Chapter 7
Forces in Two and Three Dimensions

We have now introduced a vectorized description of motion that allows us to discuss motion not only in one dimension, but also for two- and three-dimensional systems. However, in order to predict and calculate the motion, we need to extend Newton's laws to two and three dimensions, and we need to introduce force models that are applicable in two and three dimensions. This is the focus of the current chapter.

The structured problem-solving approach used to address problems in mechanics has exactly the same form for one-, two-, and three-dimensional problems (see Fig. 7.1 for an illustration). The first step is to *identify* what objects we are studying, how we characterize their position, and what reference system we use to describe the motion. Second, we *model* the system by finding the forces acting on the object, we introduce models for the force, and use Newton's second law to find the acceleration of the object. Third, we *solve* the equations of motion, and determine the position and velocity of the objects as functions of time. Finally, we *analyze* the resulting motion, use the solution to answer the questions posed, and check the validity of the solutions.

In this chapter, we discuss how to identify forces, how to apply Newton's laws in two- and three dimensions, and we generalize all force models to two- and three-dimensional motion.

7.1 Identifying Forces

In Chap. 5 we introduced a general method to identify and name the forces acting on an object, by drawing the free-body diagram. This method is the same in one-, two-, or three-dimensional systems. The only difference is that for the one-dimensional case we have so far not included all forces in order to ensure the problem was indeed one-dimensional. Now, we loosen that constraint and include all forces in the free-body diagram.

A. Malthe-Sørenssen, *Elementary Mechanics Using Matlab*,
Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-19587-2_7

Identify

What object is moving?

How is the position, $x(t)$, measured? (Origin and axes of coordinate system).

Find initial conditions, $x(t_0)$ and $v(t_0)$.

→

Model

Find the forces acting on the object.

Introduce models for the forces.

Apply Newton's second law of motion to find the acceleration, $a = a(x, v, t)$.

→

Solve

Solve the equation:

$$\frac{d^2x}{dt^2} = a(x, v, t) ,$$

with the initial conditions $x(t_0) = x_0$ and $v(t_0) = v_0$ using analytical or numerical techniques.

The solution gives the position and velocity as a function of time, $x(t)$, and $v(t)$.

→

Analyse

Check validity of $x(t)$ and $v(t)$.

Use $x(t)$ and $v(t)$ the answer questions posed.

Evaluate the answers.

Fig. 7.1 The structured problem solving approach

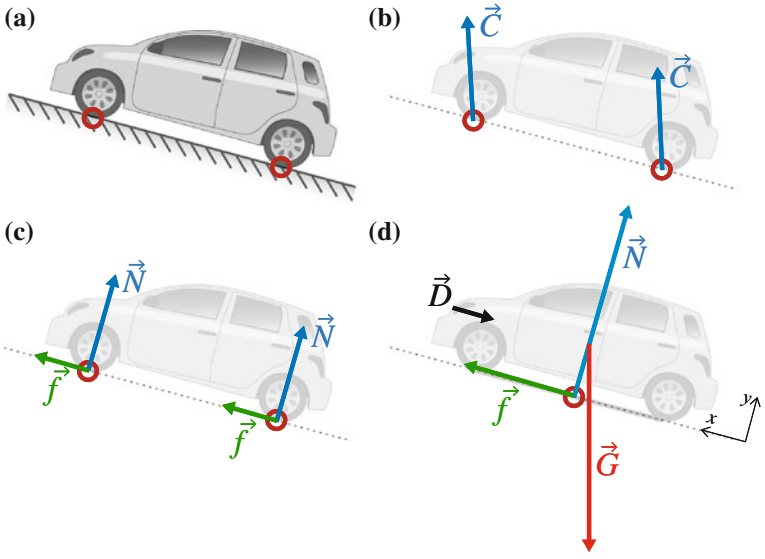

Fig. 7.2 Illustration of a car driving up an inclined slope

Let us illustrate the main principles of the free-body diagram for a fully three-dimensional problem by developing the free-body diagram of a car driving up a hill as illustrated in Fig. 7.2a. We follow the enumerated steps from the general method, while commenting on how it is applied.

1. Divide the problem into *system* and *environment*.

First, we define precisely what object we are studying the motion of—we discern between the *system* and the *environment*. We find the forces acting *on* the system. Everything that is not the *system* is the environment.

In this case the system is the car, and the environment is everything else, such as: the ground, the Earth, and the air around the car.

The external forces may occur either at the contact between the car and the environment. We call such forces *contact forces*. Or they may be *long-range forces*. Typically, we find the contact forces first and leave the long-range forces to last. We find the contact forces by addressing the physical processes occurring along the external surface of the object:

2. Draw a figure of the object and everything in contact with the object.
3. Draw a closed curve around the system.
4. Find contact points—these are the points where contact forces may act.
5. Give names and symbols to all the contact forces.

The car is in contact with the ground at the points where the four wheels are touching the ground. We have already discussed the possible ambiguity of a contact point. In many cases the contact is not in a point, but distributed over an area. However, on a large scale, we can usually still represent the force as acting in a point.

What forces are acting in these points? Your answer to this question will depend on your insight into and knowledge about force models—what kinds of macroscopic forces are acting between objects.

Contact forces: Previously, we argued that we should separate the identification of forces from the modeling of forces. Unfortunately, this is not really possible. In practice, we cannot separate the two steps. As we identify forces acting on a body, we are actually also identifying the interactions mechanisms, and we are already making assumptions on how to model the interactions.

To make this rather abstract point more concrete, let us look at the example of the car in detail. What forces are acting on the contact points between the car and the ground?

As a first approximation, we could introduce a single force vector, the contact force **C**, acting in the point of contact and having a component both normal to the ground and along the ground as illustrated in Fig. 7.2b. While this is a correct description—there is a force acting on the car from the ground, and it may have components both normal and parallel to the ground—it is not a very useful way to identify the forces. Why? Because we will not have a force model for this general contact force. We could therefore introduce this general contact force here, but we would need to decompose it into different physical models when we addressed how to model this force.

Decomposing the contact force: Instead, what we mean when we say that we identify the forces acting on the car in the contact point, is that we identify the different force models, and that we introduce an individual force for each of the models. For the contact between the ground and the car we could identify two different force models: We recognize one force as due to the deformation of the wheel and the ground. This is the normal force, **N** from the ground on the car as illustrated in Fig. 7.2c. We recognize another force as due to the sticking or sliding of the wheel relative to the

ground. This is the friction force from the ground on the car, $\mathbf{f}$. Notice that when we identify forces, we are really identifying mechanisms or processes that result in a force, which is the first step in identifying a model to describe the force.

One or many contact forces: The contact forces from the ground on the car are therefore the normal force, $\mathbf{N}$, and the friction force, $\mathbf{f}$. One force acts on each of the four contact points of the car. We may represent each of these forces individually, or we could instead describe the whole car as one block with only a single contact point with the ground, and then represent the four contact forces only by a single force. In this case we should redraw our figure so that there is only one contact point, as shown in Fig. 7.2d, and only one normal force, $\mathbf{N}$, and one friction force, $\mathbf{f}$. Later on, when we study the equilibrium of an object, we will see that we need to use more than one contact point to determine if the car is rotating or not.

Choosing the direction of the vectors: We draw the normal force as a vector $\mathbf{N}$ pointing upwards, since we expect this to be the direction of the force. If the y-axis is the direction normal to the hill, as illustrated in Fig. 7.2d, the normal force is $\mathbf{N} = N\mathbf{j}$. This simply means that if the normal force actually points in this direction, then N is positive. Does it matter if we draw the vector in the "wrong direction"? What would happen if we instead drew the normal force pointing downwards? In this case, we would have $\mathbf{N} = N(-\mathbf{j})$, and a positive value of N would mean that the normal force was acting downwards. We are therefore free to draw the vector in whatever direction we want, the resulting numerical values for the components of the vector will tell us in what direction the force is acting at a particular time. However, you have to be consistent: When you have drawn the vector in a particular direction, you need to stick to your choice.

Similarly, we draw a vector $\mathbf{f}$ pointing up along the hill to represent the friction force. With the x-axis pointing up along the hill, this means that: $\mathbf{f} = f\mathbf{i}$. The friction force may still act in the opposite direction for $f < 0$.

What other contact forces are acting on the car? The car is in contact with the surrounding air. We should therefore also add a force due to air resistance on the car, $\mathbf{D}$, as illustrated in Fig. 7.2d.

6. Identify the long-range forces.

Finally, we find and draw the long-range forces acting on the car. The only long-range force is gravity from the Earth, which acts on the car and point down towards the center of the Earth. Gravity is drawn as the force $\mathbf{G}$ in Fig. 7.2d.

Now, the next two points in the general method are:

7. Make a drawing of the *object*. Draw the forces as arrows, vectors, starting where the force is acting. The direction of the vector indicates the (positive) direction of the force. Try to make the length of the arrow indicate the relative magnitude of the forces.
8. Draw in the axes of the coordinate system. It is often convenient to make one axis parallel to the direction of motion. When you choose direction of the axis you also choose the positive direction for the axis.

When you are more experienced, you will probably follow the expert's method to find the free-body diagram, but you should not progress to this stage before you have practiced the first method on several examples and found that it is too simple and elaborate for your taste.

Expert's method for drawing a free-body diagram: Follow these steps to find, identify, and draw all the forces acting on an object in a free-body diagram.

1. Identify the *system*, and make a drawing of the system. You may sketch the environment in different colors or with a dotted line to help you.
2. Contact forces are acting at the contact points between the system and the environment. Identify, name, and draw each contact force as a vector starting at the point of contact.
3. Choose a coordinate system and draw the axes of the coordinate system in the same figure as the system. It is often convenient to make one axis parallel to the direction of motion.

7.2 Newton's Second Law

We have already introduced Newton's laws of motion on a vector form, therefore, we do not need a new formulation for two- and three-dimensional problems.

Newton's second law relates the acceleration of an object to the net force acting on the object:

$$\sum_j \mathbf{F}_j^{\text{ext}} = m\mathbf{a}, \tag{7.1}$$

where the sum is over all the *external* forces acting on the system.

The external forces are exactly the forces you have included in the free-body diagram of the object. Often we refer to the sum of all the external force as the *net external force*:

$$\mathbf{F}_{\text{net}} = \sum_j \mathbf{F}_j^{\text{ext}}. \tag{7.2}$$

Inertial system: Newton's second law is only valid in an *inertial system*. If the references system is accelerated—either linearly or rotating—we cannot use Newton's law.

Net external force: Newton's second law is related to the *net external force*. External forces are forces that have a cause outside the system, as we insisted on when you

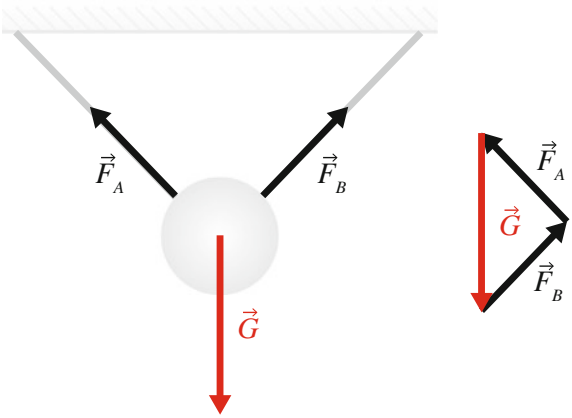

Fig. 7.3 A sphere is hanging from two ropes that are attached to the roof. In this case, the net force is zero even when none of the forces point in the same direction, as shown by the graphical vector summation to the right

drew the free-body diagram of the system. You can therefore not use the law for a single force alone—it is the net force that is causing the acceleration of the object, not one individual force acting on the system. For example, for the car driving up an inclined slope discussed above, the net force on the car is:

$$\mathbf{F}_{\text{net}} = \sum_j \mathbf{F}_j = \mathbf{N} + \mathbf{f} + \mathbf{D} + \mathbf{G}. \tag{7.3}$$

Net force is a vector sum: The net force is a *vector sum* of all the external forces. Notice that the net force can be zero even if none of the force vectors point in the same direction, as illustrated in Fig. 7.3.

Vector equation: Newton's second law is a *vector equation*. This means that it is valid for each of the vector components independently:

$$\sum_j \mathbf{F}_j^{\text{ext}} = m\mathbf{a}, \tag{7.4}$$

implies that:

$$\sum_j F_{j,x}^{\text{ext}} = ma_x, \quad \sum_j F_{j,y}^{\text{ext}} = ma_y, \quad \sum_j F_{j,z}^{\text{ext}} = ma_z, \tag{7.5}$$

where $F_{j,x} = \mathbf{F}_j \cdot \mathbf{i}$ and $a_x = \mathbf{a} \cdot \mathbf{i}$, and similarly for the other two components.

We can decompose Newton's law along any set of axes we like. We will see that it is often useful to choose the axes wisely, for example by ensuring that the net force along one of the axis direction is zero, so that there is no change in motion in this direction.

Notice that this means that the object can be accelerated in the x-direction, if the net force in this direction is non-zero, while it moves with constant velocity in the

y-direction, if the net force in this direction is zero. The behavior along orthogonal axes can therefore be completely decoupled.

Superposition principle: Forces are subject to the *superposition principle*. We can add together or decompose forces as we like. This allows us to subdivide a force, such as the surface interaction force, $\mathbf{f}$, for the car driving up the hill, into several forces, each representing a specific surface interaction term:

$$\mathbf{f} = \mathbf{f}_{\text{friction}} + \mathbf{f}_{\text{adhesion}} + \mathbf{f}_{\text{lubrication}} + \dots . \tag{7.6}$$

7.3 Force Model—Constant Gravity

According to Newton's law of gravity, there is a gravitational force between any two objects with gravitational masses. For an object close to the Earth's surface, the gravitational force on the object can be approximated as:

$$\mathbf{G} = -mg\,\mathbf{j}, \tag{7.7}$$

where m is the gravitational mass of the object, g is the acceleration of gravity, and the unit vector $\mathbf{j}$ points upwards. Upwards is indeed usually defined based on the direction of the force from gravitation. This **constant gravity** force model is valid as long as the object does not move far away from the surface of the Earth, and as long as the object does not move too far along the surface of the Earth, since this would lead to a change in the unit vector $\mathbf{j}$.

We notice that this force model is particularly simple: The force on an object due to gravity is a *constant*—both in magnitude and direction. Our discussion of the constant gravity force model may be extended to any constant force, such as the force on a charged particle in a homogeneous electric field.

If you throw a ball from the ground, the only forces acting on the ball after it has left your hand are the force from gravity, $\mathbf{G}$, and air resistance, $\mathbf{D}$. It we neglect the effect of air resistance, the only force acting on the ball is gravity. We can therefore apply Newton's second law to find the acceleration of the ball:

$$\sum_j \mathbf{F}_j = \mathbf{G} = mg\,\mathbf{j} = m\mathbf{a} \;\Rightarrow\; \mathbf{a} = -g\,\mathbf{j}. \tag{7.8}$$

Since we have wisely chosen the y-axis to correspond to the direction of gravity, the acceleration of the ball is non-zero only in y-direction. The acceleration in the x- and z-directions are zero, and the velocities in these directions do therefore not change. We call such a motion *decoupled* because the motion in the x- and y-axes are independent of each other. We will use this when we solve problems with constant forces.

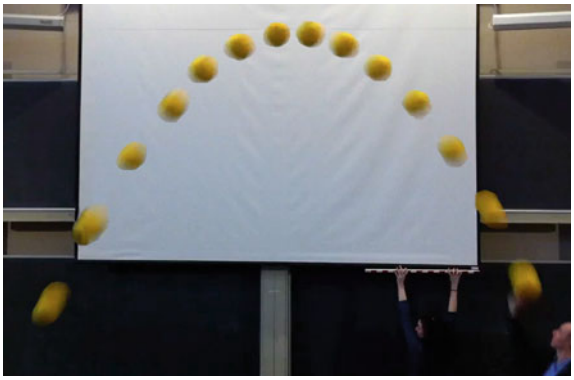

Fig. 7.4 The motion of a ball thrown across the lecture room

7.3.1 Example: Motion of a Ball with Gravity

Throughout this chapter we will follow a particular problem as we gradually increase the complexity of the physical model: The motion of a ball thrown across the classroom. The experiment is illustrated in Fig. 7.4, which illustrates the observed path of a ball in an experiment. How can we develop a realistic model for the motion of this ball? We start from the simplest description: The motion when affected by gravity alone.

Problem: A ball is thrown from a height h above the ground at an angle α with the horizontal with an initial speed v_0. Find the velocity and position of the ball at a time t.

Identify and Sketch: In this exercise we address the motion of the ball, described by the position $\mathbf{r}(t)$ as a function of time. At the time t_0, the ball was thrown. We place the coordinate system so that gravity is acting in the y-direction, and we place the x-axis so that the ball is thrown in the positive x-direction. The origin is placed at the ground, directly below the initial position of the ball at $t = t_0$. The initial position vector is therefore $\mathbf{r}(t_0) = \mathbf{r}_0 = h\,\mathbf{j}$. The initial velocity is directed at an angle α with the horizontal, this means that the initial velocity is $\mathbf{v}(t_0) = v_0 \cos(\alpha)\,\mathbf{i} + v_0 \sin(\alpha)\,\mathbf{j}$. The situation is illustrated in Fig. 7.5.

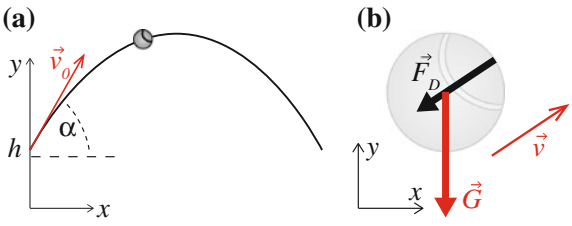

Fig. 7.5 Illustration of the motion of the ball (*left*) and free-body diagram of the ball (*right*)

Model: The motion of the ball is determined by the forces acting on it. The only contact force acting on the ball is air resistance, $\mathbf{F}_D$, but we will here assume that this force is negligible. The only long-distance force acting on the ball is gravity, $\mathbf{G}$, as illustrated in the free-body diagram in Fig. 7.5.

Newton's second law is applied to both the x- and the y-component of the forces independently. In the x-direction Newton's second law gives:

$$\sum F_x = ma_x = 0. \tag{7.9}$$

There are no horizontal forces. The sum of the forces in the horizontal, x-direction is therefore zero. Consequently, the acceleration in the x-direction, a_x is also zero.

Newton's law of motion in the y-direction gives:

$$\sum F_y = G = -mg = ma_y, \tag{7.10}$$

where we have used that the gravitational force from the Earth is mg, and that it acts in the negative y-direction.

The acceleration of the ball is therefore:

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -g\,\mathbf{j}, \tag{7.11}$$

and the initial conditions are $\mathbf{r}(t_0) = h\,\mathbf{j}$ and $\mathbf{v}(t_0) = \mathbf{v}_0$.

Solve: We find the motion of the ball by solving the differential equation in (7.11). Since the acceleration is constant, we can solve it by direct integration for each of the components.

In the x-direction, the acceleration is zero, and the velocity in this direction is therefore constant.

$$v_x(t) = v_x(t_0) = v_0\cos(\alpha). \tag{7.12}$$

The x-position is given by direct integration:

$$v_x(t) = \frac{dx}{dt} \tag{7.13}$$

$$\int_{t_0}^{t} v_x(t)dt = \int_{t_0}^{t} \frac{dx}{dt}dt \tag{7.14}$$

$$\int_{t_0}^{t} v_0\cos(\alpha)dt = \int_{x(t_0)}^{x} (t)dx \tag{7.15}$$

$$v_0\cos(\alpha)(t - t_0) = x(t) - x(t_0) \tag{7.16}$$

that is, we have recovered motion with constant velocity:

$$x(t) = x(t_0) = v_0\cos(\alpha)(t - t_0), \tag{7.17}$$

In the y-direction, the acceleration is constant, $a_y = -g$. We can find the velocity by direct integration:

$$\int_{t_0}^{t} \frac{dv_y}{dt} dt = \int_{t_0}^{t} -g dt \tag{7.18}$$

$$v_y(t) - v_y(t_0) = -g(t - t_0) \tag{7.19}$$

which gives:

$$v_y(t) = (v_0 \sin(\alpha) - g(t - t_0)) \tag{7.20}$$

We find the position by integrating once more, using that $v_y(t) = dy/dt$, and that $v_{0,y} = v_0 \sin(\alpha)$:

$$\frac{dy}{dt} = \left(v_{0,y} - g(t - t_0)\right) \tag{7.21}$$

$$\int_{t_0}^{t} \frac{dy}{dt} dt = \int_{t_0}^{t} \left(v_{0,y} - g(t - t_0)\right) dt \tag{7.22}$$

$$y(t) - y(t_0) = \int_{t_0}^{t} v_{0,y} dt - g \int_{t_0}^{t} (t - t_0) dt \tag{7.23}$$

$$y(t) - y(t_0) = v_{0,y}(t - t_0) - \frac{1}{2} g(t - t_0)^2 \tag{7.24}$$

which gives

$$y(t) = h + v_0 \sin(\alpha)(t - t_0) - \frac{1}{2} g(t - t_0)^2, \tag{7.25}$$

where $y(t_0) = h$ is the launch height of the projectile.

Analyze: We notice that the motion in the x- and y-directions are independent of each other. The motion in the x-direction is simply a motion with constant velocity. The motion in the y-direction is the same as for the one-dimensional problem. If the ground is flat, it is the motion in the y-direction that determines how long time it takes to reach the ground. We can therefore answer questions about flight time, and maximum height by just studying the one-dimensional motion along the y-direction.

We have now found the complete solution for the motion of a ball subject only to gravity. From this solution, we can answer any complicated question, such as how far the projectile travels or what choice of initial direction gives the maximum length.

7.4 Force Model—Viscous Force

For an object moving through a fluid, such as a projectile flying through the air, a meteor entering the Earth's atmosphere, or a tiny microrobot navigating through your bloodstream, there is a contact force on the object due to the motion of the

object relative to the fluid. The fluid has to flow around the object when the object moves, as a result the fluid exerts a force on the object. This force is distributed: It acts everywhere on the surface of the object, and it may also vary in magnitude and direction along the surface of the object. Usually, we will simplify the effect of the this distribution of forces into a single force acting in a single point on the object, and we will call this force the drag force or the "fluid resistance" acting on the object. For most purposes this is a sufficiently precise description of the interaction with the fluid.

The form of the drag force depends on the velocity of the object relative to the fluid. We discern between a behavior at low velocities, where the drag force is proportional to the velocity, and high velocities, where the drag force depends on the square of the velocity:

The **drag force** on an object moving at a velocity $\mathbf{v}$ relative to a fluid moving with a velocity $\mathbf{w}$ is:

$$\mathbf{F}_D \simeq \begin{cases} -k_v\,(\mathbf{v}-\mathbf{w}) & \text{at small velocities} \\ -D\,|\mathbf{v}-\mathbf{w}|\,(\mathbf{v}-\mathbf{w}) & \text{at high velocities} \end{cases} \tag{7.26}$$

The constant k_v depends on the object's size, shape and surface, as well as on the (dynamic) viscosity of the fluid. For a sphere Stokes found that

$$k_v = 6\pi R\eta \tag{7.27}$$

where R is the radius of the sphere, and η is the viscosity of the fluid. The viscosity of air is $\eta = 1.82 \times 10^{-5}\,\mathrm{Ns\,m^{-2}}$ and for water it is $\eta = 1.00 \times 10^{-3}\,\mathrm{Ns\,m^{-2}}$, both at room temperature.

Experimental data indicates an approximative value for D for a spherical object:

$$D \simeq 12.0\,\rho R^2. \tag{7.28}$$

where $v = |\mathbf{v}|$, ρ is the density of the fluid, and R is the radius of the sphere.

Versatility of the viscous force model: The viscous force model

$$\mathbf{F}_D = -k_v\mathbf{v}, \tag{7.29}$$

is much more versatile than suggested by its application to fluid drag forces. It is often used as a general damping term—a term reducing relative motion that also introduces dissipation and heat generation. For example, you will find that the viscous force model used to model damping of vibrations in solid object, to model the damping of vibrations in macroscopic objects and macroscopic springs, and to model surface

forces in nano-scale surface contact. The viscous force model is a first order model to study any velocity-dependent force that tends to reduce velocity differences.

7.4.1 Example: Path Through a Tornado

You are part of a tornado-chaser team—a group of scientists trying to discover the inner workings of tornadoes. An important part of this work is to develop methods to measure the pressure and wind velocity inside the tornado. Your plan is to use many tiny projectiles with small accelerometers inside. You plan to shoot the projectiles through the tornado, pick them up afterwards, and read the recorded accelerations. Here, we will assume that the accelerometers record the acceleration in the x, y, and z-direction during flight. Here, we will develop a model for the flight of the projectile, and calculate realistic trajectories in order to learn how to launch the projectiles.

Sketch and Identify: Our task is to determine the motion of a projectile, characterized by its position $\mathbf{r}(t)$. For the calculations, we use a coordinate system with the origin in the center of the tornado at ground level. The z-axis points in the vertical direction, with the positive direction upwards. We will launch the projectile from the position $\mathbf{r}(t_0) = \mathbf{r}_0$ with an initial velocity, $\mathbf{v}(t_0) = \mathbf{v}_0$ at $t_0 = 0.0\,\mathrm{s}$.

Model: While the projectile is in the air, the only contact force affecting the object is the force from the surrounding air, $\mathbf{F}_D$, in addition to gravity, $\mathbf{G}$, as illustrated in Fig. 7.6.

Since the projectile will be moving fast, we use the square-law force model for the air resistance. However, in this case it is important to realize that the force depends on the velocity of the projectile, $\mathbf{v}$, relative to the velocity of the wind, $\mathbf{u}(\mathbf{r})$. The square-law force model is therefore:

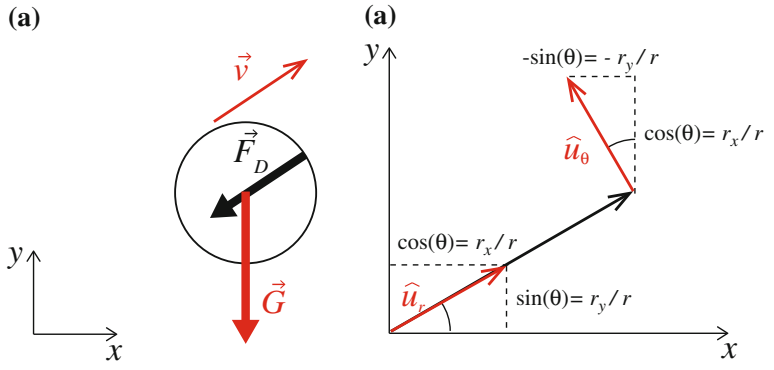

Fig. 7.6 **a** Free-body diagram of the projectile. **b** Illustration of the tangential direction in the tornado

$$\mathbf{F}_D = -D\,(\mathbf{v}-\mathbf{u})\,|\mathbf{v}-\mathbf{u}|, \tag{7.30}$$

where for a spherical object we have that the prefactor is $D \simeq 3.0\rho d^2$, where d is the diameter of the sphere and ρ is the density of the surrounding air. Here, we will assume that the density of the surrounding air does not change significantly, and we will use $\rho = 1.293\,\mathrm{k/m^3}$. Let us also assume that the projectile has a diameter of $d = 0.02\,\mathrm{m}$, and that its mass is $m = 0.1\,\mathrm{kg}$.

The force from gravity is $\mathbf{G} = -mg\,\mathbf{k}$, where $\mathbf{k}$ is the unit vector in the z-direction.

Newton's second law: Newton's second law gives the acceleration of the projectile:

$$m\mathbf{a} = \sum_j \mathbf{F}_j = \mathbf{F}_D + \mathbf{G}. \tag{7.31}$$

The acceleration is therefore:

$$\mathbf{a} = -\frac{D}{m}\,(\mathbf{v}-\mathbf{u})\,|\mathbf{v}-\mathbf{u}| - g\,\mathbf{k}. \tag{7.32}$$

and the initial conditions are $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{v}(t_0) = \mathbf{v}_0$.

Model of wind velocity: However, in order to test and analyze the path of the projectile, we need a model for the velocity $\mathbf{u}$ in the tornado. Since we do not know this velocity-field, we will here use a model for the velocity taken from an analoguous situation. We can make an experimental tornado by rotating a thin cylinder in a fluid. For this case, we know that the velocity in the fluid will have the form:

$$\mathbf{u}(\mathbf{r}) = u(r)\hat{u}_\theta, \tag{7.33}$$

where the center of the cylinder is at the origin, and the speed, $u(r)$, depends only on the distance to the center of the cylinder, and the unit vector $\hat{u}_\theta$ points in the tangential direction, as illustrated in Fig. 7.6. The speed $u(r)$ has the following form:

$$\mathbf{u}(r) = \begin{cases} u_0\,(r^*/r) & \text{for } r > r^* \\ u_0\,(r/r^*) & \text{for } r < r^* \end{cases}, \tag{7.34}$$

where r^* is the radius of the cylinder.

We use this as a model for the velocity field inside the tornado to estimate the path of the projectile. Let us assume that we study a tornado with a radius of $r^* = 10\,\mathrm{m}$, and with a maximum wind speed of $u_0 = 50.0\,\mathrm{m/s}$, which corresponds to a category F1 tornado.

Numerical solution: We can now use our theoretical model to find the motion of the projectile. We use a Euler-Cromer method to find the velocity and position vectors as function of time, starting from $t = t_0$, and continuing until the projectile hits the ground.

Euler-Cromer's method consists of the following steps:

$$\mathbf{v}(t_0 + \Delta t) \simeq \mathbf{v}(t_0) + \Delta t\, \mathbf{a}(t_0, \mathbf{r}(t_0), \mathbf{v}(t_0)) \tag{7.35}$$

$$\mathbf{r}(t_0 + \Delta t) \simeq \mathbf{r}(t_0) + \Delta t\, \mathbf{v}(t_0 + \Delta t). \tag{7.36}$$

This method is implemented in the following program:

```
m = 0.2;
diam = 0.025;
rho = 1.293;
D = 3.0*rho*diam^2;
Dm = D/m;
g = [0.0 0.0 9.8];
u0 = 50.0;
rast = 5.0;
r0 = [-100.0 0.0 0.0];
alpha = 45.0*pi/180.0;
v0 = 100.0*[cos(alpha) 0 sin(alpha)];
time = 10.0;
dt = 0.001;
n = time/dt;
r = zeros(n,3);
v = zeros(n,3);
a = zeros(n,3);
t = zeros(n,1);
r(1,:) = r0;
v(1,:) = v0;
i = 1;
while ((r(i,3)>=0.0)&&(i<n))
    rr = norm(r(i,:));
    if (rr>rast)
        U = u0*(rast/rr);
    else
        U = u0*rr/rast;
    end
    u = U*[-r(i,2)/rr r(i,1)/rr 0.0];
    vrel = v(i,:) - u;
    aa = -g - Dm*norm(vrel)*vrel;
    a(i,:) = aa;
    v(i+1,:) = v(i,:) + dt*aa;
    r(i+1,:) = r(i,:) + dt*v(i+1,:);
    t(i+1) = t(i) + dt;
    i = i + 1;
end
imax = i;
ii = (1:imax);
plot3(r(ii,1),r(ii,2),r(ii,3))
xlabel('x [m]');ylabel('y [m]');zlabel('z [m]')
axis equal
```

Notice how we have used that the tangential vector to a point at (x, y) points in the direction $(-y/r, x/r)$, where $r^2 = x^2 + y^2$.

Analysis: We now have a tool to start addressing the motion of the projectile inside the tornado. Let us use this to test the path of a projectile launched from a distance of 100 m towards the center of the tornado with an initial speed $v = 100$ m/s.

Since the tornado is symmetric, we launch the projectile from the position $\mathbf{r}_0 = -100\,\mathrm{m}\,\mathbf{i}$. We fire the projectile at an angle of 45° with the horizon, which corresponds to an angle of $\pi/4$ in radians, since this gives the maximum length when

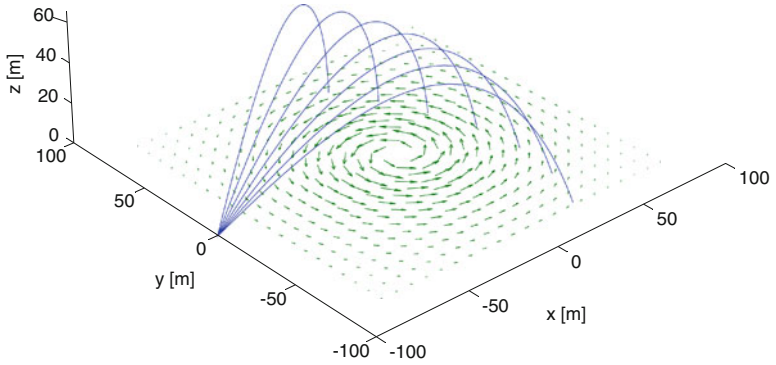

Fig. 7.7 Simulated trajectories of projectiles launched at different angles. The velocity field of the tornado is illustrated by the *arrows*

there is no air-resistance. The initial velocity is therefore $\mathbf{v}_0 = 100\,\text{m/s}\,\cos(\pi/4)\,\mathbf{i} + 100\,\text{m/s}\,\sin(\pi/4)\,\mathbf{j}$. The resulting path is shown in Fig. 7.7.

The trajectory is hardly affected by the tornado. How can we change the trajectory to make it more sensitive to the wind speed? We could shoot it at an angle with the center, and not directly towards the center. This is attempted by introducing the angle θ, which gives the deviation from the line straight into the center. The initial velocity is now.

$$\mathbf{v}_0 = v_0\,(\cos\alpha\,\cos\theta\,\mathbf{i} + \cos\alpha\,\sin\theta\,\mathbf{j} + \sin\alpha\,\mathbf{k}), \tag{7.37}$$

Simulations for various values of θ are given in Fig. 7.7.

Notice the change in trajectories for the various angles. The next step would be to see how easy it would be to find the velocity field from these trajectories or from a set of trajectories, but we will not pursue this direction here. From our discussion you should be able to see how we can use this model to address how to launch the projectile.

Test your understanding: Based on this analysis, would you recommend launching the projectile by shooting it into the tornado, or by ejecting them vertically as the tornado is passing from a container ledged to the ground?

7.5 Force Model—Spring Force

When we introduced the spring model in one dimension, it was used to represent two different concepts:

- To model the force from a linearly elastic spring
- To represent the simplest position-dependent force model

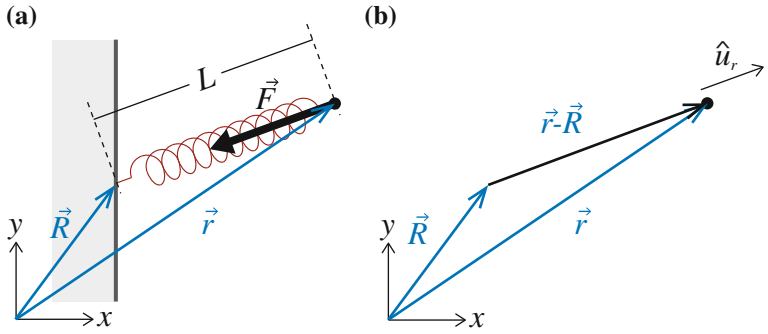

Fig. 7.8 A spring attached to a wall at **R** and to a small particle at **r**. The force **F** from the spring on the particle is shown

Both of these interpretations are still important in two- and three-dimensional systems. We may use a spring model as a model for deformation, and we may use the spring model as a simplified model for a position-dependent force. However, we have more degrees of freedom, and we can therefore define a spring model in several different ways.

Full Spring Model

Let us first see how the force due the deformation of a spring can be generalized in two- and three-dimensions. From one-dimensional experiments, we expect the force from a spring on the object attached to the spring to depend on the elongation of the spring and act in the direction of the spring. The situation is illustrated in Fig. 7.8. A spring is characterized by its equilibrium length, L_0, and its spring constant, k. The force from the spring on the object is:

$$\mathbf{F} = -k\,(L - L_0)\,\hat{u}_r, \tag{7.38}$$

where L is the length of the spring, and the unit vector $\hat{u}_r$ points from the spring towards the object. (Check for yourself that the sign is indeed correct). If the object is located at the position **r**, and the other end of the spring is attached to the point **R**, as in Fig. 7.8, the length of the spring is:

$$L = |\mathbf{r} - \mathbf{R}|, \tag{7.39}$$

and the unit vector pointing from **R** toward **r** is:

$$\hat{u}_r = \frac{\mathbf{r} - \mathbf{R}}{|\mathbf{r} - \mathbf{R}|}. \tag{7.40}$$

In the **full spring model**, the force from a spring attached to one end at $\mathbf{R}$ and at the other end to an object at $\mathbf{r}$ is:

$$\mathbf{F} = -k\,(L - L_0)\,\hat{u}_r, \tag{7.41}$$

where $L = |\mathbf{r} - \mathbf{R}|$ is the length of the spring, and L_0 is the equilibrium length of the spring.

Often, we will place the origin at the attachment point of the spring, so that $\mathbf{R} = 0$, and the full model simplifies to:

$$\mathbf{F} = -k\,(r - L_0)\,\frac{\mathbf{r}}{r}, \tag{7.42}$$

where the length of the spring, $L = r = |\mathbf{r}|$, is the distance from the origin to the particle.

We have named this model the "full model" because it most closely represents the behavior of a real, physical spring. This force model is versatile and general and can be widely applied. For example, it can be used to model the deformation of an elastic body, or the force between two atoms in a molecule. This model will be our preferred model for contact forces such as forces due to deformation in two- and three-dimensional systems.

Notice that the force model has a spherical symmetry: When we choose the origin at the attachment point ($\mathbf{R} = 0$), the force from the spring on the attached object always acts along a line through the origin, and the magnitude of the force depends on the distance r to the origin. This means that force on the particle from the spring in the x-direction is:

$$F_x = -k\,(r - L_0)\,\frac{x}{r} = -k\left(\sqrt{x^2 + y^2 + z^2} - L_0\right)\frac{x}{\sqrt{x^2 + y^2 + z^2}}. \tag{7.43}$$

That is, the force in the x-direction, depends not only on the x-position, but also on the y and z coordinates. If we apply Newton's second law of motion to such a system, the acceleration of the object in the x-direction, will depend on the x, y, and z coordinates of the object: We call such a system *coupled*, and notice that it may not be that simple to find the solution to the equations of motion in this case.

Lattice Spring Model

While the full spring model has a simple physical interpretation—it models the behavior of a physical spring—the resulting equations of motion are not that simple

from a mathematical point of view, because the motion in the x, y, and z-directions are coupled. It is therefore common to introduce a *decoupled* spring model. The physical analogue of a decoupled system is found in a crystalline lattice: A central atom is attached with linear springs to four neighboring atoms, as illustrated in Fig. 7.9. Let us assume that the neighboring atoms do not move.

In the **lattice spring model**, the force on an object attached with springs in all directions is:

$$\mathbf{F} = -k_x\,(x - x_e)\,\mathbf{i} - k_y\,(y - y_e)\,\mathbf{j}, \tag{7.44}$$

where x_e and y_e is the position of the atom when it is in equilibrium, and k_x and k_y are the spring constants in the x- and y-directions respectively. This model is often referred to as the *harmonic oscillator* model.

In this model, the forces in the x- and y-directions are independent of each other, which also means that the motion in the x- and y-directions are decoupled. For example, if the atom is only affected by the lattice spring force, Newton's second law gives:

$$\sum \mathbf{F} = k_x\,(x - x_e)\,\mathbf{i} + k_y\,(y - y_e)\,\mathbf{j} = m\left(a_x\,\mathbf{i} + a_y\,\mathbf{j}\right) \tag{7.45}$$

equating the components on each side therefore gives two, independent equations of motion for motion along the x- and y-axes:

$$k_x\,(x - x_e) = ma_x,\;\; k_y\,(y - y_e) = ma_y \tag{7.46}$$

This problem is mathematically much simpler than the full model: We can simply use the results we already found in the one-dimensional case.

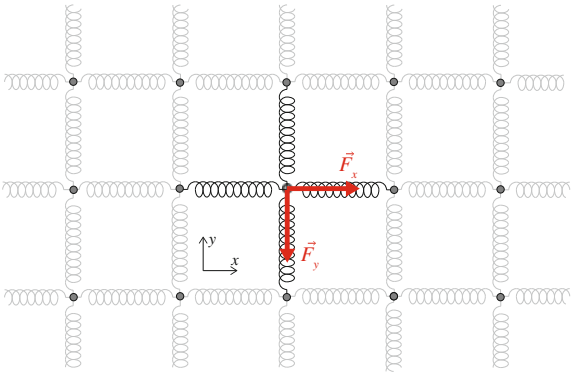

Fig. 7.9 An atom in a lattice is attached to four neighbors by linear springs, so that the springs in the x-direction only act in the x-direction, and the springs in the y-direction only act in the y-direction

7.5.1 Example: Motion of a Bouncing Ball with Air Resistance

In this example we study the motion of an object subject to a constant force, a velocity-dependent force, and a position-dependent force. We solve the problem numerically and discuss the results following a workflow similar to what you will find in many practical problems.

Let us continue to refine the description of a ball thrown in the classroom. So far we have introduced gravity and air resistance. But what happens when the ball hits the floor? We need to also include a force model for the normal force from the floor on the ball. The simplest approach to such a contact force model is a spring model: We model the interaction between the floor and the ball as a single spring. But the normal force is zero when there is no contact. In this problem we demonstrate how to include such effects in our models.

Problem: A ball is thrown from a height h above the ground with an initial velocity $\mathbf{v}_0$. Find the velocity and position of the ball as a function of time t. Include the normal force from the floor while the ball is in contact with the floor.

Identify and Sketch: We describe the position of the ball by $\mathbf{r}(t)$, measured in a coordinate system with origin at the floor. The initial position and velocity of the projectile is $\mathbf{r}(t_0) = h\,\mathbf{j}$ and $\mathbf{v}(t_0) = \mathbf{v}_0 = v_{x,0}\,\mathbf{i} + v_{y,0}\,\mathbf{j}$.

Model: The motion of the ball is determined by the forces acting: air resistance, $\mathbf{F}_D$, the normal force $\mathbf{N}$ from the floor, and gravity, $\mathbf{G} = -mg\,\mathbf{j}$, as illustrated in the free-body diagram in Fig. 7.10. We use a square law for air resistance:

$$\mathbf{F}_D = -Dv\mathbf{v}. \tag{7.47}$$

The normal force from the floor on the ball is represented by a spring force. This is a strong simplification of the actual deformation process occurring at the contact between the ball and the floor due to the deformation of both the ball and the floor. The deformed region corresponds roughly to the region of "overlap" between the ball and the floor in Fig. 7.10. The depth of this region is $\Delta y = R - y(t)$, where R is

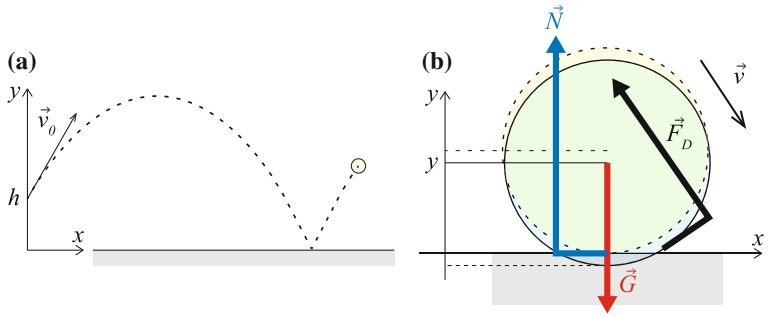

Fig. 7.10 **a** Sketch of the path of the ball. **b** Free-body diagram for the ball when in contact with the floor

the radius of the ball, which corresponds to the compression ΔL of the spring:

$$\mathbf{N} = -k\,(R - y(t))\,\mathbf{j}. \tag{7.48}$$

We check that the sign is correct: The normal force must act upward when $y < R$, hence the sign must be negative.

However, we must also ensure that the normal force only acts when the ball is in contact with the floor, otherwise the normal force is zero. The full formation of the normal force is therefore:

$$\mathbf{N} = \begin{cases} -k\,(R - y(t))\,\mathbf{j} & \text{when } y(t) < R \\ \mathbf{0} & \text{when } y(t) \geq R \end{cases}. \tag{7.49}$$

Newton's second law: Newton's second law is now

$$\sum_j \mathbf{F}_j = \mathbf{G} + \mathbf{F}_D + \mathbf{N} = m\mathbf{a}, \tag{7.50}$$

which gives

$$\mathbf{a} = -\,(D/m)\;v\,\mathbf{v} - g\,\mathbf{j} + \mathbf{N}/m, \tag{7.51}$$

with the initial conditions: $\mathbf{r}(t_0) = \mathbf{r}(0\,\mathrm{s}) = \mathbf{r}_0$ and $\mathbf{v}(t_0) = \mathbf{v}(0\,\mathrm{s}) = \mathbf{v}_0$. While it is difficult to determine the motion analytically, we may be able to find analytical solutions for parts of the motion. However, we can determine the motion numerically by integrating (7.51) using Euler-Cromer's method:

$$\mathbf{v}(t_i + \Delta t) \simeq \mathbf{v}(t_i) + \Delta t\,\mathbf{a}(t_i, \mathbf{r}(t_i), \mathbf{v}(t_i)) \tag{7.52}$$

$$\mathbf{r}(t_i + \Delta t) \simeq \mathbf{r}(t_i) + \Delta t\,\mathbf{v}(t_i + \Delta t). \tag{7.53}$$

The implementation is straight-forward:

```
m = 0.2;      % kg
g = 9.81;     % m/s^2
vT = 20.0;    % m/s
h = 2.0;      % m
R = 0.1;      % m
k = 1000.0;   % N/m
r0 = [0 h];   % m
v0 = [10.0 10.0]; % m/s
time = 20.0; % s
D = m*g/vT^2;
dt = 0.001;
n = ceil(time/dt);
r = zeros(n,2);
v = zeros(n,2);
t = zeros(n,1);
r(1,:) = r0;
v(1,:) = v0;
for i = 1:n-1
    if (r(i,2)<R)
        N = k*(R-r(i,2))*[0 1];
    else
```

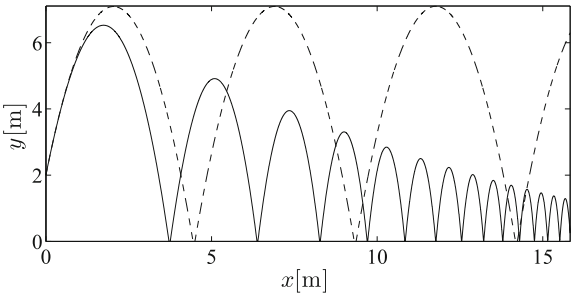

Fig. 7.11 Plot of the trajectory of the ball calculated using Euler-Cromers method for quadratic air resistance (*solid line*), compared with the trajectory without air resistance (*dashed line*)

```
        N = [0 0];
    end
    FD = - D*norm(v(i,:))*v(i,:);
    G = -m*g*[0 1];
    Fnet = N + FD + G;
    a = Fnet/m;
    v(i+1,:) = v(i,:) + dt*a;
    r(i+1,:) = r(i,:) + dt*v(i+1,:);
    t(i+1) = t(i) + dt;
end
plot(r(:,1),r(:,2));
xlabel('x [m]'); ylabel('y [m]');
```

where we have used the $v_T = 10.0\,\text{m/s}$ to calculate D, and $m = 0.2\,\text{kg}$. We have chosen to use the spring constant $k = 1000\,\text{N/m}$, a number we have largely guessed for now. (The effective spring constant may be measured by experiment or calculated if we know the material properties of the ball and the floor). The resulting path is illustrated in Fig. 7.11, where it is compared with the path of the ball without air resistance.

Detailed analysis of wall contact: While the behavior in Fig. 7.11 looks reasonable at first, a closer examination shows that something is wrong, at least according to our intuition. When there is no air resistance, the ball bounces back to the same height! We know that a real ball would not behave like this. What is wrong?

Figure 7.12 shows a magnification of the behavior of the ball during a bounce when there is no air resistance. The red lines in the figures mark the positions and the times when the ball comes in contact with the floor. We see that the horizontal velocity, v_x, does not change at all during the bounce. This is not surprising, since the normal force only has a vertical component. No horizontal forces means no horizontal acceleration. However, we also see that the vertical velocity component, v_y, simply reverses during the bounce. This is the part that strikes us as unrealistic. And indeed it is. How can we modify the force model for the normal force to ensure that the vertical speed is smaller after the collision? This question we will return to later when we discuss energy and collisions. The short answer is that we need to introduce a force that is velocity dependent: Any force model that only depends on the vertical position $y(t)$ of the ball will always result in a reversal of the velocity!

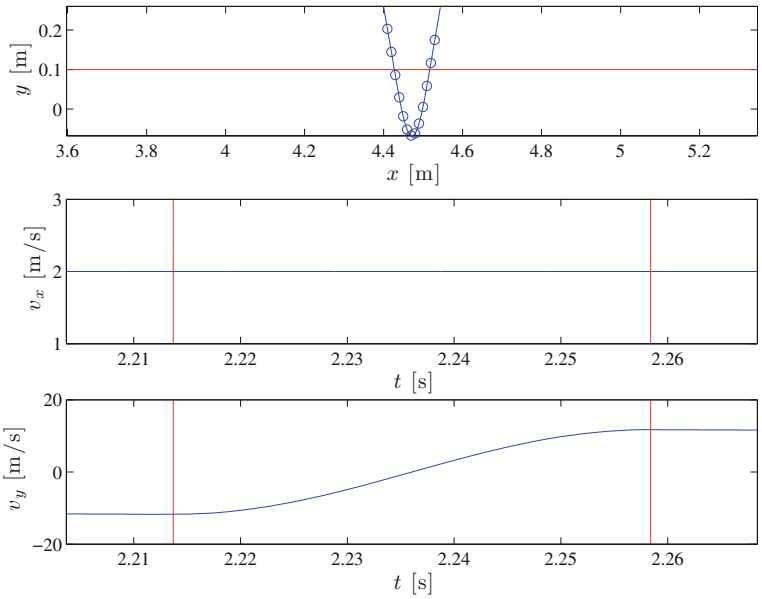

Fig. 7.12 Plot of the trajectory of the ball without air resistance during its collision with the floor. The *red lines* illustrates the points where the ball comes in contact with the floor

By introducing a viscous force, a linear velocity dependent force model, during the collision with the floor, the outgoing vertical velocity will be reduced. But more on this later! (See Chap. 12.)

Comments: We have now built a model for a bouncing ball, including both air resistance and a normal force from the floor. You should notice the simple structure followed systematically: as long as we can develop force models for the interactions, we can model the motion. This method is robust and no different in one-, two-, or three dimensions. You should also notice that the normal force changes: It does not always just balance the gravitational force—it is indeed larger than the gravitational force in parts of the collision. Otherwise the normal force would not be able to change the direction of the ball. This simple point is a clear result of the force model approach, and also of a careful physical analysis, but represents a classical misunderstanding.

7.6 Force Model—Central Force

The long-distance forces of gravitation and the electrostatic interaction (Coulomb's law) are examples of central forces: A force between two objects that:

- acts in the center of the objects
- acts along a line connecting the two objects
- depends on the distance between the two objects

A **central force** has the form:

$$\mathbf{F} = F(r)\frac{\mathbf{r}}{r} = F(r)\,\hat{u}_r, \tag{7.54}$$

where the magnitude $F(r)$ of the force is a function of the distance r only.

Both gravitation and the electrostatic force has the same form for the central force, the *inverse square law*:

$$\mathbf{F} = \frac{C}{r^2}\,\hat{u}_r = C\frac{\mathbf{r}}{r^3}. \tag{7.55}$$

where for $C = -GmM$ this corresponds to Newton's law of gravitation.

The central force model is a force between two objects, where we have placed one of the objects in the origin, and we are only interested in the motion of the other object. This corresponds to the case where the object in the origin is very massive, so that it does not move significantly, or where it is in some way attached, so that it does not move.

Notice that the full spring model also is a central force model, but it does not display the inverse square law. We can use the central force model to describe not only gravitation and electrostatic interactions, but also for many interatomic forces, and as we have seen, for spring forces.

7.6.1 Example: Comet Trajectory

A comet of mass m is affected by the gravitational force from the Sun

$$\mathbf{F}(\mathbf{r}) = -GmM\,\frac{\mathbf{r}}{r^3}, \tag{7.56}$$

where $\mathbf{r}$ is the position of the comet in a coordinate system centered on the Sun. We assume that the Sun does not move. How can we find the motion of the comet?

Sketch: We start from a simple sketch of the system. The comet starts at $\mathbf{r} = R\mathbf{i}$, with an initial velocity $\mathbf{v}_0$. But we do not still know how it will move, so we have only made a guess for its trajectory. We have also added the coordinate system in the center of the Sun (Fig. 7.13).

Newton's second law: Newton's second law gives us the acceleration of the comet:

$$m\mathbf{a} = -GmM\,\frac{\mathbf{r}}{r^3} \;\Rightarrow\; \mathbf{a} = -GM\,\frac{\mathbf{r}}{r^3}, \tag{7.57}$$

which is independent of the mass of the comet. The trajectory of a small comet and a large planet will therefore be the same with the same initial conditions.

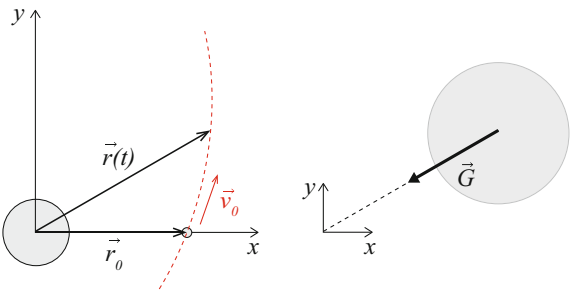

Fig. 7.13 Sketch of a comet moving around the Sun

Integration of motion: We find the motion of the comet by solving (7.57) numerically. Generally, this problem requires a more advanced numerical solution method in order to avoid inaccuracies—you should use a fourth order Runge-Kutta with adaptive time step—but we will here employ the Euler-Cromer scheme with a very small time step because of its transparent implementation.

The Euler-Cromer scheme allows us to find the velocity and time after a small time-step Δt, starting from the initial condition at time $t = t_0$:

$$\mathbf{v}(t_0 + \Delta t) \simeq \mathbf{v}(t_0) + \Delta t\, \mathbf{a}(t_0, \mathbf{r}(t_0), \mathbf{v}(t_0)) \tag{7.58}$$

$$\mathbf{r}(t_0 + \Delta t) \simeq \mathbf{r}(t_0) + \Delta t\, \mathbf{v}(t_0 + \Delta t). \tag{7.59}$$

We use realistic numbers for the masses of the Sun, and choose the initial radius R and the initial velocity to correspond to that of the Earth: v_0: $M = 1.99 \times 10^{30}$ kg, $\mathbf{R} = 1.5 \times 10^{11}$ m, $v_0 = 3 \times 10^4$ m/s, and $G = 6.673 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$. We choose the direction of the initial velocity to be along the y-axis. The implementation is then straight-forward:

```
clear all;
M = 1.99e30;    % kg
R = 2e11;       % m
v0mag = 3e4;    % m/s
G = 6.673e-11; % m^3 kg^-1 s^-2
r0 = R*[1 0];
v0 = v0mag*[0.0 1.0];
time = 60*60*24*365*5; % s
dt = 100; % s
n = ceil(time/dt);
r = zeros(n,2);
v = zeros(n,2);
t = zeros(n,1);
r(1,:) = r0;
v(1,:) = v0;
GM = G*M;
for i = 1:n-1
    rr = norm(r(i,:));
    a = -GM*r(i,:)/rr^3;
    v(i+1,:) = v(i,:) + dt*a;
    r(i+1,:) = r(i,:) + dt*v(i+1,:);
    t(i+1) = t(i) + dt;
end
plot(r(:,1),r(:,2))
xlabel('x (m)'); ylabel('y (m)'); axis equal
```

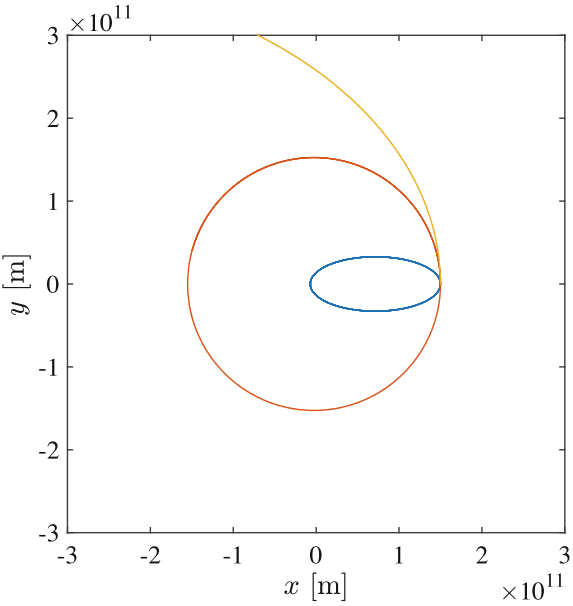

Fig. 7.14 Trajectories for a comet following Earth's trajector (*solid line*), with 1/3 of the initial velocity (*dashed line*), and with 1.5 times the initial velocity (*dotted line*)

Analysis: The resulting trajectory in Fig. 7.14 shows that the comet is moving in a circular orbit! What happens if we change the initial conditions a bit? If we reduce the initial velocity by a factor 3, the resulting trajectory is no longer a circle, but looks more like an ellipse. What if we increase the initial velocity by a factor 1.5? Then the resulting trajectory is no longer a closed loop—the comet leaves the solar system!

You will learn more about planetary motion later on. For now we realize that we can calculate the motion of planet using Newton's second law and the gravitational force model.

Summary

Newton's second law:

- Newton's second law relates the acceleration of an object to the net forces acting on it: $\sum_j \mathbf{F}_j = m\mathbf{a}$, where the sum is over all forces acting on the object, and m is the inertial mass.
- All forces acting on *a system* has a source in the *environment*.
- Forces can be *contact forces* acting on the boundary between the system and the environment.
- Forces can be *long range forces* from an object in the environment.

- Forces are drawn as vectors starting at the point where the force is acting, pointing in the direction of the force, and with a length indicating the length of the force.
- The force may be a given quantity, $\mathbf{F}$.
- The gravitational force acts between all objects. On the surface of the Earth the gravitational force on an object is $\mathbf{W} = -mg\,\mathbf{j}$, where $\mathbf{j}$ is a unit vector pointing upwards, g is the acceleration of gravity, and m is the gravitational mass. The gravitational mass is equal to the inertial mass.
- The contact force from a fluid on a moving object depends on the velocity of the object relative to the fluid. The simplest force model is the viscous force, $\mathbf{D} = -k_v\mathbf{v}$, where the constant k_v depends on the viscosity of the fluid and the size of the object.
- The contact force from a solid depends on the distance between the object and the solid. The simplest force model that depends on the position of an object is the independent spring force model: $\mathbf{F} = -k_x(x - x_e)\,\mathbf{i} - k_y(y - y_e)\,\mathbf{j} - k_z(z - z_e)\,\mathbf{k}$. Here, x_e, y_e, z_e are equilibrium positions in the x, y, and z-directions, and k_x, k_y, and k_z are spring constants. The spring model is one of the most fundamental force models because it is the first order Taylor expansion of any position-dependent force.

Problem-solving approach:

- We **identify** the object and its initial conditions.
- We **model** the behavior by find the forces acting on the object, introducing force models for all the forces, and applying Newton's second law to find an equation for the acceleration of the object.
- We **solve** the problem by finding the position and velocity from the acceleration and the initial conditions using numerical or analytical techniques.
- We **analyze** the solution to validate it, and use the solution to answer the original question posed.

Exercises

Discussion Questions

7.1 Free kick. A soccer player is making a free kick and the opposing team is making a wall to protect their goal. Is it always theoretically possible for the kicker to hit the goal in an ideal situation with only a vertical acceleration due to gravity?

7.2 Flying ball. A projectile is shot through the air. Can you think of any situation where the projectile may experience an upward acceleration?

7.3 Bouncing ball. A basket ball is thrown in a long arc and bounces off the floor. We assume that the contact with the floor can be modelled as a spring force acting

normal to the floor. Describe how the horizontal and vertical components of the velocity change during the collision.

7.4 Earth and Sun. The force from the Sun on the Earth acts directly towards the Sun, yet the Earth does not fall into the Sun. Explain.

7.5 Rope magic. You tie a long, strong rope between your car and a tree in order to exert a large force on the car. How can you pull the rope to ensure that you pull at the car with a much larger force than you can exert on the rope?

7.6 Curving the ball. You may know from soccer that you can curve a ball by spinning it. Can you explain this by the physics you have learned so far?

7.7 Suspension. The suspension of a car consists of both a spring and a dashpot. The dashpot provides a viscous force response. Why is it not sufficient with a spring alone?

Problems

7.8 Paraglider. Samantha is jumping from an airplane.
(a) Identify the forces acting on Samantha and draw a free-body diagram of her before she has pulled the chord.
(b) Identify the forces acting on Samantha and draw a free-body diagram of her after she has pulled the chord.
(c) Samantha hits the water instead of the boat she was aiming for. Identify the forces acting on Samantha and draw a free-body diagram of her as she slows down in the water.

7.9 Boat on a lake. A boat is sailing at constant velocity over a small lake. Identify the forces acting on the boat and draw a free-body diagram of the boat.

7.10 Force on the Moon. Identify the forces acting on the Moon, and draw a free-body diagram of it.

7.11 Chandelier. A chandelier of mass $m = 200\,\text{kg}$ is hanging in 4 wires of equal length. The wires are attached on the corners of a square in the ceiling. The distance between each attachment point in $L = 4\,\text{m}$. The chandelier is suspended a length h below the ceiling.
(a) Draw a free-body diagram of the chandelier.
(b) Find an expression for the wire tension if the chandelier is not moving.
(c) The wires can sustain a maximum tension of $10{,}000\,\text{N}$. How far down can the chandelier be suspended?

7.12 Three-pointer. You are throwing for a three-pointer. The ball leaves your hand with a velocity of 9.4 m/s at and angle of 60° with the horizon. You score from

a horizontal distance of 7 m. The height of the basket is 3.5 m. You can ignore air resistance.
(a) Draw a free-body diagram of the ball.
(b) Find the position and velocity of the ball as a function of time.
(c) At what height was the ball released?
(d) What is the velocity in the vertical direction as the ball hits the goal?

7.13 Hitting an apple. You are aiming your bow directly at an apple placed on top a high pole 50 m away. The arrow leaves the bow with a horizontal velocity of 50 m/s. You can ignore air resistance.
(a) Draw a free-body diagram of the arrow while in the air.
(b) Find the position and velocity of the arrow as a function of time.
(c) How far does the arrow fall below a horizontal path in the first half of the motion?
(d) How far does the arrow fall down in the second half of the motion? Why is it not the same as you found above?
(e) The apple is 4 cm in radius. How far from the apple can you stand and still hit it?

7.14 Hitting the target. You are trying to make a winning play in urban terrain golf - you are standing on top of the physics building, 10 m above ground, and try to hit a hole located 5 m out from the building. What initial speed should you give the ball in order for it to hit the hole? You can ignore air resistance.

7.15 Long jump world record. In 1991 Mike Powell beat the long-standing world record of Bob Beamon by reaching a length of 8.95 m. His maximum speed is 9.5 m/s. What is his maximum range?

7.16 Adjusting the aim of a rifle. You are adjusting the aim of your rifle by shooting at a target 100 m away. When you have adjusted your rifle at this length, your start shooting at a target 200 m away. How far above the target do you need to aim in order to hit the target? You know that the bullet leaves your rifle with a speed of 1000 m/s. You can ignore air resistance.

Projects

7.17 Ball in a spring. In this project you will study an advanced model of a pendulum. The pendulum consists of a ball in a massless rope moving in a vertical plane (Fig. 7.15). The ball has mass m. You can neglect air resistance. We describe the position of the ball by the position vector, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$. In this project we will introduce a *model* for the pendulum by assuming that the rope can be modelled as a spring with a spring constant k and an equilibrium length L_0.
(a) Identify the forces and draw a free-body diagram of the ball.
(b) Show that the net external force acting on the ball can be written as:

$$\sum_j \mathbf{F} = -mg\,\mathbf{j} - k\,(r - L_0)\,\frac{\mathbf{r}}{r}, \tag{7.60}$$

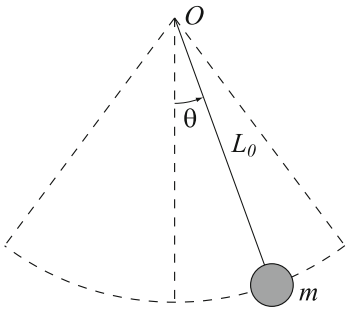

Fig. 7.15 Illustration of a pendulum consisting of a ball of mass m attached to rope of length L_0. The other end of the rope is attached at the point O

where $r = |\mathbf{r}|$ is the length of the (stretched) rope, and the origin of the coordinate system is chosen to be the attachment point, O, of the rope.

(c) Rewrite the expression of the external force on component form by writing the force components, F_x, and F_y, as functions of the components x and y of the position vector, $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$.

In this project, we will not assume that the ball is following a particular path, such as a circle, but we will instead use Newton's second law to determine the motion of the ball from the forces acting on it. Using our model, we can measure the tension in the rope, as well as the motion of the ball, and analyze these to learn about the motion.

(d) For a pendulum, it is customary to describe the position of the pendulum by its angle θ with the vertical. Does the angle θ give a sufficient description of the position of the ball in this case? Explain your answer.

(e) If the ball is at rest at $\theta = 0$ with no velocity ($\mathbf{v} = \mathbf{0}$) and no acceleration, what is the position of the ball? What happens if you increase the value of k for the rope?

We will now study a specific pendulum, consisting of a ball with a mass of 0.1 kg, and a rope of equilibrium length $L_0 = 1$ m with a spring constant $k = 200$ N/m, which corresponds to a rather elastic rope. Initially, you can assume that the ball starts with zero velocity at an angle $\theta = 30°$ at a distance L_0 from the origin. We want to study the motion of the ball by integrating the equations of motion numerically.

(f) Find an expression for the acceleration, $\mathbf{a}$, of the ball. You should write it both on vector form, where there acceleration vector is a function of the position vector $\mathbf{r}$ and its length, r, and on component form, where the components a_x and a_y are functions of the x and y components of the position vector.

(g) What is the mathematical initial value problem you need to solve in order to find the motion of the ball? Include both the differential equation you need to solve and the initial conditions in your answer.

(h) How can you solve this problem numerically? Write down a set of equations that find the position and velocity at a time $t + \Delta t$ given the position and velocity at t using Euler-Cromer's method. Insert your expression for the acceleration from above. Mark the terms in your equations that vary in time.

(i) Write a program that "solves" the problem by finding the motion of the ball. The program should plot the position of the ball in the xy-plane for the first 10 s of the motion. *Hint 1*: You may write the mathematical expression almost directly into your program if you use a vector notation and vector operations in your code. *Hint 2*: Remember that $r = r(t) = |\mathbf{r}(t)|$ varies in time! *Hint 3*: Do *not* use $\theta(t)$ to describe the position of the ball. Describe the motion using $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ and use your results from above for the acceleration.
(j) Use the program to find the behavior for the given initial conditions using a time-step of $\Delta t = 0.001$. Plot the resulting motion. Describe what you see.
(k) What happens if you increase Δt to $\Delta t = 0.01$ and $\Delta t = 0.1$? Can you explain this? Test what happens if you use Euler's method with $\Delta t = 0.001$ instead of Euler-Cromer's method.
(l) Rerun the program with $k = 20$ and $k = 2000$. Describe the motion in these cases and compare with $k = 200$ case. Are your results reasonable? Based on this, can you suggest how to use this method to model a pendulum in a stiff rope? What do you think would be the limitation of this approach? (Test what happens if you use $k = 2\,10^6$ in your program).
(m) Rewrite your program to ensure that the rope tension is zero if the spring is compressed, because the rope cannot sustain compression. Use this program to determine the motion with the initial conditions $\mathbf{v}_0 = 6.0\,\text{m/s}\,\mathbf{i}$ and $\mathbf{r}_0 = -L_0\,\mathbf{j}$. What happens?

7.18 Weather balloon. In this project we will develop a model to determine the motion of a weather balloon released from the ground. We start from a simplified model and gradually add features to make to model more realistic.

After the balloon is released, it is driven by buoyancy. Initially, we will assume that the buoyancy force is a constant, B.
(a) Draw a free-body diagram of the balloon. Identify the forces and introduce symbols. Indicate the relative magnitudes of the forces by the length of the vectors.

First, let us neglect air resistance.
(b) What is the acceleration of the balloon?
(c) Find the position and velocity of the balloon as a function of time.

Let us now introduce air resistance, using a quadratic law: $\mathbf{F}_D = -D\,v\,\mathbf{v}$.
(d) Show that the acceleration of the balloon in the upward (z) direction is $a_z = (B/m) - g - (D/m)\,|v_z|\,v_z$, where v_z is the velocity in the z-direction.
(e) Sketch the acceleration and velocity as a function of time for the model including air resistance.
(f) Find the asymptotic (terminal) velocity of the balloon.

The balloon is released on a windy day, with a wind blowing with a velocity $\mathbf{w} = w\,\mathbf{i}$ along the horizontal x-axis.
(g) How does the wind modify the air resistance force $\mathbf{F}_D$ on the balloon?
(h) Draw a free-body diagram for the balloon in this case. Indicate the magnitude of the forces by the relative lengths of the vectors.
(i) Find an expression for the acceleration $\mathbf{a}$ of the balloon. What are the initial conditions for the motion of the balloon?

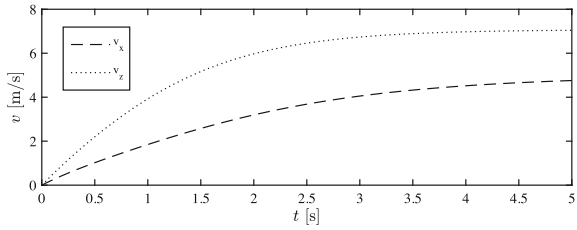

Fig. 7.16 Result of a simulation

(j) Why do we call the motion in the z and the x direction "coupled" in this case? Can you determine the motion of the balloon analytically?
(k) We can determine the motion of the balloon using numerical methods. Write a program to find the velocity $\mathbf{v}(t)$ and position $\mathbf{r}(t)$ as functions of time. (It is sufficient to only include the main integration step in your answer—that is the part that determines $\mathbf{v}(t + \Delta t)$ and $\mathbf{r}(t + \Delta t)$ given $\mathbf{v}(t)$ and $\mathbf{r}(t)$).

Figure 7.16 shows the result of a simulation.
(l) Describe the motion of the balloon. Illustrate by relevant sketches.
(m) Find the asymptotic (terminal) velocity of the balloon.

In a real situation, the wind velocity is smaller near the ground and increases gradually to the full velocity w_0 as the balloon moves upward. Typically, the velocity of the wind can be described by $\mathbf{w} = w_0\,(1 - \exp(-z/d))\,\mathbf{i}$, where $d = 10\,\mathrm{m}$ is a length determining the transition.
(n) Rewrite your program to include this effect.
(o) What is the terminal velocity of the balloon now?