# Chapter 13
# Multiparticle Systems

So far we have studied the motion of objects, but we have not been very precise in defining an object. What we really have studied is the motion of mass-points, particles, or extended objects that move as a particle. What does it mean to move as a particle? It means that all the points in the object move with the same velocity and the same acceleration, the whole object is translated as a stiff (rigid) body: The various parts of the object are not moving relative to each other.

But this does not seem to be a good description of everyday phenomena around us. If you are running, you are surely not moving both your arms and your legs with the same velocity and acceleration all the time. And if you throw a ball, it may wobble and spin on its path. Even on the microscopic level objects are not only translated: Molecules may vibrate, spin, or wobble as they move. When the world is so complex, can we still use the simple descriptions and models we have developed so far, or do we have to describe the motion of each small part of the object independently?

Fortunately, we are saved by Newton's third law: If we only define the position of the object in a particular way, by defining the position of the object as its center of mass, Newton's second law is valid for any system of particles, and therefore for any object. This wonderful consequence of Newton's second and third laws allows us to use the force models and the concepts we have developed so far also to address extended objects. The motion of an object is further simplified if the object is a rigid body: An object where the relative positions of any two points do not change, that is, the object is only translated and rotated, but does not change shape, stretch, or otherwise deform during its motion.

In this chapter, we discuss the motion of systems of particles—multiparticle systems. Before we proceed to describe the motion of rigid bodies in Chap. 15 and Chap. 16, we first introduce a general description of rotational motion in Chap. 14.



A. Malthe-Sørenssen, *Elementary Mechanics Using Matlab*,
Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-19587-2_13

## 13.1 Motion of a Multiparticle System

Let us start describing a system of many particles. For most practical purposes, two particles are many particles, but we are ambitious and start with a system of $N$ particles. We number the particles using the index $j$ running from 1 to $N$ as illustrated in Fig. 13.1. The position of each particle is given in a coordinate system $S$ as illustrated. The position of particle 1 as a function of time is $\mathbf{r}_1(t)$, and its mass is $m_1$. Similarly, the position of particle $j$ is $\mathbf{r}_j$ and its mass is $m_j$.

All the quantities we have defined so far are easily extended to each of these particles. For example, the velocity and acceleration of particle $j$ is found from the derivatives of the position vector:

$$\mathbf{v}_j = \frac{d\mathbf{r}_j}{dt} \; , \; \mathbf{a}_j = \frac{d\mathbf{v}_j}{dt} , \tag{13.1}$$

and the momentum of particle $j$ is $\mathbf{p}_j = m_j \mathbf{v}_j$.

The motion of each particle is determined from the forces acting on it. In our discussion of momentum, we already discussed that the forces acting on particle $j$ may be either from one of the other particles in the system, or from the environment. Forces from other particles in the system are called *internal forces*, and forces from the environment are called *external forces*. Newton's second law for particle $i$ is therefore:

$$\mathbf{F}_i^{\mathrm{net}} = \mathbf{F}_i^{\mathrm{ext}} + \sum_{j \neq i} \mathbf{F}_{j,i} = \frac{d}{dt} \mathbf{p}_i , \tag{13.2}$$

where $\mathbf{F}_{j,i}$ is the force from particle $j$ on particle $i$.

In order to find the motion of one of the particles inside the system, we need to know both the external forces acting on this particle and the forces from the other particles in a system. This may require a complicated force model, for example, consider the forces acting between two parts of a tennis ball as the ball wiggles and rotates. Often, we are not interested in the detailed motion of particles inside a system, but only in the motion of the system as a whole. We can address this by adding together (13.2) for each particle $i$, getting:

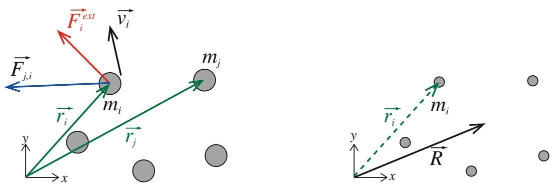


**Fig. 13.1** The motion of a system of particles is described by the position $\mathbf{r}_j$ of each of the $N$ particles in the system. Here, the system consists of 5 particles

$$\sum_i \mathbf{F}_i^{\text{net}} = \sum_i \mathbf{F}_i^{\text{ext}} + \underbrace{\sum_i \sum_{j \neq i} \mathbf{F}_{j,i}}_{=0} = \sum_i \frac{d}{dt}\mathbf{p}_i \,, \tag{13.3}$$

As we noticed when we discussed the total momentum of a system of particles, the sum of all the internal forces will always contain both the force $\mathbf{F}_{j,i}$ and the force $\mathbf{F}_{i,j}$. Since these two forces are action-reaction pairs, they are equal, but oppositely directed. Therefore, every such pair will cancel. The sum of all the internal forces is therefore zero!

If we introduce $\mathbf{P} = \sum_i \mathbf{p}_i$ as the total momentum of the system, we find:

$$\sum_i \mathbf{F}_i^{\text{ext}} = \frac{d}{dt}\mathbf{P}, \tag{13.4}$$

This looks a lot like Newton's second law, but now for the system of particles. We can make this similarity even stronger by introducing the velocity $\mathbf{V}$ of the system so that:

$$\boxed{\mathbf{P} = M\mathbf{V} = \sum_i \mathbf{p}_i = \sum_i m_i \mathbf{v}_i \,,} \tag{13.5}$$

where $M = \sum_i m_i$ is the total mass of the system.

Notice that this is a *definition* of the velocity $\mathbf{V}$ of the system. We define it this way to be able to write the total momentum, $\mathbf{P}$, of the system in the intuitive way $\mathbf{P} = M\mathbf{V}$. From (13.5) we find:

$$\boxed{\mathbf{V} = \frac{\sum_i m_i \mathbf{v}_i}{\sum_i m_i} = \frac{1}{M}\sum_i m_i \mathbf{v}_i \,.} \tag{13.6}$$

What is the *acceleration* of the system? It is natural to define the acceleration, $\mathbf{A}$, of the system as the time derivative of the velocity $\mathbf{V}$ of the system:

$$\boxed{\mathbf{A} = \frac{d}{dt}\mathbf{V} = \frac{1}{M}\sum_i m_i \mathbf{a}_i \,.} \tag{13.7}$$

Similarly, we define the position of the system, **R**, as:

**Center of mass:** The effective position of the system, or the *center of mass* of the system, is defined as

$$\mathbf{R} = \frac{1}{M}\sum_i m_i \mathbf{r}_i , \tag{13.8}$$

The velocity and acceleration of the center of mass are defined in the usual way:

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} , \ \mathbf{A} = \frac{d^2\mathbf{R}}{dt^2} . \tag{13.9}$$

With these definitions:

**Newton's second law for a system of particles:** is system as:

$$\sum_i \mathbf{F}_i^{\text{ext}} = \frac{d}{dt}\mathbf{P} = M\mathbf{A} . \tag{13.10}$$

where $M = \sum_i m_i$ and $\mathbf{P} = \sum \mathbf{p}_i$.

(Where we have assumed that the masses of the particles are constant).

This law, *Newton's second law for a system of particles*, is what we have been looking for. Equation (13.10) shows that if we define the position of the system in this particular way, we can use Newton's law exactly as we are used to, just remembering that we are not describing the motion of each particle separately, but instead we describe the motion of the effective position **R** of the system.

This law is powerful and surprisingly beautiful. It is the theoretical justification for why we do not have to care too much about whether we describe the motion of an object as a point or as a system of particles. We can describe the motion of any system of particles as a point: The center of mass **R** of the system.

In the next sections we build our intuition about the center of mass **R** of a system and we learn to apply Newton's second law for a system of particles.

## 13.2 The Center of Mass

Why do we call the effective position **R** the center of mass? Let us start by address this in a two-particle system. The effective position, **R**, of the two-particle system is:

$$\mathbf{R} = \frac{1}{M}\left(m_1\mathbf{r}_1 + m_2\mathbf{r}_2\right). \tag{13.11}$$

If the two masses are identical, we see that:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{m\mathbf{r}_1 + m\mathbf{r}_2}{m + m} = \frac{1}{2}\left(\mathbf{r}_1 + \mathbf{r}_2\right), \tag{13.12}$$

which is the midpoint between the two points.

We get a similar result for the $N$-particle system: When all the masses are the same, the effective position $\mathbf{R}$ is:

$$\mathbf{R} = \frac{1}{N}\sum_i \mathbf{r}_i, \tag{13.13}$$

as illustrated in Fig. 13.1. This is simply the arithmetic mean of the position vectors.

As long as all the masses are the same, the center of mass is what we typically would call the geometric center of the points. What happens when the masses are not equal? In this case, we weigh in the masses in the average, so that the center of mass is the mass-weighed average of the positions of the particles—which is the natural definition of the center of mass of an object.

Notice that the center of mass $\mathbf{R}$ is a **vector**. We can therefore calculate the center of mass for each component, along each axis, independently of the other axes.

### The Subdivision Principle

If we combine two systems A and B, where we know the center of mass for each these systems, how can we find the center of mass for the whole system?

The situation is illustrated in Fig. 13.2. Each of the systems could be a rigid body, or just a collection of point masses. System A has total mass $M_A$ and a center of mass at $\mathbf{R}_A$, and system B has a total mass $M_B$ and center of mass at $\mathbf{R}_B$.

Let us enumerate all the particles using the index $i$. The first $k$ particles are in system A, and the last $N - k$ particles are in system B. What is the center of mass, $\mathbf{R}$, of the whole system with mass $M = M_A + M_B$?

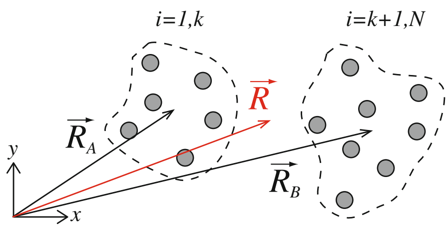


**Fig. 13.2** A system consists of two objects A and B. We find the center of mass of the whole system

The definition of **R** is:

$$M\mathbf{R} = \sum_i m_i \mathbf{r}_i = \sum_{i=1}^{k} m_i \mathbf{r}_i + \sum_{i=k+1}^{N} m_i \mathbf{r}_i = M_A \mathbf{R}_A + M_B \mathbf{R}_B \ , \tag{13.14}$$

and therefore:

$$\mathbf{R} = \frac{M_A \mathbf{R}_A + M_B \mathbf{R}_B}{M_A + M_B} . \tag{13.15}$$

We can therefore find the center of mass of a system of two objects A and B, by assuming that object A and B are point masses with the whole mass of each object located in the center of mass of each object. This means that if we want to find the center of mass of two solid bodies, we can find the center of mass of each object, and then find the center of mass of the combined object by assuming each object to be a point mass.

**Solid Bodies**

So far we have only defined the center of mass of a system of a finite number of particles. How can we find the center of mass of a solid body?

The definition of the center of mass for a continuous object follows directly from the definition for a system of many particles. We divide the solid body into small volumes $\Delta V_i = \Delta x_i \ \Delta y_i \ \Delta z_i$ at the position $\mathbf{r}_i$. The center of mass is:

$$\mathbf{R} = \frac{1}{M} \sum_i m_i \mathbf{r}_i , \tag{13.16}$$

where the mass of the volume element $\Delta V_i$ depends on the local mass density, $\rho(\mathbf{r}_i)$. When the size of the volumes goes to zero, the sum approaches the integral of the mass density of the volume, $V$, of the solid body:

$$\mathbf{R} = \frac{1}{M} \iiint_V \mathbf{r} \, \rho(\mathbf{r}) \, dV . \tag{13.17}$$

In physics, we often write this as an integral over the mass elements $dm$ instead:

$$\mathbf{R} = \frac{1}{M} \int_M \mathbf{r} \, dm . \tag{13.18}$$

In order to calculate the integral, we calculate the values for each of the components separately:

$$MX = \iiint_V x \, \rho(x, y, z) \, dx \, dy \, dz \ , \tag{13.19}$$

and similarly for the $Y$ and $Z$ components.

### 13.2.1 Example: Points on a Line

**Problem:** Find the center of mass $\mathbf{R}$ for the system of three particles illustrated in Fig. 13.3.

**Solution:** We want to determine:

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i}. \tag{13.20}$$

Since all the masses are equal, $m_i = m$. We find the $x$- and $y$-components independently:

$$X = \frac{1}{3}\sum_i x_i = \frac{1}{3}(a + 2a + 3a) = \frac{1}{3}6a = 2a, \tag{13.21}$$

$$Y = \frac{1}{3}\sum_i y_i = \frac{1}{3}(a + a + a) = \frac{1}{3}3a = a \tag{13.22}$$

The center of mass is therefore:

$$\mathbf{R} = 3a\ \mathbf{i} + a\ \mathbf{j}. \tag{13.23}$$

### 13.2.2 Example: Center of Mass of Object with Hole

**Problem:** Find the center of mass of a homogeneous disk with radius $R$, with a circular hole of radius $r$ touching the outer edge of the disk, as illustrated in Fig. 13.4.

**Solution:** This examples demonstrates that the subdivision principle also can be used in reverse to remove a part of an object, such as a circular hole in a circular disk. We start from a homogeneous disk, object AB, and remove a smaller circular portion, object B, and is left with a disk with a hole, object A.

The mass of the complete disk is $M_{AB} = \pi R^2 \rho$, where $\rho$ is the mass (area) density, and the mass of the small disk is $M_B = \pi r^2 \rho$. The subdivision principle

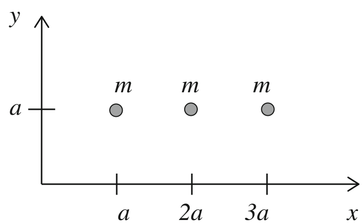


**Fig. 13.3** A system of three particles with identical masses, $m$

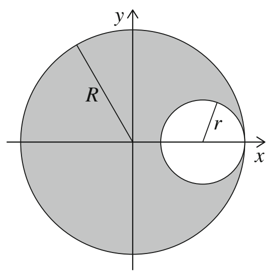


**Fig. 13.4** Illustration of the *circular disk* of radius $R$ with a *circular hole* of radius $r$

states that the center of mass of the whole disk (object AB), which is at the origin, $\mathbf{R} = 0$, can be written as:

$$M_{AB} \underbrace{\mathbf{R}}_{=0} = M_A \mathbf{R}_A + M_B \mathbf{R}_B \ . \tag{13.24}$$

We solve this equation to find $\mathbf{R}_A$, the unknown center of mass for object A.

$$\mathbf{R}_A = -\frac{M_B}{M_A}\mathbf{R}_B = -\frac{\pi r^2 \rho}{\pi R^2 \rho}\,(R - r)\,\mathbf{i} = -\frac{r^2}{R^2}\,(R - r)\,\mathbf{i}, \tag{13.25}$$

Notice the simplicity of this approach. We did not have to perform any integration. This use of symmetries is a characteristic of physics that you will meet many times during your career.

### *13.2.3 Example: Center of Mass by Integration*

**Problem:** Find the center of mass of a thin, homogeneous triangular plate with sides of length $a$ and $b$, as illustrated in Fig. 13.5. (You must be able to solve double-integrals to understand this example).

**Solution:** The center of mass for a continuous, homogeneous object is defined as:

$$M\mathbf{R} = \int_m \mathbf{r}\, dm, \tag{13.26}$$

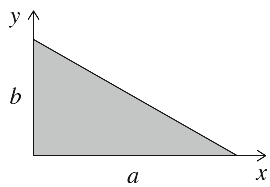


**Fig. 13.5** Illustration of a homogeneous triangle with sides of length $a$ and $b$

where we have written the integral over the mass. Instead, we may integrate over space and use the mass (area) density, $\rho$:

$$M\mathbf{R} = \iint_A \mathbf{r}\,\rho\, DA. \tag{13.27}$$

We need to find both the mass, $M$, and the position $\mathbf{R}$. Both are found by integration over the area $A$, which is the area of the triangle. First, we find the mass by integrating over the area $A$. We integrate $x$ from 0 to $a$, and $y$ from 0 and up to the line corresponding to the upper boundary of the triangle. This is a line going through the points $x = 0$, $y = b$ and $x = a$, $y = 0$. The straight line through these points has the equation $y = b(1 - x/a)$.

$$M = \iint_A \rho dA = \rho \int_0^a \int_0^{b(1-x/a)} dy\, dx = \rho \int_0^a b(1 - x/a)\, dx = \frac{1}{2}\rho ab, \tag{13.28}$$

which, of course, is the well know formula for the area of a triangle multiplied with the mass density $\rho$ of the triangle.

Now, we find the position of the center of mass by calculating the integral for $M\mathbf{R}$ for each component:

$$\begin{aligned} MX &= \iint_A x\rho dA = \rho \int_0^a \int_0^{b(1-x/a)} x dy dx = \rho \int_0^a xb(1 - x/a)\, dx \\ &= \rho \left(\frac{1}{2}a^2 b - \frac{1}{3}a^3 b/a\right) = \rho \left(\frac{1}{2}a^2 b - \frac{1}{3}a^2 b\right) = \rho \frac{1}{6}a^2 b, \end{aligned} \tag{13.29}$$

The center of mass in the $x$-direction is therefore:

$$X = \frac{MX}{M} = \frac{\rho\,(1/6)\,a^2 b}{\rho\,(1/2)\,ab} = \frac{1}{3}a. \tag{13.30}$$

We use the same method in the $y$-direction:

$$\begin{aligned} MY &= \iint_A y\rho dA = \rho \int_0^a \int_0^{b(1-x/a)} y\, dy\, dx = \rho \int_0^a \frac{1}{2}\,(b(1 - x/a))^2\, dx \\ &= \rho b^2 \frac{1}{2} \int_1^0 u^2(-1/a)\, du = \rho b^2 \frac{1}{2}\frac{1}{3} b^2 a, \end{aligned} \tag{13.31}$$

which gives

$$Y = \frac{MY}{M} = \frac{\rho\,(1/6)\,b^2 a}{\rho\,(1/2)\,ab} = \frac{1}{3}b. \tag{13.32}$$

The center of mass is therefore:

$$\mathbf{R} = \frac{1}{3} a\,\mathbf{i} + \frac{1}{3} b\,\mathbf{j}. \tag{13.33}$$

### *13.2.4 Example: Center of Mass from Image Analysis*

The center of mass is often used to describe the center of an object in an image . It may be because we are taking pictures of an object we want to track, such as the wandering behavior of a small grain of dust dancing through the air or the motion of a asteroid seen on the sky, or it may be to determine the center of mass of an irregularly shaped object.

How can we find the center of mass from an image? First, we need to read the image so that we can access it. The image is taken from a classroom experiment, where we have extracted a smaller part of the image for analysis (see Fig. 7.4, Figs. 13.6 and 13.7). We read the image `ballimage02.png` using:

```
z=imread('ballimage02.png');
```

Let us immediately display it to see if we got the right image:

```
subplot(1,2,1);
imagesc(z);
axis equal
axis tight
```

where the `axis` commands are to clean up the plotted image. Notice that matlab uses position `(1,1)` for the upper left part of the image, and that the first coordinate is the vertical coordinate and the second coordinate is the horizontal coordinate, so that `(iy,ix)` is position `(ix,iy)` in the image. We call each $(x, y)$ position for a pixel . We find the size of the image using `size`:

```
>> size(z)

ans =
    411    559     3
```

The image is stored as the matrix $z(y, x, j)$ which contain values of red (R, $j = 1$), green (G, $j = 2$), and blue (B, $j = 3$) . However, we cannot use these color values directly to find the center of mass. Instead, we need to know if a pixel at $(x, y)$ is a part of the object or not. We therefore set a threshold on the image, so that all pixels that are brighter than this threshold is included (set to value 1), and all the rest of the pixels are set to zero:

```
z2 = 1-im2bw(z,0.5);
subplot(1,2,2);
imagesc(z2);
axis equal
axis tight
```

The resulting images are shown in Fig. 13.7. The left image is the original image and the right image shows the filtered image, where all the pixels that are part of the ball are colored red.

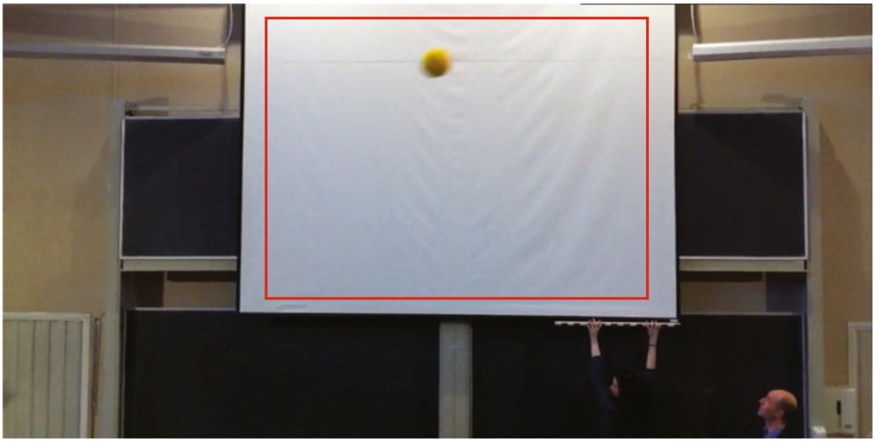

**Fig. 13.6** Image from video from classroom demonstration. The *inset* shows the image used for analysis

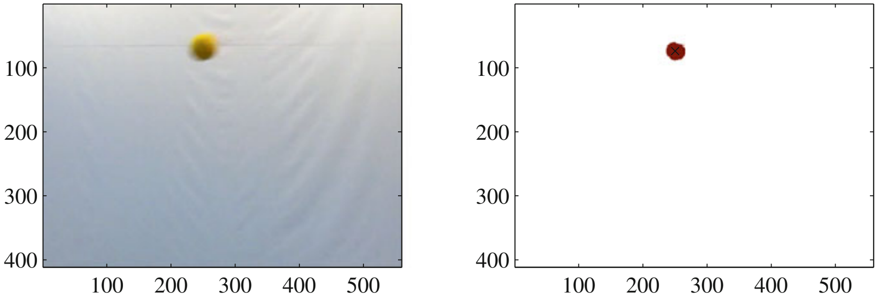


**Fig. 13.7** (*Left*) Image of ball (cut) (*Right*) Filtered image of ball

Now, we are ready to find the center of mass:

$$X = \frac{1}{M}\sum_i x_i\,,\ \ Y = \frac{1}{M}\sum_i y_i \tag{13.34}$$

These formulas can be directly converted into an algorithm: For each pixel $i$, if the pixel is a part of the object, that is if $z(x_i, y_i) = 1$, we include the positions $x_i$ and $y_i$ in the sum for the center of mass and include the pixel in the sum for the mass.

```
s = size(z2)
x = 0;
y = 0;
m = 0;
for iy = 1:s(1)
    for ix = 1:s(2)
        if (z2(iy,ix)==1)
            x = x + ix;
```

```
            y = y + iy;
            m = m + 1;
        end
    end
end xcm = x/m;
ycm = y/m;
hold on
plot(xcm,ycm,'kx');
hold off
```

where we also plot the center of mass as an "x". The last three lines ensure that empty pixels, pixels where $z(x, y) = 0$, are shown as white. (We set all three R, G, B values to 1 to generate a white entry in the colormap).

This method is used for motion tracking of an image. If we are able to automatically filter the image so that we only get the object of interest, we can use this method to find the center of mass of the object for each frame in a movie and thereby find the center of mass as a function of time. Usually this requires careful positioning of the camera and a good choice of background for the filming.

## 13.3 Newton's Second Law for Particle Systems

We have found that if we measure the position of a system of particles using the center of mass, **R**, of the system, the system behaves according to Newton's second law:

$$\sum \mathbf{F}^{\mathrm{ext}} = M\mathbf{A}, \tag{13.35}$$

where **A** is the acceleration of the center of mass of the system of particles, and the sum is over all external forces. This is true for any system of particles, from a galaxy consiting of starts, to the solar system, to a rigid body consiting of a large number of individual atoms, down to a molecule or even an atom: The acceleration of the center of mass is given by the external forces acting on the system.

It is this law that allows us to use the techniques we have developed so far on any system, a solid body or a system of particles. In the previous chapters we have strictly speaking only discussed the motion of point-particles with a mass. We have always assumed that every part of a solid body has been moving with the same velocity. We have not allowed the object to oscillate, vibrate, change shape, or rotate. We have not allowed it to do any of the things that real objects do. However, we have now been saved by Newton's second law for particle systems: If we measure the position of an object as the center of mass of the object, we can still use Newton's second law to find its motion, even if the object is vibrating, oscillating, rotating, or displaying other types of internal motion.

If I throw a ball through the room, we have previously found that the motion of the ball can be found from Newton's second law for the ball:

$$\sum \mathbf{F} = \mathbf{G} = -mg\,\mathbf{j} = m\mathbf{a}. \tag{13.36}$$

The beauty of Newton's second law for particle systems is that we can use exactly the same analysis for a spinning or oscillating ball. The motion of the center of mass of the ball only depends on the external forces acting on the ball:

$$\sum \mathbf{F} = \mathbf{G} = -mg\,\mathbf{j} = m\mathbf{A}. \tag{13.37}$$

It does not matter what happens internally in the ball—if it is deformed, spinning, or vibrating—the motion of the center of the mass is the same as for a point particle as long as the external forces acting are the same.

However, you might argue that the external forces acting on a spinning ball are different because of air resistance: It is the interaction with the air that causes a spinning ball to move sideways, which is called curving the ball. This is a valid objection. The motion of the center of mass is determined by the external forces acting on the object. But, if we can neglect the effects of air resistance, the motion of the center of mass of a rod when you throw it is the same when it is rotating as when it is not rotating. This may be surprising, but it is a result of Newton's second law for particle systems.

### *13.3.1 Example: Ballistic Motion with an Explosion*

**Problem:** A projectile is fired from the ground. Its initial velocity in the horizontal direction is $v_0$. When it reaches its maximum height of $h$, a charge is set off, splitting the projectile into two equal parts. One part moves forward with the velocity $v_1$. Find the trajectory of each of the parts, and the trajectory of their center of mass. You may neglect air resistance.

**Solution:** We have illustrated the process in Fig. 13.8. The process has three stages. In stage one, the projectile moves to its maximum height only under the influence of gravity. In the second stage, the explosion takes place, after which the projectile is split into two projectiles. In the third stage, each part propagates to the ground only affected by gravity.

**Model:** First, let us address the explosion. During the explosion, each part of the projectile is subject to large forces. But the only external force acting on the parts is gravity. Consequently, there is no external horizontal force acting on the system. The horizontal momentum is therefore conserved at all times. In particular, the momentum is the same immediately before and immediately after the explosion. We use conservation of momentum to determine the horizontal velocities of the parts after the explosion.

Before the explosion, the horizontal momentum of the system is:

$$p_0 = mv_0, \tag{13.38}$$

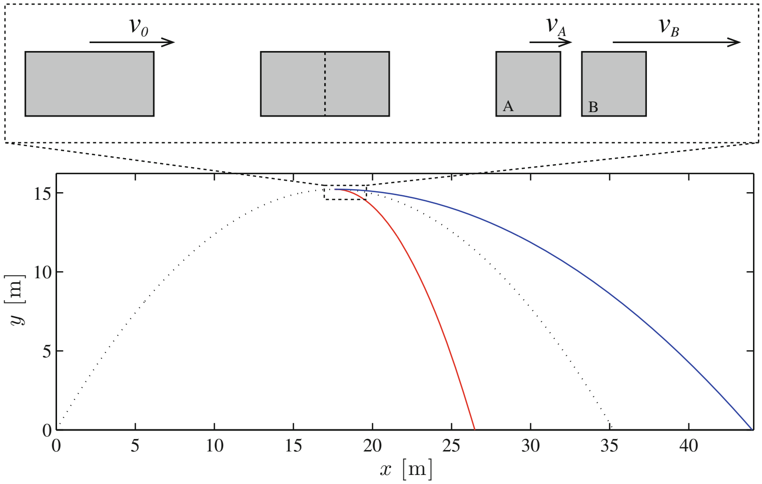


**Fig. 13.8** A projectile explodes at the *top* of its path, splitting into two equal pieces. We track the position of each part until they hit the ground

and after the collision, the horizontal momentum is:

$$p_1 = m_A v_A + m_B v_B = \frac{m}{2} v_A + \frac{m}{2} v_B . \tag{13.39}$$

Conservation of momentum gives:

$$v_0 = \frac{1}{2} v_A + \frac{1}{2} v_B , \tag{13.40}$$

where $v_B = v_1$ is the velocity of part B after the explosion, and $v_A$ is the velocity of part A, which we need to find:

$$v_A = 2v_0 - v_B = 2v_0 - v_1 . \tag{13.41}$$

We therefore know the initial conditions for the motion in the third stage. Each part is affected by gravity alone: $\mathbf{G}_A = -m_A g\,\mathbf{j}$, $\mathbf{G}_B = -m_B g\,\mathbf{j}$.

**Finding the motion of part B:** First, we find the motion of part B. Newton's second law in the $x$-direction gives:

$$\sum F_x = 0 = m_B a_B . \tag{13.42}$$

The velocity in the $x$-direction is constant. The position is therefore:

$$x_B(t) = x_B(t_0) + v_B t = v_0 t , \tag{13.43}$$

where we have placed the origin at the ground directly below the explosion. Therefore $x_B(t_0) = 0$.

The motion in the $y$-direction corresponds to the motion of a falling object, hence:

$$y_B(t) = h - \frac{1}{2}gt^2. \tag{13.44}$$

**Finding the motion of part A:** Similarly, we find the position of part A:

$$x_A(t) = v_A t = (2v_0 - v_1)\,t,\;\; y_A(t) = h - \frac{1}{2}gt^2. \tag{13.45}$$

We notice that the motion in the $y$-direction is the same for the two parts, which is as expected. The two parts will therefore strike the ground at the same time.

**Finding the motion of the center of mass:** We use these results to find the center of mass:

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i}. \tag{13.46}$$

We find the $x$- and $y$-components independently:

$$X = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{\frac{m}{2}x_A + \frac{m}{2}x_B}{\frac{m}{2} + \frac{m}{2}} = \frac{1}{2}\left(x_A + x_B\right) = \frac{1}{2}\left((2v_0 - v_1)\,t + v_1 t\right) = v_0 t. \tag{13.47}$$

This is what we get if we apply Newton's second law for a particle system. There are no external horizontal forces acting on the system, therefore the horizontal component of the center of mass moves with constant velocity:

$$\sum \mathbf{F}^{\text{ext}} = -mg\,\mathbf{j} = m\mathbf{A} \;\Rightarrow\; A_x = 0 \;\Rightarrow\; X = v_0 t. \tag{13.48}$$

We see that Newton's second law for particle systems gives the same result as when Newton's second law is applied to each object.

Similarly, we find the $y$-position of the center of mass:

$$Y = \frac{\sum_i m_i y_i}{\sum_i m_i} = \frac{\frac{m}{2}y_A + \frac{m}{2}y_B}{\frac{m}{2} + \frac{m}{2}}\frac{1}{2}\left(y_A + y_B\right) \tag{13.49}$$

$$= \frac{1}{2}\left(h - \frac{1}{2}gt^2 + h - \frac{1}{2}gt^2\right) = h - \frac{1}{2}gt^2, \tag{13.50}$$

We could also have found this directly from Newton's second law for particle systems:

$$\sum \mathbf{F}^{\text{ext}} = m\mathbf{A} = -mg\,\mathbf{j} \;\Rightarrow\; A_y = -g \;\Rightarrow\; Y(t) = Y(t_0) - \frac{1}{2}gt^2 = h - \frac{1}{2}gt^2. \tag{13.51}$$

**Analyze:** We have demonstrated that we can find the motion of the center of mass either by calculating the motion of each of the parts of the system, or we can find the motion of the center of mass by applying Newton's second law for particle systems directly. The results are of course the same. However, there are many questions we only can answer if we know the motion of each part. For example, in order to determine how far apart the two parts are when they hit the ground, we need to find the motion of each of the parts.

## 13.4 Motion in the Center of Mass System

Newton's law for multiparticle systems gives us the tool to determine the motion of the center of mass of a complex object based on the external forces acting on the system. The law is surprisingly robust. If you throw a rod through the air, and you neglect the effects of air resistance, the motion of the center of mass of the rod does not depend on how the rod is moving relative to its center of mass: The motion of the center of the mass is the same if the rod is moving as a ridig body without rotating, as in Fig. 13.9a; if the rod is rotating around its center of mass, as in Fig. 13.9b; or if the rod is rotating and wobbling. The motion of the center of mass of the rod does not depend on internal forces in the rod. Therefore, the motions of individual parts relative to the center of mass do not affect the motion of the center of mass.

However, in many cases we are interested in both the motion of the center of mass and of the motion of the individual parts relative to the center of mass. For example, we may be interested in the rotation of the rod, or its wobbling, or in how it is vibrating. Then it is useful to split the motion of the system into the motion of the center of mass and the motion of a particle relative to the center of mass.

### Laboratory and Center of Mass Systems

We have already discussed how we always measure the motion of a system relative to some reference system. For example, we may characterize the motion of the rod relative to a point on the ground. We call this system the *laboratory system*. In addition, we introduce a coordinate system located in the center of mass of the system of particles, as illustrated in Fig. 13.10. This system is called the *center of mass system*.

The position of a particle $i$ is $\mathbf{r}_i$ in the laboratory system, and the position of the center of mass of the system is $\mathbf{R}$ in the laboratory system. The position of particle $i$ in the center of mass system is $\mathbf{r}_{\mathrm{cm},i}$:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{\mathrm{cm},i} , \tag{13.52}$$

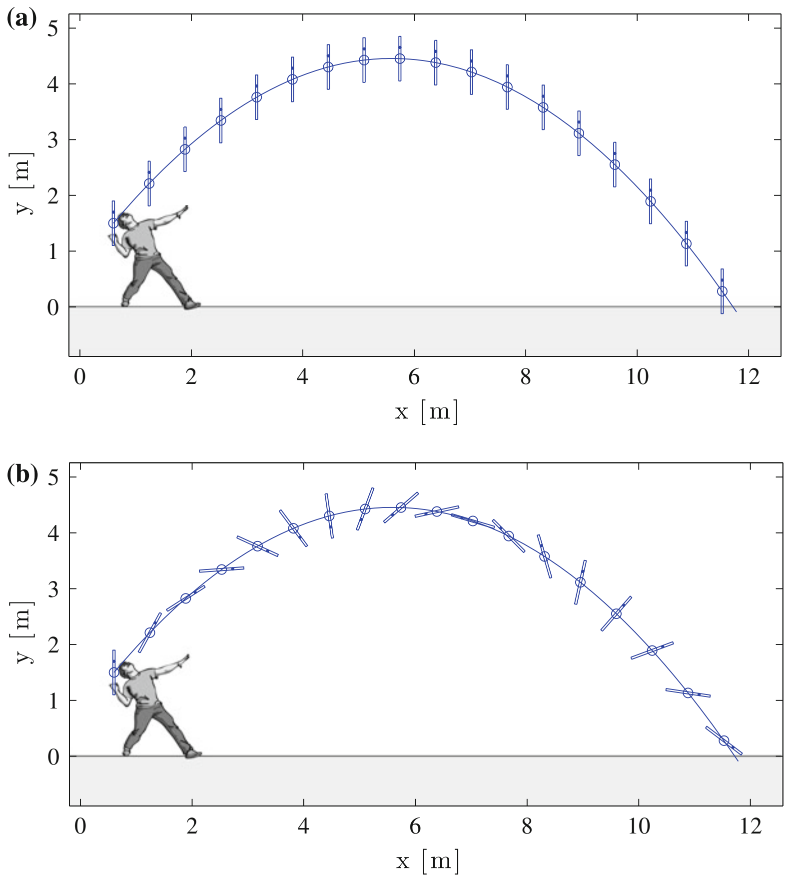


**Fig. 13.9** The motion of a rod thrown through the air **a** without rotation, and **b** rotating around its center of mass

## Center of Mass in the Center of Mass System

What is the center of mass of the particles measured in the center of mass system? The center of mass is:

$$\mathbf{R}_{\text{cm}} = \frac{1}{M}\sum_i m_i \mathbf{r}_{\text{cm},i} = \frac{1}{M}\left(\sum_i \mathbf{r}_i - \sum_i \mathbf{R}\right) = \frac{1}{M}\sum_i \mathbf{r}_i - \mathbf{R} = \mathbf{R} - \mathbf{R} = 0. \tag{13.53}$$

Not surprisingly, the center of mass measured in the center of mass system, is in the origin of the center of mass system. This was indeed the whole point of the center of mass system.

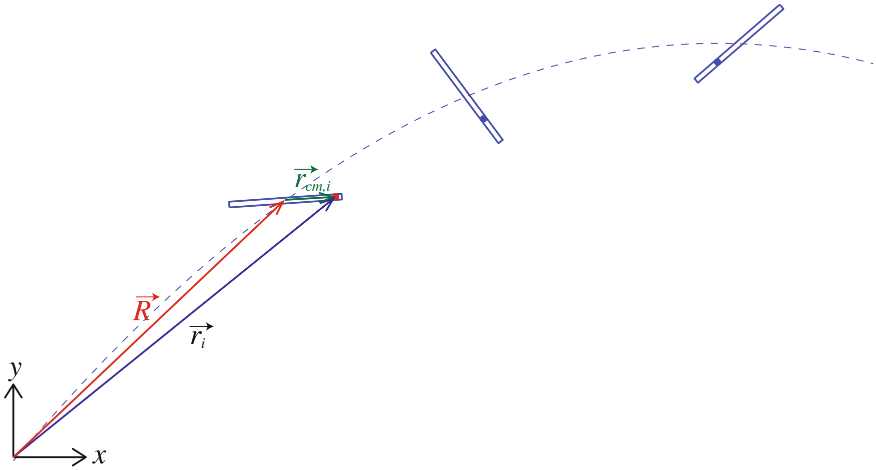


**Fig. 13.10** The position of a point on the rod $\mathbf{r}_i$ can be written as a sum of the position of the center of mass, $\mathbf{R}$, and the position of the point relative to the center of mass, $\mathbf{r}_{\text{cm},i}$

### Total Momentum in the Center of Mass System

What is the total momentum, $\mathbf{P}_{\text{cm}}$, of the system in the center of mass system? The total momentum is defined as:

$$\mathbf{P}_{\text{cm}} = \sum_i m_i \mathbf{v}_{\text{cm},i} = \sum_i m_i \frac{d}{dt}\mathbf{r}_{\text{cm},i} = \frac{d}{dt} \underbrace{\sum_i m_i \mathbf{r}_{\text{cm},i}}_{=M\mathbf{R}_{\text{cm}}=0} = 0. \tag{13.54}$$

The total momentum of the system in the center of mass system is always zero! This result does not depend on the sum of external forces being zero. This is always true. Independently of what is done to the system, it only depends on the definition of the center of mass.

## 13.5 Energy Partitioning

We can describe the motion of (the center of mass of) multiparticle systems using the concepts we developed in our studies of Newton's laws of point particles. What about the concepts of mechanical energy we used to address the behavior of single particles, can we still use energy concepts for multi-particle systems?

## *Kinetic Energy of a Multi-particle System*

What is the kinetic energy of a system of particles? The total kinetic energy is the sum of the kinetic energy of every particle in the system:

$$K = \sum_{i=1}^{N} \frac{1}{2} m \left(\mathbf{v}_i\right)^2 = \sum_{i=1}^{N} \frac{1}{2} m \left(\frac{d\mathbf{r}_i}{dt}\right)^2 . \tag{13.55}$$

Now, we want to divide the motion into the motion of the center of mass (the motion of the whole system), and the motion relative to the center of mass:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{\mathrm{cm},i} , \tag{13.56}$$

and similarly for the velocities:

$$\mathbf{v}_i = \mathbf{V} + \mathbf{v}_{\mathrm{cm},i} . \tag{13.57}$$

We use this to rewrite the total kinetic energy of the system, getting

$$K = \frac{1}{2} M \left(\mathbf{V}\right)^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \left(\mathbf{v}_{\mathrm{cm},i}\right)^2 . \tag{13.58}$$

(You find a proof in Sect. A.3). This shows that the total kinetic energy can be divided into two terms: The kinetic energy for the motion of the center of mass, the **external kinetic energy**.

$$\boxed{K_{\mathrm{cm}} = \frac{1}{2} M \mathbf{V}^2 ,} \tag{13.59}$$

and the kinetic energy due to the motion of the particles relative to the center of mass, which we call the **internal kinetic energy**:

$$\boxed{K_{\Delta\mathrm{cm}} = \frac{1}{2} \sum_{i=1}^{N} m_i \left(\mathbf{v}_{\mathrm{cm},i}\right)^2 .} \tag{13.60}$$

The total kinetic energy is therefore partitioned into a sum of the *internal* ($K_{\Delta\text{cm}}$) and *external* ($K_{\text{cm}}$) kinetic energies:

$$K = \underbrace{\frac{1}{2}M\mathbf{V}^2}_{K_{\text{cm}}} + \underbrace{\frac{1}{2}\sum_{i=1}^{N} m_i \mathbf{v}_{\text{cm},i}^2}_{K_{\Delta\text{cm}}} . \tag{13.61}$$

If the whole object is translated, as illustrated in Fig. 13.9a, there is no motion relative to the center of mass. Then the total kinetic energy is simply the kinetic energy of the center of mass. What we have done previously is therefore correct as long as we assume that the object is translated!

But what if parts of the system are moving relative to the center of mass? Then we must also include the internal kinetic energy: The energy related to the motion relative to the center of mass. For example, if a diatomic molecule is vibrating, we must also include the kinetic energy of the vibrating motion of the atoms. This means that for a system of particles we have more degrees of freedom—there are many possible ways that kinetic energy can be realized inside the system. Hence, the kinetic energy of a system may be conserved, even though the kinetic energy of the center of mass is not conserved.

This means that we need a simplified way to describe the internal motion, the motion relative to the center of mass, of a particle system. For example, as we will see later, for rigid bodies we do not allow vibrations or other deformations of the object. The only motion possible relative to the center of mass is a rotation the whole body. In this case, we need to introduce a kinetic energy term related to the rotation of the body, and this is indeed one of the main focus areas when we discuss the dynamics of rigid bodies.

### *Potential Energy of a Multi-particle System*

For a multi-particle system, the kinetic energy is partitioned into the external and internal kinetic energy. What about the potential energy of a multi-particle system?

First, we need to be more precise. The potential energy related to what force? The rod in Fig. 13.9 has a potential energy due to the gravitational force, which is the sum of the potential energy of every particle in the rod. But in addition, parts of the rod may be compressed or stretched, and as a result the rod has an internal potential energy just as a in a diatomic molecule. We therefore need to discern between **external potential energy**, the potential energy of an external force, and **internal potential energy**, the potential energy due to interactions within the system.

### Potential Energy Due to External Forces

If a conservative external force acts on all the particles, we can describe the interaction through a potential energy, $U_i(\mathbf{r}_i)$, of particle $i$ at position $\mathbf{r}_i$ due to the external force. The total potential energy of the system due to the external force is then:

$$U_{\text{TOT}} = \sum_{i=1}^{N} U_i(\mathbf{r}_i). \tag{13.62}$$

If the force is constant, such as for gravity near the Earth's surface, this expression can be further simplified. In this case, the potential energy of particle $i$ is:

$$U_i(\mathbf{r}_i) = m_i g y_i, \tag{13.63}$$

and the total potential energy is:

$$U_{\text{TOT}} = \sum_{i=1}^{N} m_i g y_i = g \sum_{i=1}^{N} m_i y_i = MgY, \tag{13.64}$$

where $Y$ is the $Y$-coordinate of the center of mass of the system. We can therefore use the well-known formula, $U = mgy$, also for the potential energy of a multiparticle system, we only need to use the center of mass position for $y$. Again, we find that we can use the concepts developed for point particles also for particle systems.

Notice that this conclusion is only true for a constant force. For a force that depends on the position of each small part of the multi-particle system, the total potential energy cannot always be expressed as a function of the position of the center of mass alone. We may have to calculate the full sum (or integral) in (13.62). But if the external force is approximately constant over the multi-particle system we may approximate the potential energy by a function that depends only on the center of mass position.

In addition, a multi-particle system may have internal degrees of freedom and a corresponding internal potential energy.

### Potential Energy Due to Internal Forces

The net force acting on a particle $i$ in a multi-particle system includes both external and internal forces:

$$\mathbf{F}_i^{\text{net}} = \mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{j,i}, \tag{13.65}$$

where $\mathbf{F}_i^{\text{ext}}$ is the net external force acting on particle $i$, external here meaning that it has its origin outside the system. The force $\mathbf{F}_{j,i}$ from particle $j$ on particle $i$ is an internal force.

If all forces, external and internal, acting on the system are conservative, we can introduce a potential energy for every force. The total potential energy of the system is the sum of all the potential energies for each of the forces. We divide the total potential energy into the external potential energy, the potential energy due to external forces, and the internal potential energy, the potential energy due to internal forces:

$$U_{\text{TOT}} = U_{\text{ext}} + U_{\text{int}}. \tag{13.66}$$

The external potential energy was found in (13.62):

$$U_{\text{ext}} = \sum_i U_i \left(\mathbf{r}_i\right). \tag{13.67}$$

The internal potential energy is a sum over all potential energies for all the interactions in the system:

$$U_{\text{int}} = \sum_{i<j} U_{i,j} \left(\mathbf{r}_i, \mathbf{r}_j\right). \tag{13.68}$$

Notice that we include a given interaction, a pair of particles $i$, $j$, only once in this sum! Here, we will not develop a general theory for this, but instead illustrate the principle by an example: The conservation of energy in a bouncing dumbbell as discussed in Sect. 13.5.

## *Conservation of Energy in a Multi-particle System*

If all the forces, both internal and external, acting on a particle system are conservative, the total mechanical energy of the system is conserved:

$$E_{\text{TOT}} = K_{\text{TOT}} + U_{\text{TOT}}, \tag{13.69}$$

where

$$K_{\text{TOT}} = K_{\text{cm}} + K_{\Delta\text{cm}}. \tag{13.70}$$

and

$$U_{\text{TOT}} = U_{\text{ext}} + U_{\text{int}}, \tag{13.71}$$

which gives:

$$E_{\text{TOT}} = K_{\text{cm}} + U_{\text{ext}} + K_{\Delta\text{cm}} + U_{\text{int}}. \tag{13.72}$$

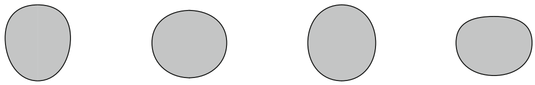

**Fig. 13.11** A football in flight filmed using a high-speed camera: The football rotates and wobbles as it moves

Based on this result we realize that in order to apply the principle of energy conservation when solving problems in mechanics, we need to have expressions for the two terms $K_{\Delta cm}$ and $U_{int}$ for the system. Unfortunately, these terms are not always simple. For example, if you kick a football, the football will both rotate and wobble during its flight, as illustrated in Fig. 13.11. The kinetic and potential energy associated with the wobbling represent internal degrees of freedom, and we do not know how to quantify these without a detailed model for the deformation of the football. For a detailed study of the motion of the football, energy concepts are therefore of limited value as means for calculation. However, the energy concepts are still useful theoretical techniques that provides us with concepts and methods to discuss the motion. In many cases, energy consideration are also the theoretical starting point, for example, for determining the deformation of the football.

For a particular type of object, what we call a rigid body, we assume that the internal deformation and the energies associated with these are negligible, and that the object moves a rigid body. In this case we neglect the internal potential energy of the system, but we still need to develop expressions for the kinetic energy of a rotating rigid body. We return to this in Chap. 15 after a discussion of rotation in Chap. 14.

## *13.5.1 Example: Bouncing Dumbbell*

*In this example we will demonstrate the main principles of energy partitioning through a simple model system.*

In this example we will address a two-particle system, where the two particles interact through a spring (see Fig. 13.12). This can be a model of a diatomic molecule consisting of two identical atoms, or for an elastic body that deforms and vibrates. Here, we simplify the problem by assuming that the that the particles move along a line, so that the motion is one-dimensional. In the next example, we extend our discussion to two-dimensional motion, opening for rotation in addition to vibrations.

**Identify and Sketch:** We address a system of two particles, each with mass $m$, connected with a spring of equilibrium length $b$ and spring constant $k$. Particle A starts on top and particle B starts on bottom, their positions are $x_A(t)$ and $x_B(t)$ respectively. We place particle B at a height $h_0$ above the floor: $x_B(t_0) = h_0$, and particle A is a distance $b$ above particle B:$x_A(t_0) = x_B(t_0) + b$. When a particle

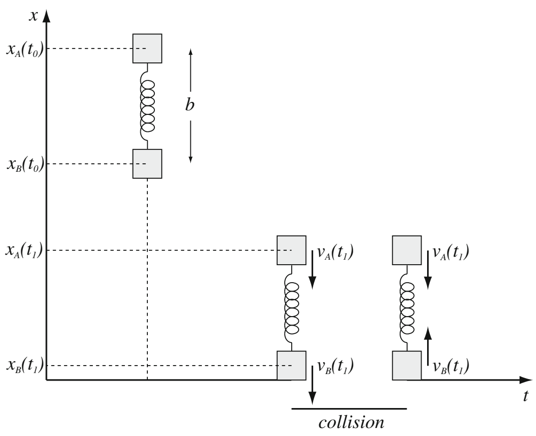


**Fig. 13.12** Illustration of the dumbbell. Two identical particles of mass $m$ are connected by a spring with equilibrium length $b$ and spring constant $k$. The system is lifted to a height $h$ above a flat floor and released. When the *bottom* particle hits the floor its velocity is reversed, as we know happens in an elastic collision with a wall

hits the floor the collision is elastic, which means that the velocity of the particle is reversed after the collision.

**Model:** We find the motion of each particle from the forces acting on it. Particle A is affected by two forces: Gravity $\mathbf{G}_A = -mg\,\mathbf{j}$ and the spring force $\mathbf{F}_A$. The spring force depends on the position of both particle A and B:

$$\mathbf{F}_A = -k\,(x_A - x_B - b)\ \mathbf{j}, \tag{13.73}$$

Similarly, the forces acting on particle B are $\mathbf{G}_B = -mg\,\mathbf{j}$ and $\mathbf{F}_B$:

$$\mathbf{F}_B = -\mathbf{F}_A = k\,(x_A - x_B - b)\ \mathbf{j}. \tag{13.74}$$

Is it sufficient to study the motion of the center of mass alone? No, since we do not have models for the external forces acting on the system, and we do not know when or where they are acting without finding the motion of the individual particles. The center of mass of the system is at:

$$X = \frac{1}{M}\sum_i x_i = \frac{1}{2m}\,(m x_A + m x_B) = \frac{1}{2}\,(x_A + x_B)\,. \tag{13.75}$$

Since the system collides with the wall when one of the particles hits the wall, and not when the center of mass hits the wall (which it never will), we must find the motion of each particle individually.

**Newton's second law:** We apply Newton's second law to each particle to find its acceleration:

$$\mathbf{a}_A = -\frac{k}{m_A}\,(x_A - x_B - b)\ \mathbf{j}\,, \tag{13.76}$$

$$\mathbf{a}_B = \frac{k}{m_B}(x_A - x_B - b)\,\mathbf{j}. \tag{13.77}$$

where we also have to include a possible collision between particle B and the floor.

**Numerical solution:** We determine the motion of the particles numerically using a Euler-Cromer method. We need to include all forces acting on each particle, and we execute a collision step whenever particle B collides with the wall. When does a collision occur? Since we are determining the positions only at discrete time intervals $\Delta t$, we do not expect the position of particle B ever to be exactly at the wall. Instead, the particle will be outside the wall at one time, $t$, and then inside the wall at a later time, $t + \Delta t$, and the only thing we know is that a collision occurred at some time during the time interval from $t$ to $t + \Delta t$. We could try to improve our estimate of when the collision occurs, but here we will simply assume that the result of a collision is to reverse the velocity at the first time step the particle is "inside" the wall. The most important part of our collision algorithm is that is conserves energy, and this is indeed achieved by this method. Because the particle is not moved during the collision, the potential energy is conserved. And since only the direction of the velocity is changed, the kinetic energy is conserved.

```
m = 0.1;
k = 200.0;
b = 0.2;
h0 = 1.0;
g = 9.8;
time = 2.0;
dt = 0.00001;
n = round(time/dt);
t = zeros(n,1);
xA = zeros(n,1); vA = zeros(n,1);
xB = zeros(n,1); vB = zeros(n,1);
xA(1) = h0 + b;  vA(1) = 0.0;
xB(1) = h0;      vB(1) = 0.0;
for i = 1:n-1
    f = k*(xA(i) - xB(i) - b);
    fA = -f - m*g;
    fB = f - m*g;
    aA = fA/m;
    vA(i+1) = vA(i) + aA*dt;
    xA(i+1) = xA(i) + vA(i+1)*dt;
    aB = fB/m;
    vB(i+1) = vB(i) + aB*dt;
    xB(i+1) = xB(i) + vB(i+1)*dt;
    t(i+1) = t(i) + dt;
    if (xB(i+1)<0.0)&&(xB(i)>=0.0)
        vB(i+1) = abs(vB(i+1));
    end
end
xcm = (xA+xB)*0.5; vcm = (vA+vB)*0.5;
Kcm = 0.5*(2.0*m)*vcm.^2;
Kcmdelta = 0.5*m*(vA - vcm).^2+0.5*m*(vB-vcm).^2;
Ug = xcm*(2.0*m)*g;
Uk = 0.5*k*(xA - xB - b).^2;
E = Kcm + Kcmdelta + Ug + Uk;
subplot(2,1,1)
plot(t,xA,'-b',t,xB,'-r',t,xcm,':k')
xlabel('t[s]'); ylabel('x [m]');
subplot(2,1,2);
```

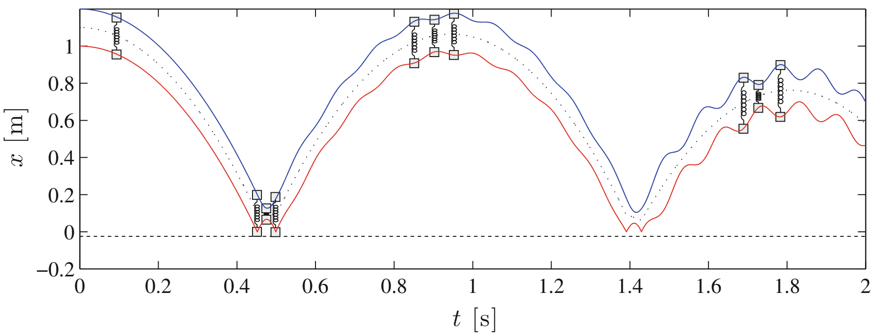


**Fig. 13.13** Illustration of the motion of the dumbbell. Here we have illustrated the path of the center of mass (*dashed line*) and the path of each of the particles. We have drawn in a system of two blocks and a spring to help your intuition

```
plot(t,Kcm,'-b',t,Kcmdelta,'-r',t,Uk,'-y',t,E,':k')
xlabel('t [s]'); ylabel('E [J]');
```

**Visualization of motion:** A visualization of the motion of the system for the first two collisions is shown in Fig. 13.13. Here, we have illustrated the positions of the two particles at a few selected times. Before the first collision, the two particles move without relative motion. After the second collision, the relative motion is significant. (Notice that we have drawn the particles as small blocks, but they should be interpreted as point-particles.)

**Motion of the center of mass:** The dashed line in Fig. 13.13 shows the motion of the center of mass. We know that the motion of the center of mass only depends on the external forces acting on the system. During the collision with the floor, the system is affected by both gravity and the contact force, but between collisions, the system is only affected by gravity. Consequently, we expect the center of mass to behave as a single object of mass $2m$ falling under the effect of gravity. That is, we expect:

$$A = -g \;\Rightarrow\; X(t) = X_0 + V_0 t - \frac{1}{2} g t^2, \tag{13.78}$$

which is seen as the parabolically shaped motion of the center of mass in Fig. 13.13. Even though the particles are oscillating back and forth, the center of mass does not display any oscillations, but shows a smooth, parabolic shape (as a function of time).

**Energy partitioning:** We notice that after the second bounce, the center of mass does not bounce up to its initial level even though the total energy in the system is conserved. If our system was a rigid ball, and the collision with the floor was elastic (and we neglected air resistance), we would expect the ball to bounce up to its initial level. Why do we expect this? Because at the top of the path, when $X$ is maximum, the velocity is zero. For a rigid ball, this would also imply that the kinetic energy is zero, and that the total mechanical energy is equal to the potential energy, which only depends on the height. For the rigid ball, the height of each bounce must be the same. But for the dumbbell system, there are also internal degrees of freedom. The kinetic energy is not zero at the top of the path, even though the velocity of the center of mass is zero, because the particles are still moving relative to the center of mass. Similarly, the total potential energy is not only equal to the potential energy in the external gravitational field, but also depends on the relative positions of the two particles.

We can use the concept of energy partitioning to analyze the behavior. The total kinetic energy consists of:

$$K = K_{\rm cm} + K_{\Delta \rm cm}, \tag{13.79}$$

where the kinetic energy of the center of mass is:

$$K_{\rm cm} = \frac{1}{2} M V^2, \tag{13.80}$$

and the kinetic energy due to the motion relative to the center of mass is:

$$K_{\Delta \rm cm} = \sum_i \frac{1}{2} m_i v_{{\rm cm},i}^2 = \frac{1}{2} m \left(v_A - V\right)^2 + \frac{1}{2} m \left(v_B - V\right)^2 . \tag{13.81}$$

The kinetic energies are plotted in Fig. 13.14. Notice that before the bottom particle hits the floor, there is no relative motion, and the internal kinetic energy ($K_{\Delta \rm cm}$) is zero. Each of the particles moves with the same velocity and the center of mass. Immediately after the collision, the internal kinetic energy increases discontinuously—it jumps to a high level. What happened? After the collision, the velocity of the bottom particle is reversed. This has two effects. First, the center of mass now has zero velocity. But the magnitudes of the velocities of each of the particles are unchanged. The result is that each of the particles suddenly has a velocity relative to the center of mass equal to the velocity they have before the collision. As a result, there is a jump in the internal kinetic energy.

Another interesting observation is the oscillation in the internal kinetic energy due to the oscillation of the particles around the center of mass.

The total potential energy of the system is the sum of all the potential energies:

$$U_{\rm TOT} = \sum_i U_i^{\rm ext} + \sum_{i<j} U_{i,j}^{\rm int} = \underbrace{\sum m_i g x_i}_{=U_{\rm ext}} + \underbrace{\frac{1}{2} k \left(x_A - x_B - b\right)^2}_{=U_{\rm int}}, \tag{13.82}$$

The two potential energies are plotted in Fig. 13.15. The potential energy of the center of mass only follows the center of mass movement, and the internal potential energy is related to the energy stored in the spring as the the spring is compressed. A fine analysis of Figs. 13.14 and 13.15 shows that the internal kinetic and potential energies

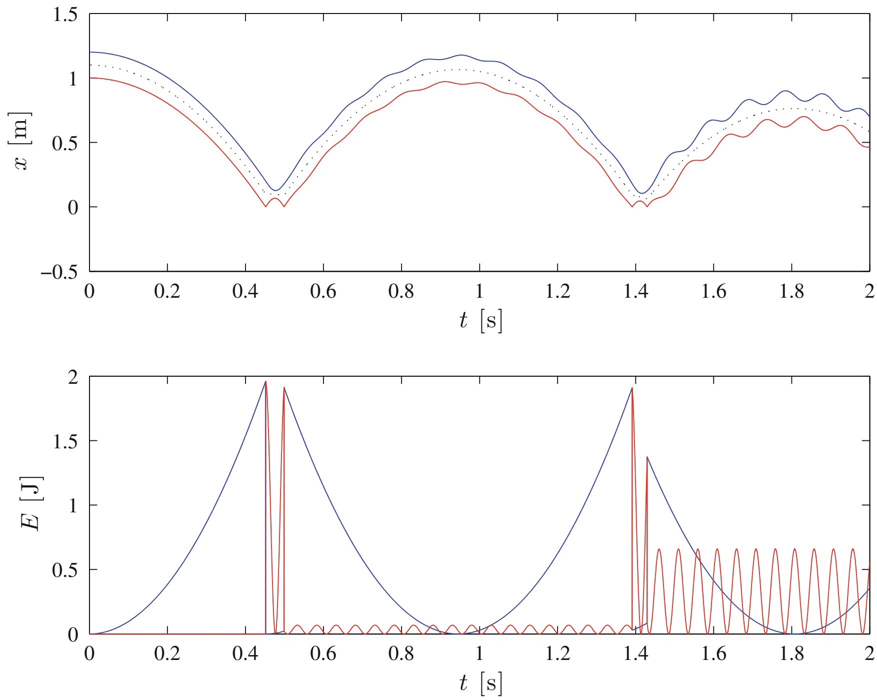


**Fig. 13.14** Illustration of the motion of the dumbbell and the kinetic energy of the system, partitioned into the kinetic energy of the center of mass and the internal kinetic energy

are oscillating with opposite phases—as we expect—so that when the internal kinetic energy is maximum, the potential energy is at its minimum and vice versa.

**Discussion:** From this example we learn that mechanical energy is conserved only if we include all the forms it may take. The mechanical energy corresponding to the kinetic and potential energy of the center of mass is generally not conserved. In addition we must take into account the many possible internal modes of motion and associated potential energies. However, if we simplify our system so that vibrational modes are not present, such as for a rigid body that does not deform, is it then sufficient only to use the kinetic and potential energy of the center of mass? No! We have looked at a too simple system! Real, two- and three-dimensional systems can be rotated without deforming. We must therefore also include the effect of rotations. This is the subject of the next chapters.

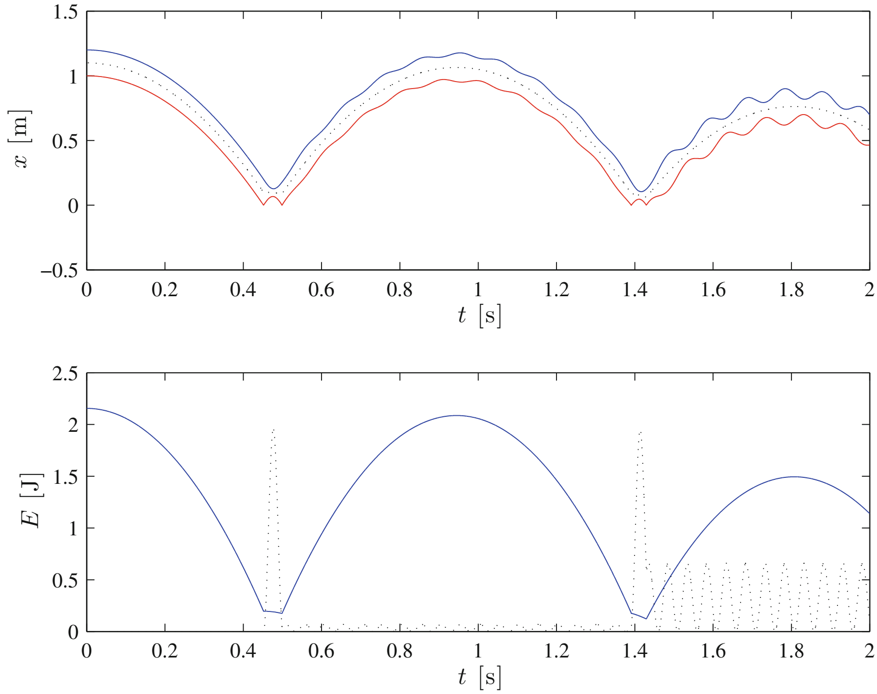


**Fig. 13.15** Illustration of the motion of the dumbbell and the potential energy of the system, partitioned into the potential energy of the center of mass and the internal potential energy

## 13.6 Energy Principle for Multi-particle Systems

There are several ways to view energy conservation. So far, we have mainly used energy conservation as a tool to determine the velocity of an object as a function of position. In this case, we include all external and internal conservative forces in the total energy of the object:

$$E_{\mathrm{TOT}} = K_{cm} + K_{\Delta cm} + U_{\mathrm{ext}} + U_{\mathrm{int}}, \tag{13.83}$$

As a method for calculation this worked very well as long as we could ignore the internal potentials and motion relative to the center of mass. Fortunately, this represents a whole class of problems, the motion of ridig bodies, where there are no internal vibrations and no internal potential energies, although the object may rotate, as we will see further on. This approach is therefore a useful approach from a practical point of view.

However, there is a different view on energy conservation which is very useful from a theoretical point of view, while not that useful for direct calculations. In this view, we consider all conservative interactions to be internal interactions. This is done by including all the interacting objects in the system. For example, if we study the motion of a ball falling towards the Earth under gravity, we include both the ball and the Earth in the system. The potential energy of the ball relative to the Earth is then an internal potential energy in a multi-particle system. Similarly, we would treat the solar system as a system consisting of all the objects in the solar system. We could also take a similar approach on an atomic scale. Indeed, we could consider any object to consist of a large set of atoms, and since all the atoms interact through position-dependent forces, usually central forces, all the internal forces are conservative and the total energy of the system should be conserved. Unless there are other, external forces acting on the system.

**Internal Energy:**

If all the internal forces are conservative, the only way we can change the total energy of the system is by the work done by an external force. For a system of a book lying on the floor, a system of the book and the Earth, we would increase the total energy of the system if we lifted the book by an external force. External work leads to a change in the total energy. We often call the total energy of such a system the *internal energy* of the system. We write this as:

$$W_{\text{ext}} = \Delta E \ . \tag{13.84}$$

The work done by external force is the change in internal energy, $\Delta E$. This formulation is call the first law of thermodynamics, and is considered a fundamental law in physics.

How can we then interpret the work done by an internal force, such as the work done by gravity in the system of the book and the Earth? In this case, the work done by gravity is given as a change in the internal potential energy of the book-Earth system. For example, if we release the book from a height $h$ above the ground and the let it fall, the book has gained a kinetic energy corresponding to the change in potential energy when book reaches the ground. This is not a change in the total energy. There is no change in internal energy. But it is an energy transfer between different forms of internal energy: It is an energy transfer from potential to kinetic energy inside the system. Since we have assumed that all internal forces are conservative, all internal processes can be considered as transfers of energy.

**The Arrow of Time**

Hmmm. What about internal forces that are not conservative? How can we treat such processes in this view? If we take the atomic view: All our systems consists of atoms and interatomic interactions are conservative, any system should therefore only have conservative forces. Where does the non-conservative interactions come from? For example, if I take the book and slide it along an inclined surface from its initial height $h$, there will also be a frictional force, and friction is not a conservative force. Is the total energy still conserved in the book-Earth system in this case? Yes! What happens is that as the book slides, the potential energy of the book-Earth system is transfered to kinetic energy of the book, as well as many internal kinetic and potential energies: The atoms in the book and the floor starts vibrating. This corresponds to an increase in the temperature of both book and Earth, which again corresponds to an increase in the internal energy. Thus, the total energy is conserved, but it is now hidden in different internal energies inside the system. So why do we then call the friction force conservative? Because it is conservative on a microscopic level: The total energy is conserved, but it is not conservative on a macroscopic level: The friction force depends on the relative velocities of the moving objects.

This transfer of energy from macroscopic kinetic energy, the kinetic energy of a sliding book, to microscopic kinetic and potential energies in the various vibrations of the atoms, effectively introduces the arrow of time. If we release the book with an initial velocity downward, it will slide down the slope and eventually come to rest. The total internal energy of the system is conserved (if we ensure that it does not interact with any external forces). It is then fully possible that the book instantaneously starts to slide upwards, back to its original position. If by accident the atomic vibrations every time pushed the book in the right direction, it could happen. None of the individual processes would be against the laws of physics. Such a processes would not change the total energy of the system. But it would be extremely unlikely. And this is the reason why it is not happening. And it is the reason why we laugh when we play movies backwards. You will learn more about this later, when you learn about statistical physics.

## Summary

**Center of mass:** The center of mass $\mathbf{R}$ of a particle system consisting of $N$ particles with masses $m_i$ located at positions $\mathbf{r}_i$ is defined as:

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{r}_i \,, \;\; M = \sum_{i=1}^{N} m_i \,.$$

**Center of mass is a vector:** The center of mass is defined in a *vector equation* which is valid for each of the coordinates:

$$X = \frac{1}{M}\sum_{i=1}^{N} m_i x_i \,, \;\; Y = \frac{1}{M}\sum_{i=1}^{N} m_i y_i \,, \;\; Z = \frac{1}{M}\sum_{i=1}^{N} m_i z_i \,.$$

**Velocity of center of mass:** The *velocity of the center of mass* is:

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{v}_i \,.$$

**Acceleration of center of mass:** The *acceleration of the center of mass* is:

$$\mathbf{A} = \frac{d^2\mathbf{R}}{dt^2} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{a}_i .$$

**Center of mass for a solid body:** The center of mass of a solid body is:

$$\mathbf{R} = \frac{1}{M}\int_m \mathbf{r}\, dm = \frac{1}{M}\int\int\int \mathbf{r}\,\rho\, dV .$$

**Newton's second law for a particle system:** Newton's second law for a particle system relates the *external forces* to the acceleration of the *center of mass* of the system:

$$\sum \mathbf{F}^{\text{ext}} = M\mathbf{A}.$$

**Motion in center of mass system:** We relate the position $\mathbf{r}_i$ of a particle in the *laboratory system* to a position $\mathbf{r}_{\text{cm},i}$ in the *center of mass system* by:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{\text{cm},i} .$$

In the center of mass system, the positions are measured relative to the center of mass.

**Partitioning of kinetic energy:** The *kinetic energy*, $K$, of a system consist of two terms: The kinetic energy of the center of mass, $K_{\text{cm}}$, and the kinetic energy of the motion relative to the center of mass, $K_{\Delta\text{cm}}$:

$$K = K_{\text{cm}} + K_{\Delta\text{cm}} = frac12MV^2 + \sum_{i=1}^{N} \frac{1}{2} m_i v_{\text{cm},i}^2$$

**Partitioning of potential energy:** Similarly, the potential energy is partitioned into potential energy due to external forces, $U_{\text{ext}}$ and potential energy due to internal forces, $U_{\text{int}}$:

$$U_{\text{TOT}} = U_{\text{ext}} + U_{\text{int}} ,$$

**Potential energy of a particle system in constant gravity:** For a particle system in a homogeneous gravitational field, the potential energy is the same as the potential energy of a point particle with the total mass of the system located in the center of mass of the system:

$$U = MgY ,$$

where $Y$ is the vertical position of the center of mass.

## Exercises

### *Discussion Questions*

**13.1 Balance center.** If you want to balance a thin rod on a needle, why should you place the needle at the center of mass?

**13.2 Jumping people.** If all the people on Earth come together in one place and jump, what happens with the center of mass of the Earth-people system?

**13.3 Rectangle.** You have to place a rectangle of area A fully inside the first quadrant. How would you place it in order to make the distance from the origin to the center of mass the smallest?

**13.4 Thor's hammer.** You throw a hammer across the lecture hall (don't try this, please). Discuss its trajectory if you (i) threw it without any rotation or (ii) threw it so that it rotated during its flight.

### *Problems*

**13.5 Two-particle system.** A 2 kg particle is placed at $x = 2$ m and a 4 kg particle is placed at $x = 6$ m.
**(a)** Where is the center of mass of this two-particle system?

**13.6 Center of mass of Earth-Moon system.**
**(a)** Estimate the position of the center of mass of the Earth-Moon system. Give your answer in units of the Earth's diameter.

**13.7 Carbon-monoxide.** For a Carbon-monoxide molecule, the mass of the Carbon atom is 12.0107 g/mol, and the mass of the Oxygen atom is 15.9994 g/mol, and the typical distance from the Carbon to the Oxygen is 112.8 pm.
**(a)** Find the center of mass of a Carbon-monoxide molecule.

**13.8 Three-particle system.** Three particles of equal mass are placed as shown in Fig. 13.16.
**(a)** What is the center of mass of this system?
**(b)** How can you add another particle to the system without changing the center of mass?

**13.9 Tetrahedron.** A tetrahedron consists of four points (vertexes) connected by six lines of equal length, with three lines originating from each vertex, as seen in the figure. A particle of mass $m$ is placed on each vertex. The coordinates of the corners are (1,1,1), $(-1, -1, 1)$, $(-1, 1, -1)$, $(1, -1, -1)$.
**(a)** What is the center of mass of this system?
**(b)** How does the center of mass change if we double the mass of the first particle?

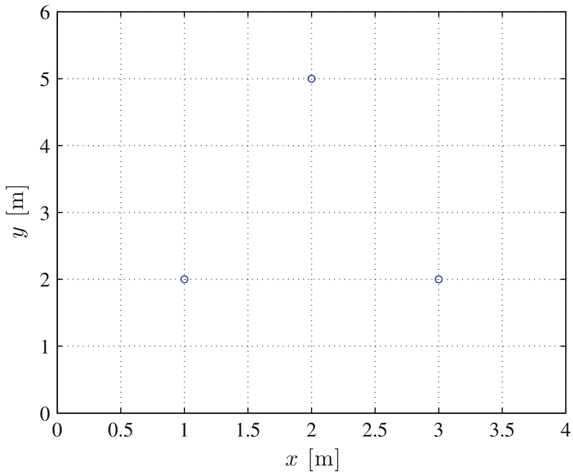


**Fig. 13.16** A system of particles

**13.10 Cubic hole.** You make a cubic hole with sides of length $d$ in the center of one of the sides of a homogeneous cube with sides of length $L$.
**(a)** Find the center of mass of the cube with the hole.

**13.11 Triangle.**
**(a)** Find the center of mass of a homogeneous isosceles triangle of base $b$ and height $a$. (An isosceles triangle have two sides of equal length.)

**13.12 Triangle.**
**(a)** Find the center of mass of a homogeneous equilateral triangle of base $b$ in Fig. 13.17. (An equilateral triangle has three sides of equal length.)

**13.13 A piece of pie.** You cut out a piece with an angle $\theta$ from a flat, homogeneous pie of radius $R$.
**(a)** Find the center of mass of the piece.

**13.14 Person in a boat.** John (80 kg) is standing in a 200 kg boat. He starts at one end and walks 6 m to the other end of the boat. You may neglect drag forces between the boat and the water.
**(a)** How far does the boat move in this process?

**13.15 Car on a train**. A 1000 kg car is standing on an inclined plane on top of a 2000 kg train cart. The cart rolls without friction on the track. The plane has an

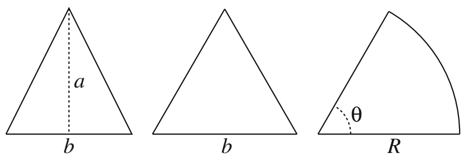


**Fig. 13.17** An isosceles triangle, an equilateral triangle, and a piece of pie

inclination of 30° with the horizontal. The car drives a horizontal distance of 10 m from one end of the cart to the other.
**(a)** How far has the cart moved in this process?

## *Projects*

**13.16 Pushing the blocks.** In this project you will learn about Newton's second law for multi-particle system and energy partitioning in multi-particle systems. We will study the a two-particle system affected by an external force. The system consists of two identical blocks, A and B, sliding on a frictionless, horizontal surface. The blocks have mass $m$ and are attached with a massless spring with spring constant $k$ and equilibrium length $d$. The blocks start from rest in their equilibrium positions at $x_A(0\text{s}) = 0\,\text{m}$ and $x_B(0\text{s}) = d$. The system is illustrated in Fig. 13.18. The left-most block, block A, is pushed with an external force $F$ for a short time $T$.
**(a)** Draw a free-body diagram for each of the blocks. Name the force.
**(b)** Introduce force models for all the forces acting on the blocks.
**(c)** Find expressions for the accelerations for each block.
**(d)** Find an expression for the acceleration of the center of the mass of the system.
**(e)** Find the velocity, $V(t)$, and position, $X(t)$, of the center of mass as functions of time.

In the following we will solve to find the motion of both blocks. It is possible to solve this problem using either analytical or numerical methods. We will here follow a numerical solution, but all results can also be obtained by analytical means. We will study a system where $F = 1000\,\text{N}$, $T = 1\,\text{s}$, $k = 5000\,\text{N/m}$, $d = 0.1\,\text{m}$, and $m = 0.1\,\text{kg}$.
**(f)** Write a program to find the positions $x_A(t)$ and $x_B(t)$ as functions of time.
**(g)** Plot the position and velocity of center of mass as calculated by the program and compare with your results from above. Plot the motion of the blocks in the same plot and comment on the results (see Fig. 13.19 for comparison).

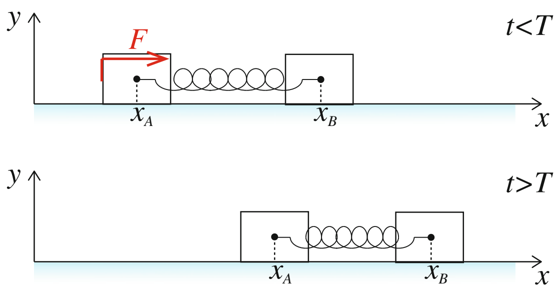


**Fig. 13.18** Illustration of two-block system

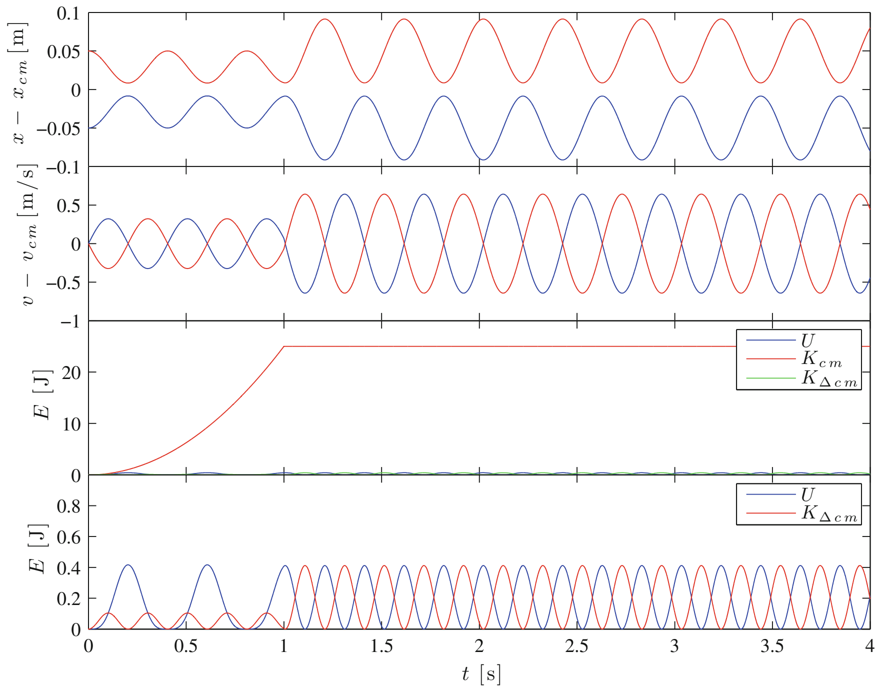


**Fig. 13.19** Plots from simulations

**(h)** Plot the kinetic energy of the center of mass, $K_{cm}$, the kinetic energy of the motion relative to the center of mass, $K_{\Delta cm}$, and the potential energy, $U$, as functions of time. Comment on the results.

**(i)** What is the maximum extension of the spring?

**(j)** How much work was done by the external force, $F$?