(t - 0) \Rightarrow v_T \ln\left(1 + \frac{v(t)}{v_T}\right) = -gt, \tag{5.37}$$

$$\ln\left(1 + \frac{v(t)}{v_T}\right) = -\frac{gt}{v_T} \Rightarrow, 1 + \frac{v(t)}{v_T} = e^{-gt/v_T} \Rightarrow v(t) = v_T\left(e^{-gt/v_T} - 1\right). \tag{5.38}$$

Full model: Symbolic solution: You can solve (5.34) using the symbolic solver in matlab. First, we define the variables `g`, `vT` and `v(t)`

```
>> syms g vT v(t)
```

Matlab can then solve this initial value problem directly by

```
dsolve(diff(v)==-g-g/vT*v,v(0)==0)
ans =
 vT*exp(-(g*t)/vT) - vT
```

This corresponds to the solution

$$v(t) = v_T\left(e^{-gt/v_T} - 1\right). \tag{5.39}$$

In most cases, machines are much better than humans at integration. In your career as a physicist it is therefore more important to be able to formulate problems so that they can be solved numerically or symbolically than to be able to solve them analytically yourself.

Test your understanding: What would happen if the drop started with a velocity $v_0 = -2v_T$?

5.7 Force Model: Spring Force

The two most common contact forces for a macroscopic object are due to

- contact with a fluid or
- contact with another solid.

We have seen that we can use a velocity-dependent force to model fluid-solid contacts. What about solid-solid contacts? The contact forces between two solid objects come from the deformation of the objects (and from surface forces such as adhesion and friction, but we will address such effects later). How can we model contact forces due to deformation?

Spring Force

First, we can *measure* the force due to deformation directly. Figure 5.11 illustrates an experiment where we pull a rubber band and measure the force needed to extend the band a distance ΔL. Figure 5.11 shows that the force, F increases linearly with ΔL:

> The force F required to extend an object by ΔL is the **spring force**:
>
> $$F = k\Delta L, \tag{5.40}$$

This relationship is valid for small deformations ΔL for practically all materials.

Why do we call this force a *spring force*? Because a coiled spring is constructed so that the deformation force is proportional to the extension ΔL also for large extensions. Coiled springs are commonly used in experimental demonstrations and in many mechanical devices. Here, we usually base our discussions on the behavior of a spring, but keep in mind that the results are valid for the deformation of any object as long as the deformation is small compared to the size of the object.

Spring constant, k: The constant k is called the *spring constant*, which is the slope of the $F(\Delta L)$ curve. Figure 5.12 shows the behavior for a weak spring (1) and for a stiff spring (2). The stiff spring has a larger spring constant than the weak spring, $k_2 > k_1$.

If we want both springs to extend the same length ΔL, we need to apply a larger force to the stiffer spring than to the weaker spring: $F_2 > F_1$ (see left side in figure).

If we apply the same force to both springs, the weak spring extends further than the stiff spring: $\Delta L_1 > \Delta L_2$ (see right side in figure). This is consistent with our intuition: If we pull with the same force on a rubber band and on a stiff rope, the

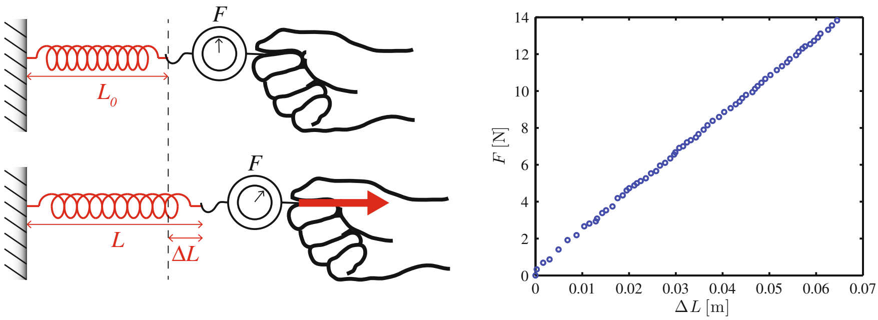


Fig. 5.11 Illustration of an experiment to measure the force needed to extend a spring a distance ΔL, and a plot of the force, $F(\Delta L)$

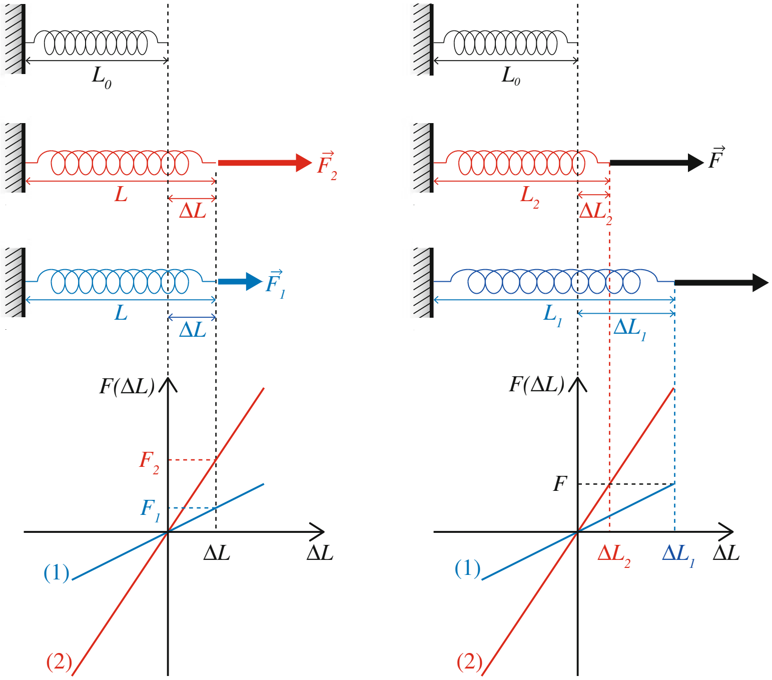


Fig. 5.12 Illustration of a stiff spring (1) and a weak spring (2) extended by the same length ΔL (on the *left*), and affected by the same force F (on the *right*)

rubber band deforms more than the rope. Hence the rubber band has a smaller spring constant than the rope.

The spring constant characterizes a particular object. If we redo the experiment with the same spring, we find the same spring constant every time.

Elongation, ΔL: A physical object such as a spring has a non-zero length when it is not stretched. We call this length the *equilibrium length*, L_0, of the spring. The elongation of the spring is the difference between the length of the spring, L, and the equilibrium length L_0:

$$\Delta L = L - L_0. \tag{5.41}$$

Various elongations are illustrated in Fig. 5.12. If the spring is stretched, ΔL is positive. We usually assume that a spring can be also compressed, even if this may not be physically realistic. (You cannot really compress a rope, it will buckle instead). For a compressed spring, the length of the spring is smaller than the equilibrium length, and ΔL is negative.

Sign: You have to determine the sign in front of $k\Delta L$ in (5.40) using your physical intuition. It is not that difficult: All you have to remember is that in order to extend the spring, you need to pull at the spring in the direction you extend it.

Spring-Block Models

We now know that the force required to extend a spring a length ΔL is $F = k\Delta L$. How can we use this to determine the motion of an object *attached to a spring*? In Fig. 5.13, we have illustrated two general cases: A block attached to a spring, and a ball in contact with a spring. Both objects slide on a frictionless surface, so that there are no other horizontal forces: The only horizontal force acting on the block (ball) is from the spring.

A Block Attached to a Spring

The force, F, on the block from the spring is the same as the force acting on the spring from the block, but it has the opposite direction. (Later, when we introduce Newton's third law, we see that these forces are action-reaction pairs.)

$$F(\Delta L) = -k\Delta L. \tag{5.42}$$

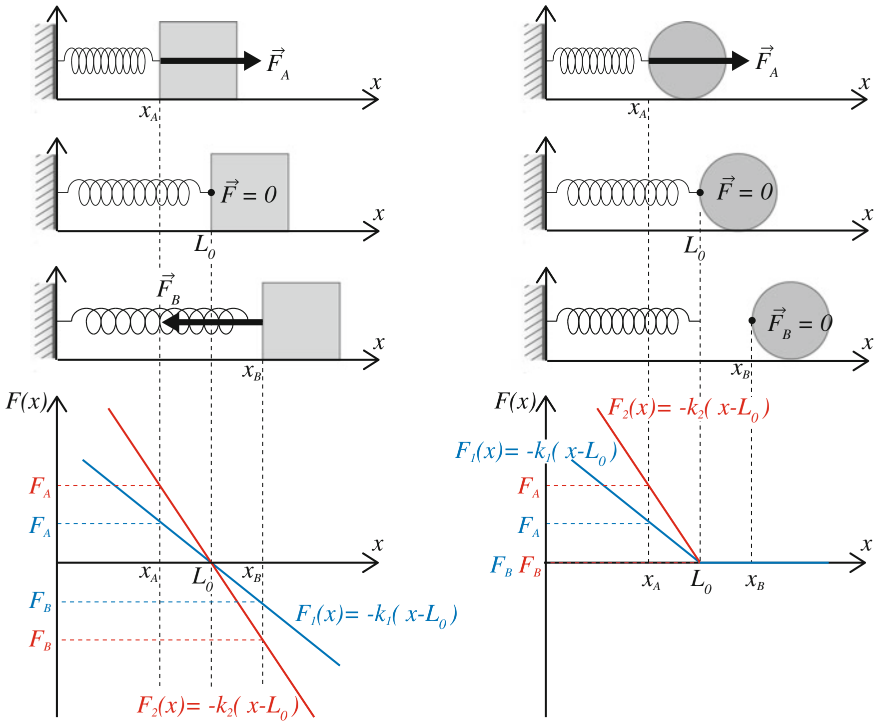


Fig. 5.13 Illustration of the force on a block attached to a spring (*left*), and on a ball colliding with (but not attached to) a spring (*right*)

The sign is chosen so that when the spring is extended—it is longer than in equilibrium—the force is negative, that it, the force acts in the negative x-direction.

We describe the position of the block by the position, x, of the left-hand side of the block, where it is attached to the spring. The other end of the spring is attached to a wall at $x = 0$. The extension of the spring depends on the position of the block:

$$\Delta L = L - L_0 , \tag{5.43}$$

where $L = (x - 0) = x$ is the length of the spring.

The force F on the block consequently depends on the position of the block:

$$F = F(x) = -k\,(x - L_0) . \tag{5.44}$$

The spring force acting on an object attached to the spring is therefore an example of a **position-dependent force**.

Why is it important if a force is position-dependent? Because this means that the position occurs on both sides of the equation of motion following from Newton's second law:

$$F = ma \Rightarrow a = \frac{d^2x}{dt^2} = -\frac{k}{m}\,(x - L_0) . \tag{5.45}$$

We need to solve a differential equation to determine the motion of the block.

This also illustrates another important lesson: If the force model includes variables that change during motion, we need to rewrite the force model to include the position of the object. A common mistake is to stop at (5.42), where the force, $F(L)$, depends on the length L, and not realize that L is a function of the position, $L = L(x)$, and that the equation must be solved as a differential equation.

A Ball Colliding with a Spring

The ball in Fig. 5.13 is not attached to the spring—it is not glued to the spring as the block was. The ball is therefore only affected by the spring force as long as the spring is compressed, that is, when $\Delta L < 0$. We must rewrite the force model to reflect this:

$$F(\Delta L) = \begin{cases} -k\Delta L \ , & \Delta L < 0 \\ 0 \ , & \Delta L \geq 0 \end{cases} \tag{5.46}$$

The rest of the analysis is exactly the same as for the ball: The spring force is position dependent, and can be written as:

$$F(\Delta L) = \begin{cases} -k\,(x - L_0)\ , & x < L_0 \\ 0\ , & x \geq L_0 \end{cases} \tag{5.47}$$

Notice that this force model is non-linear and the problem therefore requires special consideration when you solve it: You must remember that the force is zero as soon as the ball loses contact with the spring.

Notice also that the ball-spring contact is a lot like the normal force. This is not a coincidence. The spring-ball model is actually our simplest model for the normal force, as we discuss in the following section, and the spring-block model is the simplest model for a contact force between two attached objects. These are probably the models you will use the most during your physics studies.

Contact Forces

The spring force was introduced as an approximative model for the force due to deformation. It is based on experimental evidence: We find the law by measuring the force as a function of the deformation. And the law is surprisingly versatile: We can use it as a model for almost any contact force between solid objects. Let us see exactly how we map a complicated, realistic deformation problem onto the simplified spring model description.

Figure 5.14 shows a computer simulation of a collision between a soft, round disk and a solid wall that does not deform. The top figures show the deformation of the ball, where the colors indicate the magnitude of the forces inside the disk. During the collision, the ball is deformed, but how do we measure the extent of deformation? The simplest approach is to look at how much the ball has changed from its original, round shape. This measurement of ΔL is illustrated in the bottom figures, where we see that the original shape overlaps with the wall, and we define ΔL as the length of this overlap. The middle figure shows the force from the wall on the ball as a function of deformation, ΔL, and it is indeed a linear function, $F = k\Delta L$, as we found for a spring!

We can therefore use the spring model as a model for the deformation of a ball, or as a first approximation for the deformation of any deforming object. In this case we say that we use the spring model as an approximative model for the contact force. In the bottom of Fig. 5.14 we illustrate that we model the contact between the disk and the floor by representing the disk as a mass located at the center of the sphere connected to a massless spring of length $L_0 = R$ equal to the radius of the sphere.

We may also express the force in terms of the position x of the center of the ball. We notice that the spring is compressed, $\Delta L = L - L_0 = x - R < 0$, during the collision, and we expect the force on the ball to act upward while the ball is compressed, hence:

$$F = -k\Delta L = -k\,(x - R), \tag{5.48}$$

is the force on the ball from the wall.

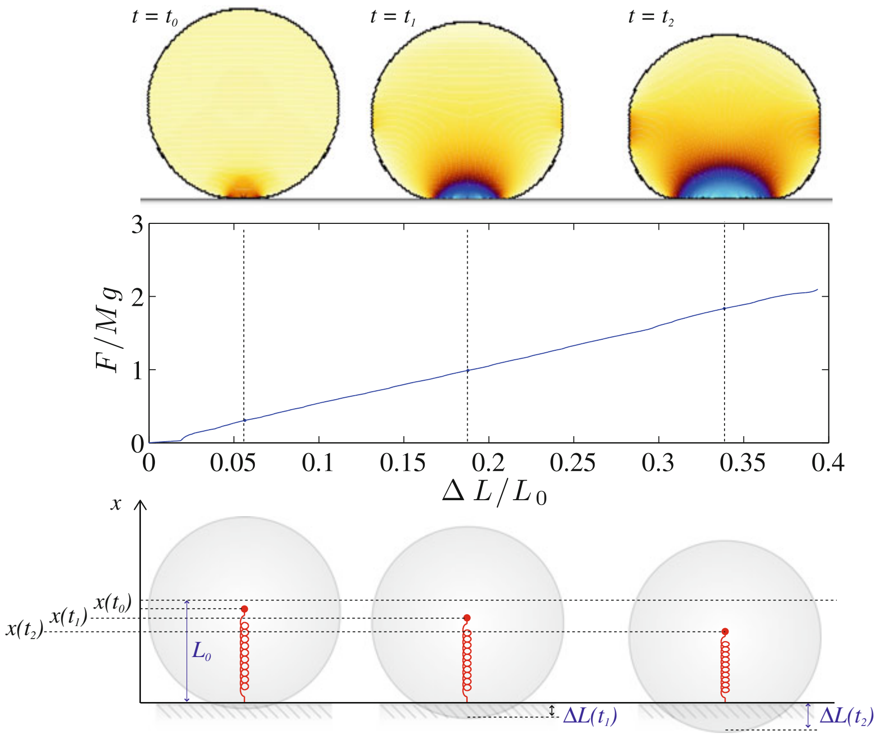


Fig. 5.14 (*Top*) Computer model of a soft ball colliding with a solid wall. (*Middle*) Plot of the force from the wall on the ball as a function of displacement. (*Bottom*) Illustration of how the displacement ΔL is measured, and an illustration of the spring-block model used to represent the soft ball

Generally, we do not know what the spring constant will be for such a model. We need either to measure the spring constant or to find the spring constant from a theoretical consideration based on for example elasticity theory. You must also ensure that you use a reasonable version of the spring model. For example, for the collision between a ball and a wall in Fig. 5.14, where the ball does not adhere to the wall, we must use a spring-ball model without attachment.

We can use the spring model both when the object itself is deformed, as illustrated in Fig. 5.14, as well as when the wall deforms while the ball remains practically undeformed—a steel ball bouncing on a mattress, or when both the ball and the wall deforms.

Test your understanding: How would you represent the force between an undeformable ball and a deformable surface? Make a drawing, and show how you introduce the spring that models the deformation.

Normal and Contact Forces

The force from the wall on the ball is the *normal force* on the ball. However, a normal force only acts as long as the objects are in contact.

We can **model a normal force** between two objects using a spring model:

$$F = \begin{cases} k\Delta L & \text{while in contact} \\ 0 & \text{otherwise} \end{cases} , \tag{5.49}$$

where ΔL describes the deformation of the two objects.

For the ball in Fig. 5.14 the normal force F is:

$$F = \begin{cases} -k(x - R) \ , & x < R \\ 0 \ , & x \geq R \end{cases} . \tag{5.50}$$

We can use this force model to find the motion of a bouncing ball if we also include the effect of gravity, $G = mg$. (We assume air resistance is negligible). The acceleration is then found from:

$$a = \frac{1}{m} F^{\text{net}} = F - G, \tag{5.51}$$

and the resulting motion is illustrated in Fig. 5.15.

The normal force only acts when two objects are pressed together. On the other hand, if the objects are *attached* to each other, the contact force acts both when the objects are pressed together and when they are pulled apart. Examples of attached objects are: two objects that are glued or welded together or attached by surface forces such as adhesion, two parts of one larger solid body such as the two lobes on a dumbbell, or two atoms in a diatomic molecule.

We can **model an attachment force** between two objects using a spring model:

$$F = k\Delta L, \tag{5.52}$$

where ΔL represents the elongation of the contact, which is typically equal to the change in distance between the centers of the two objects in contact.

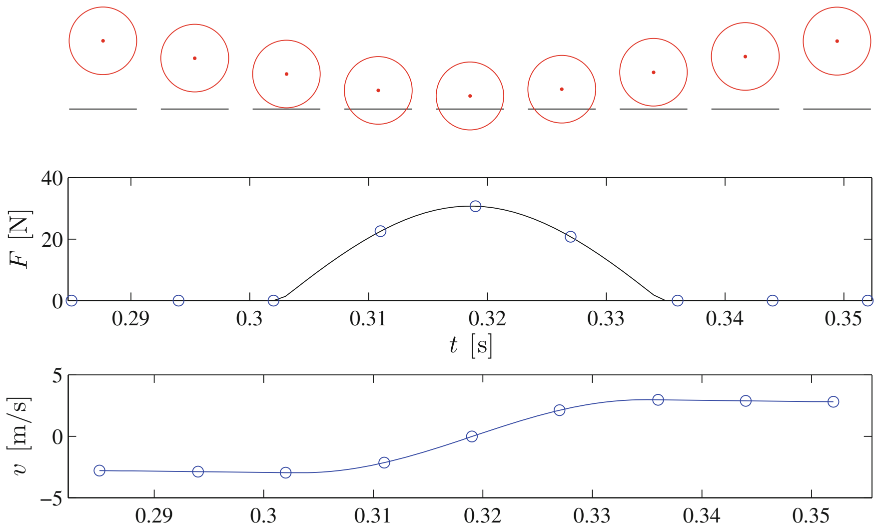


Fig. 5.15 Illustration of a ball bouncing on a floor, the normal force $F(t)$, and the velocity $v(t)$ before, during, and after the bounce

Contact Forces

The spring model is not only a model for the deformation of springs. It is an extremely versatile model that provides a good description of the deformation of almost any object, spanning length scales from the deformation of the Earth's surface to the deformation of objects only a few atom diameters in size. It describes the deformation of rubber bands, strings, wires, wheels, bars, and the interactions between molecules and atoms. However, the spring model is typically only valid for small deformations: the deformation should be small compared to the size of the system.

The spring model is probably the most powerful and common model for interactions that you will encounter in physics. Why is that? Because many types of forces depend on the position of an object relative to another object: Contact forces, gravitational forces, electromagnetic forces, and inter-atomic forces all depend on the position, x, of the object they act on: $F = F(x)$. For motion near a point $x = b$, we can approximate the force on the object with Taylor's formula for $F(x)$:

$$F(x) = F(b) + F'(b)(x - b) + \mathcal{O}(x - b)^2. \tag{5.53}$$

The force on the object at a position x near b, that is when $(x - b)$ is small, can be approximated by the first order term:

$$F(x) \simeq F(b) + F'(b)(x - b). \tag{5.54}$$

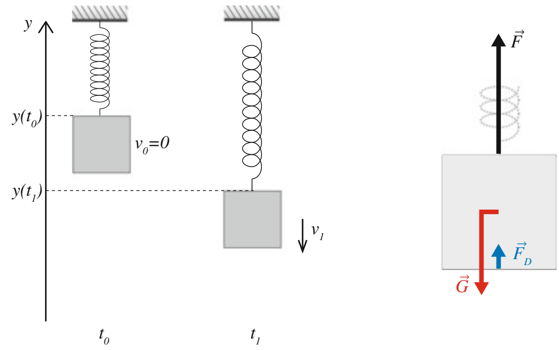


Fig. 5.16 Sketch of a block hanging in a spring (*left*), and the free-body diagram of the system (*right*)

This force model has the same form as the contact force model in (5.50) except for the constant $F(b)$. However, in most cases we measure the displacement from an equilibrium position where no force is acting and therefore $F(b) = 0$.

The spring model corresponds to a Taylor expansion of a position-dependent force $F(x)$ around an equilibrium point, b:

$$F(x) \simeq F'(b)\,(x - b) = k\,(x - b) \tag{5.55}$$

where $k = F'(b)$.

For example, for the ball in contact with the wall in Fig. 5.14, the deformation of the ball is small, and we can use the spring model as an approximation to the real deformation force.

5.7.1 Example: Motion of a Hanging Block

This example demonstrates how we can find the motion of an object affected by a spring force, using both numerical and analytical methods. The methods introduced can be applied to find the behavior of an object subject to any type of position-dependent force.

A block of mass $m = 1$ kg is hanging from a spring with spring constant $k = 100$ N/m. The other end of the spring is attached to the ceiling. We apply the structured problem-solving approach to find the motion of the block after it is released, and then make the model more realistic by adding air resistance.

Sketch and Identify: The *system* is the block, and the environment consists of the spring, the ceiling, the ground, and the surrounding air. The motion of the block is illustrated in Fig. 5.16, where $x(t)$ is used to describe the position of the block.

Model: We use the free-body diagram in Fig. 5.16 to identify the forces acting on the block: It is affected by a contact force, **F**, from the spring, air resistance, $\mathbf{F}_D$, and gravity $\mathbf{G} = -mg\mathbf{j}$. We use a spring force model for the force from the spring:

$$F = \pm k\Delta L, \tag{5.56}$$

where the elongation ΔL depends on the position, y, of the block. For simplicity, we place the coordinate system so that the spring is in equilibrium when $y = 0\,\text{m}$, so that $y = \Delta L$ as illustrated in Fig. 5.16. If the block is pulled down in the negative y-direction, the spring will act to pull the block up in the positive direction, therefore the correct sign of the spring force is:

$$\mathbf{F}(y) = -ky\,\mathbf{j}. \tag{5.57}$$

Newton's second law: Newton's second law along the y-axis gives

$$\mathbf{F}^{\text{net}} = \mathbf{F} + \mathbf{G} + \mathbf{F}_D = F\,\mathbf{j} + G\,\mathbf{j} + F_D\,\mathbf{j} = ma\,\mathbf{j}, \tag{5.58}$$

where we remove the vectors to get:

$$F^{\text{net}} = F + F_D + G = -ky + F_D - mg = ma. \tag{5.59}$$

Equilibrium model: First, let us consider the equilibrium situation—where the block does not move when released. Since the block does not move, air resistance is zero, $F_D = 0$, and the acceleration is also zero, which gives:

$$-ky + F_D - mg = -ky - mg = ma = 0, \tag{5.60}$$

which gives

$$y_{eq} = -\frac{mg}{k}, \tag{5.61}$$

for the equilibrium position of the block.

What would you expect to happen if you instead released the block from a position above this equilibrium position? We would expect the block to oscillate, but the oscillations would grow smaller, and the block would eventually stop at the equilibrum position. Does our model system reproduce this behavior?

Simplified model—No air resistance: We start from a simplified model, where we assume that air resistance is negligible, so that $F_D = 0$. From (5.59) we see that the acceleration is

$$a = \frac{d^2x}{dt^2} = -g - \frac{k}{m}y, \tag{5.62}$$

and the block starts at rest, $v(t_0) = 0\,\text{m/s}$, at $y(t_0) = 0\,\text{m}$, where $t_0 = 0\,\text{s}$.

Simplified model—Numerical solution: We can find the motion of the block using an Euler-Cromer scheme. Generally, we advice you to use a fourth-order Runge-Kutta method for oscillator problems, but we use Euler-Cromer here to make the programming transparent. (Notice that the direct Euler scheme is not stable for this equation). The Euler-Cromer scheme for (5.62) reads:

$$\begin{aligned} v(t_i + \Delta t) &= v(t_i) - \frac{k}{m} x(t_i) - g\,\Delta t \\ x(t_i + \Delta t) &= x(t_i) + v(t_i + \Delta t)\,\Delta t \end{aligned} \tag{5.63}$$

which is implemented in the following program:

```
m = 1.0;     % kg
k = 100.0; % N/m
g = 9.8; % m/s^2
v0 = 0.0;    % in m/s
time = 2.0; % s
dt = 0.0001; % s
n = ceil(time/dt);
t = zeros(n,1);
y = zeros(n,1);
v = zeros(n,1);
y(1) = 0.0;
v(1) = v0;
for i = 1:n-1
    F = -k*y(i) - m*g;
    a = F/m;
    v(i+1) = v(i) + a*dt;
    y(i+1) = y(i) + v(i+1)*dt;
    t(i+1) = t(i) + dt;
end
```

Here we chose a particular value for the time step, Δt, but how was this value chosen? Generally, we try to choose Δt as small as is practically possible: Small enough to ensure that the error is small, but not so small that the calculation takes too long time. In this case, the time step must be much smaller than time it takes for the block to swing back a forth one time, otherwise the results will not make any sense. However, it is a good rule to check your results by reducing your time-step by a factor 10 and observing if your solution is stable to such a change.

While the resulting motion is illustrated in the plots in Fig. 5.17, the motion becomes clearer by visualizing the dynamics using the simplest possible tool: the plotting function. We use a sequence of `plot` commands to give an impression of the dynamics by adding the following program lines

```
for i = 1:10:n-1
    subplot(2,1,1)
    plot(t,y,'-b',t(i),y(i),'ob');
    xlabel('t [s]')
    ylabel('y [m]')
    drawnow
    subplot(2,1,2)
    plot(t,v,'-b',t(i),v(i),'ob');
    xlabel('t [s]')
    ylabel('v [m/s]')
    drawnow
end
```

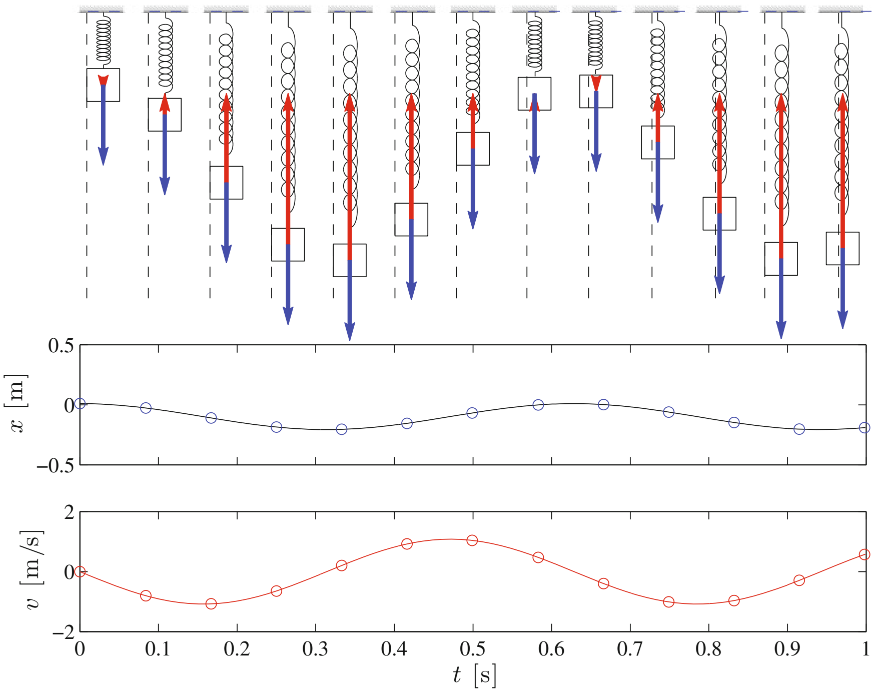


Fig. 5.17 Illustration of block position and plot of $y(t)$ and $v(t)$ from a numerical solution. The *top figures* illustrates the position of the block, and the forces acting on the block. *Red arrow* shows the spring force and *blue arrow* shows gravity

Notice that we are not showing every time step, but jump in steps of 10 using `1:10:n` in the `for`-loop. You should tune this to a number that gives a suitable dynamics on your computer. Using this tool, you can build your intuition of the impact of various variables. For example, you can check what happens if you change the initial velocity, v_0, or the spring constant k, for example, you could try changing the initial velocity to $v_0 = -0.1$ m/s, or you could set $y_0 = 0.1$ m and $v_0 = 0.0$ m/s.

If you want to make a more realistic visualization of the motion of the block, we can draw a block with a reasonable size around the center of the block, and then show the motion of the block by going through the sequence of time step. This is done in the following program, where we have used the function `rectangle` to draw the block.

```
figure(3);
d = 0.1; % Block size
for i = 1:10:n-1
    plot(y(i),0,'o');
    xlabel('x [m]')
    ylabel('y [m]')
    mycurvature = 0.0;
    rectangle('Position',[y(i)-d -d 2*d 2*d],...
        'Curvature',[mycurvature mycurvature],'EdgeColor','blue');
    axis equal
```

```
    axis([-1 1 -1 1]);
    drawnow
end
```

Notice that you must use the command `drawnow` to force matlab to draw the plot immediately. Also notice that we have changed the axes to ensure that block fits inside the axes all the time, otherwise the axes will move during the simulation and the visualization will be confusing.

Simplified model—Analytical solution: (*You may skip this section without loss of continuity—solving differential equations often involves tricks that require experience, often are non-intuitive and not simple to follow*.)

For this particular problem, we can also find the exact solution, which means finding a function $y(t)$ that satisfies (5.62) and the initial conditions. The problem is simplified by rewriting the equation using the equilibrium position $y_{eq} = -mg/k$

$$a = -g - (k/m)\,y = -\,(k/m)\left(y - y_{eq}\right), \tag{5.64}$$

where we introduce a new variable, $u = y - y_{eq}$. Since y_{eq} is a constant, this does not change the second derivative with respect to time:

$$\frac{d^2u}{dt^2} = \frac{d^2y}{dt^2} = -\,(k/m)\,u, \tag{5.65}$$

where the initial conditions now are $u(0) = y(0) - y_{eq} = -y_{eq}$ and $du/dt(0) = v(0)$.

Finding an analytical solution means finding a function $y(t)$ that satisfies (5.65) and the initial conditions. If we have found one such function, we can be sure this is *the* solution, because there is a uniqueness theorem in the mathematics of ordinary differential equations.

What functions become minus themselves after being derived twice? You may know (if you have already learned this trick), that this is true for the trigonometric functions sin and cos. The general solution to (5.65) is:

$$u(t) = A\cos(\omega t) + B\sin(\omega t). \tag{5.66}$$

If we insert (5.66) in (5.65) we find that $\omega = \sqrt{k/m}$. The two prefactors A and B must be determined from the initial conditions:

$$u(0) = A\cos(0) = A = -y_{eq} = mg/k, \tag{5.67}$$

and

$$du/dt(0) = B\omega\cos(0) = 0 \;\Rightarrow\; B = 0 \tag{5.68}$$

So the complete solution is

$$y(t) = u(t) + y_{eq} = (mg/k)\cos\omega t - (mg/k). \tag{5.69}$$

(Please do not be discouraged if you did not understand how we found the solution in this case. You will solve this equation many times in your career, and each time you will learn to know it better. Eventually, it will become a natural part of your knowledge base.)

Simplified model—Symbolic solution: You can solve (5.62) using the symbolic package in matlab. First, you need to define all the relavant variables in the problem, where we introduce $q = k/m$ for simplicity:

```
>> syms g q y0 y(t)
```

Then, we solve the differential equation, which includes the second derivative of $y(t)$, which is written as `diff(y,2)`. In addition, we need to provide the initial conditions at both $y(0) = y_0$ and $dy/dt(0) = 0$. This is done by first introducing a function for the derivative of $y(t)$

```
>> Dy = diff(y);
```

We can then use `dsolve` to find the analytical solution:

```
>> dsolve(diff(y,2)==-g-q*y,y(0)==y0,Dy(0)==0)
ans = (exp((-q)^(1/2)*t)*(g + q*y0))/(2*q) -
    ...  g/q + (exp(-(-q)^(1/2)*t)*(g + q*y0))/(2*q)
```

This is the correct result, but it is in a form which you may not recognize, unless you have some experience with complex numbers. The key lies in the $(-q)^{1/2}$ factor. In our case, q is a positive number, but we have not told the symbolic solver this. We can redefine `q` and try again:

```
>> q = sym('q','positive');
>> dsolve(diff(y,2)==-g-q*y,y(0)==y0,Dy(0)==0)
ans = cos(q^(1/2)*t)*(y0 + g/q) - g/q
```

This is an answer that is much easier to recognize as it, of course, corresponds to the solution in (5.69).

Simplified model—Analysis: The solution we have found so far show an everlasting, oscillating motion:

$$y(t) = (mg/k)\cos(\omega t) - (mg/k). \tag{5.70}$$

The block oscillates up and down with a period $T = 2\pi/\omega$. However, this is not what we would expect in a realistic situation, where we expect the motion to be damped and that the block eventually comes to rest. We can study one way this may happen by introducing air resistance.

Full model—Numerical solution: Now, we assume that the block also is affected by a non-negligible air resistance force, F_D described by an advanced model for air resistance: At large velocities the air resistance force is described by a quadratic law with drag coefficient $D = 0.15\,\text{m}^{-1}$, and at low velocities it is described by a viscous drag force with drag constant k_v. We assume that the transition occurs at $v_t = 0.01$ m/s:

$$F_D(v) = \begin{cases} -Dv(t)|v(t)| \ , & v > v_t \\ -k_v v(t) \quad , & v < v_t \end{cases} \tag{5.71}$$

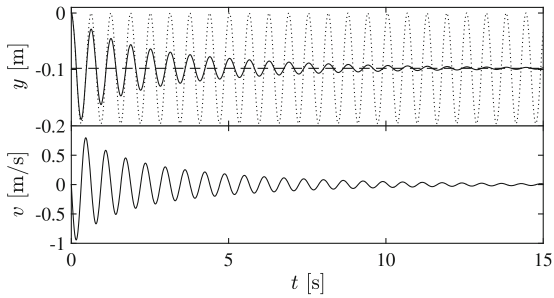


Fig. 5.18 Plot of the position $y(t)$ and velocity $v(t)$ of the block when air resistance forces are included

A continuous force requires the two behaviors to be the same at $v = v_t$, that is:

$$-Dv_t^2 = -k_v v_t \Rightarrow k_v = Dv_t. \tag{5.72}$$

(You are not yet expected to come up with a law like this, but you should be able to solve problems if you are given such a law).

This gives us a force model for F_D and we can find the motion of the block using a numerical scheme such as Euler-Cromer, as implemented in the following program. Notice the use of the `if`-statement in order to test if the ball is experiencing the high-velocity or the low-velocity air resistance force:

```
m = 1.0;      % kg
k = 100.0;    % N/m
g = 9.8;      % m/s^2
v0 = 0.0;     % in m/s
time = 15.0; % s
D = 2.5;      % m^-1
vt = 0.2;     % m/s
kv = D*vt;
dt = 0.0001; % s
n = ceil(time/dt);
t = zeros(n,1);
y = zeros(n,1);
v = zeros(n,1);
y(1) = 0.0;
v(1) = v0;
for i = 1:n-1
    if (v(i)<vt)
        FD = -kv*v(i);
    else
        FD = -D*v(i)*abs(v(i));
    end
    F = -k*y(i) - m*g + FD;
    a = F/m;
    v(i+1) = v(i) + a*dt;
    y(i+1) = y(i) + v(i+1)*dt;
    t(i+1) = t(i) + dt;
end
```

The resulting behavior is shown in Fig. 5.18. The dashed line shows the equilibrium solution, $y = y_{eq}$, and the numerical solution does indeed converge towards this. The dotted line shows the solution without air resistance, which demonstrates that the air resistance does not affect the oscillation period significantly.

5.8 Newton's First Law

What happens if the net external force on a body is zero? Applying Newton's second law, we find:

$$\mathbf{F}_{\text{net}} = 0 = m\mathbf{a} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = 0. \quad (5.73)$$

The acceleration is zero, which means that the velocity of the object is constant. This is often referred to as Newton's first law:

> **Newton's first law**: An object in a state of uniform motion tends to remain in that state unless an external force changes its state of motion.

Why do we need a separate law for this? Is it not simply a special case of Newton's second law? Yes, Newton's first law can be deduced from the second law as we have illustrated. However, the first law is often used for a different purpose: Newton's First Law tells us about the limit of applicability of Newton's Second law. Newton's Second law can only be used in reference systems where the First law is obeyed. But is not the First law always valid? No! The First law is only valid in reference systems that are not accelerated. If you observe the motion of a ball from an accelerating car, the ball will appear to accelerate even if there are no forces acting on it. We call systems that are not accelerating *inertial systems*, and Newton's first law is often called the law of inertia. Newton's first and second laws of motion are only valid in inertial systems. We will discuss references systems and inertial systems in detail when we discuss motion in two and three dimensions in Chap. 6.

A system is an inertial system if it is not accelerated—that is, the reference system must not be accelerating linearly or rotating. Unfortunately, this means that most systems we know are not really inertial systems. For example, the surface of the Earth is clearly not an inertial system, because the Earth is rotating. The Earth is also not an inertial system, because it is moving in a curved path around the Sun. However, even if the surface of the Earth is not strictly an inertial system, it may be considered to be approximately an inertial system for many laboratory-size experiments.

5.9 Newton's Third Law

So far, we have studied the motion of a single object by considering the interactions between the object (the system) and the environment (everything else). But most problems we deal with include an interplay between several objects. How can we address such systems?

We already started to address systems with several components when we addressed forces between a system and its environment. When you press your finger toward the table, you experience a force, and the table experiences a force. How are these forces related?

First, we realize that they act on different objects. There is a force from the finger on the table, and a force from the table on the finger. If the finger is the system, there is a contact force on the finger from the table, which is in the environment. On the other hand, if the table is the system, there is a contact force on the table from the finger, which is the environment in this case.

Could we get around this by introducing a more precise description of the interaction between the table and the finger? We argued that the top of the table is really not flat but rather rough on a microscopic scale—its surface consists of microscopic bumps. When I press my finger on the table, I press onto these small bumps, so that the bumps are deformed, and the bumps push down on the table. However, when my finger pushes toward the bumps, there is still a force from the finger on the bumps and a force from the bumps on my finger. Similarly, the bumps press down on the rest of the table, and the rest of the table pushes up on the bumps.

We realize that all forces, all interactions between objects, come in pairs. If there is a force from object A on object B, there is also a force from object B on object A. This fundamental principle of interactions is called Newton's third law. We do not know of any force that do not obey this law: All forces appear in pairs. Newton's third law is usually formulated as:

Newton's third law of motion: For every action there is an equal and opposite reaction.

This is a classical formulation of Newton's third law. The words "action" and "reaction" here means force and counter-force. If you push with your finger (F) on the table (T), there is a force, an action, $\mathbf{F}_{\text{from F on T}}$, from the finger on the table. Typically we write this by subindicies:

$$\mathbf{F}_{\text{from F on T}}, \tag{5.74}$$

for the force from F on T.

Newton's third law then states that there is an equal and oppositely directed force from the table on the finger. That is:

$$\mathbf{F}_{\text{from T on F}} = -\mathbf{F}_{\text{from F on T}}, \tag{5.75}$$

It is important to realize that the two forces in the force-pair *act on different objects*.

Application of Newton's Third Law

Let us introduce and apply Newton's third law through a simple example. We address a two crates lying on top of each other on the ground, as illustrated in Fig. 5.19a. The system consists of three bodies: the top crate (A), the bottom crate (B), and the ground (E). The ground is really the whole Earth, and we therefore use the symbol (E). We use the standard procedure to establish the free-body diagram for the compound system.

Drawing a free-body diagram for a compound system:
Follow these steps to find and identify all the forces acting on an each object, and then to draw the free-body diagram for each of the objects the system.

- Draw each object as separate systems.
- Find all forces on all objects, and draw them as vectors.
- Express the forces as $F_{\text{A on B}}$.
- Identify action-reaction pairs.
- Check that every force has a unique reaction, and that they act on different objects. Draw in the axes of the coordinate system.

1. **Draw all objects as separate systems**. This is done in Fig. 5.19b. We draw the systems apart, so that we have room to fill in more details such as forces and coordinate systems for the various systems.
2. **Find all forces on all objects, and draw them as vectors**. This is done in Fig. 5.19c. For each object, we find where it is in contact with other objects, or with the environment, and draw in the contact forces. Finally, we add the long-range forces. Here, the only long-range force is gravity. Note that gravity acts between all the objects, but we have only included the gravitational forces between the Earth and each of the two crates, since the gravitational force between the two crates are negligible.
3. **Express the forces as** $F_{\text{from A on B}}$. For each of the forces we find what object it is acting on, and in what object the force has it origin. This is shown in Fig. 5.19d.
4. **Identify action-reaction pairs**. We draw a dotted line between each action-reaction pair. All force in our figure should have an origin either in one of the other objects, or in the environment. For clarity, we have placed the gravitational force from the crate on the Earth, in a point close to the surface of the Earth. In reality, these forces act in the center of the Earth. We have illustrated this process in Fig. 5.19e
5. Check that **every force has a unique reaction**, and that they act on different objects. Draw in the axes of the coordinate system.

The result of this analysis is a free-body diagram for each of the objects. The proposed method is elaborate. You may argue that it is too elaborate. As you become experienced, you no longer need to be this rigorous in your approach. An expert

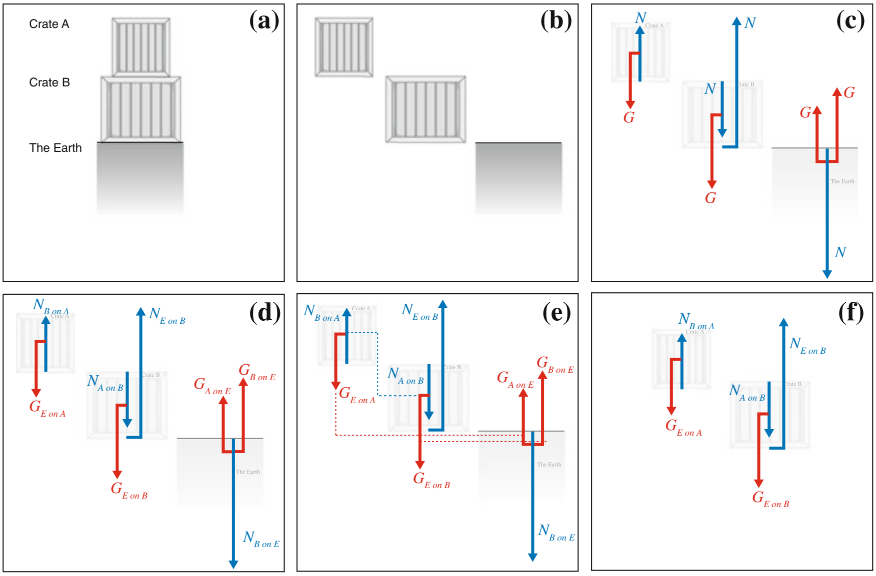


Fig. 5.19 Illustration of two crates lying on the ground. Various steps in the process of designing the free-body diagram are illustrated

would draw the free-body diagram directly, but would still recognize action-reaction pairs in the drawing, even though he will probably not mark them.

When you become more experienced in recognizing action-reaction pairs, you may also exclude The Earth from the drawing in Fig. 5.19e. Instead, you will draw two objects, and consider everything outside the two objects to be the environment. You can do this, because you are typically not interested in the motion of the Earth, but only the motion and in the forces on the two crates. In this case, you will have two types of forces in your drawing of one of the objects: either a force has its origin in one of the other objects, in this case your free-body diagrams include the action-reaction pair, or a force has its origin in the environment. This is illustrated in Fig. 5.19f. Here, the two gravitational forces on the two crates, and the normal force on the lower crate, all have their origin in the environment, and their reactions are therefore not included in the figure.

One of the most common mistakes when applying Newton's third law is to assign an incorrect action-reaction pair. For example, it is common to assign the normal force $F_{\text{from B on A}}$ to be the reaction force to $G_{\text{from E on A}}$ in Fig. 5.19d. Such an error can be spotted in two ways: First, the action-reaction pair in this case acts on the same object. They must always act on different objects according to Newton's third law. Second, the two forces are due to quite different mechanisms and contacts. For contact forces, the action and reaction forces act in the same point. For long-range forces, the action-reaction pair is due to the same type of interaction between the two objects. In Fig. 5.19d the reaction force to the gravity from the Earth on crate A must therefore also be a gravitational force and also between the same two objects.

Structured Approach to Compound Problems

With the addition of Newton's third law, we now have sufficient tools to address the forces between objects in problems with several moving components. We call such systems *compound systems*. Solving problems with several components are not more complicated than solving problems with just a single object—we simply apply the structured problem-solving approach to each of the objects.

5.9.1 Example: Weight in an Elevator

In this example you learn to analyze problems with several bodies, using Newton's third law to relate force pairs, and apply Newton's second law to find the motion of each body in the system.

Problem: A person of mass m is standing in an elevator of mass M. The person is standing on spring-weight with negligible mass and spring constant k. Find the force shown by the weight and the force in the wire if both the elevator and the person is moving with a constant acceleration a. Discuss what really happens as the elevator starts moving from rest.

Our plan is to find the forces acting on each object and use Newton's second law to find the acceleration of each object. If all objects are moving with the same acceleration, we can use this to find the forces between the object. In the second part the objects do not necessarily have the same acceleration, and we need to introduce force models for the internal forces to relate the positions.

Sketch and identify: In this problem we are studying the motion of the elevator (E), the weight (W), and person (P) as illustrated in Fig. 5.20, described by the corresponding positions, $x_E(t)$, $x_W(t)$, and $x_P(t)$.

Model: We find the forces acting on each object separately in three free-body diagrams (see Fig. 5.20). The *elevator* is affected by the external force T from the wire, the contact force $N_{\mathrm{W\ on\ E}}$ from the weight, gravity, $G_E = m_E g$. The *weight* is affected by the contact force $N_{\mathrm{E\ on\ W}}$ from the elevator and the contact force $N_{\mathrm{P\ on\ W}}$ from the person. The gravitational force is negligible, $G_W = 0$. The *person* is affected by the contact force $N_{\mathrm{W\ on\ P}}$ from the weight and by gravity, $G_P = m_P g$. We notice that the contact forces $N_{\mathrm{E\ on\ W}}$ and $N_{\mathrm{W\ on\ E}}$ are action-reaction pairs, and similarly for $N_{\mathrm{P\ on\ W}}$ and $N_{\mathrm{E\ on\ W}}$. We apply Newton's second law in the vertical direction for each object:

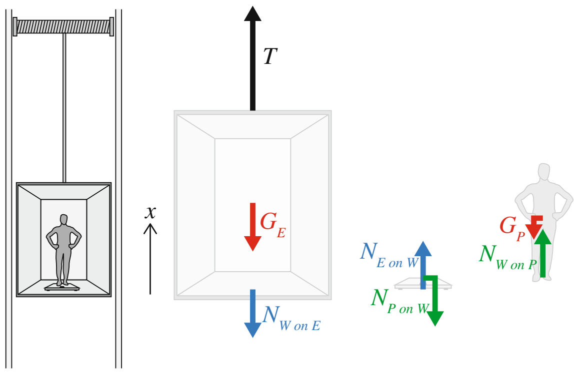


Fig. 5.20 Illustration of the elevator system and free-body diagrams for the elevator, the weight, and the person

Elevator:

$$\sum_j F_{j,E} = T - N_{\text{W on E}} - G_E = m_E a_E . \tag{5.76}$$

Weight:

$$\sum_j F_{j,W} = N_{\text{E on W}} - N_{\text{P on W}} = \underbrace{m_W}_{=0} a_W = 0 . \tag{5.77}$$

Person:

$$\sum_j F_{j,P} = N_{\text{W on P}} - G_P = m_P a_P . \tag{5.78}$$

Notice that the directions of the forces are included in the choice of signs. This means that $N_{\text{W on P}}$ is the magnitude of this contact force. Newton's third law therefore gives that $N_{\text{W on P}} = N_{\text{P on W}} = N_P$, where we have written N_P for simplicity, and similarly $N_{\text{W on E}} = N_{\text{E on W}} = N_E$.

Simplified model—No motion: First, let us address the case where the elevator is not moving. In this case all the accelerations are zero, and we find from (5.78) that

$$N_{\text{W on P}} - G_P = m_P a_P = 0 \Rightarrow N_P = G_P . \tag{5.79}$$

Since what we read from the weight is the normal force applied to the weight, this means that we can read the gravitational force on the person from the weight, which is what we call the weight of the person!

The force, T, from the wire can be found from (5.76):

$$T - N_{\text{W on E}} - G_E = m_E a_E = 0 , \tag{5.80}$$

which gives

$$T = N_E + G_E , \tag{5.81}$$

where (5.77) always gives $N_E = N_P$, which we combine with $N_P = G_P$, getting:

$$T = N_E + G_E = G_E + G_P = (m_E + m_P)\, g. \tag{5.82}$$

When the system is at rest, the force from the wire is equal to the sum of the gravitational forces.

Simplified model — No relative motion: Second, we assume that the elevator, the weight, and the person do not move relative to each other. They will therefore have the same acceleration, $a_E = a_W = a_P = a$. In this case, the weight still displays the contact force N_P, but this force is now given by (5.78), which gives:

$$N_P = G_P + m_P a_P = m_P g + m_P a = m_P\,(a + g). \tag{5.83}$$

So the number you read on the weight is larger if the elevator is accelerating upward.

Full model—Force models: Finally, what happens if we open for the possibility that the objects move relative to each other? As the elevator starts to move, the weight starts to compress, increasing the force on the person, until the person starts moving, changing the compression in the spring in the weight, and so on. We must "Solve" to find the motion of each of the objects based on force models for the interactions. We can still use (5.76), since the mass of the weight is negligible, which means that $N_E = N_P = N$, and the forces acting on each end of the spring are the same!

How do we model the contact force N? It depends on the compression of the spring inside the weight. Let us describe the position of each side of the weight. If we describe the motion of the person by the position of her feet, they are in contact with the top of the weight, and the position of the top of the weight is therefore $x_P(t)$. Similarly, we can describe the position of the elevator by the position of the point where the weight is in contact with the elevator, so that the bottom of the weight is at $x_E(t)$. The change in the extension of the spring inside the weight is therefore

$$\Delta L = L - L_0 = x_P - x_E - L_0, \tag{5.84}$$

where L_0 is the equilibrium distance between the top and bottom of the weight. The weight is compressed by the contact force, N, which is given as a spring force:

$$N = \pm k \Delta L \tag{5.85}$$

where we must choose the sign to ensure that a compressive force must be applied to compress the spring. The normal force must be positive if the spring is compressed. For compression $\Delta L < 0$. The sign must therefore be negative to ensure a positive contact force:

$$N = -k\Delta L = -k\,(x_P - x_E - L_0), \tag{5.86}$$

Notice that the contact force cannot be negative, since the person is not glued to the weight. A weight can push, but not pull, on a person standing on it.

Full model—Newton's second law: Finally, we have a force model for the contact forces, and we can use Newton's second law to find equations for the positions of the person and the elevator. Newton's second law for the elevator from (5.76) gives:

$$T - N - G_E = m_E a_E , \tag{5.87}$$

and similarly Newton's second law for the person from (5.78) gives:

$$N - G_P = m_P a_P , \tag{5.88}$$

where N from (5.86) is a function of both x_E and x_P.

We can rewrite these equations as:

$$a_E = \frac{d^2 x_E}{dt^2} = \frac{1}{m_E} \left(T + k \left(x_P - x_E - L_0 \right) - m_E g \right) , \tag{5.89}$$

$$a_P = \frac{d^2 x_P}{dt^2} = \frac{1}{m_P} \left(-k \left(x_P - x_E - L_0 \right) - m_P g \right) . \tag{5.90}$$

While these differential equations may seem daunting, they are not that difficult to solve analytically (but we will not do that here), and they are simple to solve numerically (which we will do here).

Full model—Initial conditions: What are the initial conditions? We know that the person and elevator are starting from rest. Then an additional force F was added to the elevator. Since they are starting from rest, we know that the initial velocities are zero: $v_E(t_0) = 0\,\text{m/s}$ and $v_P(t_0) = 0\,\text{m/s}$. What about the initial positions? We know that initially the person is standing on the weight. The weight must therefore be compressed. We find the compression from Newton's second law for the person, in the case where the acceleration is zero:

$$a_P(t_0) = -k \left(x_P(t_0) - x_E(t_0) - L_0 \right) - m_P g = 0 , \tag{5.91}$$

which gives

$$x_P(t_0) = x_E(t_0) + L_0 - \frac{m_P g}{k} , \tag{5.92}$$

where we are free to choose the coordinate system so that $x_E(t_0) = 0\,\text{m}$.

Notice that the elevator was started by adding a force F to the tension T already in the wire when the system is at rest. We find the initial value for T from Newton's second law for the elevator in the case where the acceleration is zero:

$$a_E(t_0) = T + k \left(x_P - x_E - L_0 \right) - m_E g = 0 , \tag{5.93}$$

where (5.91) gives $k(x_P - x_E - L_0) = m_P g$, which inserted in (5.93) gives:

$$T = m_E g + m_P g = (m_E + m_P)g. \tag{5.94}$$

This is not surprising: At rest, the force T from the wire must balance the total gravitational force on the elevator with the person, $T = G_P + G_E$, just as we found above. And the result would be the same if we assumed the elevator, weight, and person to be a single object with mass $M = m_E + m_P$.

Full model—Numerical solution: While this problem can be solved analytically, the numerical solution and discussion is simpler, and provides the most essential insights into the process. We therefore apply Euler-Cromer's method to find the positions and velocities of the person and the elevator starting from $t = t_0 = 0.0$ s. The method is implemented in the following program:

```
mP = 70.0; % kg
mE = 2000.0; % kg
k = 20000.0; % N/m
L0 = 0.1; % m
g = 9.8;  % m/s^2
T = (mP+mE)*g;
deltaT = 4000.0; % N
T = T + deltaT;
time = 1.0; % s
dt = 0.0001; % s
n = round(time/dt);
xE = zeros(n,1); vE = zeros(n,1);
aE = zeros(n,1); xP = zeros(n,1);
vP = zeros(n,1); aP = zeros(n,1);
t = zeros(n,1);
xE(1) = 0.0; vE(1) = 0.0;
xP(1) = xE(1)+L0-mP*g/k; vP(1) = 0.0;
for i = 1:n-1
    aE(i) = (T-mE*g+k*(xP(i)-xE(i)-L0))/mE;
    vE(i+1) = vE(i) + aE(i)*dt;
    xE(i+1) = xE(i) + vE(i+1)*dt;
    aP(i) = (-k*(xP(i)-xE(i)-L0)-mP*g)/mP;
    vP(i+1) = vP(i) + aP(i)*dt;
    xP(i+1) = xP(i) + vP(i+1)*dt;
    t(i+1) = t(i) + dt;
end
```

The positions x_E and x_P are compared with the positions found using the acceleration of the whole system found previously:

$$a = \frac{T - (m_E + m_P)g}{m_E + m_P}. \tag{5.95}$$

The resulting behavior is shown in Fig. 5.21. We see that the position of the person on the weight oscillates around the stationary solution. The position of the elevator also oscillates, but the oscillations are much smaller and are therefore not visible on this plot.

Analysis: What we read off the weight is the spring force, which is:

$$N = -k(x_P - x_E - L_0) \tag{5.96}$$

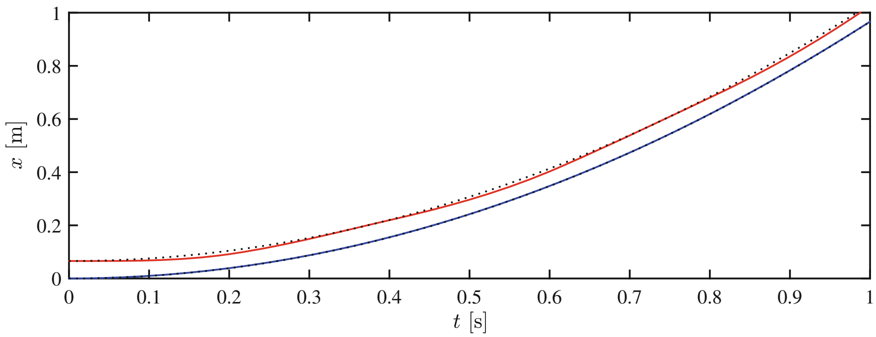
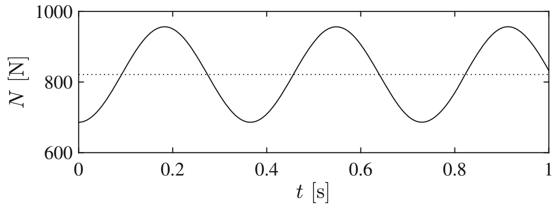


Fig. 5.21 Plot of the positions x_E (*blue*) and x_P (*red*) as a function oftime

Fig. 5.22 Plot of the force N recorded by the weight as a function of time



The spring force is plotted as a function of time in Fig. 5.22. The spring force, which is the force read off the weight, oscillates around the expected value, $N = m_P(g+a) = 823.2$ N, indicated by the straight line in the plot.

Final comment: Here, we have found an alternative behavior of the weight-spring system: The weight is oscillating! Such a behavior is to be expected in a real system, but there will also be additional forces that will tend to reduce the oscillations, so that after a few oscillations, the weight will stop at the expected value, $F = 823.2$ N

Summary

Newton's Second Law:

- Newton's second law relates the acceleration of an object to the net forces acting on it: $\sum_j \mathbf{F}_j = m\mathbf{a}$, where the sum is over all forces acting on the object, and m is the inertial mass.
- All forces acting on *a system* has a source in *the environment*.
- Forces can be *contact forces* acting on the boundary between the system and the environment.
- Forces can be *long range forces* from an object in the environment.
- Forces are drawn as vectors starting at the point where the force is acting, pointing in the direction of the force, and with a length indicating the length of the force.

- The force may be a given quantity, $\mathbf{F}$.
- The gravitational force acts between all objects. On the surface of the Earth the gravitational force on an object is $\mathbf{W} = -mg\,\mathbf{j}$, where $\mathbf{j}$ is a unit vector pointing upward, g is the acceleration of gravity, and m is the gravitational mass. The gravitational mass is equal to the inertial mass.
- The contact force from a fluid on a moving object depends on the velocity of the object relative to the fluid. The simplest force model is the viscous force, $\mathbf{D} = -k_v\mathbf{v}$, where the constant k_v depends on the viscosity of the fluid and the size of the object.
- The contact force from a solid depends on the distance between the object and the solid. The simplest force model is the spring force, $\mathbf{F} = -k\Delta L\,\mathbf{i}$. Where ΔL is the elongation, and k is the spring constant which gives the stiffness of the contact. The spring model is one of the most fundamental force models because it is the first order Taylor expansion of any position-dependent force.

Problem-solving approach:

- We **identify** the object and its initial conditions.
- We **model** the behavior by find the forces acting on the object, introducing force models for all the forces, and applying Newton's second law to find an equation for the acceleration of the object.
- We **solve** the problem by finding the position and velocity from the acceleration and the initial conditions using numerical or analytical techniques.
- We **analyze** the solution to validate it, and use the solution to answer the original question posed.

Exercises

Exercises

5.1 Single force. Can an object affected only by a single force have zero acceleration?

5.2 Zero velocity. If you throw a ball vertically it has zero velocity at its maximum point. Does it also have zero acceleration at this point?

5.3 Acceleration of gravity. You measure the acceleration of gravity in an elevator moving at a velocity of 9.8 m/s downwards. What will you measure?

5.4 Hammerhead. The head of your hammer is loose. How would you hit the shaft in order to fasten the hammerhead? Does this work if you are an astronaut working in space?

5.5 In the army. You are told by a friend in the army that the force you feel when you fire a gun is the same as the force felt by a sandbag hit by the bullet because the two forces are action-reactions pairs. Is this true?

5.6 Car with trailer. A car pulls on a trailer with a given force, but the trailed pulls back at the car with the same force. So why does not the trailer remain at rest, independently of how hard the car pulls?

5.7 Wet dog. When a dog is wet it shakes its body violently to get dry. Explain how this works.

5.8 Whiplash. Explain why a properly adjusted headrest will reduce the chance of whiplash injury if your car is hit from behind.

5.9 Seat belts. Explain why seat belts reduce the risk of injury if you are involved in a car accident. How would you improve seatbelt design?

5.10 Parachute. If you jump from a plane you quickly reach the terminal velocity. Why do you die if you hit the ground at terminal velocity, but not if you open your parachute at the same velocity?

5.11 Tug-of-war. Two persons are pulling at each end of a long rope. If the rope is effectively massless, the tension in the rope is the same all along the rope, and the force on each end of the rope must therefore be of equal magnitude. How can then one of them win in a tug-of-war?

5.12 Bouncing ball. You drop a ball onto a weight, and it bounces back up. Does the value displayed by the weight change during the bounce, and if it does, how does it change? Explain.

5.13 Air resistance. You throw a ball straight up and measure the velocity as it passes you on its way down. Will the velocity be larger, the same, or smaller if you did the same experiment in vacuum?

Problems

5.14 Parachuter. A person jumps from an airplane, falling freely for several seconds before he pulls the cord of his parachute and the parachute unfolds.
(a) Identify the forces acting on the parachuter and draw a free-body diagram of the parachuter before he has pulled the chord.
(b) Identify the forces acting on the parachuter and draw a free-body diagram of the parachuter after he has pulled the chord.
(c) Sketch the net force acting on the parachuter as a function of time, $F(t)$.

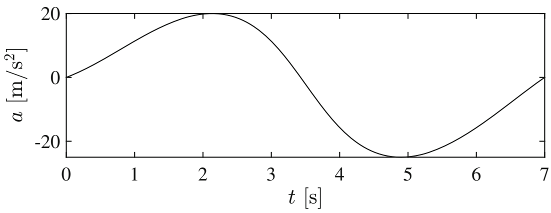


Fig. 5.23 Acceleration of a car

5.15 Forces on a drop of water. A drop of water is hanging from a faucet.
(a) Identify the forces acting on the drop and draw a free-body diagram of the drop.
The drop finally falls down towards the sink.
(b) Identify the forces acting on the drop and draw a free-body diagram of the drop.

5.16 Forces on an anchor. Susan is standing on the floor in a boat, pulling a rope attached to an achor.
(a) Identify the forces acting on the anchor and draw a free-body diagram of the anchor.
(b) Identify the forces acting on Susan and draw a free-body diagram of Susan.
(c) Identify the forces acting of the boat and draw a free-body diagram of the boat.

5.17 Force on a car. A car driver needs to make a rapid maneuver. You have access the an accelerometer fitted in the car. The acceleration is shown in Fig. 5.23. You can assume that the only horizontal force on the car is from the ground on the wheels of the car. The mass of the car is $m = 1000\,\text{kg}$.
(a) Identify in what time interval the car is speeding up and when it is slowing down.
(b) Draw the force on the car from the ground as a function of time.
(c) What is the maximum and minimum force on the car?
(d) Make of drawing of the car, and draw the force on the car from the ground when the car is speeding up and when the car is slowing down.
Alan is sitting on his seat in the car. He does not move relative to the seat throughout the maneuver. His mass is $m = 70\,\text{kg}$.
(e) Draw a free-body diagram for Alan during the maneuver, including only horizontal forces. Can you use the same free-body diagram throughout the whole maneuver, also when the force on the car is negative?
(f) What is the maximum and minimum force acting on Alan during the maneuver?

5.18 Pulling a train. A locomotive exerts a force $F = 20{,}000\,\text{N}$ on a train cart loaded with automobiles. The mass of the cart, including its load, is 10,000 kg.
(a) What is the acceleration of the train cart?
Another car with mass 2000 kg is added to the load on the train cart.
(b) What is the acceleration of the train cart now?

5.19 Firing a bullet. A bullet of mass 0.1 kg is fired through a 1 m rifle barrel. Assume a constant force $F = 1000\,\text{N}$ is applied on the bullet while in the barrel. What is the velocity of the bullet as it leaves the barrel?

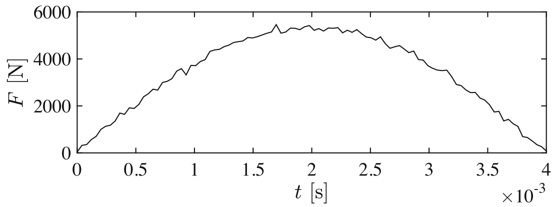


Fig. 5.24 Force on baseball

5.20 Jumping into snow. Albert jumps from the roof of his house into a deep pile of snow. He starts 5 m above the snow, and stops 1 m into the snow. What force was exerted on Albert? Albert's mass is $m = 70$ kg. You can ignore air resistance, and you can assume that the force on Albert from the snow is constant.

5.21 Force sensor reading. A small force sensor has been fitted on the surface of a baseball, measuring the force from the bat on the ball. The recorded force is shown in Fig. 5.24, and you can find the data in the file baseballforce.d.[6]. The mass of the baseball is $m = 0.145$ kg.
(a) Draw a free-body diagram of the baseball.
(b) Find the acceleration of the baseball as a function of time.
(c) Find the velocity of the baseball as a function of time.
(d) What is the velocity of the baseball as it leaves the bat?

5.22 Vertical throw. You stand at the top of a bridge, throwing a rock directly downwards towards the water underneath. You give the rock an initial velocity, v_0, and the rock starts at a height h_0 above the surface. Air resistance is negligible.
(a) Draw a free-body diagram of the rock after you have thrown it.
(b) What is the acceleration of the rock?
(c) Find the time it takes until the rock hits the water.
(d) What is the velocity of the rock as it hits the water.
(e) What is the velocity of the rock as it hits the water if you instead threw the rock upwards with velocity v_0?

5.23 Reaction time. Your reaction time can be measured with the help of a fried using a ruler. Your friend holds the ruler vertically between your thumb and index finger. When he releases the ruler, you grab it as soon as you can. If the ruler is placed with the 0 cm mark initially between your fingers, how can you use how far the ruler has fallen to find your reaction time? You can assume that you use negligible time to actually grab the ruler as soon as you start moving your finger.
(a) Draw a free-body diagram for the ruler when it is falling.
(b) Find the position of the ruler as a function of time.
(c) Find your reaction time, if the ruler fell a vertical distance h before you grabbed it.
(d) If you are driving in your car at 80 km/h, how far do you travel during your reaction time?

[6]http://folk.uio.no/malthe/mechbook/baseballforce.d

5.24 Terminal velocity of heavy and large objects. You drop two spheres from a high tower. First, assume that the spheres have the same diameter, d, and surface properties, so that they have the same air resistance, but they have different masses, m_A and m_B. The air resistance is described using a quadratic law with the coefficient D for both spheres.
(a) Draw a free-body diagram for a sphere as it is falling.
(b) Find an expression for the acceleration of either sphere.
(c) Which object has the largest acceleration - the object with the largest or with the smallest mass?

Now, let us modify the experiment. We now drop two spheres of different diameter, d_1 and d_2, but the spheres are solid and made of the same materials, for example steel. They will therefore have different masses, m_1 and m_2. Still, air resistance for both spheres are described using a quadratic law, but the coefficient D depends on cross-sectional area of the sphere, and therefore on the diameter: $D = C_0 d^2$, where C_0 is a constant.
(d) Find an expression for the acceleration of such a sphere as a function of the diameter of the sphere.
(e) Which object has the largest acceleration—the object with the largest or with the smallest diameter?

5.25 Space shuttle with air resistance. During lift-off of the space shuttle the engines provide a force of 35 million Newtons. The mass of the shuttle is approximately 2 million kg.
(a) Draw a free-body diagram of the space shuttle immediately after lift-off.
(b) Find an expression for the acceleration of the space shuttle immediately after lift-off.

Let us assume that the force from the engines is constant, and that the mass of the space shuttle does not change significantly over the first 20 s.
(c) Find the velocity and position of the space shuttle after 20 s if you ignore air resistance.

Let us assume that we can describe the air resistance force on the space shuttle with a square law, $F = -Dv|v|$, where $D \simeq 388$ kg/m.
(d) Develop a program to find the velocity and position of the space shuttle using numerical methods.
(e) Find the velocity and position of the space shuttle after 20 s if you include air resistance.
(f) Plot the velocity and position and compare with the results without air resistance. Comment on the results.

Notice that D depends on the density of the surrounding air, and the density falls when as the shuttle ascends, hence D actually depends on the height of the shuttle.

5.26 Experiments in Pisa. On a visit to Pisa, you decide to redo Galileo's original experiment based on your knowledge of physics. You bring to steel spheres of the same size to the top of the tower. One sphere is hollow and the other is solid.
(a) Draw a free-body diagram for one of the spheres.

(b) How would you describe air resistance for each of the spheres?
(c) Find an expression for the acceleration of the sphere as a function of its mass.
(d) Which of the two spheres have the largest acceleration?

5.27 Stretching an aluminum wire. A thin aluminum wire is stretched 1 mm when a 10 kg weight is suspended from it. Assume the wire can be modelled as a spring, what is the spring constant for the wire?

5.28 Two masses and a spring. Two particles of $m = 0.1$ kg are attached with a spring with spring constant $k = 100$ N/m and equilibrium length $b = 0.01$ m. Both particles start at rest and the spring is at equilibrium. An external force $F = 1000$ N acts during 1 second on one of the particles in the direction of the other particle. Find the position of both particles as a function of time from the time $t = 0$ s when the external force starts acting. (You may solve this problem analytically or numerically).

Projects

5.29 Modeling a 100 m race. In this project we will develop an advanced model for the motion of a sprinter during a 100 m race. We will build the model gradually, adding complications one at a time to develop a realistic model for the race.
(a) A sprinter is accelerating along the track. Draw a free-body diagram of the sprinter, including only horizontal forces. Try to make the length of the vectors correspond to the relative magnitudes of the forces.

Let us assume that the sprinter is accelerated by a constant horizontal driving force, $F = 400$ N, from the ground all the way from the start to the 100 m line (averaged over a few steps). The mass of the sprinter is $m = 80$ kg.
(b) Find the position, $x(t)$, of the sprinter as a function of time.
(c) Show that the sprinter uses $t = 6.3$ s to reach the 100 m line.

This is a bit fast compared with real races. However, real sprinters are limited by air resistance. Let us introduce a model for air resistance by assuming that the air resistance force is described by a square law:

$$D = (1/2)\rho C_D A(v - w)^2 \tag{5.97}$$

where ρ is the density of air, A is the cross-sectional area of the runner, C_D is the drag coefficient, v is the velocity of the runner, and w is the velocity of the air. At sea level $\rho = 1.293$ kg/m^3, and for the runner we can assume $A = 0.45$ m^2, and $C_D = 1.2$. You can initially assume that there is no wind: $w = 0$ m/s.

Assume that the runner is only affected by the constant driving force, F, and the air resistance force, D. (d) Find an expression for the acceleration of the runner.
(e) Use Euler's method to find the velocity, $v(t)$, and position, $x(t)$ as a function of time for the runner. The runner starts from rest at the time $t = 0$ s. Plot the position, velocity and acceleration of the runner as a function of time. How did you decide

on the time-step Δt? (Your answer should include the program used to solve the problem and the resulting plots).
(f) Use the results to find the race time for the 100 m race.
(g) Show that the (theoretical) maximum velocity of a runner driven by these forces is:

$$v_T = \sqrt{2F/(\rho\, C_D\, A)}. \tag{5.98}$$

The runner may have to run more than 100 m to reach this velocity. (We often call this maximum velocity the terminal velocity—"terminal" because the velocity increases until it reaches the terminal velocity, where the acceleration becomes zero). Find the numerical value of the terminal velocity for the runner. Do you think this is realistic?

So far the model only includes a constant driving force and air resistance. This is clearly a too simplified model to be realistic. Let us make the model more realistic by adding a few features.

First, there is a physiological limit to how fast you can run. The driving force from the runner should therefore decrease with velocity, so that there is a maximum velocity at which the acceleration is zero even without air resistance. While we do not know the detailed physiological mechanisms for this effect, we can make a simplified force model to implement the effect by introducing a driving force, F_D, with two terms: a constant term, F, and a term that decreases with increasing velocity, F_V: $F_V = -f_V\, v$, so that the driving force is:

$$F_D = F + F_V = F - f_V v. \tag{5.99}$$

Reasonable values for the parameters are $F = 400$ N, and $f_v = 25.8$ s N/m. (These values are chosen to make the maximum velocity reasonable—they are not based on a physiological consideration).
(h) If you assume that the runner is subject only to these two driving forces, what is his maximum velocity? (You can ignore the drag term, D, in this calculation).

In addition, during the first few seconds the runner is crouched and accelerating rapidly. In this phase, his cross-sectional area is smaller because he is crouched, and the driving force exerted by the runner is larger than later. Let us also introduce these aspects into our model.

First, let us assume that the crouched phase lasts approximately for a time, t_c. We do not expect this phase to end abruptly at a specific time. Instead, we expect the driving force to decrease gradually (and the cross-sectional area to increase gradually) as the runner is going from a crouched to an upright running position. A common way to approximate such a change is through an exponential function that depends on the time and the characteristic time, t_c. For example, by introducing an initial driving force, F_C:

$$F_C = f_c \exp(-(t/t_c)^2). \tag{5.100}$$

When $t = 0$, the force is equal to f_c, but as time increases, the force decreases rapidly. When the time has reached t_c, the force has dropped to $1/e \simeq 0.37$ of the value at

$t = 0$, and after a time $4t_c$ this contribution to the driving force has dropped to less than 2 % of its initial value.

Notice that we do not have any experimental or theoretical reason to use this particular form for the time dependence. We have simply chosen a convenient form as a first approximation, and then we use this form and try to get reasonable results with it. A better approach would be to have experimental data on how the force varied during the first few seconds, but unfortunately we do not know this. Making rough estimates that you can subsequently improve by better measurements, calculations, or theory will be an important part of how you apply physics in practice.

The total *driving* force is then:

$$F_D = F + f_c \exp(-(t/t_c)^2) - f_v v. \tag{5.101}$$

where reasonable values for the parameters are $f_c = 488$ N and $t_c = 0.67$ s. (These values are chosen so that the total race-time becomes reasonable).

In addition, we need to modify the air resistance force because the runner is crouched in the initial phase, so that the cross-sectional area is reduced. We therefore need to replace the cross-sectional area A in the expression for D with a time-dependent expression, $A(t)$, with the properties that: (1) when time is zero, the area should be reduced to 75 % of the area during upright running (again, we guess reasonable values); and (2) after a time much larger than t_c, the runner is upright, and the cross-sectional area should be A. Again, we introduce a modification to the area that depends on the exponential factor used above:

$$A(t) = A - 0.25A \exp(-(t/t_c)^2) = A\left(1 - 0.25 \exp(-(t/t_c)^2)\right). \tag{5.102}$$

The air resistance force therefore becomes:

$$D = \frac{1}{2}A(t)\rho C_D(v - w)^2 = \frac{1}{2}A\left(1 - 0.25e^{-(t/t_c)^2}\right)\rho C_D(v - w)^2 \tag{5.103}$$

The total force on the runner is:

$$F_{\text{net}} = F + F_C - F_V - D = F + f_c e^{-(t/t_c)^2} - f_v v - D, \tag{5.104}$$

where $F = 400$ N is a constant driving force, and the other terms have been addressed above.

(i) Modify your numerical method to include these new forces. Find and plot $x(t)$, $v(t)$, and $a(t)$ for the motion.

(j) How fast does he run 100 m?

(k) Compare the magnitudes of the various forces acting on the runner by plotting F (which is constant), F_C, F_V and D as a function of time for a 100 m race. Discuss how important the various effects are.

(l) Use the model to test how the resulting time on 100 m would change if the runner had a hind wind with a wind speed of $w = 1$ m/s. What if he was running into a wind with a wind speed of $w = 1$ m/s?

5.30 Modelling Bungee Jumping Numerically. In this exercise we will study a person bungee jumping. The bungee cord acts as an ideal spring with a spring constant k when it is stretched, but it has no strength when pushed together. The cord's equilibrium length is d. There is also a form of dampening in the cord, which we will model as a force which is dependent on the speed of the cord's deformation. When the cord is stretched a length x, and is being stretched with the instantaneous speed v, the force from the spring is given as

$$F(x, v) = \begin{cases} -k(x-d) - c_v v & \text{when} \quad x > d \\ 0 & \text{when} \quad x \leq d \end{cases}$$

where c_v is a constant that describes the dampening in the cord, and k is the spring constant.

We set $x = 0$ to be where the bungee cord is attached and let the positive direction of the x-axis point downwards. A person with a mass m places the cord around the waist and jumps from the point where it is attached. The initial velocity is $v_0 = 0$. You can neglect air resistance and assume that the bungee cord is massless. The motion is solely vertical. The acceleration of gravity is g.

(a) Draw a free-body diagram of the person when the bungee cord is taut. Name all the forces.

(b) At what height is the person hanging when the motion has stopped?

(c) Write a numerical algorithm that finds the persons position and velocity at the time $t + \Delta t$ given the persons position and velocity at a time t. And implement this algorithm in a program that finds the motion of a person bungee jumping.

(d) Use your program to plot the height as a function of time, $x(t)$, for a person of mass $m = 70$ kg jumping with a bungee cord of equilibrium length $d = 20$ m and spring constant $k = 150$ N/m, for $T = 60$ s with a timestep of $dt = 0.01$ s. The acceleration of gravity is $g = 9.8$ m/s^2. What is a reasonable choice for c_v? Explain your choice.

(e) Is the system conservative during the whole motion, parts of the motion, or not at all? Explain.

(f) How would our model be different if we included air resistance?