Chapter 6
Motion in Two and Three Dimensions

You have now learned to use Newton's second law. This is the main tool you need to solve problems in mechanics. However, so far, we have only addressed motion and forces in one dimension. Fortunately, the description of motion and Newton's laws do not change when we go to higher dimensions. We can continue to apply the structured problem-solving approach in exactly the same way as we did before. But in order to address motion in two- and three dimensions, we need to extend our mathematical and numerical methods to address two- and three-dimensional motion. This will be done in two parts: In this chapter we introduce general two- and three-dimensional motion. Later we will introduce constrained motion—motion constrained to occur along a specific path in the same way a rollercoaster cart is constrained to follow the rollercoaster track, or in the way a part of a rotating body is following the rotation of the body.

6.1 Vectors

The use of vectors to describe positions in two- and three-dimensional systems is essential in order to describe general motion. If you are familiar with the use of vectors you can safely jump to the next section.

Scalars and Vectors

A scalar is a single number (with notation), such as a length, a mass, or a temperature. In order to describe physical quantities such as a displacement or a force, we need to describe both a magnitude and a direction: A displacement is described by a distance and a direction; A force is described by the magnitude of the force and the direction it acts in.

A. Malthe-Sørenssen, *Elementary Mechanics Using Matlab*,
Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-19587-2_6

A **vector** is a quantity with both direction and length. A vector can have physical units.

We indicate that a quantity is a vector by drawing a small arrow above it: $\mathbf{a}$.[1] It will be important for you to use a notation that clearly separates vector and scalar quantities. Because one of the most common mistakes made by students is to mix up vectors and scalars in their calculations, with dire results, we strongly urge you to use the arrow notation for vectors and to stick with it.

Vector Addition

Vector addition is intuitive for the addition of displacements (see Fig. 6.1b): If you first move along the vector $\mathbf{a}$ from A to B, and then along the vector $\mathbf{b}$ from B to C, the net displacement is the vector:

$$\mathbf{c} = \mathbf{a} + \mathbf{b}, \tag{6.1}$$

from point A to C.

This geometric definition of vector addition is general, and we use it also for vectors that are not displacements: We find the sum of two vectors $\mathbf{a}$ and $\mathbf{b}$ geometrically by placing the tail of vector $\mathbf{b}$ at the tip of vector $\mathbf{a}$. The sum is called the **resultant** vector.

From this definition, we realize that vector addition is **commutative**, the order of addition is arbitrary, and associative:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}, \ \text{(commutative law)} \tag{6.2}$$

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}, \ \text{(associative law)} \tag{6.3}$$

[1] Our definition of a vector is rather limited compared to the more general definition of vector spaces you may be used to in mathematics. It means we usually limit ourselves to vectors in one, two, and three Cartesian dimensions. Usually, we illustrate a vector by an arrow, as shown in Fig. 6.1a. The length of the arrow indicates the magnitude, and the direction shows the direction of the vector. Notice that it does not matter where we start drawing a vector. The vector $\mathbf{a}$ in Fig. 6.1 is the same even if it is drawn in position A, B, or C, but the vector $\mathbf{b}$ in position D is different, because it has a different direction than $\mathbf{a}$, and the vector $\mathbf{c}$ in position E is different from both $\mathbf{a}$ and $\mathbf{b}$ since it has a different magnitude. Remember: The only thing that matters for a vector is its magnitude and direction—not where it starts.

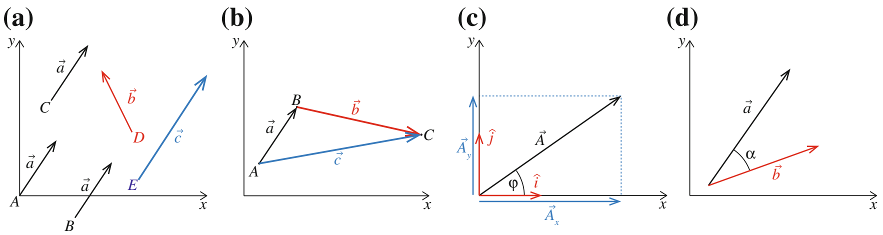


Fig. 6.1 **a** Illustration of vectors, **b** vector addition, **c** units vectors, decomposition and angle, **d** dot product

Scalar Multiplication

We can rescale the length of a vector by multiplying it with a scalar:

$$\mathbf{b} = 2\mathbf{a}. \tag{6.4}$$

Vector **b** is twice as long as vector **a**, but still pointing in the same direction. By multiplying a vector with a scalar we change the magnitude, but not the direction of the vector.

If we multiply a vector by 0, we get a vector of zero length, called the **zero vector**:

$$0\mathbf{a} = \mathbf{0}. \tag{6.5}$$

If we multiply a vector by a negative number, we change the direction of the vector to point in the opposite direction. For example, if we multiply with a vector with -1, we get the negative vector—a vector of the same length, but which points in the opposite direction.

Vector Components

A coordinate system is a grid you choose to describe the world in numbers. You are free to choose any coordinate system you like: You may choose where to place the origin and how to orient the axis. When you have decided on this, we use the coordinate system to describe a vector by *decomposing* the vector in the given coordinate system.

Here, we use *Cartesian* coordinate systems, where the axes are orthogonal to each other. We describe the coordinate system by the position of the origin, O, and by unit vectors pointing along each axis: the x-, y-, and z-axis. The unit vectors are of unit length, of length 1, and do not have any unit. The unit vectors are orthogonal, they form 90° angles with each other, as illustrated in Fig. 6.1c. It is common to use the symbol **i**, **j**, and **k** for the unit vectors along the x, y, and z-axis respectively.

Any vector can be uniquely decomposed into a set of component vectors along each of the axes:

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y , \tag{6.6}$$

as illustrated in Fig. 6.1c, where each of the component vectors can be written in terms of the unit vector along the axis:

$$\mathbf{A}_x = A_x \, \mathbf{i}, \;\; \mathbf{A}_y = A_y \, \mathbf{j}. \tag{6.7}$$

Here, the units of the vectors are in the scalar numbers A_x and A_y.

If you know the magnitude and direction of a vector, you can find the component of the vector from trigonometrical considerations. For example, the vector **A** shown in Fig. 6.1c, may be decomposed into its x- and y-components by:

$$\mathbf{A} = A_x \, \mathbf{i} + A_y \, \mathbf{j} = |\mathbf{A}| \cos\phi \, \mathbf{i} + |\mathbf{A}| \sin\phi \, \mathbf{j}. \tag{6.8}$$

We may write a vector by using the unit vectors or by writing the vector directly in coordinate form:

$$\mathbf{A} = A_x \, \mathbf{i} + A_y \, \mathbf{j} + A_z \, \mathbf{k} = \left(A_x, A_y, A_z\right) . \tag{6.9}$$

The Magnitude of a Vector Using Components

If the coordinate system is orthogonal, then all the axis are orthogonal to each other, and we can use Pythagoras' theorem to relate the magnitude to the vector to the components:

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}. \tag{6.10}$$

Addition, Subtraction and Scalar Multiplication Using Components

A particular advantage of the component form is that addition, subtraction, and scalar multiplications can be done for each component independently.

$$\underbrace{A_x \, \mathbf{i} + A_y \, \mathbf{j}}_{\mathbf{A}} + \underbrace{B_x \, \mathbf{i} + B_y \, \mathbf{j}}_{\mathbf{B}} = \underbrace{(A_x + B_x) \, \mathbf{i} + \left(A_y + B_y\right) \mathbf{j}}_{=\mathbf{A}+\mathbf{B}} . \tag{6.11}$$

$$c\mathbf{A} = c \left(A_x \, \mathbf{i} + A_y \, \mathbf{j}\right) = cA_x \, \mathbf{i} + cA_y \, \mathbf{j}. \tag{6.12}$$

The Dot Product

The **dot product** between two vectors **A** and **B** is defined as:

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \alpha, \tag{6.13}$$

where α is the angle between the two vectors, as illustrated in Fig. 6.1d.

The dot product is **linear**:

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}, \tag{6.14}$$

and **commutative**:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}. \tag{6.15}$$

The dot product depends on the angle α, as illustrated in Fig. 6.1d. When two vectors are parallel and point in the same direction, the dot product is equal to the product of the magnitudes. A particular useful property of the dot product is that the dot product of two orthogonal vectors is zero. As a result the dot product is simple on component form

$$\begin{aligned}
\mathbf{A} \cdot \mathbf{B} &= \left(A_x\,\mathbf{i} + A_y\,\mathbf{j}\right) \cdot \left(B_x\,\mathbf{i} + B_y\,\mathbf{j}\right) \\
&= A_x B_x \underbrace{\mathbf{i} \cdot \mathbf{i}}_{=1} + A_x B_y \underbrace{\mathbf{i} \cdot \mathbf{j}}_{=0} + A_y B_x \underbrace{\mathbf{j} \cdot \mathbf{i}}_{=0} + A_y B_y \underbrace{\mathbf{j} \cdot \mathbf{j}}_{=1} \\
&= A_x B_x + A_y B_y.
\end{aligned} \tag{6.16}$$

The value of the dot product is independent of the unit vectors used to decompose the vectors. We say that the dot product is invariant under a change of coordinate system.

What makes the dot product so useful, is that it can be used to decompose a vector onto a given set of unit vector—it can be used to find the components of a vector in any given coordinate system. A vector **A** can be written in component form as:

$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}. \tag{6.17}$$

How can we determine the components A_x, A_y, and A_z? We find them by using to dot product, and remembering that the unit vectors are orthogonal. We find component A_x from:

$$\mathbf{A} \cdot \mathbf{i} = A_x \underbrace{\mathbf{i} \cdot \mathbf{i}}_{=1} + A_y \underbrace{\mathbf{j} \cdot \mathbf{i}}_{=0} + A_z \underbrace{\mathbf{k} \cdot \mathbf{i}}_{=0} = A_x, \tag{6.18}$$

and similarly for the other components:

The component of a vector **A** can be found by dot-multiplication with the unit vectors **i**, **j**, and **k**:

$$A_x = \mathbf{A} \cdot \mathbf{i}, \ A_y = \mathbf{A} \cdot \mathbf{j}, \ A_z = \mathbf{A} \cdot \mathbf{k}, \tag{6.19}$$

so that

$$\mathbf{A} = (\mathbf{A} \cdot \mathbf{i})\ \mathbf{i} + (\mathbf{A} \cdot \mathbf{j})\ \mathbf{j} + (\mathbf{A} \cdot \mathbf{k})\ \mathbf{k}. \tag{6.20}$$

Numerical Representation of Vectors

In matlab a vector is represented by its component form. The vector **a**:

$$\mathbf{a} = 1\,\mathbf{i} + 2\,\mathbf{j} = (1, 2, 0)\ , \tag{6.21}$$

is generated by the following command:

```
a = [1 2 0];
```

Addition and Subtraction

All the ordinary mathematical operations can be applied to vectors just as you would apply them to scalars. For example, vector addition is achieved by:

```
b = [2 -4 0];
c = a + b;
echo,c

c =      3    -2      0
```

You can decide if you want to use a vector in two- or three dimensions. For example, you could instead have defined the vector **a** as:

```
a = [1 2];
```

But notice that you cannot add two vectors that do not have the same number of components.

Scalar Multiplication

Scalar multiplication is similarly naturally implemented:

```
d = 3*a;
echo,d

c =      3      6      0
```

Componentwise Operations

Notice that there is room for error because of the way commands are interpreted. For example, if you add a scalar to a vector, this is interpreted as a componentwise addition: The scalar is added to each of the components:

```
e = a + 3;
echo,e

e =        4      5      3
```

Dot Product

The dot product is found by applying the function `dot`, which returns a scalar:

```
f = dot(a,b)
echo, e

f =     -6
```

A common application of the dot product is to find the component of a vector **a** along the direction given by a vector **b**. In general, **b**, is not a unit vector. We therefore first need to find a unit vector in the direction of **b**:

$$\mathbf{u}_b = \frac{\mathbf{b}}{|\mathbf{b}|}, \tag{6.22}$$

and the component of **a** in this direction is given by the dot product:

$$a_b = \mathbf{a} \cdot \mathbf{u}_b = \mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|}. \tag{6.23}$$

Numerically, this is done in exactly the same way:

```
ab = dot(a,b)/sqrt(dot(b,b));
echo,ab
```

Notice that we use the relation:

$$|\mathbf{b}| = \sqrt{\mathbf{b} \cdot \mathbf{b}}, \tag{6.24}$$

for the magnitude of **b**.

Time Sequences of Vectors

We will often work with a sequence of vectors, corresponding to the time evolution of a vector. For example, we may be interested in the position, **r**, or the force, **F**, as a function of time, t:

$$\mathbf{r}(t) \text{ and } \mathbf{F}(t). \tag{6.25}$$

Numerically, we will have a corresponding sequence of positions or forces at discrete times, t_i:

$$\mathbf{r}_i = \mathbf{r}(t_i) \text{ and } \mathbf{F}_i = \mathbf{F}(t_i). \tag{6.26}$$

Fortunately, it is simple to both represent and apply mathematical operations to an element in a sequence.

We generate a sequence of n vectors $\mathbf{r}_i$ with x, y, and z coordinates by:

```
n = 10;
r = zeros(n,3);
```

We can use mathematical vector operations directly on element in the sequence, as illustrated in the following example:

```
v = [1.0 -2.0 2.0];
n = 10;
r = zeros(n,3);
r(1,:) = [0 0 0];
dt = 0.1;
r(2,:) = r(1,:) + v*dt;
```

6.2 Description of Motion

The cheetah is the world fastest land animal. How can we characterize its motion as it chases its prey? How fast does it run and how fast does it turn?

Motion Diagram and Position Vector

Figure 6.2 shows a few frames from a movie of a cheetah chasing a Thompson gazelle. To quantify the motion we generate a motion diagram: We mark the position of the cheetah at regular time intervals and record the positions $\mathbf{r}(t_i)$ of the cheetah relative to the origin at time t_i.

We are free to choose the origin and the axes of the coordinate system. The origin determines where we measure the positions from. In Fig. 6.2 we have chosen a stationary point on the ground as the origin. In addition to the origin, a coordinate system consists of a set of axes that we use to decompose the position vector. The directions of the axes indicate the positive direction of the corresponding unit vector. The position can be decomposed along the x, y, and z-axes respectively:

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \tag{6.27}$$

where $x(t)$, $y(t)$, and $z(t)$ are lengths along the axes and hence have units of length. For example, the position at $t = 0.5$ s is:

$$\mathbf{r}(1.0\,\mathrm{s}) = x(1.0\,\mathrm{s})\,\mathbf{i} + y(1.0\,\mathrm{s})\,\mathbf{j} = 29.9\,\mathrm{m}\,\mathbf{i} + 19.0\,\mathrm{m}\,\mathbf{j}, \tag{6.28}$$

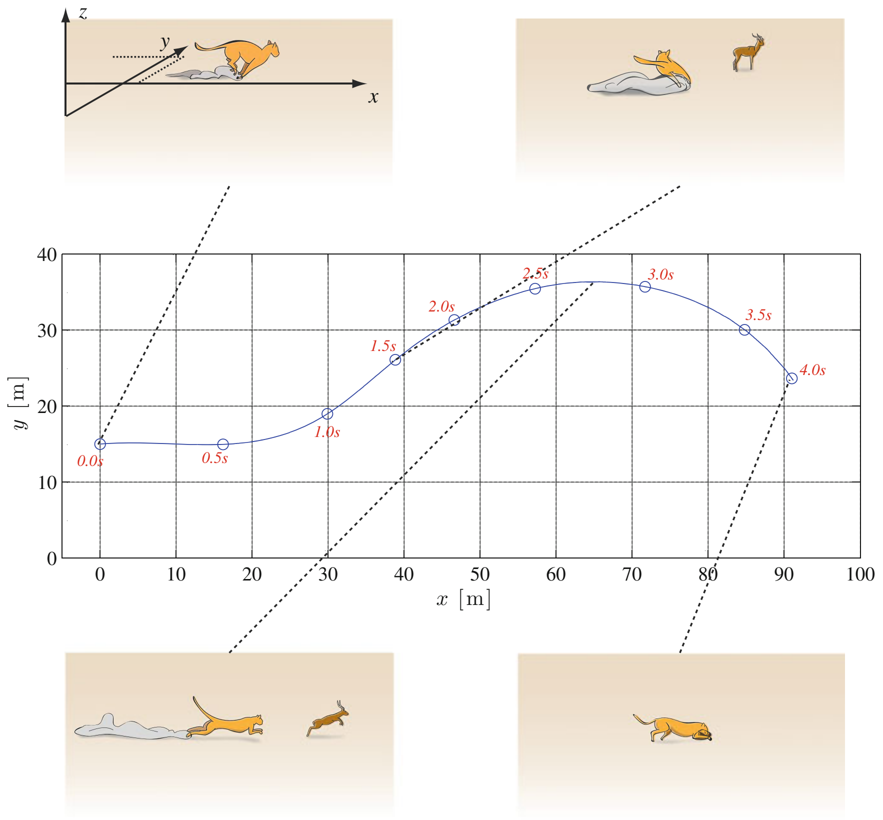


Fig. 6.2 Illustration of a cheetah chasing a Thompson gazelle, and an illustration of the two-dimensional motion and the two-dimensional motion diagram. (Illustration by S. B. Skattum)

Table 6.1 Positions of the running cheetah

t_i (s)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
x_i (m)	0.0	16.2	29.9	38.9	46.6	57.2	71.7	84.8
y_i (m)	15.0	14.95	19.0	26.1	31.3	35.4	35.7	30.0

where we have skipped the z-coordinate since all the motion is in the xy-plane. You find the complete dataset in the file cheetah.d.[2] Each line gives t_i, x_i, y_i, where t_i is measured in seconds, and x_i and y_i are measured in meters. We have tabulated the positions at $\Delta t = 0.5$ s intervals in Table 6.1.

[2]http://folk.uio.no/malthe/mechbook/cheetah.d.

Velocity Vector

Figure 6.2 shows how the position changes over a time interval Δt. The change in position is also a vector and is called the **displacement**. The displacement from $t = 1.0\,\text{s}$ to $t = 2.0\,\text{s}$ is denoted $\Delta\mathbf{r}(1.0\,\text{s})$:

$$\begin{aligned}\Delta\mathbf{r}(1.0\,\text{s}) &= \mathbf{r}(2.0\,\text{s}) - \mathbf{r}(1.0\,\text{s}) = (46.6\,\text{m}\,\mathbf{i} + 31.3\,\text{m}\,\mathbf{j}) - (29.9\,\text{m}\,\mathbf{i} + 19.0\,\text{m}\,\mathbf{j}) \\ &= 16.7\,\text{m}\,\mathbf{i} + 12.3\,\text{m}\,\mathbf{j}. \end{aligned} \tag{6.29}$$

We can read the displacement directly from the motion diagram in Fig. 6.2 as the vector from $\mathbf{r}(1.0\,\text{s})$ to $\mathbf{r}(2.0\,\text{s})$. Because the displacement depends on a difference between two positions, it does not depend on the choice of origin.

We see from Fig. 6.2 that both the length and the direction of the displacement vectors are changing throughout the motion. The rate of change of the displacement, the velocity, must therefore also be a vector:

The **average velocity** from a time $t = t_0$ to a time $t = t_0 + \Delta t$ is defined as:

$$\bar{\mathbf{v}}(t_0) = \frac{\mathbf{r}(t_0 + \Delta t) - \mathbf{r}(t_0)}{\Delta t} = \frac{\Delta\mathbf{r}(t_0)}{\Delta t}. \tag{6.30}$$

It is measured in meters per second, m/s.

We find the average velocity at $t = 1.0\,\text{s}$ using the data in the table above:

$$\bar{\mathbf{v}}(1.0\,\text{s}) = \frac{\Delta\mathbf{r}(1.0\,\text{s})}{1.0\,\text{s}} = \frac{16.7\,\text{m}}{1.0\,\text{s}}\,\mathbf{i} + \frac{12.3\,\text{m}}{1.0\,\text{s}}\,\mathbf{j} = 16.7\,\text{m/s}\,\mathbf{i} + 12.3\,\text{m/s}\,\mathbf{j}. \tag{6.31}$$

If we instead use a time interval $\Delta t = 0.5\,\text{s}$ to find the average velocity at $t = 1.0\,\text{s}$ we find:

$$\begin{aligned}\Delta\mathbf{r}(1.0\,\text{s}) &= \mathbf{r}(1.5\,\text{s}) - \mathbf{r}(1.0\,\text{s}) = (38.9\,\text{m}\,\mathbf{i} + 26.1\,\text{m}\,\mathbf{j}) - (29.9\,\text{m}\,\mathbf{i} + 19.0\,\text{m}\,\mathbf{j}) \\ &= 9.0\,\text{m}\,\mathbf{i} + 7.1\,\text{m}\,\mathbf{j}, \end{aligned} \tag{6.32}$$

$$\bar{\mathbf{v}}(1.0\,\text{s}) = \frac{\Delta\mathbf{r}(1.0\,\text{s})}{\Delta t} = \frac{1}{0.5\,\text{s}}\,(9.0\,\text{m}\,\mathbf{i} + 7.1\,\text{m}\,\mathbf{j}) = 18.0\,\text{m/s}\,\mathbf{i} + 14.2\,\text{m/s}\,\mathbf{j}. \tag{6.33}$$

We see that the average velocity depends on the time interval Δt, just as we saw in the one-dimensional case. Again, we can understand this better by studying the displacement at $t = 1.0\,\text{s}$ for smaller and smaller time intervals Δt, as shown in Fig. 6.3. We see that as Δt becomes smaller, the displacement also becomes smaller, but its direction approaches the tangent to the curve describing the motion around $t = 1.0\,\text{s}$.

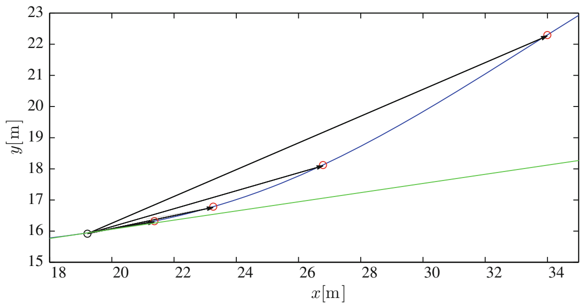


Fig. 6.3 Illustration of the average velocity, $\bar{\mathbf{r}}(2.0\text{s})$ for decreasing time intervals Δt for the motion of the cheetah

The **instantaneous velocity** at the time t is defined as the limit of the average velocity when the time interval Δt goes to zero, that is, the time derivative of the position vector, $\mathbf{r}(t)$.

$$\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{d\mathbf{r}}{dt} . \tag{6.34}$$

Whenever we use the term velocity, we will mean the instantaneous velocity. The velocity vector is tangential to the trajectory.

Speed

The magnitude of the velocity vector is called the *speed*, v, defined as:

$$v(t) = |\mathbf{v}(t)| . \tag{6.35}$$

We use the word velocity for the velocity vector, and the word speed for the magnitude of the vector velocity.

Time Derivatives of Vector Functions

How do we find the derivative of a vector function such as $\mathbf{r}(t)$? The simplest approach is to write the vector in terms of the unit vectors for the coordinate system, and then take the derivative of each component:

$$\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t) = \frac{d}{dt}\left(x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}\right) = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k} , \tag{6.36}$$

where[3] we define the component-wise velocities as

$$v_x(t) = \frac{dx}{dt}, \; v_y(t) = \frac{dy}{dt}, \; v_z(t) = \frac{dz}{dt}. \tag{6.37}$$

The velocity vector can therefore also be written:

$$\mathbf{v}(t) = v_x(t)\,\mathbf{i} + v_y(t)\,\mathbf{j} + v_z(t)\,\mathbf{k} = \left(v_x(t), v_y(t), v_z(t)\right). \tag{6.38}$$

Acceleration

It is clear from Fig. 6.2 that the average velocity of the cheetah is not constant throughout the motion—it is varying both in direction and magnitude. Just as we introduced velocity to characterize the change in the position vector, we introduce the acceleration vector to characterize the change in the velocity vector:

The **average acceleration vector** over a time interval Δt from t to $t + \Delta t$ is defined as:

$$\bar{\mathbf{a}} = \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t}. \tag{6.39}$$

We define the **instantaneous acceleration vector**, or simply the instantaneous acceleration, as the limit of the average acceleration vector when the time interval approaches zero:

$$\mathbf{a}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}}. \tag{6.40}$$

The acceleration vector is the time derivative of the velocity vector.

We find the acceleration in the vector component representation by componentwise derivation:

$$\begin{aligned} \mathbf{a}(t) &= \frac{d}{dt}\mathbf{v} = \frac{d}{dt}\left(v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}\right) \\ &= \frac{dv_x}{dt}\,\mathbf{i} + \frac{dv_y}{dt}\,\mathbf{j} + \frac{dv_z}{dt}\,\mathbf{k} = a_x(t)\,\mathbf{i} + a_y(t)\,\mathbf{j} + a_z(t)\,\mathbf{k}, \end{aligned} \tag{6.41}$$

[3]Here we have implicitly assumed that the time derivatives of the unit vectors are zero. This is not necessarily the case: The unit vectors vary with time for rotating reference systems.

where we see that:

$$a_x(t) = \frac{dv_x}{dt}, \ a_y(t) = \frac{dv_y}{dt}, \ a_z(t) = \frac{dv_z}{dt}. \tag{6.42}$$

Since the velocity vector is the time derivative of the position vector

$$\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t), \tag{6.43}$$

we can write the acceleration vector as the second time derivative of the position vector:

$$\mathbf{a}(t) = \frac{d}{dt}\mathbf{v} = \frac{d}{dt}\frac{d}{dt}\mathbf{r} = \frac{d^2\mathbf{r}}{dt^2}, \tag{6.44}$$

which can be written on component form:

$$a_x(t) = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \ a_y(t) = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}, \ a_z(t) = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}. \tag{6.45}$$

Notation for Time Derivatives

In physics, we use both the differential form, d/dt and the dot notation for time derivatives:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}, \tag{6.46}$$

and

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}, \tag{6.47}$$

but we *do not* use the "marked" notation, $v_x = x'(t)$, often used in mathematics because we often use $x'(t)$ to mean the position x measured in another coordinate system. We therefore strongly recommend you to use the notations introduced here, either the d/dt notation or the dot-notation.

Interpretation of Motion Diagrams

It is often takes time to gain a good intuition for acceleration, in particular for two- and three-dimensional motions. Motion diagrams can help in developing that intuition by visualizing velocities and accelerations.

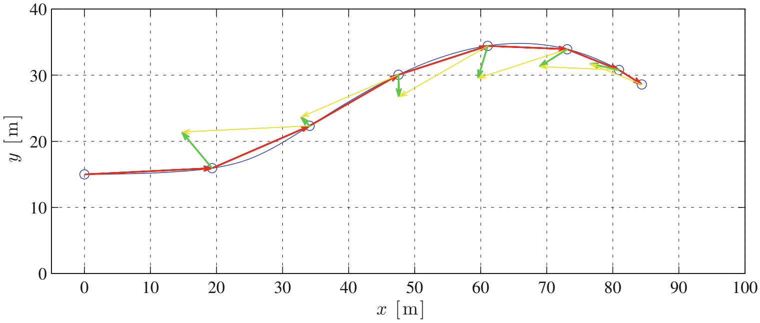


Fig. 6.4 Motion diagrams for the cheetah with $\Delta t = 0.5$ s illustrating both the displacements, interpreted as velocities, and the change in displacements, interpreted as accelerations. The constructions of the accelerations are illustrated

If the motion diagram is drawn using a constant time interval Δt, we can use the displacement vector as a visualization of the velocity, since the velocity is proportional to the displacement:

$$\bar{\mathbf{v}}(t_i) = \frac{1}{\Delta t} \Delta \mathbf{r}(t_i). \tag{6.48}$$

The displacement vectors are illustrated by red vectors at intervals of 0.5 s and 0.25 s in Fig. 6.4.

Notice that if we want to look at the *change* in velocity at the time $t = 1.0$ s, we would like to compare the velocity before the time $t = 1.0$ s with the velocity after the time $t = 1.0$ s. Now, the average velocity at the time $t = 1.0$ s is really the average velocity over the time interval from 1.0 to 1.5 s. And the average velocity at the time $t = 0.5$ s is the average velocity over the time interval from 0.5 to 1.0 s. Therefore, a reasonable way to characterize the change in velocity at $t = 1.0$ s is to characterize it as the change in velocity over the time interval from 0.5 t to 1.5 s:

$$\Delta \bar{\mathbf{v}}(1.0\,\text{s}) = \bar{\mathbf{v}}(1.0\,\text{s}) - \bar{\mathbf{v}}(0.5\,\text{s}). \tag{6.49}$$

We can interpret this as the average acceleration of the cheetah at $t = 1.0$ s, since it is (approximately) proportional to the average acceleration:

$$\bar{\mathbf{a}}(1.0\,\text{s}) \simeq \frac{1}{\Delta t} \left(\bar{\mathbf{v}}(1.0\,\text{s}) - \bar{\mathbf{v}}(0.5\,\text{s}) \right). \tag{6.50}$$

This method therefore provides a way to use the motion diagram to find approximations for the acceleration vectors in each point on the motion diagram. Incidentally, this method is the same as the simplest numerical method to find the second order time derivative of the position vector.

1. We find the displacements, $\Delta \mathbf{r}$:

$$\Delta \mathbf{r}(t) = \mathbf{r}(t + \Delta t) - \mathbf{r}(t), \tag{6.51}$$

   and interpret these as average velocity vectors for the motion.
2. We find the change in displacements, $\Delta\Delta \mathbf{r}$:

$$\Delta\Delta \mathbf{r}(t) = \Delta \mathbf{r}(t) - \Delta \mathbf{r}(t - \Delta t), \tag{6.52}$$

   and interpret these as average acceleration vectors for the motion.

We have illustrated the motion diagram of the cheetah using this approach in Fig. 6.4. Notice that we can generate the average acceleration vectors by a graphical vector subtraction, as illustrated at $t = 1.0$ s in Fig. 6.4.

6.2.1 Example: Mars Express

This example demonstrates how we can find the velocity and acceleration from both real data and mathematical representations of the position—based on real data provided by ESA.

The Mars Express probe was launched on June 2nd 2003 and reached Mars in December 2003 (see ESA's site for more information on Mars Express). We want to use the data provided by ESA to illustrate the motion of the Mars Express, analyze the velocity and acceleration of the module, and compare with the motion of the Earth and Mars (see Fig. 6.5).

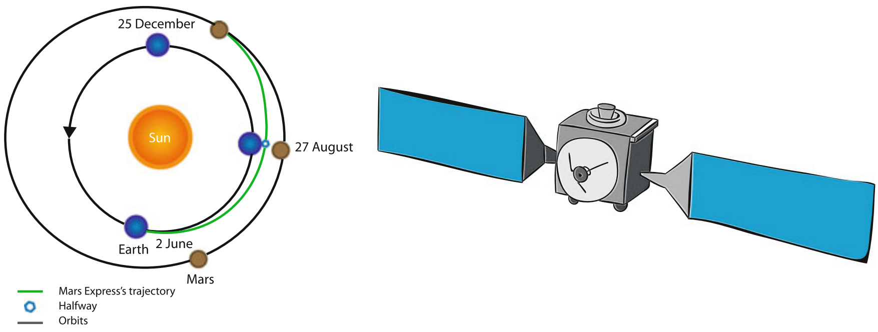


Fig. 6.5 The trajectory of the Mars Express spacecraft illustrated as a motion diagram and the Mars Express spacecraft. (Illustration by S. B. Skattum)

Reading and converting the data: The path of the Mars Express from Earth to Mars is approximately given by the data set marsexpresslr.d,[4] which has been reduced to be strictly two-dimensional. Each line in the data set contains the following information:

t_1	x_1	y_1	z_1
t_2	x_2	y_2	z_2
...	...	...	...

The time is given in days, and the spatial coordinates are measured in kilometers. We read the data set into matlab to examine the motion, using `load`:

```
load -ascii marsexpresslr.d
t = marsexpresslr(:,1);
r = [marsexpresslr(:,2) marsexpresslr(:,3)];
```

Since the `load` command generates the array `marsexpresslr` which has 4 columns, one for each column in the data file, we need to convert this array to an array with time, corresponding to t_i, and an array with positions, corresponding to $\mathbf{r}_i$. We convert by creating new variables `t` and `r` by extraction from the array. Notice the use of **:** to get all the elements along one dimension in the array. This provides us with a vector $\mathbf{r}(t_i)$, which contains the x- and y-components

$$\mathbf{r}(t_i) = x(t_i)\,\mathbf{i} + y(t_i)\,\mathbf{j}. \tag{6.53}$$

The vector representation in matlab is useful and allows us to make operations on the whole vector in a very similar way to how we write the operations mathematically. We demonstrate this by converting the lengths to astronomical units. The x- and y-coordinates in the data-set are measured in kilometers, but we would like to measure lengths in Astronomical Units, where $1\text{AU} = 149,598,000\,\text{km}$. An astronomical unit (1 AU) roughly corresponds to the average radius of the orbit of the Earth around the Sun—it is therefore a useful unit for describing planetary motion. We can convert the data by dividing the length by 1 AU. Mathematically, we would write this as:

$$\mathbf{r}'(t) = \frac{1}{1\,\text{AU}}\,\mathbf{r}(t). \tag{6.54}$$

Using the vector representation, the implementation in matlab is almost identical:

```
AU = 149598000.0;
r = r/AU;
```

How does this work? The command r/a divides each element in the array by $a = 1\,\text{AU}$—both the x and the y coordinate for all times i. This is a very intuitive and efficient way of coding, and you should learn to utilize the power of vectorization.

[4] http://folk.uio.no/malthe/mechbook/marsexpresslr.d.

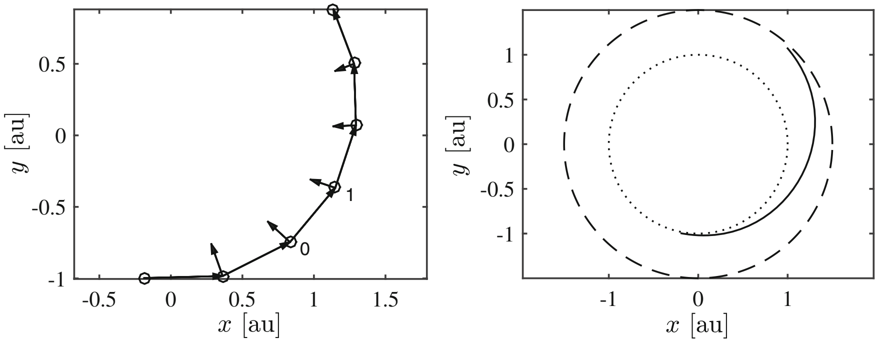


Fig. 6.6 The trajectory of the Mars Express spacecraft illustrated as a motion diagram. The *circles* shows the position of the module at 30 day intervals, the *arrows* indicate the velocities (displacements) and the accelerations (change in displacements)

Plotting the trajectory—Low resolution data: The data-set provides the positions $\mathbf{r}_i$ at times t_i. We can visualize the trajectory by plotting all the positions of the module:

```
plot(r(:,1),r(:,2),'-ko')
axis equal
xlabel('x [au]');
ylabel('y [au]');
```

where we have used `axis equal` to ensure that the scaling of the x- and y-axis are the same, so that a circle will appear as a circle and not as an ellipse. Notice that `r(:,1)` gives the x_i for all i, and similarly for the y_i.

The resulting trajectory is shown by a dotted line in Fig. 6.6a. Unfortunately, this data-set only contains the position of the module every 30 days. This only gives us a coarse illustration of the motion, but we are still able to see the main features.

Motion diagram: The plot of the trajectory itself does not provide much insight into the motion of the module. We can gain more insight through a motion diagram or by calculating the velocity and acceleration of the module along the trajectory. Since the data is so coarse, we start by illustrating the velocities and the accelerations in the diagrams.

The average velocity is proportional to the displacement:

$$\bar{\mathbf{v}}_i = \frac{1}{\Delta t}\Delta\mathbf{r}_i = \frac{1}{\Delta t}\left(\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)\right) . \tag{6.55}$$

We can therefore illustrate the velocities by drawing the displacement vectors from $\mathbf{r}_i$ to $\mathbf{r}_{i+1}$. This is done by drawing an arrow from `r(i,:)` to `r(i+1,:)` using the `quiver` command, which draws an arrow from a point $\mathbf{r}_i$ and in a distance $\Delta\mathbf{r}_i = \mathbf{r}_{i+1} - \mathbf{r}_i$. Notice the use of `hold on` and `hold off` to ensure all the arrows are drawn.

```
hold on
n = length(r);
for i = 1:n-1
    dr = r(i+1,:) - r(i,:);
    quiver(r(i,1),r(i,2),dr(1),dr(2),0);
end
hold off
```

The loop stops at $n-1$, since we cannot find the displacement from $i = n$ to $i = n+1$. The arrow illustrating the (average) velocities are shown in Fig. 6.6a.

Similarly, the acceleration is approximately given by the change in velocities:

$$\mathbf{a} \simeq \frac{1}{\Delta t}\left(\mathbf{v}(t_i) - \mathbf{v}(t_{i-1})\right), \tag{6.56}$$

We use the velocities in t_i and t_{i-1} because the velocity in t_i is calculated from the points t_i and t_{i+1} and the velocity in t_{i-1} is calculated from the points t_{i-1} and t_i. In this way, the acceleration is properly centered. This becomes clear by reviewing the motion diagram in Fig. 6.6a. To find the acceleration in point 1, we need to use the velocities (displacements) from point 0 to 1.

We can therefore illustrate the accelerations by:

$$\mathbf{a} \simeq \frac{1}{\Delta t^2}\left((\mathbf{r}_{i+1} - \mathbf{r}_i) - (\mathbf{r}_i - \mathbf{r}_{i-1})\right). \tag{6.57}$$

which is implemented as:

```
hold on
for i = 2:n-1
    dr = (r(i+1,:)-r(i,:))-(r(i,:)-r(i-1,:));
    quiver(r(i,1),r(i,2),dr(1),dr(2),0);
end
hold off
```

The accelerations are shown in Fig. 6.6a. Already from this very simple figure we gain insight into the motion: The acceleration is toward the center of the trajectory and the acceleration is decreasing in magnitude since the length of the arrows are decreasing.

Plotting the trajectory—High resolution data: Fortunately, we have access to data with much higher time resolution in the file marsexpresshr.d.[5] We read and plot the data using the same method as before:

```
load -ascii marsexpresshr.d
t = marsexpresshr(:,1);
r = [marsexpresshr(:,2) marsexpresshr(:,3)];
AU = 149598000.0;
r = r/AU;
plot(r(:,1),r(:,2),'-'), axis equal
xlabel('x [au]');
ylabel('y [au]');
```

The resulting trajectory is shown in Fig. 6.6b. In this case, the density of points is so high that it does not make sense to plot the displacements—they are all very small.

[5] http://folk.uio.no/malthe/mechbook/marsexpresshr.d.

Calculating velocity and acceleration: The velocity at t_i is the time derivative of $\mathbf{r}(t)$ at t_i, which we approximate by the numerical derivative over the time interval Δt, when Δt is small:

$$\mathbf{v}(t_i) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)}{\Delta t} \simeq \frac{\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)}{\Delta t} . \tag{6.58}$$

Notice that this is exactly the same form we used to calculate the displacement in the motion diagram. We can therefore calculate the velocities numerically using the same approach:

```
n = length(r);
dt = t(2)-t(1);
v = zeros(n,2);
for i = 1:n-1
    v(i,:) = (r(i+1,:)-r(i,:))/dt;
end
```

The magnitude of the velocity, $v(t)$, is shown in Fig. 6.7.

Similarly, we approximate the acceleration using the numerical derivative of the velocity:

$$\mathbf{a}(t_i) \simeq \frac{\mathbf{v}(t_i) - \mathbf{v}(t_{i-1})}{\Delta t} , \tag{6.59}$$

which again is analogous to what we calculated in the motion diagrams:

```
a = zeros(n,2);
for i = 2:n-1
    a(i,:) = (v(i,:) - v(i-1,:))/dt;
end
```

Figure 6.7 shows that the acceleration is decreasing in magnitude as the module moves from the Earth to Mars, but the data is noisier for the acceleration. The magnitudes of velocity and acceleration are calculated using the `norm`-function.

```
vv = zeros(n,1); aa = zeros(n,1);
for i = 1:n
    vv(i) = norm(v(i,:));
    aa(i) = norm(a(i,:));
end
```

Mathematical models: Since we do not have any intuition for velocities or accelerations of space travel such as for the Mars Express, we need to compare with other relevant motions, such as the motion of the Earth and Mars around the Sun. For simplicity, we assume that both the Earth and Mars follow circular orbits with radi $R_E = 1$ au and $R_M = 1.5$ au, and periods (the time a complete revolution takes) of $T_E = 1$ year $= 365.25$ days and $T_M = 2$ years $= 730.5$ days. The trajectory for a circular orbit with radius R and period T can be described by

$$\mathbf{r}(t) = R\,(\mathbf{i}\,\cos\omega t + \mathbf{j}\,\sin\omega t) , \tag{6.60}$$

where $\omega = 2\pi/T$. This describes a circle with radius R and the trajectory is at $\mathbf{r} = R\mathbf{i}$ at $t = 0$ and at $t = T$, hence the period is T. If you are not familiar with this representation of circles, you can plot the trajectory by

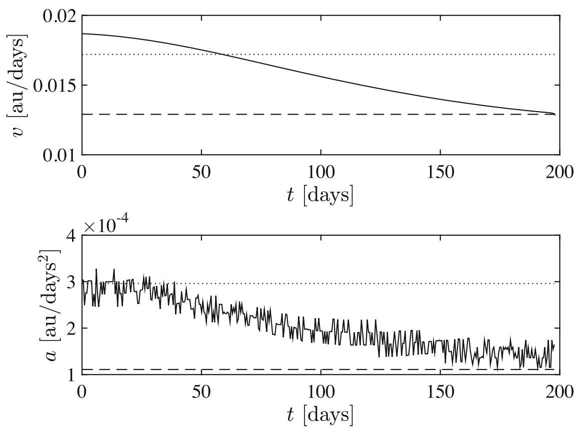


Fig. 6.7 The magnitude of the velocity and the acceleration of the module as function of time. (Notice the noise in the accelerations)

```
>> R = 1.0;
>> T = 365.25;
>> t = linspace(0,T,1000);
>> r = R*[cos(2*pi*t/T) sin(2*pi*t/T)];
>> plot(r(1,:),r(2,:)), axis equal
```

We are here primarily interested in the velocity and acceleration of a planet in the trajectory, and not in when the planet is where along the trajectory. Therefore we have not taken care to ensure that the initial positions at $t = 0$ for Mars or the Earth are correct relative to each other. (That is, we do not care where the planets are at $t = 0$).

We use this representation for both the Earth and Mars. The velocity of the planet in this circular orbit is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt} R\,(\mathbf{i}\,\cos\omega t + \mathbf{j}\,\sin\omega t) = R\omega\,(-\mathbf{i}\,\sin\omega t + \mathbf{j}\,\cos\omega t)\,. \tag{6.61}$$

The magnitude of the velocity is therefore a constant:

$$|\mathbf{v}| = R\omega\,|-\mathbf{i}\,\sin\omega t + \mathbf{j}\,\cos\omega t| = R\omega = 2\pi R/T, \tag{6.62}$$

which corresponds to the distance traveled during one complete revolution divided by the time it takes to make a revolution. We therefore now have numbers we can use to compare with the results from our calculations. For the Earth (v_E) and for Mars (v_M) we find:

$$v_E = 2\,\pi\;1.0\,\text{au}/\,(365.25\,\text{days}) = 0.017\,\text{au/days}, \tag{6.63}$$

$$v_M = 2\,\pi\;1.5\,\text{au}/\,(2\,365.25\,\text{days}) = 0.013\,\text{au/days}. \tag{6.64}$$

These values are illustrated by a dotted and a dashed line in Fig. 6.7. Indeed, we see that the Mars Express holds approximately the same velocity as Mars when the module approaches Mars, and that the module starts with a velocity significantly larger than that of the Earth in its orbit.

We find the acceleration for these trajectories from the derivative of the velocity:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt} R\omega \left(-\mathbf{i} \sin \omega t + \mathbf{j} \cos \omega t\right) = -R\omega^2 \left(\mathbf{i} \cos \omega t + \mathbf{j} \sin \omega t\right), \quad (6.65)$$

which shows that the acceleration of the Earth (and Mars) always points in toward the center of the orbit. The magnitude of the acceleration is a constant, $a = R\omega^2$, which gives

$$a_E = \left(2^2 \, \pi^2 \, 1.0 \, \mathrm{au}\right) / \left(365.25 \, \mathrm{days}\right)^2 = 3.00 \times 10^{-4} \, \mathrm{au/days}^2, \quad (6.66)$$

$$a_M = \left(2^2 \, \pi^2 \, 1.5 \, \mathrm{au}\right) / \left(2 \, 365.25 \, \mathrm{days}\right)^2 = 1.11 \times 10^{-4} \, \mathrm{au/days}^2. \quad (6.67)$$

We have plotted these values as dotted (Earth) and dashed (Mars) lines in Fig. 6.7. We see that the acceleration of the module is the same as the acceleration of the Earth when it is near the Earth and Mars when it is near Mars.

We can also plot the trajectories of the Earth and Mars into the plot of the trajectory of the Mars Express. First, we calculate the trajectories and then plot them in the same plot:

```
hold on
RE = 1.0;
TE = 365.25;
omegaE = 2*pi/TE;
tt = linspace(0,TE,1000);
rE = RE*[cos(omegaE*tt) sin(omegaE*tt)]';
plot(rE(:,1),rE(:,2),':');
RM = 1.5;
TM = 2*365.25;
omegaM = 2*pi/TM;
tt = linspace(0,TM,1000);
rM = RM*[cos(omegaM*tt) sin(omegaM*tt)]';
plot(rM(:,1),rM(:,2),'--');
hold off;
```

Notice the compact way of calculating the trajectory. First, we generate an array `tt` of time from 0 days to the period T, and then we generate the x and y positions. Notice also the use of transpose (using the `'` command) to ensure that the vectors have the shape $(N, 2)$ and not $(2, N)$. The resulting trajectories are shown in Fig. 6.6.

Analysis of acceleration: We can use the data-set to find out how the magnitude of the acceleration depends on the distance, r, to the Sun. The distance, r, is the magnitude of the position vector, since all the positions are measured with the Sun in the origin of the coordinate system. The magnitude of the position vector is $r(t) = |\mathbf{r}|$. We have already found the magnitude of the velocity and the acceleration, and we can find the norm of the position in the same way:

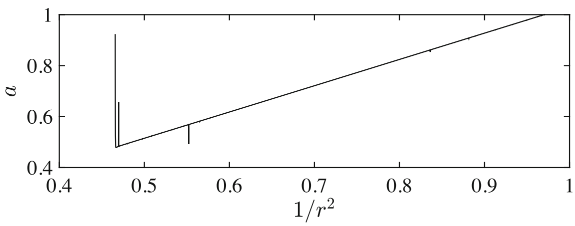


Fig. 6.8 A plot of the acceleration a as a function of $1/r^2$, where r is the distance from the Mars Express to the Sun

```
rr = zeros(n,1);
for i = 1:n
    rr(i) = norm(r(i,:));
end
```

Let us now test the hypothesis that the acceleration is inversely proportional to the distance to the Sun, which is the essence of Newton's law of gravitation:

$$a(r) = \frac{c}{r^2}, \tag{6.68}$$

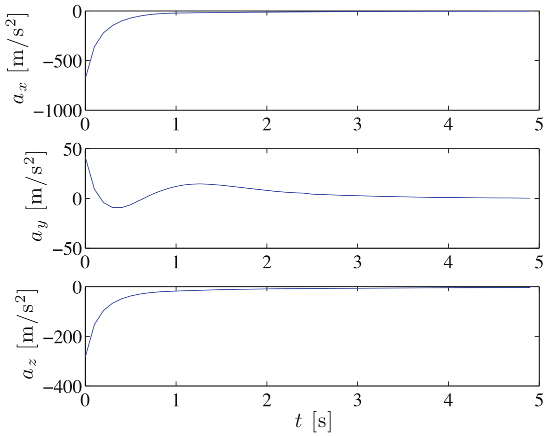
where c is a constant. We test this idea by plotting a as a function of $1/r^2$. If the resulting graph is a straight line, we have shown that the acceleration indeed is described by Newton's law of gravitation. The resulting plot is shown in Fig. 6.8. The plot shows that the data is consistent with Newton's law of gravitation, except for in the initial phase. However, in this case we expect the spacecraft to be affected by other effects, such as the effect of the engine driving it, and the gravitational force from the Earth.

6.3 Calculation of Motion

We are typically not given a description of the motion, instead we have measured the velocity or the acceleration or we have a theory for the acceleration, and we want to determine the motion. This requires us to integrate the equations of motion. Thus we need methods to determine the motion of an object in two and three dimensions given a set of measurements of the velocity or the acceleration, given a mathematical expression for the velocity or the acceleration, or given a differential equation for the velocity or acceleration. Here we equip you with the methods needed to actually solve mechanics problems.

Discrete Integration

In your line of work as a tornado chaser you have just developed a tornado probe, a small spherical ball you plan to shoot through a tornado to measure the wind



t_i	$a_x(t_i)$	$a_y(t_i)$	$a_z(t_i)$
0.00s	-683.1m/s^2	41.1m/s^2	-283.0m/s^2
0.01s	-634.8m/s^2	36.6m/s^2	-263.3m/s^2
0.02s	-591.4m/s^2	32.3m/s^2	-245.6m/s^2
0.03s	-552.3m/s^2	28.4m/s^2	-229.9m/s^2
0.04s	-516.9m/s^2	25.0m/s^2	-215.6m/s^2
0.05s	-484.8m/s^2	21.7m/s^2	-202.4m/s^2
0.06s	-455.6m/s^2	18.8m/s^2	-190.5m/s^2
0.07s	-428.9m/s^2	15.9m/s^2	-179.9m/s^2
0.08s	-404.4m/s^2	13.7m/s^2	-169.9m/s^2
0.09s	-382.1m/s^2	11.4m/s^2	-160.9m/s^2

Fig. 6.9 Plot of the components of the accelerations recorded by the accelerometer in the probe, and a table listing the first 10 vales

velocities. The probe is fitted with a tiny accelerometer based on a MEMS-device (a micro-electromechanical system) that records the acceleration of the probe in the x, y, and z-directions every 0.01 s. After the flight, you recover the probe and read out the accelerations. How can you use these readings to find the velocity and position of the probe during its flight?

The acceleration was recorded at a sequence of times, t_i, with constant time intervals, $\Delta t = 0.01$ s, as shown in Fig. 6.9. We want to use this sequence of accelerations, $\mathbf{a}(t_i)$ to find both the sequence of velocities, $\mathbf{v}(t_i)$, and the sequence of positions, $\mathbf{r}(t_i)$ of the probe. Just as for one-dimensional motion, we use the expression for the numerical derivative of the velocity, the average acceleration, to determine the velocities. The average acceleration vector from the time t_i to $t_{i+1} = t_i + \Delta t$ is:

$$\bar{\mathbf{a}}(t_i) = \frac{\mathbf{v}(t_i + \Delta t) - \mathbf{v}(t_i)}{\Delta t}. \tag{6.69}$$

We solve for $\mathbf{v}(t_i + \Delta t)$ to step forward in time:

$$\begin{aligned} \mathbf{v}(t_i + \Delta t) - \mathbf{v}(t_i) &= \Delta t \cdot \bar{\mathbf{a}}(t_i) \\ \mathbf{v}(t_i + \Delta t) &= \mathbf{v}(t_i) + \Delta t \cdot \bar{\mathbf{a}}(t_i)\ . \end{aligned} \tag{6.70}$$

Even though the accelerometer does not provide the average acceleration during a time interval, but rather the instantaneous acceleration at the time, t_0, we get a reasonable approximation by assuming that the average acceleration is equal to the instantaneous acceleration:

$$\bar{\mathbf{a}}(t_0) \simeq \mathbf{a}(t_i). \tag{6.71}$$

Indeed, this corresponds to the approximation used in the first order numerical derivative.

This produces a sequence of velocities, $\mathbf{v}(t_i)$. We can now use exactly the same procedure to find the positions from the velocities, since the velocity is the time derivative of the position. The velocity from time t_i to $t_{i+1} = t_i + \Delta t$ is approximately gives as the average velocity:

$$\mathbf{v}(t_i) \simeq \frac{\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)}{\Delta t}, \tag{6.72}$$

which we again can solve for $\mathbf{r}(t_i + \Delta t)$ getting:

$$\begin{aligned} \mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i) &\simeq \Delta t \cdot \mathbf{v}(t_i) \\ \mathbf{r}(t_i \Delta t) &\simeq \mathbf{r}(t_i) + \Delta t \cdot \mathbf{v}(t_i). \end{aligned} \tag{6.73}$$

We can now find the position and velocities of the probe, starting at $t_0 = 0\,\text{s}$:

1. At $t = t_0 = 0\,\text{s}$ the probe is launched from $\mathbf{r}_0 = -80.0\,\text{m}\,\mathbf{i}$ with a velocity $\mathbf{v}(t_0) = \mathbf{v}_0 = 184.9\,\text{m/s}\,\mathbf{i} - 18.5\,\text{m/s}\,\mathbf{j} + 74.0\,\text{m/s}\,\mathbf{k}$:

$$\mathbf{v}(0.0\,\text{s}) = 184.9\,\text{m/s}\,\mathbf{i} - 18.5\,\text{m/s}\,\mathbf{j} + 74.0\,\text{m/s}\,\mathbf{k}, \tag{6.74}$$

$$\mathbf{r}(0.0\,\text{s}) = -80.0\,\text{m}\,\mathbf{i} + 0.0\,\text{m}\,\mathbf{j} + 0.0\,\text{m}\,\mathbf{k}. \tag{6.75}$$

2. At $t_1 = t_0 + \Delta t = 0.01\,\text{s}$, the velocity vector is:

$$\mathbf{v}(0.01\,\text{s}) \simeq \mathbf{v}(0.0\,\text{s}) + \Delta t\,\mathbf{a}(0.0\,\text{s}) = 178.0\,\text{m/s}\,\mathbf{i} - 18.1\,\text{m/s}\,\mathbf{j} + 71.1\,textm/s\,\mathbf{k}, \tag{6.76}$$

where the acceleration $\mathbf{a}(0.0\,\text{s}) = -683.1\,\text{m/s}^2\,\mathbf{i} + 41.1\,\text{m/s}^2\,\mathbf{j} - 283.0\,\text{m/s}^2\,\mathbf{k}$ is listed in the table in Fig. 6.9. The position vector of the probe is:

$$\mathbf{r}(0.01\,\text{s}) \simeq \mathbf{r}(0.0\,\text{s}) + \Delta t\,\mathbf{v}(0.0\,\text{s}) = -78.2\,\text{m}\,\mathbf{i} - 0.18\,\text{m}\,\mathbf{j} + 0.73\,\text{m}\,\mathbf{k}. \tag{6.77}$$

This trajectory is illustration in Fig. 6.10, where we have also illustrated the velocity field of the tornado. We will return to this example with a more physical approach in the next chapter.

The method we have presented here is called Euler's method for integration. However, just as for one-dimensional motion, a small modification of the method makes it stronger: If we instead use the newly calculated velocity, $\mathbf{v}(t_i + \Delta t)$ when we calculate the new positions, we get Euler-Cromer's method , which usually has higher precision and is more stable than Euler method. We will therefore usually use this method here.

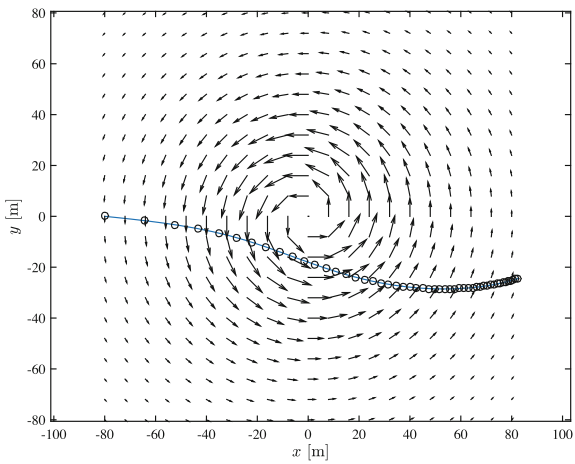


Fig. 6.10 Motion diagram for the probe. The *circles* illustrates the positions of the probe at 0.1s intervals. The velocity field of the tornado is included for illustration

In **Euler-Cromer's method** we find the position vector, $\mathbf{r}(t_i)$, and the velocity vector, $\mathbf{v}(t_i)$, as a function of time by a stepwise summation of the acceleration vectors $\mathbf{a}(t_i)$, starting from $\mathbf{v}(t_0) = \mathbf{v}_0$ and $\mathbf{r}(t_0) = \mathbf{r}_0$:

$$\mathbf{v}(t_i + \Delta t) \simeq \mathbf{v}(t_i) + \Delta t\, \mathbf{a}(t_i)$$
$$\mathbf{r}(t_i + \Delta t) \simeq \mathbf{r}(t_i) + \Delta t\, \mathbf{v}(t_i + \Delta t) \tag{6.78}$$

We have implemented this method in a short program that finds the velocities and positions of the probe. You can find the components of the acceleration vector in the file tornado.d[6] recorded at an interval $\Delta t = 0.01$ s, where each line contains a time (in seconds) and the x- and y-components of the accelerations (a_x and a_y) given in m/s^2:

```
0.0000000e+00  -6.8310810e+02   4.1438567e+01  -2.8305553e+02
1.0000000e-02  -6.3476299e+02   3.6551439e+01  -2.6340926e+02
2.0000000e-02  -5.9144164e+02   3.2315870e+01  -2.4574236e+02
```

We read this data into matlab, calculate the timestep from the first few timesteps, $\Delta t = t_2 - t_1$, and apply Euler-Cromer's method to find the trajectory:

```
load -ascii tornado.d
t = tornado(:,1);
dt = t(2)-t(1);
n = length(t);
a = zeros(n,3);
a(:,1) = tornado(:,2);
a(:,2) = tornado(:,3);
a(:,3) = tornado(:,4);
r = zeros(n,3);
v = zeros(n,3);
```

[6] http://folk.uio.no/malthe/mechbook/tornado.d.

```
r(1,:) = [-80.0 0.0 0.0];
v(1,:) = [184.9 -18.49 73.96];
for i = 1:n-1
    v(i+1,:) = v(i,:) + a(i,:)*dt;
    r(i+1,:) = r(i,:) + 0.5*(v(i+1,:)+v(i,:))*dt;
    t(i+1) = t(i) + dt;
end
i = (1:10:n);
plot(r(:,1),r(:,2),'-',r(i,1),r(i,2),'o')
```

The resulting motion is shown as motion diagram and trajectory in Fig. 6.10.

Formal Integration

If we know a mathematical expression for the acceleration vector, we can find the velocity and position vector by integration. We start from the acceleration and integrate from t_0 to t to find the velocity:

$$\frac{d\mathbf{v}}{dt} = \mathbf{a}(t) \;\Rightarrow\; \int_{t_0}^{t} \frac{d\mathbf{v}}{dt}\,dt = \mathbf{v}(t) - \mathbf{v}(t_0) = \int_{t_0}^{t} \mathbf{a}(t)\,dt. \tag{6.79}$$

This allows you to find the velocity from the acceleration. If the acceleration is on component form, you integrate each component to find the velocity:

$$\begin{aligned}\int_{t_0}^{t} \mathbf{a}(t)\,dt &= \int_{t_0}^{t} \left(a_x(t)\,\mathbf{i} + a_y(t)\,\mathbf{j} + a_z(t)\,\mathbf{k}\right)\,dt \\ &= \left(\int_{t_0}^{t} a_x(t)\,dt\right)\mathbf{i} + \left(\int_{t_0}^{t} a_y(t)\,dt\right)\mathbf{j} + \left(\int_{t_0}^{t} a_z(t)\,dt\right)\mathbf{k}.\end{aligned} \tag{6.80}$$

When we have the velocity, we find the position by integration:

$$\int_{t_0}^{t} \frac{d\mathbf{r}}{dt}dt = \mathbf{r}(t) - \mathbf{r}(t_0) = \int_{t_0}^{t} \mathbf{v}(t)dt \;\Rightarrow\; \mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(t)dt. \tag{6.81}$$

We insert the result for $\mathbf{v}(t)$ from (6.79) to find the position:

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(t)\,dt \\ &= \mathbf{r}(t_0) + \int_{t_0}^{t} \left(\mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(t)dt\right)dt \\ &= \mathbf{r}(t_0) + \mathbf{v}(t_0)\int_{t_0}^{t} dt + \int_{t_0}^{t} \left(\int_{t_0}^{t} \mathbf{a}(t)dt\right)dt \\ &= \mathbf{r}(t_0) + \mathbf{v}(t_0)\,(t - t_0) + \int_{t_0}^{t} \left(\int_{t_0}^{t} \mathbf{a}(t)dt\right)dt\end{aligned} \tag{6.82}$$

These two equations provide the **integration method** to solve the equations of motion and find the position $\mathbf{r}(t)$ and velocity $\mathbf{v}(t)$ vectors from the acceleration vector, $\mathbf{a}(t)$:

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(t)dt$$

$$\mathbf{r}(t) = \mathbf{r}(t_0) + \mathbf{v}(t_0)\,(t - t_0) + \int_{t_0}^{t} \left(\int_{t_0}^{t} \mathbf{a}(t)dt \right) dt \qquad (6.83)$$

Again there is no need to memorize these equations. They follow from your knowledge of calculus. Instead, you should learn how to use your experience from calculus to find these equations when you need them. In this way, it is simpler for you to remember this result simply as a special case: The solution of the initial value problem when the acceleration is only a function of time.

Motion with Constant Acceleration

The integration method can use used to find the motion of an object moving with a constant acceleration, $\mathbf{a}(t) = \mathbf{a}_0$, starting from $\mathbf{r}(t_0) = \mathbf{r}_0$ and with initial velocity $\mathbf{v}(t_0) = \mathbf{v}_0$:

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}_0\,dt = \mathbf{v}_0 + \mathbf{a}_0\,(t - t_0)\,. \qquad (6.84)$$

and

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(t)\,dt \\
&= \mathbf{r}_0 + \int_{t_0}^{t} \left(\mathbf{v_0} + \mathbf{a}_0\,(t - t_0)\right)\,dt \\
&= \mathbf{r}_0 + \int_{t_0}^{t} \mathbf{v}_0\,dt + \int_{t_0}^{t} \mathbf{a}_0\,(t - t_0)\,dt \\
&= \mathbf{r}_0 + \mathbf{v}_0\,(t - t_0) + \frac{1}{2}\mathbf{a}_0\,(t - t_0)^2\,. \qquad (6.85)
\end{aligned}$$

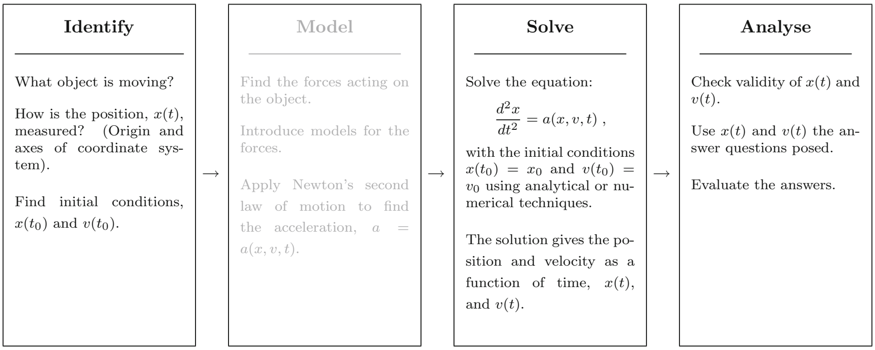


Fig. 6.11 Illustration of the structured problem-solving approach

Differential Equations

In mechanics we want to calculate the motion of an object based on the forces acting on the object. Therefore, you learn to gradually build more advanced models for the forces acting on an object. Using these models, you can apply Newton's second law to find the acceleration of the object. Finally, you find the velocity and position vector of the object as a function of time based on the expression you have for the acceleration and the initial values of the position and velocity. We called this general procedure "the structured problem-solving approach", and it is illustrated in Fig. 6.11.

So far, we have addressed the case when you have measured the acceleration vector, $\mathbf{a}(t)$, either for only a discrete number of time steps, or you can have a theoretical prediction for the acceleration for all times. However, usually, we do not know how the acceleration of an object varies in time, but we rather have models for how the acceleration depends on the position of the object, or the velocity of the object. For a probe scrambling through a tornado, it is the position and velocity of the probe relative to the tornado that determine the forces acting on it and therefore its acceleration. In this case, we cannot integrate the acceleration, because the acceleration also depends on velocity and position.

Generally, modeling the forces and applying Newton's second law leads to an expression:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}(\mathbf{r}, \mathbf{v}, t) = \mathbf{a}\left(\mathbf{r}, \frac{d\mathbf{r}}{dt}, t\right) \tag{6.86}$$

For example, a viscous force model leads to an acceleration:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a} = -c\mathbf{v} = -c\frac{d\mathbf{r}}{dt}, \tag{6.87}$$

and a spring force model leads to an acceleration on the form:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a} = -Cr\frac{\mathbf{r}}{r}. \tag{6.88}$$

We see that the unknown function, $\mathbf{r}(t)$, occurs on both sides of the equality—therefore we cannot simply integrate over time to find the solution. These are examples of differential equations. In some cases they can be solved using analytical techniques, but in most cases we will need to turn to numerical methods. Fortunately, in many cases the numerical solution of these differential equations can be done using methods identical to the methods we have developed for direct integration.

One of the major steps in the structured problem-solving approach is the "Solver", where you find the time evolution of the velocity and position vectors from an equation for the acceleration and the initial conditions:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}\left(\mathbf{r}, \frac{d\mathbf{r}}{dt}, t\right), \quad \mathbf{v}(t_0) = \mathbf{v}_0, \quad \mathbf{r}(t_0) = \mathbf{r}_0. \tag{6.89}$$

The result of the "Solver" step is the velocity and position as a function of time, either as continuous, mathematical functions, $\mathbf{v}(t)$, and $\mathbf{r}(t)$, or as the numerical solutions of the equations calculated at discrete time steps, $\mathbf{v}(t_i)$ and $\mathbf{r}(t_i)$. However, unless the time resolution, t_i, is determined by the time intervals of an experimental data-set, your numerical solution can be found at any time resolution you like.

Numerical Solution

We have seen several examples of how to find the numerical solution when the acceleration or velocity vectors are given functions of time. Let us now illustrate a general approach that also works for a differential equation of the form in (6.89). We know the initial position and velocity at $t = t_0$. The acceleration at $t = t_0$ is therefore given by (6.89). Let us to find the velocity and position at $t = t_1$ after a small time interval Δt at $t_1 = t_0 + \Delta t$. We use Euler-Cromer's method to find the new velocity vector based on the acceleration vector at $t = t_0$, and then to find the new position based on the position at $t = t_0$ and the velocity at $t = t_1$:

$$\mathbf{v}(t_0 + \Delta t) \simeq \mathbf{v}(t_0) + \Delta t\ \mathbf{a}(t_0, \mathbf{r}(t_0), \mathbf{v}(t_0)) \tag{6.90}$$

$$\mathbf{r}(t_0 + \Delta t) \simeq \mathbf{r}(t_0) + \Delta t\ \mathbf{v}(t_0 + \Delta t). \tag{6.91}$$

We can continue this method iteratively, finding in sequence first t_1, then t_2 and so on, until we have reached the time t.

In **Euler-Cromer's** method we solve the (second order) initial value problem:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}\left(\mathbf{r}, \frac{d\mathbf{r}}{dt}, t\right), \quad \mathbf{v}(t_0) = \mathbf{v}_0, \ \mathbf{r}(t_0) = \mathbf{r}_0. \tag{6.92}$$

using the following iterative procedure:

$$\begin{aligned} \mathbf{v}(t_i + \Delta t) &\simeq \mathbf{v}(t_i) + \Delta t\, \mathbf{a}(t_i, \mathbf{r}(t_i), \mathbf{v}(t_i)) \\ \mathbf{r}(t_i + \Delta t) &\simeq \mathbf{r}(t_i) + \Delta t\, \mathbf{v}(t_i + \Delta t) \end{aligned} \tag{6.93}$$

You should in general think of the "Solver" as a call to a numerical function that returns the position and velocity vectors given the functional form of the acceleration and the initial conditions. When you grow up to become a professional physicist, you will have built your own set of methods and tools, numerical and analytical, to "Solve" problems. In particular, we advice you to use a fourth order Runge-Kutta method as your preferred method of numerical integration, although in this book we will focus on clarity and simplicity instead, and typically use Euler-Cromer's method, unless this produces significant errors.

6.3.1 Example: Feather in the Wind

In this example you learn to find the position and velocity from the acceleration, when the acceleration is given by a simplified mathematical model, and when it is given by a differential equation.

You are planning to reproduce the introductory film in Forrest Gump—by capturing the motion of a feather caught in the wind. You plan to start with a feather dropped from a lift, at a height h above the ground. A slight wind is blowing. Our task is to find the motion of the feather, given its acceleration.

Sketch: As always, a good sketch solves half the problem. For simplicity we assume that the motion is two-dimensional. Figure 6.12 shows a sketch of the path of the feather, including the velocity of the wind.

Simplified model—Free fall: The simplest, and least realistic, model for the falling feather is to assume that it falls without air resistance. We release the feather with a horizontal velocity equal to that of the wind, $\mathbf{v}_0 = \mathbf{w}$, and then assume that it has constant acceleration, $\mathbf{a} = -g\mathbf{j}$. We can then find the motion by integration.

$$\mathbf{a} = -g\,\mathbf{j} \tag{6.94}$$

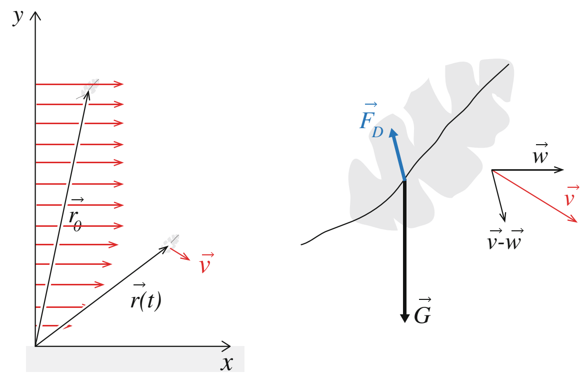


Fig. 6.12 Sketch of a feather falling to the ground. The velocity field, **w**, of the air is illustrated by the *arrows*

and we release the feather at time $t_0 = 0\,\text{s}$ wt $\mathbf{r}(0) = h\,\mathbf{j}$ with $\mathbf{v}(0) = \mathbf{v}_0 = \mathbf{w}$. We find the velocity by direct integration:

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}\,dt = \mathbf{w} + \int_0^t -g\,\mathbf{j}\,dt = \mathbf{w} - gt\,\mathbf{j}. \tag{6.95}$$

We can simplify this a bit further if the wind is horizontal, $\mathbf{w} = w\,\mathbf{i}$:

$$\mathbf{v}(t) = w\,\mathbf{i} - gt\,\mathbf{j}. \tag{6.96}$$

We find the position by integrating once more over time:

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(t)\,dt = \mathbf{r}_0 + \int_0^t (w\,\mathbf{i} - gt\,\mathbf{j})\;dt \\ &= h\,\mathbf{j} + wt\,\mathbf{i} - \frac{1}{2}gt^2\,\mathbf{j}.\end{aligned} \tag{6.97}$$

This gives us a complete solution for this simplified model, which is useful as a comparison when we address the full model.

Realistic model: A more realistic model includes two additional effects: Air resistance means that the acceleration depends on the velocity of the feather and the velocity of the wind, and the wind typically varies near the ground. A better model for the acceleration of the feather is:

$$\mathbf{a} = -g\,\mathbf{j} - C|\mathbf{v} - \mathbf{w}|\,(\mathbf{v} - \mathbf{w})\,, \tag{6.98}$$

where **w** is the velocity of the wind. A realistic model for the wind is

$$\mathbf{w}(\mathbf{r}) = \mathbf{w}_0 \left(1 - e^{-y/b}\right), \tag{6.99}$$

In this case, $b = 5\,\mathrm{m}$ and $\mathbf{w}_0 = 3\,\mathrm{m/s}\,\mathbf{i}$ is the velocity of the wind. We drop the feather from rest from a position $\mathbf{r}_0 = h\,\mathbf{j}$, with $h = 10\,\mathrm{m}$.

Unfortunately, we cannot solve this equation exactly, but it is not difficult to solve numerically. We apply Euler-Cromer's method. We find the velocity and position at time $t_i + \Delta t$ from the position and velocity at t_i using:

$$\mathbf{v}(t_i + \Delta t) \simeq \mathbf{v}(t_i) + \Delta t\,\mathbf{a}(t_i, \mathbf{v}_i) \tag{6.100}$$

$$\mathbf{r}(t_i + \Delta t) \simeq \mathbf{r}(t_i) + \Delta t\,\mathbf{v}(t_i + \Delta t). \tag{6.101}$$

In addition, you are told that $C = 30.0\,\mathrm{m}^{-1}$ for the feather.

Notice that the acceleration depends not only on the velocity $\mathbf{v}(t)$ of the feather, but also on its position, because the velocity of the air, $\mathbf{w}$, depends on the position of the feather. We therefore first calculate $\mathbf{w}(\mathbf{r})$ and then calculate $\mathbf{a}(t, \mathbf{r}, \mathbf{v})$. This is implemented in the following program. Notice the use of vector notation to find the acceleration.

```
h = 10.0; % m
C = 30.0; % m^-1
w0 = 3.0; % m/s
b = 5.0;  % m
g = 9.8;  % m/s
time = 20.0;
dt = 0.001;
n = round(time/dt);
t = zeros(n,1);
r = zeros(n,2);
v = zeros(n,2);
a = zeros(n,2);
t(1) = 0.0;
r(1,:) = [0.0 h];
v(1,:) = [0.0 0.0];
for i = 1:n-1
    w = w0*(1.0-exp(-r(i,2)/b))*[1 0];
    a(i,:) = -g*[0 1]-C*norm(v(i,:)-w)*(v(i,:)-w);
    v(i+1,:) = v(i,:) + a(i,:)*dt;
    r(i+1,:) = r(i,:) + v(i+1,:)*dt;
    t(i+1) = t(i) + dt;
end
i = find(r(:,2)>0.0);
plot(r(i,1),r(i,2),'-k');
axis equal; xlabel('x [m]'); ylabel('y [m]')
```

The expression we have used for $\mathbf{w}(\mathbf{r})$ is only value when $y > 0$, and the feather hits the ground when $y = 0\,\mathrm{m}$. We therefore only plot the trajectory when $y > 0$. This is ensured using the `find` command.

The resulting path of the feather is illustrated in Fig. 6.13. Notice that the feather almost immediately follows the wind—the drag forces rapidly reduces differences between the velocity of the feather and the velocity of the surrounding air.

We can compare these results with the simplified, analytical solution we found above. The simplified solution is illustrated by a dotted line, which clearly was not a good solution, because the vertical acceleration is greatly reduced when the feather moves fast in the horizontal direction.

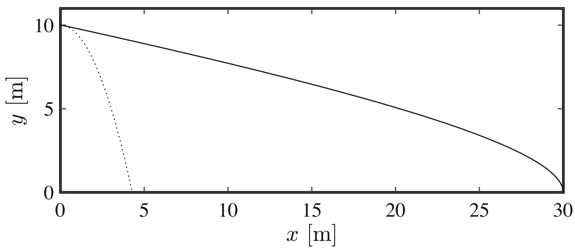


Fig. 6.13 The trajectory of the falling feather

6.4 Frames of Reference

Consider a person driving in an open convertible at constant velocity. She throws a ball straight up into the air. According to her, the ball moves straight up along the y-axis and falls straight down along the y-axis into her arms. A simple feat. However, a spectator standing by the road tells a different story. He observed the person in the car throwing the ball with an initial horizontal as well as vertical velocity component in a curved, parabolic path, and then she caught the ball exactly as it fell back to its original height. A more impressive feat as seen from the sidewalk. The two stories are illustrated in Fig. 6.14. In this section we address how we can relate these two observations.

The motion of the ball can be described in a reference system S, on the sidewalk, and in a reference system S' which moves with the car. The position of the ball as measured in system S at a time t (measured in S) is $\mathbf{r}(t)$. Similar, the position of the ball as measured in the system S' at a time t' (measured in S') is $\mathbf{r}'(t')$. The position of the system S' measured in system S is $\mathbf{R}$. We can therefore relate the positions in the two systems by:

$$\mathbf{r} = \mathbf{R} + \mathbf{r}' . \tag{6.102}$$

Let us assume that the time is the same in all reference systems. That is, we assume there is a universal time so that the time measured in system S is the same as the time measured in system S'. As we will learn later, this is only approximately true as long as the reference systems are not moving very fast relative to each other.

What is the velocity of the object measured in system S?

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left(\mathbf{R} + \mathbf{r}'\right) = \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{r}'}{dt} . \tag{6.103}$$

System S' is moving with a velocity $\mathbf{u} = d\mathbf{R}/dt$ relative to system S—this is the velocity of the car (system S') in Fig. 6.14 relative to the sidewalk (system S). Since we have assumed that the time is universal, we have that:

$$\frac{d\mathbf{r}'(t')}{dt'} = \mathbf{v}' = \frac{\mathbf{r}'(t)}{dt} , \tag{6.104}$$

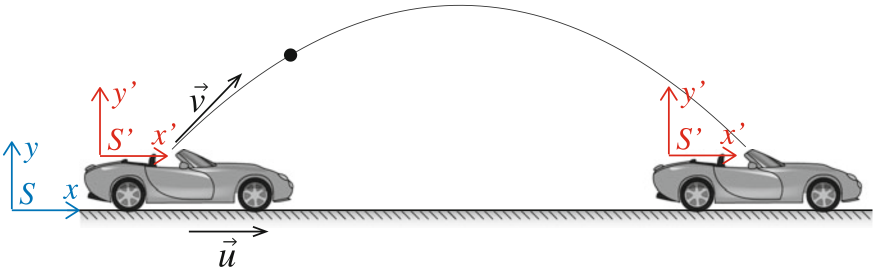


Fig. 6.14 An illustration of a person throwing a ball straight up from a convertible driving at a constant velocity as seen from a person on the sidewalk. The coordinate system S is fixed to the ground, and the coordinate system S' follows the car

which is the vector velocity of the object measured in system S'. This corresponds to the vector velocity of the ball measured from the car in Fig. 6.14.

The relation between the velocities is therefore

$$\mathbf{v} = \mathbf{u} + \mathbf{v}' . \tag{6.105}$$

If system S' moves at a constant velocity relative to system S, $d\mathbf{u}/dt = 0$, and the accelerations of the object are the same inboth systems:

$$\mathbf{a} = \mathbf{a}' . \tag{6.106}$$

This transformation is called the Gallileo-transformation. The acceleration of an object is the same in all systems moving at constant velocities relative to each other. This means that the acceleration is an invariant in the Gallileo-transformations. We call all systems moving at constant velocities **inertial systems**. Because Newton's laws determine the accelerations, Newton's laws are only valid in inertial systems.

If we now return to the case of the car, we can explain the behavior. The acceleration of the ball is the same in a system on the ground and in all systems moving at constant velocity relative to the ground. The car is such a system, moving with a constant velocity $\mathbf{u}$ relative to the ground. The acceleration of the ball is therefore the same in the sidewalk-system S and in the car's system S'. However, the initial velocities are different in the two systems. In system S' the ball is thrown straight up, which means that the initial velocity only has a component along the y'-axis of the S' system: $\mathbf{v}_0' = v_0\,\mathbf{j}$. In system S, the initial velocity has both a vertical and horizontal component, because the relation between the two velocities are given by (6.105):

$$\mathbf{v} = \mathbf{u} + \mathbf{v}' = u\,\mathbf{i} + v_0\,\mathbf{j} , \tag{6.107}$$

where u is the speed of the car in the horizontal direction. Seen from the car, it is obvious that the ball falls down into the car, since it was thrown with no horizontal velocity component relative to the car.

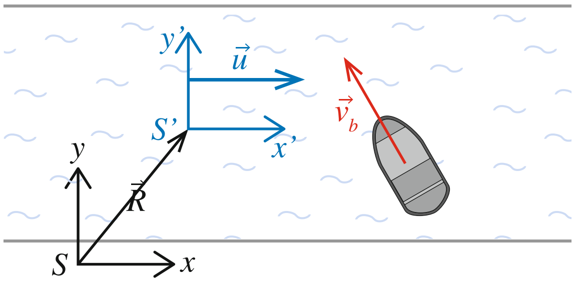


Fig. 6.15 Sketch of a boat driving in a flowing river. The velocity of the river is $\mathbf{u}$ and the velocity of the boat is $\mathbf{v}_b$

6.4.1 Example: Motion of a Boat on a Flowing River

In this example you will learn how to use transformations between frames of reference to address the motion of a boat moving relative to a moving background—a river.

You are driving a boat on a river that flows with a velocity of 1.0 m/s southwards. We will now discuss the motion of the boat in several scenarios: driving with a constant speed relative to the water and driving with a speed given by the boat speedometer, recorded at intervals of $\Delta t = 0.1$ s in the file boatvelocity.d.[7]

Sketch: We start by sketching the system: We sketch the boat and the surrounding river in Fig. 6.15. It is important to draw all the coordinate systems we will be using: We use a coordinate system S with axes x and y, which is on the riverbank, and a system S', with axes x' and y' which is following the motion of the river. The river is flowing in the x-direction, and the x and x' directions are parallel.

Position of the boat: The position of the boat in system S, which is stationary on the riverbank, is $\mathbf{r}(t)$. The position of the boat in the system S', which is following the water in the river, is $\mathbf{r}'(t)$ and the two positions are related by:

$$\mathbf{r}(t) = \mathbf{R}(t) + \mathbf{r}'(t), \tag{6.108}$$

where $\mathbf{R}(t)$ is the position of system S' measured relative to system S.

Velocity of the boat: If we know the velocity of the boat relative to the water, $\mathbf{v}'_b$, then we find the velocity of the boat relative to the bank by taking the time derivative of (6.108):

$$\mathbf{v}_b(t) = \frac{d\mathbf{r}_b}{dt} = \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{r}'_b}{dt} = \mathbf{u} + \mathbf{v}'_b. \tag{6.109}$$

Driving straight across the river: We can use this result to find what velocity the boat must have in order to drive straight across the river. To drive straight across the river means that the boat should have no velocity along the riverbank, that is,

[7] http://folk.uio.no/malthe/mechbook/boatvelocity.d.

$v_{b,x} = 0$. Since we know that $\mathbf{u}$ is directed along the x-axis, $\mathbf{u} = u\,\mathbf{i}$, the velocity of the boat is:

$$\mathbf{v}_b = u\,\mathbf{i} + \mathbf{v}'_b, \tag{6.110}$$

and on component form:

$$v_{b,x} = u + v'_{b,x}\,,\ v_{b,y} = v'_{b,y}. \tag{6.111}$$

Straight across means that $v_{b,x} = 0$, that is that

$$u + v'_{b,x} = 0 \ \Rightarrow\ v'_{b,x} = -u, \tag{6.112}$$

and that $v'_{b,y}$ can have any value you like.

Driving with constant velocity: If the boat is driving with a constant velocity $\mathbf{v}'_b$, what is the position of the boat after a time t? We can find the position in either system S or in system S'—the result will be the same. If we find the position in system S' we need to transform the velocity first. The velocity of the boat in system S is:

$$\mathbf{v}_b = \mathbf{u} + \mathbf{v}'_b, \tag{6.113}$$

and the position is therefore:

$$\begin{aligned}\mathbf{r}_b(t) &= \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}_b(t)\,dt = \mathbf{r}(t_0) + \int_{t_0}^{t} \left(\mathbf{u} + \mathbf{v}'_b\right)\,dt \\ &= \mathbf{r}(t_0) + \left(\mathbf{u} + \mathbf{v}'_b\right)(t - t_0)\,. \end{aligned} \tag{6.114}$$

Driving with varying velocity: The velocity of the boat relative to the river, $\mathbf{v}'_b(t)$, is recorded by the speedometer. We can therefore use (6.108) to find the velocity of the boat relative to the ground. We notice that the velocity of the river is:

$$\mathbf{u} = -1.0\,\text{m/s}\,\mathbf{j}. \tag{6.115}$$

We integrate $\mathbf{v}(t_i)$ numerically to find the position of the boat relative to land using Euler's method:

$$\mathbf{r}_b(t_i + \Delta t) = \mathbf{r}_b(t_i) + \Delta t\,\mathbf{v}_b(t_i), \tag{6.116}$$

where

$$\mathbf{v}_b(t_i) = \mathbf{u} + \mathbf{v}'_b(t_i), \tag{6.117}$$

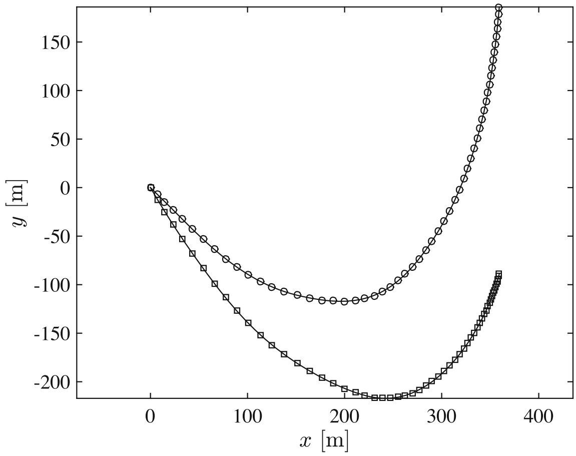


Fig. 6.16 The trajectory of the boat on the river (*blue*) compared with the trajector of the boat on a stationary lake (*red*). The *circles* illustrates the motion diagram of the boat, with a time interval of $\Delta t = 1$ s between each point

where we find $\mathbf{v}'_b(t_i)$ in the data set in boatvelocity.d.[8] This is implemented in the following program:

```
load -ascii boatvelocity.d
u = [0.0 -5.0];
t = boatvelocity(:,1);
n = length(t);
dt = t(2)-t(1);
v = zeros(n,2);
v(:,1) = boatvelocity(:,2);
v(:,2) = boatvelocity(:,3);
r = zeros(n,2);
r(1,:) = [0.0 0.0]; for i = 2:n
  r(i,:) = r(i-1,:) + (v(i-1,:)+u)*dt;
end
plot(r(:,1),r(:,2),'-k');
axis equal; xlabel('$x$ [m]'); ylabel('$y$ [m]');
```

The trajectory of the resulting motion is compared with the trajectory of the motion of a boat on a stationary lake in Fig. 6.16.

Summary

Motion:

- The motion of an object is described by the position vector, $\mathbf{r}(t)$, as a function of time, measured in a specified coordinate system.
- The velocity vector of the object is defined as

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}.$$

[8] http://folk.uio.no/malthe/mechbook/boatvelocity.d.

- The acceleration vector of the object is defined as

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}.$$

Problem-solving approach:

- We solve problems in kinematics using a structured approach.
- In the "Solver" we solve the equation:

$$\frac{d^2\mathbf{r}}{dt^2} = a\left(t, \mathbf{r}, \frac{d\mathbf{r}}{dt}\right).$$

  with the initial conditions $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{v}(t_0) = \mathbf{v}_0$.
- *Numerically*: We solve the equation using an iterative approach starting from the initial conditions. For example, we can use Euler-Cromer's method:

$$\mathbf{v}(t_i + \Delta t) = \mathbf{v}(t_i) + \Delta t\, \mathbf{a}(\mathbf{r}(t_i), \mathbf{v}(t_i), t_i)$$
$$\mathbf{r}(t_i + \Delta t) = \mathbf{r}(t_i) + \Delta t\, \mathbf{v}(t_i + \Delta t).$$

- *Analytically*: When the acceleration, $\mathbf{a} = \mathbf{a}(t)$, is only a function of time, t, we can solve the equations by direct integration:

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(t)\, dt,$$

$$\mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(t)\, dt.$$

  A typical example is motion with constant acceleration.
- The derivative of a vector function such as $\mathbf{r}(t)$ is found by componentwise derivation. If

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k},$$

  the derivative is:

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}.$$

- The integral of a vector function is found by componentwise integration. For

$$\mathbf{v}(t) = v_x(t)\,\mathbf{i} + v_y(t)\,\mathbf{j} + v_z(t)\,\mathbf{k},$$

the integral is

$$\int_{t_0}^{t} \mathbf{v}(t)dt = \left(\int_{t_0}^{t} v_x(t)dt\right)\mathbf{i} + \left(\int_{t_0}^{t} v_y(t)dt\right)\mathbf{j} + \left(\int_{t_0}^{t} v_z(t)dt\right)\mathbf{k}.$$

Exercises

Discussion Questions

6.1 Pendulum motion. A pendulum oscillates back and forth following the path of a vertical circle. What is the direction of the acceleration of the pendulum when the pendulum is at its upper position? At its lower? Explain.

6.2 Packet from a plane. A plan flies at constant velocity and altitude, and drops a packet. Describe trajectory of the packet as seen from the ground and from the plane.

6.3 U-turn. A train drives along a U-shaped turn at constant speed. When does it have maximum acceleration?

6.4 Magic hat. A magic hat is placed on a trolley that moves at a constant velocity along a straight track. How would you shoot a doll-rabbit from the trolley so that you can be sure that it falls into the hat?

6.5 Constant velocity. Can a motion occur at constant speed (constant magnitude of the velocity vector) and non-zero acceleration? Can the acceleration vector also be constant?

6.6 Remote robot. You are remote-controlling a Mars rover from a simplified control panel with four button that makes the rover move a given length North, South, East or West. You are given a complicated route consisting of a sequence of button to be pressed but make a mistake in the order. Does this matter?

6.7 No motion. If the average velocity for a motion is zero, is the average displacement also zero? If the average acceleration for a motion is zero, is the average displacement zero?

Problems

6.8 Curving swallow. A swallow is making a quick turn to catch a fly. Its motion in the horizontal plane is captured by a camera attached to a balloon. The horizontal positions were recorded at 0.1 s intervals:
(a) Draw the motion diagram and the displacements for this motion.

t (s)	0.0	0.1	0.2	0.3	0.4	0.5
x (m)	10.00	11.00	11.75	12.25	13.00	14.00
y (m)	15.0	15.00	15.50	16.50	17.00	17.00

(b) Use the motion diagram to find the average velocity vectors.
(c) Use the motion diagram to find the average acceleration vectors.
(d) When is the speed and the acceleration maximum?

6.9 Penalty shot. As a research assistant for the national soccer team, you have mounted a videocamera to record the path of the soccer ball during a penalty shot. You use a video-analysis software to extract the position of the football in the horizontal plane for each picture frame, taken at $\Delta t = 0.01$s intervals. You find an example of a penalty shot in penalty.d.[9] Each line in the file consists of a time, t_i, followed by x_i and y_i, the x- and y-positions of the ball respectively.
(a) Draw a motion diagram of the motion.
(b) Plot the x- and y-positions as function of time.
(c) Does the ball hit the goal, located at $x = 0$ m between $y = 25.0$ m and $y = 36.0$ m?
(d) Plot the components of the average velocities as functions of time. When is the x- and y-components of the velocity largest?
(e) Estimate the initial speed of the ball. And the speed of the ball when it reaches $x = 0$ m.
(f) Plot the components of the average accelerations as functions of time.
(g) Draw some of the accelerations in the same figure as the motion diagram. What do you think causes the acceleration?

6.10 Vertical loop. A small airplane is making a vertical loop. Sketch a motion diagram describing the motion of the airplane.

6.11 Unknown motion. A motion is described by the data-set in the file discretemotion06.d.[10] Each line in the file consists of a time, t_i, followed by x_i, y_i and z_i, the x-, y-, and z-positions of the object respectively.
(a) Read in the data, and plot a motion diagram of the motion.
(b) What physical process do you think this motion describes?

6.12 Alpha particle. An alpha particle is ejected from an atom. The alpha particle moves with a constant velocity $\mathbf{v} = 1000\,\text{m/s}\,\mathbf{i} + 2000\,\text{m/s}\,\mathbf{j}$. After one second it hits another atom. We use the atom it is ejected from as the origin.
(a) What is the speed of the alpha particle?
(b) Find the position of the alpha particle as a function of time.
(c) How far have the alpha particle travelled in one second?

[9] http://folk.uio.no/malthe/mechbook/penalty.d.
[10] http://folk.uio.no/malthe/mechbook/discretemotion06.d.

6.13 Airplane collision. An F-16 jet fighter is leaving from Rygge airfield, which we use as the origin of our coordinate system, at $t = 0.0$ s, and travels with a constant velocity $\mathbf{v}_1 = 1700.0\,\text{km/h}\,\mathbf{j}$ towards the North. At the same time, an Airbus A310 airplane is passing over Oslo, which is located at $\mathbf{r}_1 = -10\,\text{km}\,\mathbf{i} + 80\,\text{km}\,\mathbf{j}$. The Airbus travels with a constant velocity of $\mathbf{v}_2 = 105\,\text{km/h}\,\mathbf{i} + 905\,\text{km/h}\,\mathbf{j}$. They are both travelling at the same height.
(a) Find the position of the jet fighter as a function of time.
(b) Find the position of the Airbus as a function of time.
(c) Sketch the trajectories of both planes in the same diagram. (You can do this on your computer if you like.)
(d) Will the airplanes collide?
(e) If the airplanes are within a distance of 1 km of each other, an alarm will sound in the plane, and an evasive maneuver will be attempted. Will the planes pass that close to each other?

6.14 Motion of spaceship. A spaceship is floating in free space with an initial velocity $\mathbf{v}_0 = 1000\,\text{m/s}\,\mathbf{i}$. Suddenly, the spaceship turns its thrusters on, giving the spaceship a constant acceleration $\mathbf{a}_0 = 10\,\text{m/s}^2\,\mathbf{j}$ for 10 s. (The acceleration does not change during this time).
(a) Sketch the path of the spaceship without doing a detailed calculation.
(b) Find the velocity of the spaceship as a function of time.
(c) Find the position of the spaceship as a function of time.
(d) Plot the path of the spaceship and compare with your initial sketch.

6.15 Controlling the electron beam. An electron is shot through a varying electrical field. Initially, the electron is moving in the x-direction with a velocity $v_x = 100$ m/s. The electron enters the field when it passes the origin. The field varies with time, causing an acceleration of the electron that varies in time:

$$\mathbf{a}(t) = \left(-20\,\text{m/s}^2 - 10\,\text{m/s}^3\,t\right)\,\mathbf{j}. \tag{6.118}$$

(a) Find the velocity as a function of time for the electron.
(b) Find the position as a function of time for the electron.
The field is only acting inside a box of length $L = 2$ m.
(c) How long time is the electron inside the field?
(d) What is the displacement in the y-direction when the electron leaves the box. (We call this the deflection of the electron).
(e) Find the angle the velocity vector forms with the horizontal as the electron leaves the box.

6.16 Accelerometer reading. Your high precision pedometer contains a very precise accelerometer that measures the acceleration of your body as you are running. The reading from the accelerometer is recorded in the file pedometer.d,[11] where each line

[11] http://folk.uio.no/malthe/mechbook/pedometer.d.

contains the time, t_i, measured in seconds, followed by the acceleration in the x and y direction respectively, measured in m/s^2.
(a) Read the data from the file. Find the velocity vector as a function of time.
(b) Find the position vector as a function of time. Plot the results.
(c) Given an interpretation of the motion in its two distinct phases.

6.17 Running inside a bus. A bus is driving with constant velocity $v_x = 50$ km/h in the x-direction.
(a) If you are running towards the back of the bus at a speed of 10 km/h. How fast are you running relative to the ground?
(b) If you are running towards the front of the bus at a speed of 10 km/h. How fast are you running relative to the ground?

6.18 Jumping onto a running train. In your early career as a stuntwoman, you task was to jump from a bridge onto a running train. The train was running at 36 km/h.
(a) What was your velocity relative to the train when you landed on the train?

You were rolling and stumbling for 2 before coming to rest at the train. You can assume that you were experiencing a constant acceleration in this time.
(b) Find the acceleration.
(c) Find your velocity as a function of time relative to the train from you landed and for the first 10 s.
(d) Find your velocity as a function of time relative to the ground from you landed and for the first 10 s.

6.19 A plane in crosswinds. You are trying to steer an airplane towards the north. The airspeed of your plane is 300 km/h. However, there is a strong wind from the west, with a wind speed of 60 km/h.
(a) In what direction should you direct the plane so that it travels towards the north? Illustrate your argument with a diagram.
(b) What is the speed of the plane relative to the ground?

Projects

6.20 Motion capture. In this project we will study the motion of an object that is fitted with an accelerometer, and it is your task to figure out what physical phenomenon we are observing. You will need to find the motion from the acceleration of the object, and then interpret the motion in physical terms.

First, we will guide your intuition by starting with an introductory, analytical exercise.

A car rolling down a hill with an inclination θ with the horizon experiences an acceleration $a = g\sin(\theta)$ along the surface of the hill. Here $g = 9.81$ m/s^2 is the acceleration of gravity. The car is released from rest at the time $t = t_0 = 0$ s.
(a) Sketch a motion diagram of the motion of the car.
(b) Find the position s and the velocity v of the car along the hill after a time t.

We will now introduce a reference system S oriented with the x-axis in the horizontal direction and the y-axis in the vertical direction—that is in the direction gravity is acting. Let us assume that the car starts in the position $x = 0$ m, $y = h$, where h is the height of the car, and that the car moves in the positive x-direction.
(c) Sketch the system and the coordinate system.
(d) Find the position $\mathbf{r}(t)$ and velocity $\mathbf{v}(t)$ of the car after a time t.

We will now address the motion captured by the accelerometer. The data-set motion1.d[12] contains time (in s) and acceleration (in m/s^2) of an object, given as a sequence of points t_i, $a_{x,i}$ and $a_{y,i}$ taken at regular time intervals Δt.

```
t0 ax0 ay0
t1 ax1 ay1
```

(e) Find the velocity $\mathbf{v}$ and position $\mathbf{r}$ of the motion using numerical methods. (For example using a simple Euler scheme for integration). Plot the path of the object. (Your answer should include a listing of the program used.)
(f) Can you give a physical interpretation of the motion, that is, can you describe a physical system that you would expect to behave in this manner?
(g) Where is the magnitude of the acceleration the maximum?

Another, similar, experiment was performed, giving the motion data in motion2.d.[13]
(h) Find the position $\mathbf{r}_2$ of the motion, and plot it in the same plot as the motion in `motion1.d`.
(i) Can you give a physical interpretation of the motion?
(j) Where is the magnitude of the acceleration the maximum? How do you interpret this?

[12] http://folk.uio.no/malthe/mechbook/motion1.d.
[13] http://folk.uio.no/malthe/mechbook/motion2.d.