Chapter 4
Motion in One Dimension

As a professional physicist you will be expected to be able to determine how things move: What is the path of a proton through a curved particle accelerator? What is the motion of a passenger in a car during a collision? How does a blood cell move through the micro-capillaries in your body? Professionally and privately, you will be expected to be able to solve any such problem your friends or your employer may come up with. How can you pull it off?

Fortunately, there is a simple method to determine the motion of an object. Objects move due to the forces acting on them. As soon as you have figured out what forces are acting on them, and you have found a model that predicts the magnitude and direction of the force during the motion, you can find the acceleration of the object. From the acceleration you can determine the motion of the object given its starting position and velocity. You will work through this procedure repeatedly over the next chapters, gradually filling all the concepts with meaning, until the procedure becomes a natural part of your way of thinking.

In this chapter we concentrate on developing our intuition of motion, on finding methods to formulate mathematical equations that determine the motion, and on developing analytical and numerical methods to solve the equations of motion.

You will learn to describe the motion of an object by its position as a function of time. We introduce the velocity and the acceleration of an object, which are the first and second time-derivatives of the position of the object. We also show how to find expressions for the motion from the velocity or acceleration—finding the equations of motion for the object.

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4.1 Description of Motion

In a fantastic race in the 100 m finals of the 2008 Olympic Games in Beijing, Usain Bolt set a new world record of 9.69 s. He even took the time to celebrate his victory over the last 20 m of the race. But did this affect his winning time? Could he have run even faster?

In order to answer such a question, we need a quantitative description of the race. We already know something: He ran 100 m in 9.69 s. But we want more detail—a finer resolution of the motion. We want to know where he was at any intermediate time from he started until he finished the race.

Motion Diagram

The first few seconds of the race are illustrated by the four pictures in Fig. 4.1. How can we describe the motion of Usain Bolt in lane four? One method is to define his position by the front of his chest. For each image, we draw a dot on the ground directly below his chest, resulting in a sequence of dots along lane four. We can now describe the race by measuring the distance, x, from the starting line to each dot—giving us a sequence of *positions*, x_i, at times t_i, for $i = 0, 1, 2, \ldots$ (Table 4.1).

We plot a point at the position x_i along the x-axis to illustrate the motion in a *motion diagram* (Fig. 4.1):

> A **motion diagram** illustrates the motion by a sequence of positions x_i at subsequent times t_i for $i = 0, 1, 2, \ldots$, preferably at times $t_i = t_0 + i\,\Delta t$, where Δt is the time interval.

Position and Time

From Fig. 4.1 we see that the runner is at $x(0\,\mathrm{s}) = 0.0\,\mathrm{m}$ when $t = 0\,\mathrm{s}$ and at $x(3\,\mathrm{s}) = 21.3\,\mathrm{m}$ when $t = 3\,\mathrm{s}$. Even though we have only measured the position at discrete times t_i, we expect the position of the runner to vary continuously with time, as illustrated by the plot of $x(t)$ in Fig. 4.2. This is indeed how we characterize motion:

> The motion of an object is described by the **position**, $x(t)$, as a function of time, t, measured in a given reference system.

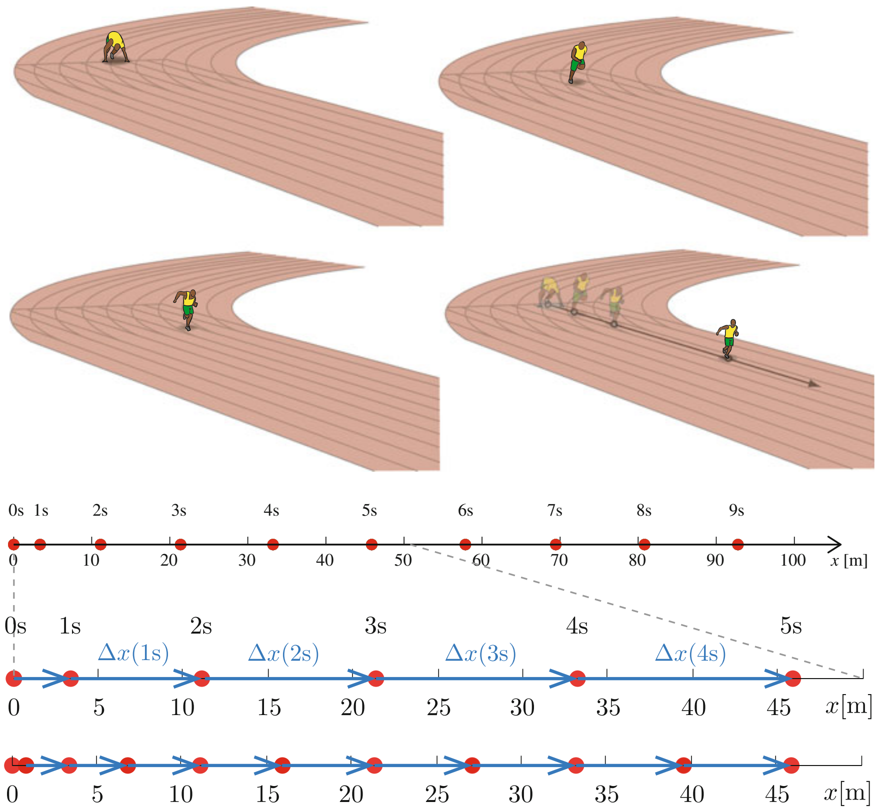

Fig. 4.1 *Top* Illustrations from the 100 m final in the 2008 Olympic Games in Beijing, showing the position of the Usain Bolt during the first 3 s. The *dots* in the 3 s image illustrate the position of the runner in lane 4 after 0, 1, 2, and 3 s. *Bottom* The position $x(t_i)$ of the runner is shown at 1 and 0.5 s intervals. Displacements Δx are drawn in *blue*

Table 4.1 Data from Usain Bolt's race

i	0	1	2	3	4	5	6
t_i (s)	0.0	1.0	2.0	3.0	4.0	5.0	6.0
x_i (m)	0.0	3.4	11.1	21.3	33.2	45.8	57.9

Reference System and Origin

We have chosen to measure the position x along the lane. We call this direction the x-axis. The position x is measured from the starting line, which we call the origin—the point where x is zero. The choice of an origin and an axis is called a *reference system*. The axis has a direction which tells us in what direction x is increasing—this

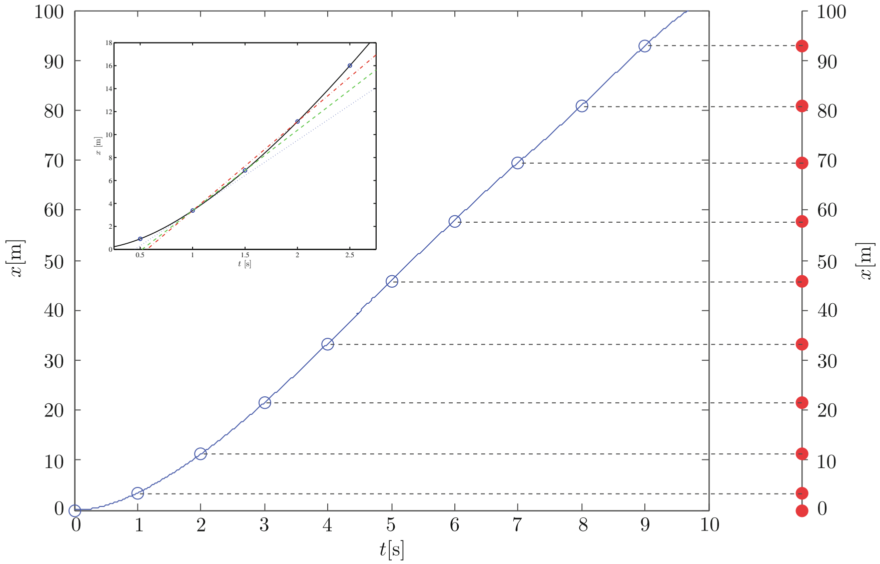

Fig. 4.2 A plot of the position x as a function of time for Usain Bolt. The *circles* along the *curve* show the position at time intervals of 1 s, corresponding to the positions in the motion diagram. The correspondence between the two representations of the motion is shown by inserting a rotated motion diagram to the right of the plot. *Inset* A magnification of $x(t)$. The average velocities at $t = 1$ s for time intervals $\Delta t = 1$ s and $\Delta t = 0.5$ s are illustrated by the slopes of the *red* and *green lines* respectively. The instantaneous velocity is illustrated by the slope of the *dotted blue line*, which corresponds to the slope of the tangent to the curve at $t = 1$ s

is indicated by the arrow on the axis. The axis is directed from the starting line to the finishing line, so that the position of the runner increases during the race.

You are free to choose the axes and the origin of your reference system as you like, but we usually try to choose so that measurements become simple, as we have done here.

Velocity

The motion diagram in Fig. 4.1 visualizes the change in position over a time interval Δt. The change in position from time $t = 1$ s to $t = 2$ s is:

$$x(2\,\text{s}) - x(1\,\text{s}) = 11.1\,\text{m} - 3.4\,\text{m} = 7.7\,\text{m} \tag{4.1}$$

We call this change the *displacement*, $\Delta x(1\,\text{s})$:

The **displacement** $\Delta x(t_1)$ over the time interval from $t = t_1$ to $t = t_1 + \Delta t$ is defined as:

$$\Delta x(t_1) = x(t_1 + \Delta t) - x(t_1). \tag{4.2}$$

The displacement is read directly from the motion diagram as the length of the line from $x(1\,\mathrm{s})$ to $x(2\,\mathrm{s})$. The displacement has a direction—it is the displacement *from* $x(t_i)$ *to* $x(t_i + \Delta t)$—and it is therefore drawn as an arrow in Fig. 4.1.

The first few displacements in Fig. 4.1 are increasing. This means that he is running faster. But how fast is he running? This cannot be described by displacement alone, because the displacements become smaller when we decrease the time interval as shown in Fig. 4.1. It is the displacement per time that describes how fast he is running:

The **average velocity** from $t = t_1$ to $t = t_1 + \Delta t$ is:

$$\bar{v}(t_1) = \frac{x(t_1 + \Delta t) - x(t_1)}{\Delta t} = \frac{\Delta x(t_1)}{\Delta t}. \tag{4.3}$$

The average velocity has units meters per second, m/s.

The average velocities for the runner in Fig. 4.1 at $t = 1$ s and $t = 2$ s over the time interval $\Delta t = 1$ s are:

$$\bar{v}(1\,\mathrm{s}) = \frac{7.7\,\mathrm{m}}{1\,\mathrm{s}} = 7.7\,\mathrm{m/s}, \tag{4.4}$$

$$\bar{v}(2\,\mathrm{s}) = \frac{10.2\,\mathrm{m}}{1\,\mathrm{s}} = 10.2\,\mathrm{m/s}, \tag{4.5}$$

However, if we calculate the average velocity from the bottom-most diagram in Fig. 4.1, the time interval is $\Delta t = 0.5$ s, and the velocities are:

$$\bar{v}(1\,\mathrm{s}) = \frac{3.5\,\mathrm{m}}{0.5\,\mathrm{s}} = 7.0\,\mathrm{m/s}, \tag{4.6}$$

$$\bar{v}(2\,\mathrm{s}) = \frac{4.9\,\mathrm{m}}{1\,\mathrm{s}} = 9.8\,\mathrm{m/s}, \tag{4.7}$$

We see that the average velocities depend on the time interval Δt! We can understand this from the inset in Fig. 4.2. First, we notice that we can read the average velocity $\bar{v}(1\,\mathrm{s})$ directly from the curve, $x(t)$, as the slope of the curve from the point $x(1\,\mathrm{s})$ to the point $x(1\,\mathrm{s} + \Delta t)$. From the figure, we see that $\bar{v}$ changes slightly as we change the time interval from $\Delta t = 1$ s to $\Delta t = 0.5$ s because the function $x(t)$ is curving. However, we also see that when the time interval Δt becomes smaller and smaller, the average velocity approaches a specific value given as the slope of the curve in the point $t = 1$ s. We call the velocity in this limit the *instantaneous velocity* at the time t, $v(t)$:

The **instantaneous velocity** is defined as the time derivative of the position:

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}. \tag{4.8}$$

In the following, whenever we use the term velocity, we will mean the instantaneous velocity.

Notation for Time Derivatives

Notice that the notation $x'(t)$ for the derivative that you may be used to from calculus, is not commonly used in physics. This is to avoid confusion with x', which is often used to represent a length in a coordinate system called the "marked" coordinate system. The notation $x'(t)$ can therefore be ambiguous: it may be interpreted as the position x' as a function of time, or as the time derivative of the position x. Instead, we denote the time derivative of a quantity by the placing a dot over it. The velocity is therefore often written as:

$$v(t) = \frac{dx}{dt} = \dot{x}. \tag{4.9}$$

Visualizing the Velocity $v(t)$

The velocity $v(t)$ represents the slope of the curve, $x(t)$. In many cases it may be useful to visualize the motion by looking at both the plot of $x(t)$ and the plot of $v(t)$, as shown in Fig. 4.3. In this case, it is evident that the velocity is changing throughout the motion. Initially, the velocity is increasing as the runner sprints out from the starting line. In the middle of the race the velocity is approximately constant, while at the end of the race, the runner is slowing down, and the velocity is falling.

Acceleration

The velocity may also vary throughout the motion. From Fig. 4.3 we see that the runner starts at rest and increases his velocity with time. Just as we introduced the velocity to characterize the rate of change of position, we introduce the acceleration to characterize the rate of change of the velocity:

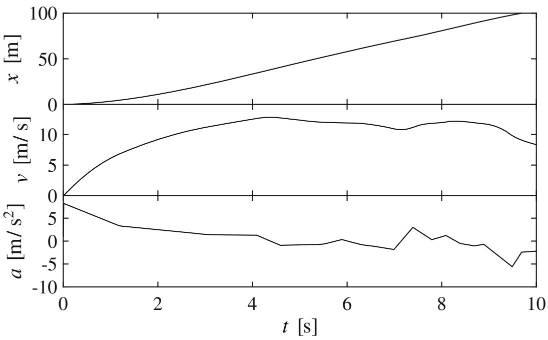

Fig. 4.3 A plot of the position $x(t)$, velocity, $v(t)$, and acceleration, $a(t)$, as a function of time for Usain Bolt

The **average acceleration** over a time interval Δt from t to $t + \Delta t$ is:

$$\bar{a}(t) = \frac{v(t + \Delta t) - v(t)}{\Delta t}. \tag{4.10}$$

The instantaneous acceleration is the limit of the average acceleration when the time interval approaches zero:

The **instantaneous acceleration** is defined as:

$$a(t) = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} = \frac{dv}{dt} = \dot{v}. \tag{4.11}$$

When we use the term acceleration we mean the instantaneous acceleration.

The acceleration can be found as the slope of the $v(t)$ curve. Figure 4.3 shows a plot of $a(t)$ together with both position $x(t)$ and velocity $v(t)$. Notice that the acceleration curve is "noisy" and consists of clear steps. This is not a physical effect, but rather an effect of how the data was gathered and interpolated. Real data often have noise from various sources—so you should expect noisy curves when you look at real systems. (You can learn more about how this data was measured in boltdatabox[1]).

Because the velocity is given as the time derivative of the position $x(t)$, we can also write the acceleration as the time derivative of the position $x(t)$ by inserting (4.9) into (4.11):

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}\frac{dx}{dt} = \frac{d^2x}{dt^2}. \tag{4.12}$$

[1] http://folk.uio.no/malthe/mechbook/boltdatabox.

Using the dot-notation, we can write this as:

$$a(t) = \dot{v}(t) = \ddot{x}(t), \tag{4.13}$$

or in shorthand

$$a = \dot{v} = \ddot{x}. \tag{4.14}$$

Interpretation of Motion Diagrams

It is often difficult to obtain a good intuition for acceleration, in particular for two- and three-dimensional motions, but sometimes also for one-dimensional motions. Experience shows that motion diagrams are useful tools for developing a good intuition for accelerations—this is why we include them here.

As long as all the time intervals in a motion diagram are identical, the displacements in the motion diagram may be interpreted as average velocities. In Fig. 4.4 the displacements and therefore the average velocities, are initially increasing, until at $t = 4$ s they are approximately constant. The change in average velocity from $t = 1$ s to $t = 2$ s is:

$$\Delta\bar{v}(1\,\text{s}) = \bar{v}(2\,\text{s}) - \bar{v}(1\,\text{s}) = 5\,\text{m/s} \tag{4.15}$$

We introduce the average acceleration[2] as:

$$\bar{a} = \frac{\Delta\bar{v}}{\Delta t} \tag{4.16}$$

The average acceleration can be constructed geometrically from the motion diagram by subtracting two subsequent (average) velocities in the diagram, as illustrated in Fig. 4.4.

4.1.1 *Example: Motion of a Falling Tennis Ball*

This example demonstrates how we can find the velocity and acceleration from the motion diagram of a falling tennis ball, both by hand calculation, using matlab, and from a mathematical model of the motion.

Motion diagram: The motion of a falling tennis ball was captured with a digital camera. The first few images were combined into one picture as shown in Fig. 4.5.

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[2] The attentive reader may realize that the average acceleration should really be defined in terms of the change in instantaneous velocity: $\bar{a} = (v(t + \Delta t) - v(t))/\Delta t$ and not in terms of the average velocity as done here. However, this small difference in definitions becomes insignificant when the time interval becomes sufficiently small.

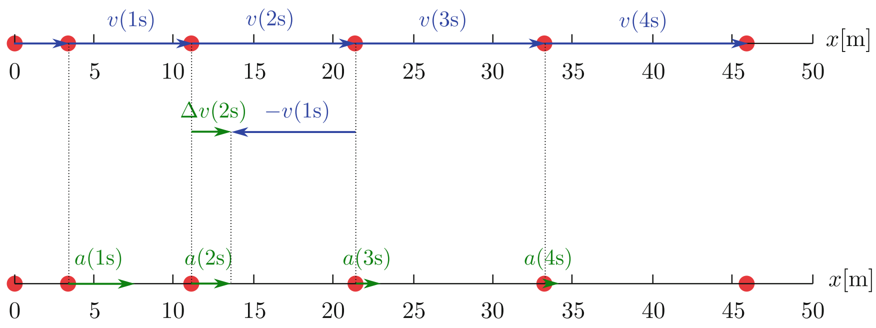

Fig. 4.4 Motion diagram for Usain Bolt. The *top figure* shows the velocities at time intervals of 1 s. The displacements are interpreted as velocities. The *top figure* shows how the change in velocity at $t = 2$ s is constructed from the velocity at $t = 1$s and the velocity at $t = 2$ s. The resulting difference, $\Delta v(2\,\text{s})$ is interpreted as the average acceleration. The *bottom figure* shows the accelerations estimated from the motion diagram

From the sequence of images, we measure the vertical position of the ball by comparing the height of the center of the ball to the ruler seen in the images. The positions are shown in Table 4.2.

We draw the motion diagram by marking the positions y_i with dots along the vertically oriented y-axis as illustrated in Fig. 4.5. We illustrate the velocities by the displacements, which are drawn as arrows from point to point. The average velocities can be calculated from the data: For each i in Table 4.2 we calculate the average velocity from t_i to t_{i+1} using:

$$\bar{v}_i = \frac{y_{i+1} - y_i}{\Delta t}. \tag{4.17}$$

The corresponding results are shown in the table. However, we cannot use this method to find a value for $i = 6$ since we do not know y_7. We find that all the velocities are negative. Since we have chosen the positive direction to be up (the arrow on the y-axis points upward) this means that the ball is falling down—as expected.

The velocities are increasing in magnitude since the ball is accelerating downward. We estimate the average accelerations by

$$\bar{a}_i = \frac{\bar{v}_i - \bar{v}_{i-1}}{\Delta t}, \tag{4.18}$$

and the results are shown in Table 4.2. For the accelerations, we cannot find a value for $\bar{a}_i$ for $i = 1$ or for $i = 6$, since the velocity are not defined at $i = 0$ or at $i = 6$. If you look at Fig. 4.5 you can also see how to construct the accelerations directly from the motion diagram.

The data shows that the acceleration is approximately constant $a \simeq -9.5 \pm 0.5\,\text{m/s}^2$ throughout the fall. This experiment therefore tells us that a tennis ball

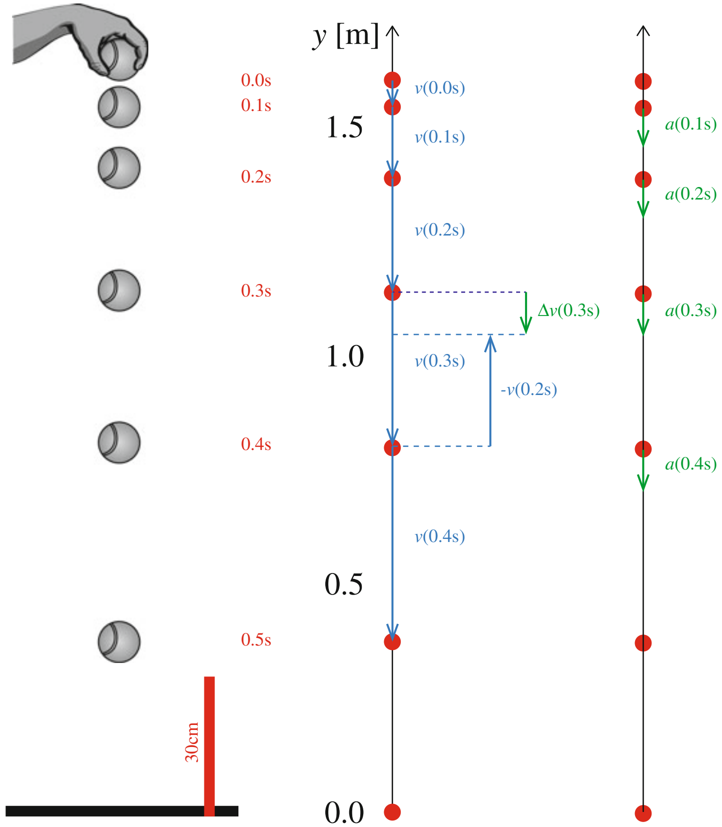

Fig. 4.5 *Left* Digital images from a falling tennis ball—we have made an artistic rendering of the ball for clarity. *Right* Motion diagram for the tennis ball. The *left* diagram shows the positions and velocities, and the *right* diagram illustrates the accelerations

Table 4.2 Table with calculated values

i	t_i (s)	y_i (m)	Δy_i (m)	$\bar{v}_i$ (m/s)	$\bar{a}_i$ (m/s^2)
1	0.0	1.60	−0.05	−0.5	
2	0.1	1.55	−0.15	−1.5	−10.0
3	0.2	1.40	−0.24	−2.4	−9.0
4	0.3	1.16	−0.34	−3.4	−10.0
5	0.4	0.82	−0.43	−4.3	−9.0
6	0.5	0.39			

falls with a constant acceleration—which is close to what you may recognize as the acceleration of gravity, $g = 9.8\,\text{m/s}^2$.

Mathematical model: A physicist friend of yours tells you that there is a mathematical model for the motion of a falling tennis ball when there is no air resistance

$$y(t) = y_0 - \frac{1}{2}gt^2, \tag{4.19}$$

where $g = 9.8\,\text{m/s}^2$ is a constant and y_0 is the position of the tennis ball at $t = 0\,\text{s}$. Let us see how this model matches up with the observed data.

We calculate the position of the ball for various times. From the experimental data, we see that $y(0\,\text{s}) = 1.6\,\text{m}$. We use matlab as a calculator to find the positions for all the times in Table 4.2 with a single line of code:

```
>> g = 9.8;
>> t = [0.0 0.1 0.2 0.3 0.4 0.5];
>> y = 1.6 - 0.5*g*t.^2
y =    1.6000    1.5510    1.4040    1.1590    0.8160    0.3750
```

Notice that the command `t. ^2` tells matlab to apply the operation for each element in the array `t`, generating an `y`-array of 6 elements. This vectorized notation allows us write the code in a similar way to the mathematics. We can output the data in a form that looks more like Table 4.2:

```
>> [t;y]'
ans =
         0    1.6000
    0.1000    1.5510
    0.2000    1.4040
    0.3000    1.1590
    0.4000    0.8160
    0.5000    0.3750
```

where the `'` means transpose. Without it, the table would have been oriented differently. Try it!

The resulting values for $y(t)$ are similar to the experimental data, but in the experiment the ball falls a bit slower than in the mathematical model: In the experiment the ball is at $y = 0.39\,\text{m}$ at $t = 0.5\,\text{s}$, whereas the mathematical model predicts $y = 0.375\,\text{m}$.

We can compare the results better by studying the velocities and accelerations. In the mathematical model, we know $y(t)$, and we can calculate the *instantaneous* velocity and acceleration by applying the definitions directly. The velocity of the ball is defined as:

$$v = \frac{dy}{dt}, \tag{4.20}$$

and if we insert $y(t)$ from (4.19) we get

$$v = \frac{d}{dt}\left(y_0 - \frac{1}{2}gt^2\right) = -gt. \tag{4.21}$$

Similarly, the acceleration is defined as

$$a = \frac{dv}{dt}, \tag{4.22}$$

where we insert $v(t)$ from (4.21) and get

$$a = -g = -9.8\,\text{m/s}. \tag{4.23}$$

The acceleration in the mathematical model is a constant. But we cannot really compare with the experimental data, since they have too low precision. We need better data!

High resolution data: To study the process in more detail, the motion of the falling tennis ball was also recorded by a motion detector placed directly above the ball. The detector provides the vertical position y of the ball, but at a much higher time resolution that the images: The detectors measures y at a time interval of $\Delta t = 0.001$s. The data is stored in the file fallingtennisball02.d.[3] The first few lines of the file looks like:

```
0.0000000000000000e+00   1.6000000000000001e+00
1.0000000000000020e-03   1.5999950510001959e+00
2.0000000000000044e-03   1.5999803020031378e+00
3.0000000000000070e-03   1.5999557530158828e+00
...                      ...
```

where each line contains the time t_i in seconds and the position y_i in meters (given in scientific notation, but with no unit). We read the data-set from file, using `load`

```
load -ascii fallingtennisball02.d
t = fallingtennisball02(:,1);
y = fallingtennisball02(:,2);
```

The command `load` generates the array `fallingtennisball10` which has 2 columns. Then, we create variables for the time, `t`, and the position `y`.

We see what is in the data-set by plotting the position as a function of time, $y(t)$, using:

```
plot(t,y)
xlabel('t [s]')
ylabel('y [m]')
```

What does the resulting plot in Fig. 4.6a show? From the plot, we see that the ball falls down, bounces up from the surface to reach a lower height than the first time, and so on. The first 0.5 s of the motion resembles what we found by analyzing the images: the position decreases with time. And we see that ball is falling faster with time—it accelerates. But it is difficult to see details of the motion directly from this plot. Could you say if the acceleration is constant or not for the first 0.5 s from this plot? To gain more insight, we need to analyze the velocity and acceleration of the ball.

[3] http://folk.uio.no/malthe/mechbook/fallingtennisball02.d.

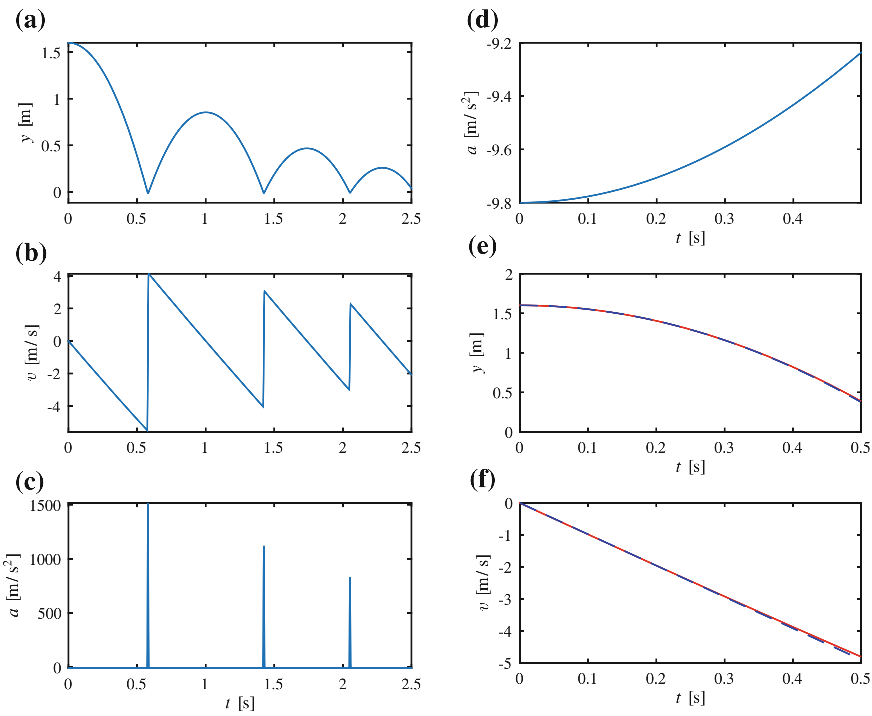

Fig. 4.6 **a–c** Plot of the position $y(t)$, velocity, $v(t)$, and acceleration, $a(t)$, of the ball as a function of time t; **d** plot of $a(t)$ for $t < 0.5$ s; **e**, **f** comparison of $y(t)$ and $v(t)$ for the experimental data (*red*, *solid line*) and the mathematical model (*blue*, *dashed lined*)

Numerical derivatives: Because we do not know $y(t)$ for all values of t, but only the measured values of $y(t_i)$, we cannot find an exact, analytical expression for the derivative of $y(t)$ as we did when we had a mathematical model. However, we can follow the procedure we used for the image data in (4.17): We can approximate the instantaneous velocity by the average velocity from t_i to $t_i + \Delta t$:

$$\frac{dy}{dt} = v(t_i) \simeq \bar{v}(t_i) = \frac{y(t_i + \Delta t) - y(t_i)}{\Delta t} . \tag{4.24}$$

The average velocity is an example of a *numerical derivative* of the position—a numerical method to calculate the derivative. This method is easily implemented numerically by directly converting the mathematical formula to code:

```
v(i) = (y(i+1)-y(i))/dt;
```

We need to apply this rule to each element i from 1 to $n - 1$, where n is the number of data points $y(t_i)$. This is done using a `for`-loop:

```
n = length(y);
dt = t(2) - t(1);
```

```
v = zeros(n-1,1);
for i = 1:n-1
   v(i) = (y(i+1) - y(i))/dt;
end
```

Here, we find `n`, the number of elements in the `y`-array, and the time difference `dt`, which we calculate from the first two times since the time intervals are regular. We also prepare an empty array `v`, which we will fill with velocities. But why do we only make it $n-1$ elements long? Because the formula `v(i) = (y(i+1) - y(i))/dt`, cannot be applied to the last element in the array, since we would then have no data for $i+1$. (We saw the same in Table 4.2). For the same reason, we must stop the loop at $n-1$.

Similarly, we find the acceleration by using the numerical derivative of the velocity:

$$a(t_i) \simeq \bar{a}(t_i) = \frac{v(t_i) - v(t_{i-1})}{\Delta t}. \quad (4.25)$$

We apply this mathematical definition of the derivative directly to the data:

```
a = zeros(n-1,1);
for i = 2:n-1
   a(i) = (v(i) - v(i-1))/dt;
end
```

For the acceleration, the formula `a(i) = (v(i) - v(i-1))/dt` cannot be applied to the first element in the array, since we have no data for $i = 0$. The loop therefore starts at $i = 2$ (Again, this is the same as in Table 4.2).

Plotting: We plot $x(t)$, $v(t)$, $a(t)$ by:

```
subplot(3,1,1)
plot(t,y)
ylabel('y [m]')
subplot(3,1,2)
plot(t(1:n-1),v)
ylabel('v [m/s]')
subplot(3,1,3)
plot(t(2:n-1),a(2:n-1))
xlabel('t [s]')
ylabel('a [m/s^2]')
```

Here we have used the `subplot` command to generate a set of plots. (Consult matlab to find out how the plots are numbered using `help subplot`.) Notice that the velocity is only defined for i from 1 to $n-1$. We therefore only include the corresponding values of t_i in the plot. Similarly, the acceleration is defined from 2 to $n-1$, and we only plot the corresponding values of t_i.

Plotting parts of the data: It is difficult to see the acceleration of the ball while it is falling from Fig. 4.6c. How can we plot only the first 0.5 s of the motion? We find the value for i where t_i goes from begin smaller than 0.5 to larger than 0.5 using `find`:

```
imax = max(find(t<=0.5));
plot(t(2:imax),a(2:imax));
xlabel('t [s]')
ylabel('a [m/s^2]')
```

and plot $a(t)$ for this range of t-values in Fig. 4.6d. (You could also have made this plot by using the zoom button in the plotting window). The acceleration is clearly *not* a constant in this case. It starts at $-9.8\,\text{m/s}^2$, but its magnitude becomes smaller with time. (This is due to air resistance).

Comparison with mathematical model: How large are the differences between the experimental data and the mathematical model for motion without air resistance? A good way to compare, is to plot the model in the same plot as the data. The model was:

$$y(t) = y_0 - \frac{1}{2}gt^2 \text{ and } v(t) = -gt. \tag{4.26}$$

We implement these formulas directly in the program, and plot both data and model:

```
g = 9.8; % m/s^2
y0 = 1.6; % m
vt = -g*t;
yt = y0 - 0.5*g*t.^2;
subplot(2,1,1)
plot(t(1:imax),y(1:imax),'-r');
hold on
plot(t(1:imax),yt(1:imax),'--b');
hold off
xlabel('t [s]')
ylabel('y [m]')
subplot(2,1,2)
plot(t(1:imax),v(1:imax),'-r');
hold on
plot(t(1:imax),vt(1:imax),'--b');
hold off
xlabel('t [s]')
ylabel('v [m/s]')
```

We use `hold on` to get both plots in the same figure (see Fig. 4.6e, f) Here we notice that the differences in $y(t)$ and $v(t)$ are more difficult to spot. Using the acceleration for comparisons was therefore a better approach to spot the differences. And an approach with a sound, physical basis, since we will later learn that differences in physics appear in differences in the accelerations.

Further work: We leave it to you to look more carefully at what happens during the bounce. Can you zoom in on the relevant area?

Comment: Notice that the data in this example were based on numerical results and not experimental data in order to get clear results. Experimental data will typically contain significant noise, which we did not want to include here. The program used to generate the data-set is makefallingtennisball.m.[4]

[4] http://folk.uio.no/malthe/mechbookmakefallingtennisball.m.

4.2 Calculation of Motion

Mechanics is about the motion of objects. Usually, we do not know the position as a function of time. Instead, we want to determine the motion based on measurements of the acceleration (or velocity); based on a mathematical expression for the acceleration; or based on a differential equation for the position. We therefore need tools to do the opposite of what we did above: We need tools to find the motion, $x(t)$, from the acceleration, $a(t)$, of an object.

Discrete Integration

As lead developer of "The Rocket", a new roller-coaster ride at a major theme-park, you have fitted an accelerometer into a test-cart. The accelerometer records the acceleration of the cart at regular time intervals of 0.1 s (Table 4.3). How can you use this data to determine the velocity and position of the test cart?

The problem is how to find the sequence of positions, $x(t_i)$, from the sequence of accelerations, $a(t_i)$? This is the reverse of what we have been doing so far, where we have estimated first the velocities and then the accelerations from the positions using numerical derivatives. Can we simply use the methods we have developed for numerical derivatives "in reverse"? The average acceleration from $t_1 = 0.0$ s to $t_2 = 0.1$ s is

$$\bar{a}(t_i) = \frac{v(t_i + \Delta t) - v(t_i)}{\Delta t} . \tag{4.27}$$

(So far this is an *exact* result—we have not done any approximations yet). We can "reverse" (4.27) to find an equation for the velocity at the time $t = t_i + \Delta t$:

$$v(t_i + \Delta t) - v(t_i) = \bar{a}(t_i)\,\Delta t$$
$$v(t_i + \Delta t) = v(t_i) + \bar{a}(t_i)\,\Delta t \tag{4.28}$$

This method would allow us to step one step forward in time from the time $t = t_i$ to the time $t = t_i + \Delta t$, if only we knew the average acceleration of the time interval. Unfortunately, the accelerometer does not give the average, but rather the instantaneous acceleration of the cart, $a(t_i)$. Let us ignore this distinction and approximate the average acceleration over the time interval by the instantaneous acceleration at the beginning of the time interval:

Table 4.3 Data from the motion of "The Rocket"

i	0	1	2	3	4	5
t_i (s)	0.0	0.1	0.2	0.3	0.4	0.5
a_i (m/s^2)	0.00	1.43	2.80	4.13	5.62	7.21

$$\bar{a}(t_i) \simeq a(t_i), \tag{4.29}$$

We are now in a position to use (4.28) to step forward in small steps of Δt, calculating the changes in the velocities of the cart as we go. However, finding the velocities only takes us part of the way—we also need to determine the positions, $x(t_i)$, of the cart, from the velocities, $v(t_i)$, calculated using (4.28). This time, we "reverse" the numerical derivative of the position:

$$\begin{aligned} x(t_i + \Delta t) - x(t_i) &= \bar{v}(t_i)\,\Delta t \\ x(t_i + \Delta t) &= x(t_i) + \bar{v}(t_i)\,\Delta t. \end{aligned} \tag{4.30}$$

where we again assume that the average velocity is approximately the same as the velocity we calculated in (4.28): $\bar{v}(t_i) \simeq v(t_i)$. We are now ready to use (4.28) and (4.30) to move forwards in steps of Δt. However, since these methods only give the increments in the velocity and the position, we need to know the first velocity of the cart, $v(t_0) = v_0$ and where the cart starts from, $x(t_0) = x_0$. This is called the initial conditions of the problem.[5]

We are now ready to find the velocities and positions, starting at the time $t = t_0 = 0.0\,\text{s}$:

1. At $t = t_0 = 0.0\,\text{s}$, the velocity and position of the cart is given $v(t_0) = v(0.0\,\text{s}) = 0.0\,\text{m/s}$, $x(t_0) = x(0.0\,\text{s}) = 0.0\,\text{m}$.
2. At $t = t_0 + \Delta t = 0.1\,\text{s}$, the velocity of the cart is:

$$v(0.1\,\text{s}) \simeq v(0.0\,\text{s}) + a(0.0\,\text{s})\,\Delta t = 0.5\,\text{m/s}, \tag{4.31}$$

where the acceleration $a(0.0\,\text{s}) = 5.0\,\text{m/s}^2$ is listed in the table Fig. 4.7. The position of the cart is:

$$x(0.1\,\text{s}) \simeq x(0.0\,\text{s}) + v(0.0\,\text{s})\,\Delta t = 0.0\,\text{m} \tag{4.32}$$

3. At $t = t_1 + \Delta t = 0.2\,\text{s}$, the velocity of the cart is:

$$v(0.2\,\text{s}) \simeq v(0.1\,\text{s}) + a(0.1\,\text{s})\,\Delta t = 0.9\,\text{m/s}, \tag{4.33}$$

where the acceleration $a(0.1\,\text{s}) = 7.0\,\text{m/s}$ is listed in the table in Fig. 4.7. The position of the cart is:

$$x(0.2\,\text{s}) \simeq x(0.1\,\text{s}) + v(0.1\,\text{s})\,\Delta t = 0.05\,\text{m}, \tag{4.34}$$

[5]Notice that we need two initial conditions, $v(t_0) = v_0$ and $x(t_0) = x_0$ to determine the position from the acceleration: This is because we need to first calculate the velocities, and this requires an initial condition of the velocities, and then calculate the position, and this also requires an initial condition, this time on the position.

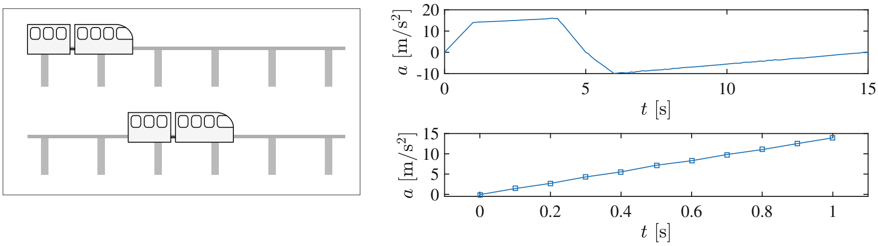

Fig. 4.7 Illustration of the motion of "The Rocket". The accelerations are illustrated for the whole time interval (*top figure*) and the time-resolution is shown by the *squares* representing the measurement points (*bottom figure*)

where the velocity $v(0.1\,\text{s}) = 0.5\,\text{m/s}$ was found in the previous step of the calculation.

This method is called *Euler's method* for numerical integration, and it is sufficiently flexible and robust to solve most problems presented in this book!

In **Euler's method** we find the position, $x(t_i)$, and velocity, $v(t_i)$, of an object as a function of time by a stepwise summation of the acceleration, $a(t_i)$, and the velocity, $v(t_i)$:

$$\begin{aligned} v(t_0) &= v_0 \\ x(t_0) &= x_0 \\ &\ldots \\ v(t_i + \Delta t) &= v(t_i) + a(t_i)\,\Delta t \\ x(t_i + \Delta t) &= x(t_i) + v(t_i)\,\Delta t \end{aligned} \tag{4.35}$$

We apply this method to find the position and velocities for the motion of "The Rocket". The accelerations for the cart are stored in the file therocket.dat,[6] where each line contains a time (in seconds) and an acceleration (in m/s^2):

```
0.0000000e+000  2.7316440e-001
1.0000000e-001  1.4411079e+000
2.0000000e-001  2.6693138e+000
3.0000000e-001  4.2383806e+000
```

We read the data into matlab, find the time-step Δt from $t_2 - t_1$, and apply Euler's algorithm from (4.35) for each i starting from the initial condition $x(t_0) = 0\,\text{m}$ and $v(t_0) = 0\,\text{m/s}$ using a `for`-loop.

```
temp  = load('therocket.dat');
t = temp(:,1);
```

[6]http://folk.uio.no/malthe/mechbook/therocket.dat.

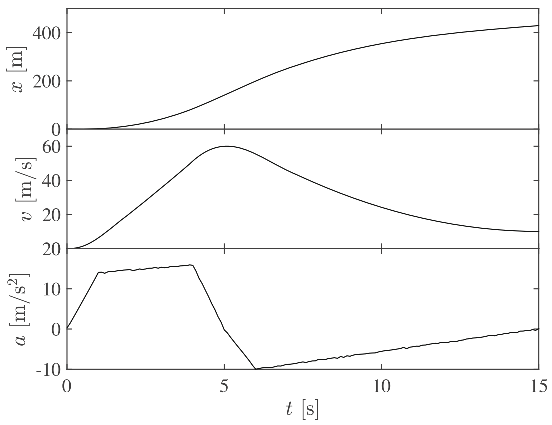

Fig. 4.8 Illustration of the motion of "The Rocket", showing the measured acceleration, and the calculated velocity and position

```
a = temp(:,2);
dt = t(2) - t(1);
n = length(t);
v = zeros(n,1);
v(1) = 0.0; % v_0
x = zeros(n,1);
x(1) = 0.0; % x_0
for i = 1:n-1
    v(i+1) = v(i) + a(i)*dt;
    x(i+1) = x(i) + v(i)*dt;
end
```

The resulting position and velocity plots are shown in Fig. 4.8.

The procedure presented here covers the most important topic in kinematics: How to determine the motion of an object given the acceleration of the object. This is important because you will later learn that the physics of a problem—the interactions between the object and other objects—gives the acceleration of the object. Given the acceleration it will be up to you to determine the motion—and you can do this using the methods provided here: Either by using Euler's method (or more advanced techniques) to solve the problem numerically, or by finding a solution to the problem based on the specialized techniques you have learned in calculus.

Formal Integration

A more formal formulation of the problem is to assume that we know the acceleration $a(t)$ of an object as a function of time. How do we find the position and velocity of the object in this case?

Again, we realize that we have already solved the "reverse" problem—we know that the acceleration is the time derivative of the velocity and that the velocity is the time derivative of the position. We find the velocity by integrating the definition of acceleration:

$$a(t) = \frac{dv}{dt} \Rightarrow \int_{t_0}^{t} a(t)\,dt = \int_{t_0}^{t} \frac{dv}{dt}\,dt = v(t) - v(t_0), \tag{4.36}$$

$$v(t) = v(t_0) + \int_{t_0}^{t} a(t)\,dt. \tag{4.37}$$

When we know the velocity as a function of time, we can find the position by integrating the velocity, starting from the definition of velocity:

$$v(t) = \frac{dx}{dt} \Rightarrow \int_{t_0}^{t} v(t)\,dt = \int_{t_0}^{t} \frac{dx}{dt}\,dt = x(t) - x(t_0) \tag{4.38}$$

$$x(t) = x(t_0) + \int_{t_0}^{t} v(t)\,dt. \tag{4.39}$$

If we insert $v(t)$ from (4.37), we get:

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} [v(t_0) + \int_{t_0}^{t} a(t)\,dt]\,dt \\ &= x(t_0) + v(t_0)(t - t_0) + \int_{t_0}^{t} [\int_{t_0}^{t} a(t)\,dt]\,dt. \end{aligned} \tag{4.40}$$

These equations constitute the **integration method** to find the position $x(t)$ and velocity $v(t)$ given the acceleration $a(t)$ of an object:

$$v(t) = v(t_0) + \int_{t_0}^{t} a(t)\,dt, \tag{4.41}$$

$$x(t) = x(t_0) + \int_{t_0}^{t} v(t)\,dt = x(t_0) + v(t_0)(t - t_0) + \int_{t_0}^{t} [\int_{t_0}^{t} a(t)\,dt]\,dt. \tag{4.42}$$

There is no need to memorize these equations. They follow from your knowledge of calculus. You only need to remember the definitions of the velocity as the time derivative of the position, and the acceleration as the time derivative of the velocity.

We can apply this method to find the motion for constant acceleration, $a(t) = a_0$, with initial conditions $x(t_0) = x_0$ and $v(t_0) = v_0$:

$$v(t) = v(t_0) + \int_{t_0}^{t} a_0\,dt = v0 + a_0\,(t - t_0). \tag{4.43}$$

and

$$x(t) = x(t_0) + \int_{t_0}^{t} v(t)\,dt = x_0 + v_0\,(t - t_0) + \frac{1}{2} a_0\,(t - t_0)^2\,. \qquad (4.44)$$

Differential Equations

Usually, we do not have a set of measurements or a mathematical expression for the acceleration. Instead, we find an expression for the acceleration based on a physical model of the forces acting on the object, and from the forces we find the acceleration. Given this expression for the acceleration, we determine the velocity and position of the object. But this sounds exactly like what we did above? We integrate the acceleration to find the velocity, and then integrate again to find the position. Unfortunately, direct integration only works if the acceleration is *only* a function of time. In most cases, we do not have an expression of the acceleration as a function of time, but instead we know how the acceleration varies with velocity and position. For example, a tiny grain of sand sinking in water has an acceleration on the form:

$$\frac{d^2x}{dt^2} = a = -a_0 - c \cdot v\,, \qquad (4.45)$$

where the acceleration depends on the velocity of the grain! And a ball suspended in a vertical spring has an acceleration:

$$\frac{d^2x}{dt^2} = a = -C \cdot x\,, \qquad (4.46)$$

that depends on the position of the ball. Such problems cannot be solved by direct integration, because the function $x(t)$ and its derivatives occur on both sides of the equation. Such equations are called differential equations. Finding analytical solutions of differential equations require some skill and experience, but, fortunately, we can solve them numerically in exactly the same way we did above.

Numerical Solution

In most mechanics problems, we want to find the position, $x(t)$, which satisfies an equation on the form:

$$\frac{d^2x}{dt^2} = a\left(t, x, \frac{dx}{dt}\right),\; v(t_0) = v_0,\; x(t_0) = x_0\,, \qquad (4.47)$$

We find the solution by moving forwards in time in small increments Δt. We start from the initial values $x(t_0) = x_0$ and $v(t_0) = v_0$. We find the velocity and position after a small time-step Δt using Euler's method (4.28):

$$v(t_0 + \Delta t) \simeq v(t_0) + a(t_0, x(t_0), v(t_0))\, \Delta t, \tag{4.48}$$

$$x(t_0 + \Delta t) \simeq x(t_0) + v(t_0)\, \Delta t, \tag{4.49}$$

where $a(t_0, x(t_0), v(t_0))$ is the acceleration we get when we put the values at $t = t_0$ into the expression we have for the acceleration in (4.47). We can now continue to step forward in time, finding subsequent values $x(t_i)$ and $v(t_i)$ in steps of Δt. This method is called Euler's method. It is definitely not the best numerical method of integration—actually we strongly advice against using Euler's method. It's strength is rather in the simple, intuitive implementation. Surprisingly, changing the step in (4.49) to the following:

$$x(t_0 + \Delta t) \simeq x(t_0) + v(t_0 + \Delta t)\, \Delta t, \tag{4.50}$$

gives significantly better solutions for many problems. This improved method is called Euler-Cromer's method, and you can use this method safely for most problems you encounter.

In **Euler-Cromer's method** to solve the (second order) differential equation of motion:

$$\frac{d^2x}{dt^2} = a\left(t, x, \frac{dx}{dt}\right), \;\; v(t_0) = v_0, \;\; x(t_0) = x_0, \tag{4.51}$$

we perform the following steps:

$$\begin{aligned} v(t_i + \Delta t) &\simeq v(t_i) + a(t_i, x(t_i), v(t_i))\, \Delta t \\ x(t_i + \Delta t) &\simeq x(t_i) + v(t_i + \Delta t)\, \Delta t \end{aligned} \tag{4.52}$$

4.2.1 Example: Modeling the Motion of a Falling Tennis Ball

This example demonstrates how we can calculate the motion of a falling tennis ball given an expression for the acceleration.

Background: In Sect. 4.1.1 we studied the motion of a falling tennis ball based on measurements of its motion. However, in physics we do not only want to observe motion, we want to predict it. We do this by first analyzing the problem to find the

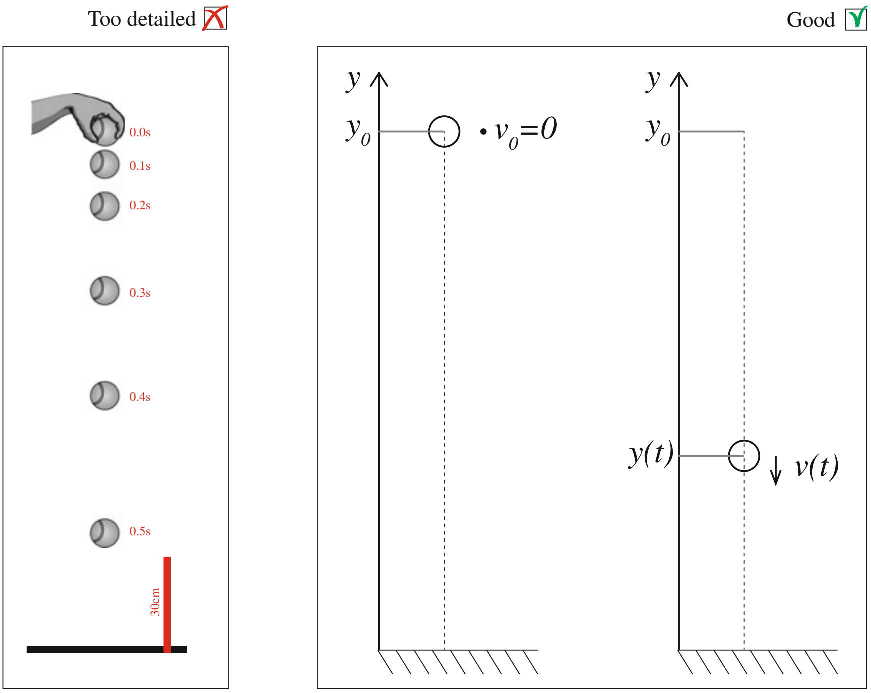

Fig. 4.9 *Left* Too detailed illustration. *Right* Correct, simple sketch

forces acting on the object, and from the forces we find a mathematical model of the acceleration of the object. (You will learn to do this in the next chapter. For now we will assume that the acceleration is given). From the acceleration, we find the position and velocity by analytical or numerical integration. We call this recipe the structured problem-solving approach.

System sketch: Your first step should always be to make a sketch the process. In physics, our sketches are vessels for our thoughts. A good, functional sketch is therefore an important part of solving a problem. While the left part of Fig. 4.9 has a nice artistic appeal and also illustrates the motion in detail, we do not encourage such detailed sketches. Instead, you should make a sketch that only focuses on the most important features of the process, as in the rightmost figure. Here we illustrate *the object* (the tennis ball), *its surroundings* (most importantly the floor), and *the coordinate system* with a clearly marked axis. We have also illustrated the initial position and velocity of the ball, and its position and velocity at a time t. Drawing a simplified illustration helps you discern the important from the unimportant, and it helps you convert a physical situation into a mathematical problem: The figure shows the axis and the position of the ball, $y(t)$, and nothing else.

Simplified model: From an analysis of the physics of the system, we have found that the acceleration of the ball is a constant:

$$a = -g = -9.8\,\mathrm{m/s^2}. \tag{4.53}$$

(You will learn where this model comes from later. Now we only want to address the consequences of such a model). In addition, we know that the ball starts from rest at the position $y_0 = 2.0\,\mathrm{m}$ at the time $t_0 = 0\,\mathrm{s}$:

$$y(0\,\mathrm{s}) = 2.0\,\mathrm{m},\ \ v(0\,\mathrm{s}) = 0\,\mathrm{m/s}. \tag{4.54}$$

We have now formulated a mathematical description of the problem we want to solve:

$$a = \frac{dv}{dt} = \frac{d^2y}{dt^2} = -g,\ \ v(0) = v_0,\ \ y(0) = y_0. \tag{4.55}$$

Solving this equation means to find the velocity $v(t)$ and the position $y(t)$ of the ball for any time t. We call this *the modeling step*, finding the mathematical problem to solve, and the next step is *to solve* this problem—to find $v(t)$ and $y(t)$.

Solving the simplified model: Since the acceleration is given and a constant, we can find the velocity by direct integration of the acceleration:

$$\frac{dv}{dt} = -g; \tag{4.56}$$

$$\int_{t_0}^{t} \frac{dv}{dt}\,dt = \int_{t_0}^{t} -g\,dt, \tag{4.57}$$

$$v(t) - \underbrace{v(t_0)}_{=0\,\mathrm{m/s}} = -g\,t + g\,\underbrace{t_0}_{=0\,\mathrm{s}}, \tag{4.58}$$

which gives

$$v(t) = -gt. \tag{4.59}$$

Similarly, we find the position by integrating the velocity:

$$\frac{dy}{dt} = v(t), \tag{4.60}$$

$$\int_0^t \frac{dy}{dt}\,dt = \int_0^t v(t)\,dt, \tag{4.61}$$

$$y(t) - y(0) = \int_0^t -gt\,dt = -\frac{1}{2}gt^2, \tag{4.62}$$

which gives

$$y(t) = y(0) - \frac{1}{2}gt^2. \tag{4.63}$$

Analysis of the simplified model: This is the complete solution to the problem. We know the position and velocity as a function of time. When you have this solution, you are prepared to answer any question about the motion. For example, you can find out when the ball hits the ground and you can find the velocity of the ball when it hits the ground. How would you do that? You need to translate the question into a mathematical problem. We do this by stating the condition "when the ball hits the ground" in mathematical terms: The ball hits the ground when its position is that of the ground, that is, when $y(t) = 0\,\text{m}$. (Notice, we have ignored the extent of the ball here). We can use our solution in (4.63) to find the corresponding time:

$$y(t) = y(0) - \frac{1}{2}gt^2 = 0\,\text{m} \;\Rightarrow\; t = \sqrt{\frac{2\,y(0)}{g}}. \tag{4.64}$$

A more realistic model: Unfortunately, data for the motion of the tennis ball, shown in Fig. 4.6b, show that the ball does not have a constant acceleration. This is due to air resistance—an effect not included in the simplified model. Fortunately, we have good models for air resistance. For a falling ball in air, a more realistic model that includes the effect of air resistance is:

$$a = -g - Dv|v|, \tag{4.65}$$

where $v = v(t)$ is the velocity of the ball, $g = 9.8\,\text{m/s}^2$ is the same constant as above, and the constant D depends on details of the ball. For a tennis ball $D = 0.0245\,\text{m}^{-1}$ is a reasonable value. (You will learn about the background for this model and how to determine values for D later). We can now formulate a mathematical problem:

$$a = \frac{dv}{dt} = -g - Dv|v|, \tag{4.66}$$

with initial conditions $v(0\,\text{s}) = 0\,\text{m/s}$ and $y(0\,\text{s}) = 2.0\,\text{m}$.

Solution of the realistic model: Our task is to solve this problem, which means to find $v(t)$ and $y(t)$ for the ball. This can be done either numerically or analytically. The numerical solution is straightforward, using the approach we have derived, but the analytical solution requires some knowledge of differential equations.

Numerical solution: We apply Euler-Cromer's method to find the positions and velocities by stepwise integration starting from the initial conditions. The integration step in Euler-Cromer's method is:

$$v(t_i + \Delta t) = v(t_0) + a(t_i, v_i, y_i)\,\Delta t \tag{4.67}$$

$$y(t_i + \Delta t) = y(t_0) + v(t_i + \Delta t)\,\Delta t, \tag{4.68}$$

where we insert the acceleration from (4.65):

$$a(t_i, v_i, y_i) = -g - Dv(t_i)|v(t_i)|, \tag{4.69}$$

This is implemented as follows: We open a new script file, and start our script by clearing all variables. This is a good habit to ensure that your previous activities do not affect your new calculations:

```
clear all; clf;
```

We define the physical constants and values given in the problem: g, D, $y(0)$ and $v(0)$:

```
D = 0.0245; % m^-1
g = 9.8; % m/s^2
y0 = 2.0;
v0 = 0.0;
```

We need to determine for how long we want to calculate the motion: What will be our maximum value of t? There are typically two strategies: We can make an initial guess for the duration of the simulation, or we can determine when the simulation should stop during the simulation. First, we make a guess for the duration of the simulation. Based on the existing data from Fig. 4.6 we guess that $t = 0.5$ s is a reasonable simulation time:

```
time = 0.5;
```

Next, we need to decide the time-step Δt. This needs to be small enough to ensure a good precision of the result, but not too small or the simulation takes too long. We try a value of $\Delta t = 0.00001$ s:

```
dt = 0.00001;
```

Based on this, we calculate how many simulation steps we need, $n = t/\Delta t$, and generate arrays for the positions, velocities, accelerations and time for the simulation. All values are initially set to zero:

```
n = ceil(time/dt);
y = zeros(n,1);
v = zeros(n,1);
a = zeros(n,1);
t = zeros(n,1);
```

Then we set the initial conditions:

```
y(1) = y0;
v(1) = v0;
```

Before, finally, the Euler-Cromer steps are implemented in an integration loop. The whole program is given in the following:

```
clear all; clf;
D = 0.0245; % m^-1
g = 9.8; % m/s^2
y0 = 2.0;
v0 = 0.0;
time = 0.5;
dt = 0.00001;
n = ceil(time/dt);
```

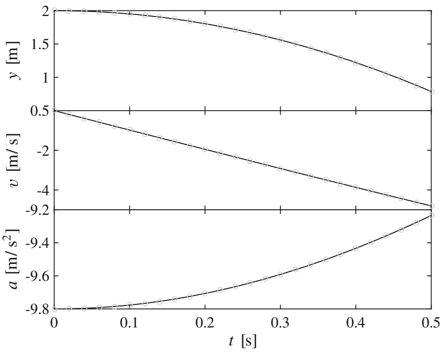

Fig. 4.10 Plots of $y(t)$, $v(t)$, and $a(t)$ calculated using the model for air resistance (*black line*) compared with analytical model (*gray circles*)

```
y = zeros(n,1);
v = zeros(n,1);
a = zeros(n,1);
t = zeros(n,1);
y(1) = y0;
v(1) = v0;
for i = 1:n-1
    a(i) = -g -D*v(i)*abs(v(i));
    v(i+1) = v(i) + a(i)*dt;
    y(i+1) = y(i) + v(i+1)*dt;
    t(i+1) = t(i) + dt;
end
```

The resulting plots of $x(t)$, $v(t)$, and $a(t)$ are shown in Fig. 4.10.

Analysis of realistic model results: We can now use this result to answer questions like how long does it take until the ball hits the ground? Again, we answer the question by translating it into a mathematical question: The ball hits the ground when $y(t) = 0$ m. However, in this case, we must find the solution numerically. The simplest approach to this would be to find when y becomes zero during the simulation. It is tempting to do this by checking when $y(t) = 0$ m:

```
if (y(i)==0.0)
  t(i)
end
```

But this will not work, because $y(t_i)$ will usually not be zero for any i. Typically, the program will step right past $y = 0$ going from a small positive value at some t_i to a small negative value at t_{i+1}. We should instead find the first time $y(t)$ passes 0, that is, we should find the first t_{i+1} when $y(t_{i+1}) < 0$. Then we know that $y(t) = 0$ somewhere in the interval $t_i < t < t_{i+1}$. We can then estimate a precise value for t using interpolation, or we can simply use the value t_{i+1}, if we find that this gives us sufficient precision. This is implemented in the following modification to the program, where we have also stopped the calculation when the ball hits the ground:

```
for i = 1:n-1
    a(i) = -g -D*v(i)*abs(v(i));
    v(i+1) = v(i) + a(i)*dt;
    y(i+1) = y(i) + v(i+1)*dt;
    if (y(i+1)<0)
        break
```

```
    end
    t(i+1) = t(i) + dt;
end
v(i+1)
plot(t(1:i),a(1:i))
xlabel('t [s]');
ylabel('a [m/s^2]');
```

where we have used `break` to stop the loop when the condition is met. Notice that we should now only plot the values up to `i`, because we have not calculated any more values. The values from $i+2$ to n were set to zero initially for y, v, and a and will make your plot confusing if you include them. (Try it and see).

Test your understanding: What would happen if we considered that the ball had an initial velocity $v_0 = -2v_T$ when it started? Sketch the resulting position, velocity and acceleration as a function of time.

Analytical solution: The differential equation in (4.66) is one of a few equations we can solve analytically as long as the velocity does not change sign. When the ball is falling down, the velocity is negative, and we can replace $|v|$ by $-v$:

$$\frac{dv}{dt} = -g - Dv\,(-v) = -g + D\,v^2. \tag{4.70}$$

This equation can be solved using separation of variables.

We separate the variables, so that all v's are on the left side and all t's are on the right:

$$\frac{dv}{g - Dv^2} = -1\,dt. \tag{4.71}$$

The differential equation can now be solved by integrating each side from $v_0 = 0$ m/s to v and from $t_0 = 0$ s to t:

$$\int_{v_0}^{v} \frac{dv}{g - Dv^2} = \int_{t_0}^{t} -1\,dt = -t, \tag{4.72}$$

The left-side integral can be solved using your knowledge from calculus (or by using the symbolic solver in matlab) giving:

$$\int_0^v \frac{dv}{g - Dv^2} = \frac{1}{g} v_T \tanh^{-1}\left(\frac{v}{v_T}\right), \tag{4.73}$$

where we have introduced the quantity $v_T = \sqrt{g/D}$ to simplify the notation. We notice that v_T has dimensions m/s, and we may therefore call it a velocity. We insert (4.73) back into (4.72), getting

$$v_T \tanh^{-1}(v/v_T) = -gt \;\Rightarrow v = v_T \tanh(-gt/v_T)\,. \tag{4.74}$$

We have now found the velocity on the form $v = v(t)$, and we can simply integrate the velocity from t_0 to t to find $y(t)$:

$$y(t) - y(t_0) = \int_{t_0}^{t} v(t)\,dt = \int_0^t v_T \tanh\left(-\frac{gt}{v_T}\right) dt. \tag{4.75}$$

This integral can solved by the symbolic integrator in matlab, giving:

$$y(t) = y(0) - v_T^2/g \log\cosh\frac{gt}{v_T}. \tag{4.76}$$

Figure 4.10 shows that the analytical solutions (given by circles) are identical to the numerical solutions (lines).

Symbolic solution: The differential equation in (4.66) can also be solved directly using the symbolic solver in matlab. We can solve the differential equation for the velocity, $v(t)$:

$$\frac{dv}{dt} = -g + D\,v^2,\ \ v(0) = 0. \tag{4.77}$$

First, we define the variables `g` and `D` as symbolic variables, and the function `v(t)` as a symbolic functions:

```
>> syms g D y(t)
```

Matlab can then solve the equation with the initial condition by

```
>> dsolve(diff(v)==-g+D*v^2,v(0)==0)
ans =  -(g^(1/2)*tanh(D^(1/2)*g^(1/2)*t))/D^(1/2)
```

We can then find the position by symbolic integration of this equation:

```
>> int(ans,t)
ans =   log(tanh(D^(1/2)*g^(1/2)*t) + 1)/D - (g^(1/2)*t)/D^(1/2)
```

Where the initial condition $y(0) = 0$ sets the second term to zero.

Alternatively, you can solve the differential equation for the position directly:

$$\frac{d^2y}{dt^2} = -g - D\left(\frac{dy}{dt}\right)^2,\ \ y(0) = 0,\ \ v(0) = 0. \tag{4.78}$$

by using the symbolic toolbox in matlab:

```
>> syms g D y(t)
>> Dy = diff(y);
>> dsolve(diff(y,2)==-g-D*Dy^2,y(0)==0,Dy(0)==0)
ans =   -log(tan(D^(1/2)*g^(1/2)*t)^2 + 1)/(2*D)
```

Simple and powerful! You should learn to use the symbolic toolboxes, as they can greatly simplify your worklife as a physicist.

Analysis of analytical solution: We can now use the analytical solution to solve problems of interest, such as finding out when the ball hits the floor, which occurs at $y(t) = 0$ m, that is, when

$$y(0) = v_T^2/g \log \cosh \frac{gt}{v_T} \Rightarrow \frac{gt}{v_T} = \cosh^{-1} \exp \frac{y(0)g}{v_T^2} \tag{4.79}$$

that is:

$$t = \frac{v_T}{g} \cosh^{-1} \exp \frac{y(0)g}{v_T^2} \ . \tag{4.80}$$

Summary

Motion: The motion of an object is described by:

- the position, $x(t)$, as a function of time, measured in a specified coordinate system
- the velocity $v(t) = dx/dt$
- the acceleration $a(t) = dv/dt = d^2x/dt^2$

Structured problem-solving approach:

- The structured problem-solving approach is illustrated in Fig. 4.11.

Solution methods: In the "Solver" we solve the equation:

$$\frac{d^2x}{dt^2} = a(t, x, \frac{dx}{dt}).$$

with the initial conditions $x(t_0) = x_0$ and $v(t_0) = v_0$.

- *Numerically*, we solve the equation using an iterative approach starting from the initial conditions. For example, we can use Euler-Cromer's method:

$$v(t_i + \Delta t) = v(t_i) + \Delta t \cdot a(x(t_i), v(t_i), t_i),$$

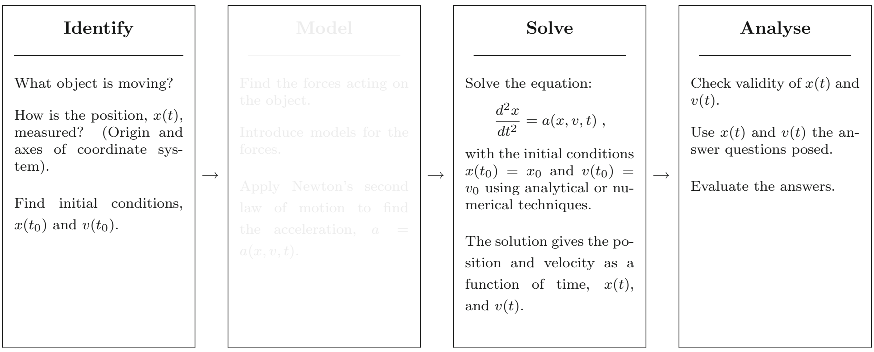

Fig. 4.11 Structured problem-solving approach. The second box, model, will be filled in Chap. 5

$$x(t_i + \Delta t) = x(t_i) + \Delta t \cdot v(t_i + \Delta t).$$

- *Analytically*, when the acceleration, $a = a(t)$, is only a function of time, t, we can solve the equations by direct integration:

$$v(t) = v(t_0) + \int_{t_0}^{t} a(t)dt, \; x(t) = x(t_0) + \int_{t_0}^{t} v(t)dt,$$

  A typical example is motion with constant acceleration.
- When the acceleration has a general form, $a = a(t, x, v)$, we need to solve the differential equation. In this case, there are no general approaches that always work. Instead, you must rely on your experience and your knowledge of calculus.

Exercises

Discussion Questions

4.1 Pedometer. Can you use the accelerometer in your phone as a pedometer? Explain.

4.2 Error in speedometer. If your speedometer overestimates your velocity by 10 %, how will that affect your measurement of your cars acceleration?

4.3 Speed of the clouds. Is it possible to use your camera to measure the speed of the clouds? What would you need to know to do that?

4.4 The slow trip. Is is possible to go for a trip (in one dimension) where the total displacement is zero, but your average velocity is non-zero?

4.5 Driving backwards. You drive in a train that is subject to constant acceleration. Can the train reverse its direction of motion?

4.6 No motion. Is is possible to envision a motion where you for a period have no displacement, but non-zero velocity? (You may use an $x(t)$ plot for illustration).

4.7 Non-falling ball. You throw a ball downwards from a high building. Can you think of a situation where the ball would have an acceleration upwards? What would happen?

4.8 Travels by sea. A boat is sailing north. Is it possible for the boat to have a velocity toward the north, but still have an acceleration toward the south?

4.9 Acceleration during throw. You throw a ball upwards as far as you can. The ball reaches its maximum height far above you. When was the magnitude of the acceleration the largest? While in your hand while throwing it or during its subsequent motion through the air?

4.10 Passing objects. A disgruntled physics student drops his pc from a window onto the ground. (You should not try this at home). At the same time as she lets the pc go, another student throws a ball upwards. The ball reaches its maximum position at the exact height where the pc was released. At what height do the pc and the ball pass each other? At the midpoint, above the midpoint or below the midpoint? Do they have the same magnitudes of their velocities at this point?

Problems

4.11 Space shuttle launch. When the space shuttle is lifting off, the vertical positions for the first 10 s in 1 s intervals are given as

t (s)	0	1	2	3	4	5	6	7	8	9
y (m)	0	15	60	135	240	375	540	735	960	1215

(a) Draw the motion diagram and the displacements for this motion.
(b) Use the motion diagram to find the average velocity as a function of time after lift-off.
(c) Use the motion diagram to find the average acceleration as a function of time after lift-off.

4.12 Capturing the motion of a falling ball. We use an ultra-sonic motion detector to measure the vertical position of a small ball. We throw the ball upwards, and measure the position until it hits the ground. You find the measured data in the file ballmotion.d.[7] Each line in the file consists of a time, t_i, measured in seconds, and a distance, x_i, measured in meters.

(a) Plot the position as a function of time for the ball.
(b) How long time does it take until the ball hits the ground?
(c) Plot the average velocity as a function of time for the ball.
(d) What is the maximum and minimum velocity of the ball?
(e) What is the initial velocity: The velocity of the ball at the start of the motion?
(f) Plot the average acceleration as a function of time for the ball.
(g) When is the maximum and minimum accelerations? Does this correspond with your physical intuition?

4.13 Motion graphs. A car is driving along a straight road. Sketch the position and velocity as a function of time for the car if:

(a) The car drives with constant velocity.
(b) The car accelerates with a constant acceleration.
(c) The car brakes with a constant acceleration.

[7] http://folk.uio.no/malthe/mechbook/ballmotion.d.

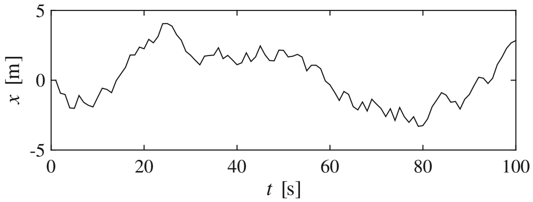

Fig. 4.12 Random motion of a grain of dust

4.14 Random walker. Figure 4.12 shows the motion of a tiny grain of dust bouncing randomly around in an air chamber.

(a) When is the grain to the left of the origin?
(b) When is the grain to the right of the origin?
(c) Is the grain ever exactly at the origin?

4.15 Motion diagram for a car. Figure 4.13 shows the motion diagram for a car driving along a straight road.

(a) Describe the motion of the car.
(b) Sketch the position as a function of time.
(c) Estimate the velocity of the car throughout the motion.
(d) Estimate the acceleration of the car throughout the motion.

4.16 Discover the motion. Figure 4.14 shows the motion diagram for a motion.

(a) Describe the motion qualitatively.
(b) Suggest a process that leads to this motion diagram.

4.17 The fastest indian. In the film "The World's Fastest Indian" Anthony Hopkins plays Burt Munro who reaches a velocity of 201 mph in his 1920 Indian motorcycle.

(a) At this velocity, how far does the Indian travel in 10 s?
(b) How long time does the Indian need to travel 1 km?

4.18 Meeting trains. A freight train travels from Oslo to Drammen at a velocity of 50 km/h. An express train travels from Drammen to Oslo at 200 km/h. Assume that the trains leave at the same time. The distance from Oslo to Drammen along the railway track is 50 km. You can assume the motion to a long a line.

(a) When do the trains meet?
(b) How far from Oslo do the trains meet?

4.19 Catching up. Your roommate sets off early to school, walking leisurely at 0.5 m/s. Thirty minutes after she left, you realize that she forgot her lecture notes. You decide to run after her to give her the notes. You run at a healthy 3 m/s.

(a) What is her position when you start running?
(b) What is your position when $t < t_1$?
(c) Sketch the position of you and your roommate as functions of time and indicate in the figure where you catch up with her.

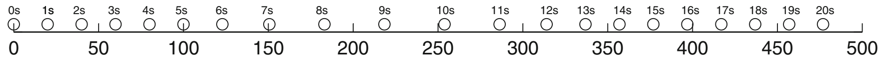

Fig. 4.13 Motion diagram for a car

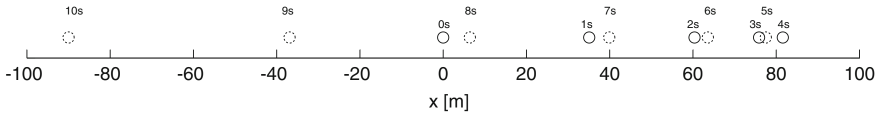

Fig. 4.14 Can you describe the motion?

(d) How long time does it take until you catch up with her?
(e) How far has she come when you catch up with her?

Now you have developed a strategy to solve such a problem, let us make the problem more complicated and see if you still can use your strategy. First, let us assume that you start off at $v_0 = 5\,\text{m/s}$, but then you tire gradually, so that your speed drops off with distance, x, reducing your speed by 1 m/s for every 100 m you run, until you reach a speed of $v_1 = 2\,\text{m/s}$, which you can keep for a long time.
(f) Show that your velocity as a function of position can be written as:

$$v(x) = \begin{cases} v_0 - b\,x & \text{when } v < v_1 = 2\,\text{m/s} \\ v_1 & \text{otherwise} \end{cases} \tag{4.81}$$

where $b = 1\,\text{m/s}/100\,\text{m}$.
(g) Plot or sketch $v(x)$.
(h) If you know your position and velocity at a time t, how can you find the position and velocity at $t + \Delta t$, a small time-step later?
(i) Write a program to find your position as a function of time. (Remember that you first start running at the time $t = t_1 = 1800, texts$. Before this you are standing still.)
(j) Validate your program by setting $b = 0$ and comparing the calculated $x(t)$ with the exact result, $x_e(t) = v_0(t - t_1)$ when $t > t_1$.
(k) How can you use this result to find where you catch up with your roommate?
(l) Where do you catch up with your roommate?
(m) What parts of your solution strategy are general, that is, what parts of your strategy do not change if we change how either person moves?

4.20 Electron in electric field. An electron is shot through a box containing a constant electric field, getting accelerated in the process. The acceleration inside the box is $a = 2000\,\text{m/s}^2$. The width of the box is 1m and the electron enters the box with a velocity of 100 m/s.

(a) What is the velocity of the electron when it exits the box?

4.21 Archery. As an expert archer you are able to fire off an arrow with a maximum velocity of 50 m/s when you pull the string a length of 70 cm.

(a) If you assume that the acceleration of the arrow is constant from you release the arrow until it leaves the bow, what is the acceleration of the arrow?

4.22 Collision. A car travelling at 36 km/h crashes into a mountainside. The crunch-zone of the car deforms in the collision, so that the car effectively stops over a distance of 1 m.

(a) Let us assume that the acceleration is constant during the collision, what is the acceleration of the car during the collision?
(b) Compare with the acceleration of gravity, which is $g = 9.8\,\text{m/s}^2$.

4.23 Braking distance. When you brake your car with your brand new tyres, your acceleration is $5\,\text{m/s}^2$.

(a) Find an expression for the distance you need to stop the car as a function of the starting velocity.

With your old tires, the acceleration is only two thirds of the acceleration with the new tyres.
(b) How does this affect the braking distance?
(c) Your reaction time is 0.5 s. If a child jumps into the street 30 m ahead of you when you are driving 50 km/h, are you able to stop with your new tires? What would happen if you did not change tyres?

4.24 Motion with constant acceleration. An object starts at $x = x_0$ with a velocity $v = v_0$ at the time $t = t_0$ and moves with a constant acceleration a_0. Show that the velocity v when the object has moved to a position x is $v^2 - v_0^2 = 2a_0(x - x_0)$.

4.25 Position plots. The position $x(t)$ of a particle moving along the x-axis is given in Fig. 4.15.

(a) Indicate in the figure where the velocity of the particle is positive, negative, and zero?
(b) Indicate in the figure where the velocity is maximal and minimal.
(c) Indicate in the figure where the acceleration is positive, negative, and zero?

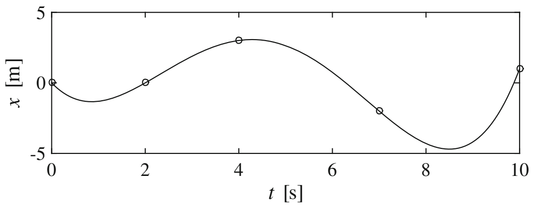

Fig. 4.15 The position of a particle moving along the x-axis.

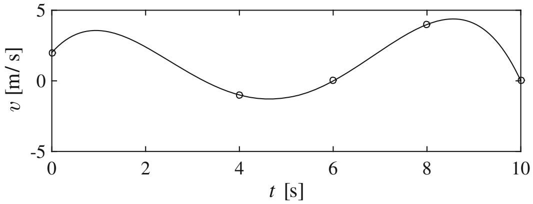

Fig. 4.16 The velocity of a particle moving along the x-axis

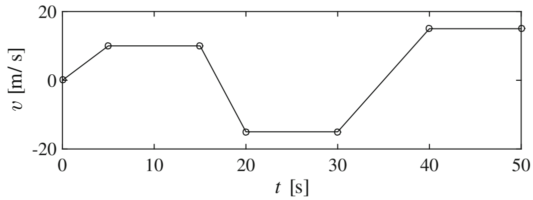

Fig. 4.17 The velocity of a particle moving along the x-axis

4.26 Velocity plots. The velocity $v(t)$ of a particle moving along the x-axis is given in Fig. 4.16.

(a) Indicate in the figure where the velocity of the particle is positive, negative, and zero?
(b) Indicate in the figure where the velocity is maximal and minimal.
(c) Indicate in the figure where the acceleration is positive, negative, and zero?
(d) Indicate in the figure where the acceleration is maximal and minimal.

4.27 Velocity plots. The velocity $v(t)$ of a particle moving along the x-axis is given in Fig. 4.17.

(a) Indicate in the figure where the velocity of the particle is positive, negative, and zero?
(b) Indicate in the figure where the particle speeds up and slows down.
(c) Indicate in the figure where the particle is stationary—that is, where it does not move.
(d) Indicate in the figure where acceleration is the largest and the smallest.
(e) Sketch the position as a function of time, $x(t)$.

4.28 A swimming bacterium. When the heliobacter bacteria swims, it is driven by the rotational motion of its tiny tail. It swims almost at a constant velocity, with small fluctuations due to variations in the rotational motion. As a simple model for the motion, we assume that the bacteria starts with the velocity $v = 10\,\mu$ m/s at the time $t = 0$ s, and is then subject to the acceleration, $a(t) = a_0 \sin(2\pi t/T)$, where $a_0 = 1\,\mu$ m/s^2, and $T = 1$ ms.

(a) Find the velocity of the bacterium as a function of time.
(b) Find the position of the bacterium as a function of time.
(c) Find the average velocity of the bacterium after a time $t = 10\,T$.

4.29 Resistance. (This problem requires some knowledge of statistics). An electron is moving with a constant acceleration, a_0, through a conductor. However, there are many small irregularities in the conductor—called scattering centers. If the electron hits a scattering center it stops, that is, its velocity immediately becomes zero. The scatting centers have a constant density. The probability for the electron to hit a scattering center when it moves a distance Δx is $P = \Delta x/b$, where b is a length describing the typical length between two scattering centers. Assume that the electron starts from rest. (For simplicity, we measure lengths in nm and time in ns, and you can assume that $b = 1$ nm and that $a_0 = 1$ nm/ns^2). First, we address the case without scattering.

(a) Write a program to find the motion of the electron using Euler-Cromer's method to find the velocity and position from the acceleration. Plot the position, $x(t)$, and velocity, $v(t)$, of the electron as functions of time and compare with the exact result.
(b) During the time interval Δt, the electron moves from $x(t)$ to $x(t + \Delta t)$. The probability for the electron to stop during this interval is $P = (x(t + \Delta t) - x(t))/b$. Explain why the following method models a collision:

```
dx = x(i+1) - x(i);
p = dx/b;
if (rand(1,1)<p)
   v(i+1) = 0.0;
end
```

where `rand(1,1)` produces a random number uniformly distributed between 0 and 1.
(c) Rewrite your program to include the effect of collisions using the algorithm described above. Plot the position, $x(t)$, and the velocity, $v(t)$, as functions of time. What do you see? Comment
(d) Find the average velocity v_{avg} for the electron.
(e) How does v_{avg} depend on a_0 and b? Can you make a theory that gives the value of v_{avg}?
(f) (Requires knowledge of statistics). What is the probability density for the distance, X, between two collisions?

4.30 Ball on vibrating surface. A ball is falling vertically through air over a vibrating surface. The position of the surface is $x_w(t) = A \cos \omega t$, where $A = 1$ cm and ω is called the angular frequency of the vibrations. The ball starts from a position $x = 10$ cm at $t = 0$ s. The acceleration of the ball is given as:

$$a(x, v, t) = \begin{cases} -g & x > x_w \\ -g - C(x - x_w) & x \le x_w \end{cases} . \tag{4.82}$$

where $g = 9.81$ m/s^2 and $C = 10000.0$ s^{-2}.

(a) Write down the equation you need to solve to find the motion of the ball. Include initial conditions for the ball.
(b) Write down the algorithm to find the position and velocity at $t_{i+1} = t_i + \Delta t$ given the position and velocity at t_i. Use Euler-Cromer's scheme.

(c) Write a program to find the position and velocity of the ball as a function of time.
(d) Check your program by comparing the initial motion of the ball with the exact solution when the acceleration is constant. Plot the results.
(e) Check your program by first studying the behavior when the vibrating surface is stationary, that is, when $A = 0\,\text{m}$ and $x_w = 0\,\text{m}$. Plot the resulting behavior. Ensure that your timestep is small enough, $\Delta t = 10^{-5}\,\text{s}$. What happens if you increase the timestep to $\Delta t = 0.02\,\text{s}$?
(f) Finally, use your program to model the motion of the ball when the surface is vibrating. Use $A = 0.01\,\text{m}$, $\omega = 10\,\text{s}^{-1}$, and simulate 5 s of motion. Plot the results. What is happening?
(g) What happens if you increase the vibrational frequency to $\omega = 30\,\text{s}^{-1}$? Plot the results. Can you explain the difference from $\omega = 10\,\text{s}^{-1}$?

Projects

4.31 Sliding on snow. In this project we address the motion of an object sliding on a slippery surface—such as a ski sliding in a snowy track. You will learn how to find the equation of motion for sliding systems both analytically and numerically, and to interpret the results.

We start by studying a simplified situation called frictional motion: A block is sliding on a surface. moving with a velocity v in the positive x-direction. The forces from the interactions with the surface results in an acceleration:

$$a = \begin{cases} -\mu(|v|)g & v > 0 \\ 0 & v = 0 \\ \mu(|v|)g & v < 0 \end{cases}, \tag{4.83}$$

where $g = 9.8\,\text{m/s}^2$ is the acceleration of gravity. Let us first assume that $\mu(v) = \mu = 0.1$ for the surface. That is, we assume that the coefficient of friction does not depend on the velocity of the block. We give the block a push and release it with a velocity of 5 m/s.
(a) Find the velocity, $v(t)$, of the block.
(b) How long time does it take until the block stops?
(c) Write a program where you find $v(t)$ numerically using Euler's or Euler-Cromer's method. (Hint: You can find a program example in the textbook.) Use the program to plot $v(t)$ and compare with your analytical solution. Use a timestep of $\Delta t = 0.01$.

The description of friction provided above is too simplified. The coefficient of friction is generally not independent of velocity. For dry friction, the coefficient of friction can in some cases be approximated by the following formula:

$$\mu(v) = \mu_d + \frac{\mu_s - \mu_d}{1 + v/v^*}, \tag{4.84}$$

where $\mu_d = 0.1$ often is called the dynamic coefficient of friction, $\mu_s = 0.2$ is called the static coefficient of friction, and $v^* = 0.5$ m/s is a characteristic velocity for the contact between the block and the surface.
(d) Show that the acceleration of the block is:

$$a(v) = -\mu_d g - g\frac{\mu_s - \mu_d}{1 + v/v^*}, \tag{4.85}$$

for $v > 0$.
(e) Use your program to find $v(t)$ for the more realistic model, with the same starting velocity, and compare with your previous results. Are your results reasonable? Explain.

The model we have presented so far is only relevant at small velocities. At higher velocities the snow or ice melts, and the coefficient of friction displays a different dependency on velocity:

$$\mu(v) = \mu_m \left(\frac{v}{v_m}\right)^{-\frac{1}{2}} \quad \text{when } v > v_m, \tag{4.86}$$

where v_m is the velocity where melting becomes important. For lower velocities the model presented above with static and dynamic friction is still valid.
(f) Show that

$$\mu_m = \mu_d + \frac{\mu_s - \mu_d}{1 + v_m/v^*}, \tag{4.87}$$

in order for the coefficient of friction to be continuous at $v = v_m$.
(g) Modify your program to find the time development of v for the block when $v_m = 1.5$ m/s. Compare with the two other models above: The model without velocity dependence and the model for dry friction. Comment on the results.
(h) The process may be clearer if you plot the acceleration for all the three models in the same plot. Modify your program to plot $a(t)$, plot the results, and comment on the results. What would happen if the initial velocity was much higher or much lower than 5 m/s?