Chapter 15
Rotation of Rigid Bodies

We have found that the motion of the center of mass of a multi-particle system can be determined from the external forces acting on the system. The general form of Newton's second law allows us to find the motion from our knowledge of forces and force models for multi-particle systems, just as we have systematically done for particle systems previously. We therefore have a well developed framework to determine the motion of the center of mass.

But what about the internal motion of the system, the internal motion relative to the center of mass? If you kick a football, it spins and wobbles on its path. How can we determine the configuration of the football—how it is rotated and deformed at a particular point along its path? Unfortunately, we generally cannot determine the configuration without having a detailed model for the particle system. We need to model the internal motion of the various parts of the football to address its deformation and rotation. Even in the simplest multi-particle system we can think of, the diatomic molecule, energy is partitioned in a non-trivial manner, including both kinetic and potential energy contributions for the deformation of the molecule.

Fortunately, in many cases we can simplify the system significantly by assuming that an object behaves as a rigid body that does not deform. A solid sphere, a stiff rod, a ball with little deformation, or a molecule with little internal deformation, may in many cases be approximated as a rigid body—as a body without any internal deformation. This simplifies our description significantly: If the body does not deform, there are no internal energies associated with the deformation either. A rigid body can only be translated or rotated. In this chapter we will introduce the kinetic and potential energy of a rigid body. We find that it depends on the rotational inertia of a rigid body, called the moment of inertia. Rigid bodies with large moments of inertia require more energy to spin than rigid bodies with smaller moments of inertia. This gives us the tools we need to apply energy considerations to systems with rotating parts.

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15.1 Rigid Bodies

A rigid body is a solid body that is not deformed. What does this mean? It means that:

> The distance between any two points in a **rigid body** does not change.

What possibilities does that leave us with?

- We may **translate** the body. If we move all the points by the same amount, we have not changed the distanced between any points.
- We may **rotate** the body, because any rotation does not deform the body—it only changes its orientation in space.

A rigid body can have any shape, and it may also be a very simple object. For example, an object consisting of two point masses at a fixed distance d is a rigid body, and a reasonable model for a diatomic molecule if the internal deformation is negligible. A rigid body is therefore a good approximation for a solid sphere, but not for a soap bubble.

15.2 Kinetic Energy of a Rotating Rigid Body

For a rigid body, we can find a simplified expression for the kinetic energy. Previously (see (13.61)), we found that the kinetic energy of a body can be expressed as:

$$K = \frac{1}{2}MV^2 + \sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{v}_{\mathrm{cm},i}^2 , \tag{15.1}$$

where the first term is related to the motion of the center of mass and describes the kinetic energy for the translational motion. The second term is related to the motion relative to the center of mass. For a rigid body, the only way a part of the body can move relative to the center of mass is by rotation. We therefore interpret the second term as kinetic energy for the rotational motion, and we will here introduce the kinetic energy for rotating objects. An object can rotate either around a fixed axis or around a moving axis such as a moving center of mass. Here, we first address the behavior of a rigid body rotating around a fixed axis, and then generalize to the case of an axis following the center of mass motion.

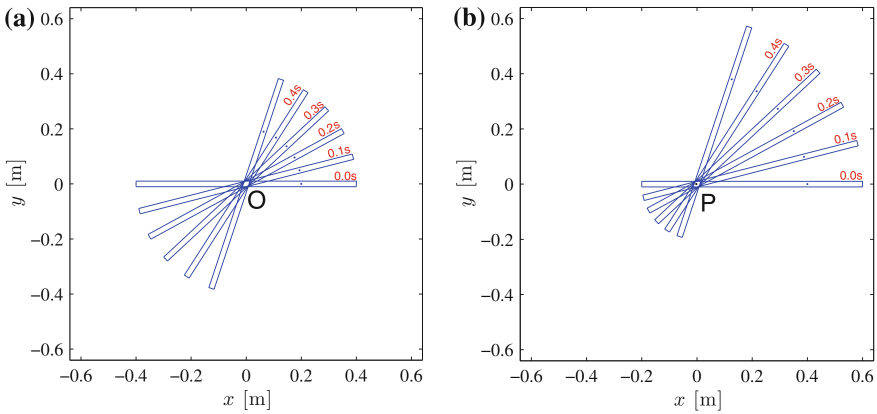

Fig. 15.1 **a** A rod is rotating around an axis O through its center of mass. **b** A rod is rotating around an axis P through another point P along the rod

Rotation Around a Fixed Axis

The rotation of an object around a fixed axis is illustrated by the motion of a rotating rod in Fig. 15.1. As we saw when we discussed rotations, the rod may rotate around an axis through its center of mass (the axis O in the figure), or it may rotate around an axis through some other point along the rod (the axis P in the figure). These two rotations are clearly different. It is therefore important to realize what we mean by a rotation around an *axis*: We mean that the object is rotating around a line. A line can be specified by a point and a vector. It is not sufficient to specify the rotation by a vector alone. In both cases in Fig. 15.1 the object is rotating with the angular velocity $\boldsymbol{\omega}$. The two axes are parallel, but they go through different points. You should therefore always ensure that when you make a sketch of a rotating system, you clearly mark what point or what axis the object is rotating about.

In order to determine the kinetic energy of the rigid body we assume that the body consists of small mass-points with masses m_i and positions, $\mathbf{r}_i$, where the positions are measured relative to an origin placed on the rotation axis. The whole body is rotating with the angular velocity $\boldsymbol{\omega}$ around the fixed axis through the origin, O. We place our coordinate system so that the z-axis points in the direction of $\boldsymbol{\omega}$: $\boldsymbol{\omega} = \omega\,\mathbf{k}$,

What is the velocity of point i on the body? From our discussion of rotations, we know that the velocity of a point on a rotating body at a position $\mathbf{r}_i$ is:

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i . \tag{15.2}$$

We can simplify the description by decomposing the position into two vectors: the vector $\boldsymbol{\rho}_i$, directed normal to the z-axis, and the vector $z_i\,\mathbf{k}$ directed along the z-axis as illustrated in Fig. 15.2:

$$\mathbf{r}_i = \boldsymbol{\rho}_i + z_i\,\mathbf{k} . \tag{15.3}$$

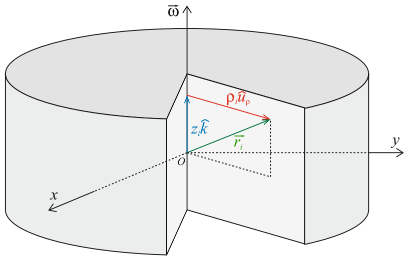

Fig. 15.2 Illustration of an object rotating around the axis O. We choose our coordinate system so that the axis O goes through the origin and it parallel to the z-axis

Using this expression, the velocity is:

$$\mathbf{v}_i = \omega\,\mathbf{k} \times (\boldsymbol{\rho}_i + z_i\,\mathbf{k}) = \omega\,\mathbf{k} \times \boldsymbol{\rho}_i + \omega z_i \underbrace{\mathbf{k} \times \mathbf{k}}_{=0} = \omega\,\mathbf{k} \times \boldsymbol{\rho}_i . \tag{15.4}$$

We write:

$$\boldsymbol{\rho}_i = \rho_i\,\hat{u}_\rho , \tag{15.5}$$

where $\hat{u}_\rho$ is a unit vector pointing radially out from the axis to the point at $\mathbf{r}_i$. The velocity is therefore:

$$\mathbf{v}_i = \omega\rho_i \underbrace{\left(\mathbf{k} \times \hat{u}_\rho\right)}_{=\hat{u}_T} = \omega\rho_i\,\hat{u}_T , \tag{15.6}$$

where the unit vector $\hat{u}_T$ points in the tangential direction.

In order to find the kinetic energy, we only need the magnitude of the velocity for each point:

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\omega\rho_i\right)^2 = \frac{1}{2} \underbrace{\left(\sum_{i=1}^{N} m_i \rho_i^2\right)}_{=I_O} \omega^2 , \tag{15.7}$$

where we have introduced the quantity I_O, which we call the *moment of inertia* of the rigid body about the axis O:

$$I_O = \sum_{i=1}^{N} m_i \rho_i^2 . \tag{15.8}$$

And we can write the kinetic energy for the rotating body as:

Kinetic energy for rotation about the axis 0:

$$K = \frac{1}{2} I_O \omega^2 . \tag{15.9}$$

Notice that the moment of inertia is a property of the object and the axis of rotation: It does not depend on how the object rotates around this axis. We interpret the moment of inertia as the **rotational inertia** of an object around an axis. The rotational inertia tells us how difficult it is to spin an object: The larger the moment of inertia, the larger energy is needed to obtain a certain angular velocity.

The expression $K = (1/2)I_O\omega^2$ in (15.9) is analogous to the expression for the translation kinetic energy for an object, $K = (1/2)Mv^2$. The mass M appears in one equation, while I_O, which involves both the mass and how it is distributed around an axis, occurs in the other.

Rotation Around an Axis Through the Center of Mass

Figure 15.3 illustrates the motion of a rod thrown across a room: An example we used to introduce rotational motion in Chap. 14. In this case the object is subject to both translational and rotational motion. The kinetic energy of this object is given by (15.1) as:

$$K = \frac{1}{2} M V^2 + \sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{v}_{\mathrm{cm},i}^2 . \tag{15.10}$$

If the rod is a rigid body, the only possible motion relative to the center of mass is a rotational motion, as illustrated by the motion of the rod seen in the center of mass system in the bottom right part of Fig. 15.3. We can then use exactly the same argument we used above for rotation around a fixed axis, and find the kinetic energy for rotation around an axis through the center of mass is:

$$\sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{v}_{\mathrm{cm},i}^2 = \frac{1}{2} I_{cm} \omega^2 . \tag{15.11}$$

The total kinetic energy is therefore:

$$K = \frac{1}{2} M V^2 + \frac{1}{2} I_{cm} \omega^2 . \tag{15.12}$$

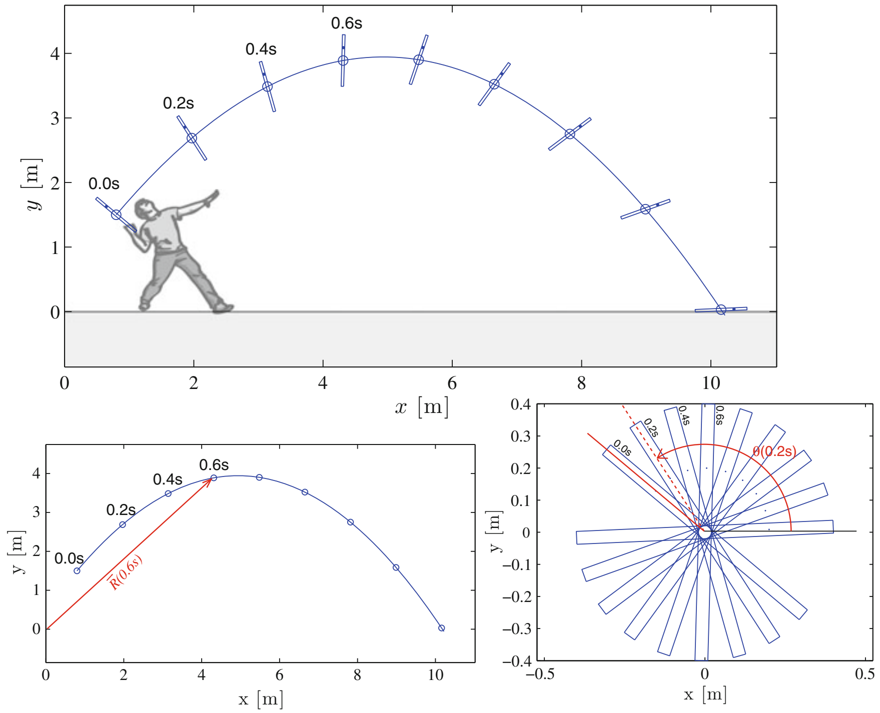

Fig. 15.3 A rod is thrown across the room. The *top figure* shows the motion of the rod. The *bottom left figure* shows the motion of the center of mass of the rod, and the *bottom right figure* shows the motion of the rod relative to the center of mass

15.3 Calculating the Moment of Inertia

The moment of inertia around an axis O of a system of particles with masses m_i at positions $\mathbf{r}_i$ is:

The **moment of inertia around an axis** O: is defined as

$$I_O = \sum_i m_i \rho_i^2 . \tag{15.13}$$

where ρ_i is the distance from particle i to the axis O.

Notice that the moment of inertia is not only a property of the object, it also depends on the axis of rotation, and the moment of inertia of a given object around different axes may be different, as illustrated below. A common mistake is to forget that the

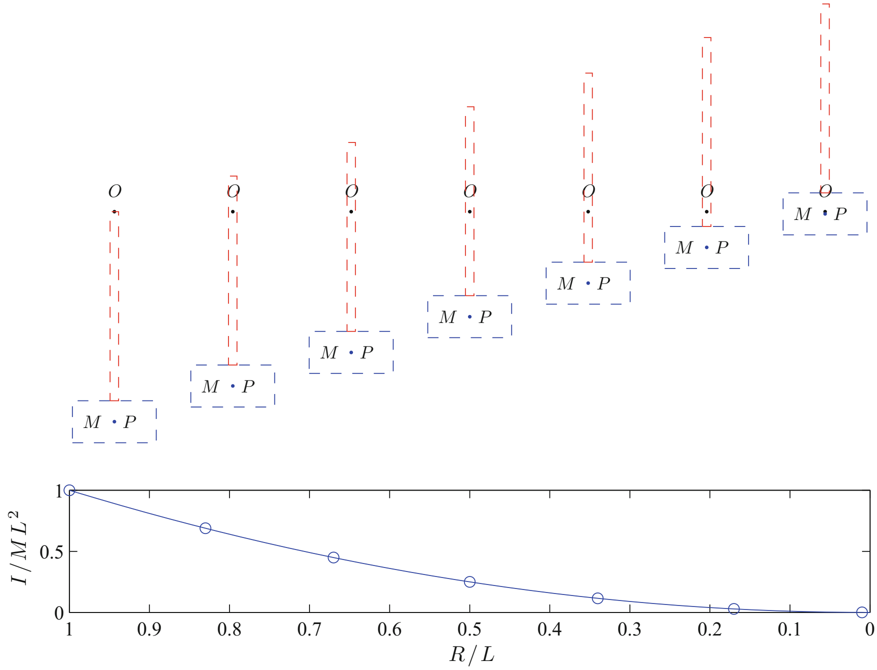

Fig. 15.4 Illustration of a hammer where all the mass is located in the point P in the head of the hammer. The hammer rotates around the point O, and the moment of inertia of a hammer depends on the distance R from the rotation axis to the hammer head

moment of inertia depends both on the object and on the axis, and that the moment of inertia changes if we change the axis.

The moment of inertia depends on both the mass and how it is distributed around the rotation axis. Figure 15.4 illustrates a hammer consisting of a heavy head of mass M attached to an effectively massless rod. (You can assume that all of the mass of the head is located in the marked point, P). The moment of inertia of the hammer is

$$I_O = MR^2, \tag{15.14}$$

where R is the distance from the head, P, to the rotation axis O. We can therefore change the moment of inertia by changing the mass of the head, or by changing the distance from the head to the rotation axis. If you rotate it as a pendulum from your hand this corresponds to changing your grip (point O), since this will change the position of the rotation axis. As you move your grip closer to the head, the moment of inertia becomes smaller, and it becomes easier (requires less energy) to give the hammer a certain angular velocity. Pulling mass in towards the axis of rotation reduces the moment of inertia, pushing it further out from the axis increases it.

Solid Bodies

The mass of the hammerhead is not really located in a point P, it is distributed in space around this point. The moment of inertia of the hammerhead is therefore not zero even if the rotation axis is located at the center of the head. To calculate the moment of inertia of a continuous body—a solid—such as the hammerhead, we must sum all the contributions from a continuum of points corresponding to small volume elements ΔV_i with mass $\Delta m_i = \rho_M \Delta V_i$, where ρ_M is the local mass density of the solid. The sum over all points i in the solid becomes an integral when the size of a each element goes to zero:

$$I_O = \sum_i \Delta m_i \, \rho_i^2 = \sum_i \rho_M \, \Delta V_i \, \rho_i^2 \rightarrow \iiint \rho_M \, \rho^2 \, dV , \tag{15.15}$$

where the integral is over the solid body. The moments of inertia around the center of mass of various solid bodies are shown in Fig. 15.5. Examples of how to calculate the moment of inertia of an object using integration are shown below.

The moment of inertia of a hammerhead of dimensions $h \times w \times d$ (h is height, w is width, and d is depth) around its center O can be found from Fig. 15.5, where we see that the moment of inertia of a plate around its center is

$$I_{cm} = \frac{1}{12} M \left(h^2 + w^2 \right) . \tag{15.16}$$

This is therefore the real moment of inertia of the hammer when $R = 0\,\mathrm{m}$ in the example above.

Parallel-Axis Theorem

We have now found a way to calculate the moment of inertia for a continuous object such as the hammer. But what if we now want to know the moment of inertia of the hammerhead for a different rotation axis—what if we want to know it as a function of the distance, R, from the center of the hammerhead to the rotation axis as illustrated in Fig. 15.6?

We could of course calculate the moment of inertia using the integral formulation in (15.15) for each position of the rotation axis. But this is cumbersome. Fortunately, it turns out that it is sufficient to calculate the moment of inertia around an axis through the center of mass. From this moment of inertia, we can find the moment of inertia around any other parallel axis using the parallel-axis theorem:

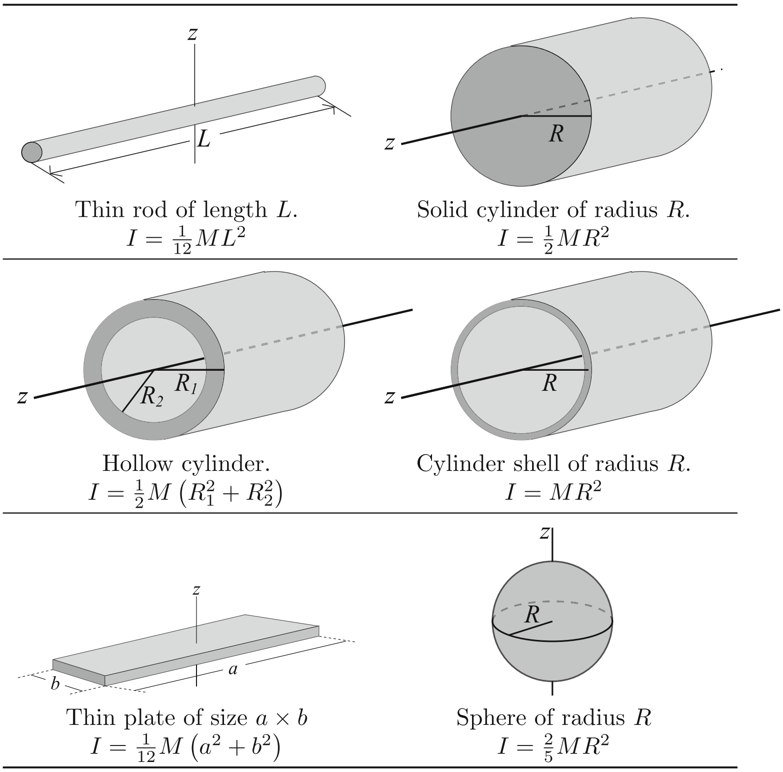

Fig. 15.5 Moments of inertia for various solid bodies

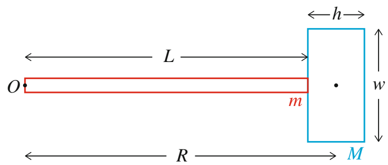

Fig. 15.6 A hammer consisting of a hammerhead (*blue*) of mass M and a thin rod (either massless or of a mass m)

Parallel-axis theorem: The moment of inertia, I_O, of an object around an axis O is related to the moment of inertia, I_{cm}, of the object around a parallel axis through the center of mass of the object by:

$$I_O = I_{cm} + Ms^2, \tag{15.17}$$

where M is the mass of the object, and s is the distance between the axis O and the parallel axis through the center of mass.

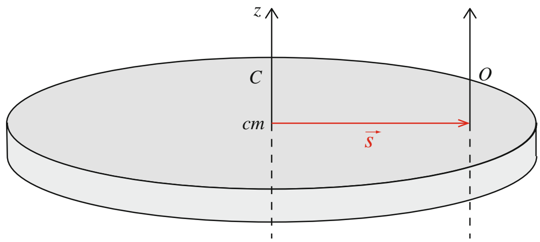

Fig. 15.7 Illustration of the parallel-axis theorem. The moment of inertia around an axis C through the center of mass can be used to find the moment of inertia around an axis O—if the axis O is parallel to the axis C through the center of mass. The vector **s** is perpendicular to both axes, and points from the origin of axis C to the origin of axis O

(You can find a proof in Sect. A.5). The theorem is illustrated in Fig. 15.7. We can use the parallel-axis theorem to find the moment of inertia around any axis if we only know the moment of inertia around the center of mass. This is why you usually only find tabulated the moment of inertia of an object around its center of mass: Given this, you can use the parallel-axis theorem to find the moment of inertia for any other parallel axis.

We can use the parallel-axis theorem to find the moment of inertia of the hammer as a function of the distance R from the center of the hammerhead to the rotation axis—it is simply given by:

$$I_O = I_{cm} + MR^2, \tag{15.18}$$

as long as the two rotation axes are parallel. This allows us to calculate the moment of inertia for any R—as illustrated in Fig. 15.4.

Superposition Principle

Still we are not satisfied with our characterization of the hammer, because it consists of two pieces: hammerhead and shaft. So far we have assumed that the mass of the shaft is negligible and that we therefore can neglects its moment of inertia. What if cannot neglect it—what to do then? We are saved by a principle called the superposition principle: For an object that consists of several parts, such as the hammerhead and the shaft of a hammer, we can find the moment of inertia around an axis by summing the moments of inertia for each part around the same axis:

The moment of inertia of two systems A and B round the axis O is the sum of the moments of inertia for each part of the systems:

$$I_{O,AB} = I_{O,A} + I_{O,B} . \tag{15.19}$$

(You can find a proof in Sect. A.4). We can therefore find the moment of inertia of a compound object by summing the moments of inertia for each of the object. Just be careful to ensure that you sum moments of inertia *around the same axis*.

This allows us to find the moment of inertia of the hammer illustrated in Fig. 15.6 as the sum of the moment of inertia of the hammerhead and the shaft:

$$I_{TOT,O} = I_{h,O} + I_{s,O} , \tag{15.20}$$

where we have already found that the moment of inertia of the hammer is:

$$I_{h,O} = \frac{1}{12} M \left(h^2 + w^2 \right) + MR^2 . \tag{15.21}$$

The moment of inertia of the shaft around the point O can be found just as for the hammerhead. The moment of inertia around the center of mass for the shaft is found from Fig. 15.5:

$$I_{s,cm} = \frac{1}{12} mL^2 , \tag{15.22}$$

and we use the parallel-axis theorem to find its moment of inertia around point O, which is at a distance $L/2$ from the center of mass:

$$I_{s,O} = \frac{1}{12} mL^2 + m \left(\frac{L}{2} \right)^2 = \frac{1}{3} mL^2 . \tag{15.23}$$

The total moment of inertia of the hammer around the point O is therefore:

$$I_{TOT,O} = \frac{1}{12} mL^2 + \frac{1}{12} M \left(h^2 + w^2 \right) + MR^2 , \tag{15.24}$$

where $R = L + h/2$ for this configuration.

15.3.1 Example: Moment of Inertia of Two-Particle System

Problem: Find the moment of inertia around the indicated axes in Fig. 15.8.

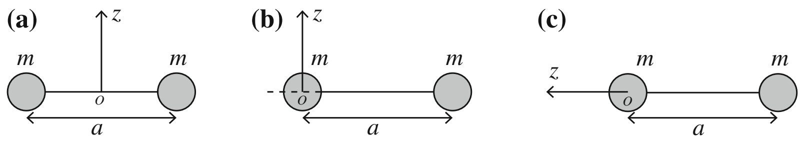

Fig. 15.8 A system of two identical point particles with masses m are attached with a rigid, massless rod of length a. **a** The rotation axis goes through the center of mass. **b** The rotation axis goes through the left-most particles. **c** The rotation axis goes through both particles

Solution: In case **(a)** the moment of inertia around the axis z is:

$$I_z = \sum_{i=1}^{2} m_i \rho_i^2 = m\left(-\frac{1}{2}a\right)^2 + m\left(\frac{1}{2}a\right)^2 = \frac{1}{2}ma^2. \tag{15.25}$$

In case **(b)** the moment of inertia around the axis z is:

$$I_z = \sum_{i=1}^{2} m_i \rho_i^2 = m0^2 + ma^2 = ma^2. \tag{15.26}$$

Notice that the point on the axis does not give any contribution to I_z, but that the moment of inertia is larger, because the other particle is further away from the origin.

In case **(c)**, the moment of inertia around the axis z is:

$$I_z = \sum_{i=1}^{2} m_i \rho_i^2 = m0^2 + m0^2 = 0. \tag{15.27}$$

15.3.2 Example: Moment of Inertia of a Plate

Problem: Find the moment of inertia of a homogeneous plate with dimensions a and b and of thickness h around an axis z through the center of mass, as illustrated in Fig. 15.9.

Solution: The moment of inertia of a solid body around the axis O is defined as:

$$I_z = \int_M \rho^2 \, dm = \int\int\int \rho_M \rho^2 \, dV. \tag{15.28}$$

Since the place is homogeneous, the mass density is uniform – constant in space. The distance ρ from the axis to the point $\mathbf{r}$ is simply the length of the horizontal (xy-plane) component of $\mathbf{r}$:

$$\rho^2 = x^2 + y^2. \tag{15.29}$$

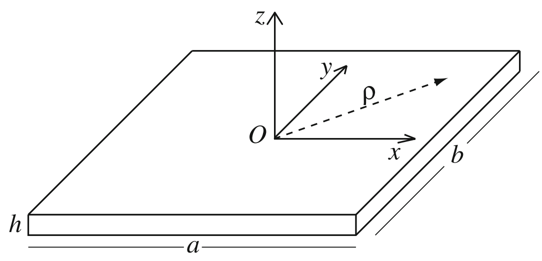

Fig. 15.9 A thin plate with sides a and b and thickness h. The coordinate system is placed at the center of mass, with the axis z in the direction of the thickness of the plate

The moment of inertia is found from the integral:

$$I_z = \rho_M \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \int_0^h x^2 + y^2 \, dz \, dy \, dx = \rho_M h \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} x^2 + y^2 \, dy \, dx, \quad (15.30)$$

where we integrate the two parts of the sum individually:

$$\begin{aligned} I_z &= \rho_M h \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} x^2 \, dy \, dx + \rho_M \, h \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} y^2 \, dy \, dx \\ &= \rho_M hb \int_{-a/2}^{a/2} x^2 dx + \rho_M \, h \, a \int_{-b/2}^{b/2} y^2 \, dy \qquad (15.31) \\ &= \rho_M hb \frac{1}{3} \left[\left(\left(\frac{a}{2}\right)^3 - \left(\frac{-a}{2}\right)^3 \right) + \left(\left(\frac{b}{2}\right)^3 - \left(\frac{-b}{2}\right)^3 \right) \right], \end{aligned}$$

which we simplify to:

$$I_z = \frac{1}{3} \rho_M hba \left(\frac{a^2}{4} + \frac{b^2}{4} \right) = \frac{1}{12} \rho_M V \left(a^2 + b^2 \right) = \frac{1}{12} M \left(a^2 + b^2 \right), \quad (15.32)$$

where we used that $V = abh$ and $M = \rho_M V$.

15.4 Conservation of Energy for Rigid Bodies

We have found the kinetic energy of a rigid body that is either rotating around a fixed axis or that is rotating around its moving center of mass. If the solid body is only affected by conservative forces, we may use energy conservation principles to determine the relation between angular velocity and position, just as we have done with translational velocity and position previously. But in order to employ energy conservation, we need to determine the potential energy of a rigid body.

Potential Energy for a Constant Gravitational Force

What is the potential energy for a rigid body due to the gravitational force? We found that for a point particle with mass m_i at the position $\mathbf{r}_i$, the potential energy due to a constant gravitational force directed along the y-axis is:

$$U_i = m_i g y_i . \tag{15.33}$$

We often call a constant gravitational force a homogeneous gravity, since gravity is the same everywhere. The total potential energy for multiparticle system of N particles is therefore:

$$U = \sum_{i=1}^{N} U_i = \sum_{i=1}^{N} m_i g y_i = g \underbrace{\sum_{i=1} m_i y_i}_{=MY} = M\, g\, Y , \tag{15.34}$$

where Y is the position of the center of mass. This result is general, and is also valid for a rigid body:

Potential energy of a rigid body for a constant gravitational force

$$U = M\, g\, Y . \tag{15.35}$$

Potential Energy Due to a Spring Force

A rigid body may also be affected by a force modelled by a spring force. This force may have several origins. It may represent a contact force due to an elastic contact; a contact force due to a rope or string attachment; an approximation for a more complex force acting on a single part of the rigid body; or as a force acting on all parts of the rigid body.

First, we start from the simplest case, where a spring is acting on a particular point on the rigid body. We illustrate this situation by the contact between a sphere and an elastic floor for example while the rod is bouncing off the floor (see Fig. 15.10). We can then model the force from the floor on the sphere using a spring model, where we assume that the force depends on the position of a particular point, the point P, on the sphere. If the surface is horizontal at $y = y_f$, the spring models a vertical normal force from the floor:

$$\mathbf{F} = -k(p_y - y_f) , \tag{15.36}$$

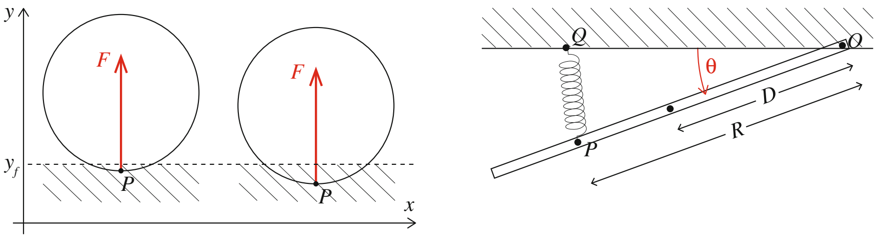

Fig. 15.10 (*Left*) Illustration of contact between a sphere and a surface. (*Right*) Illustration of a rod that can rotate around point O, but it is attached with a spring at point P. The other end of the spring is attached at the point Q

when the sphere is in contact with the floor, that is when $p_y < y_f$, where p_y is the vertical position of the point P. The potential energy of the sphere due to this spring force is then:

$$U(p_y) = \frac{1}{2}k\left(p_y - y_f\right)^2 . \tag{15.37}$$

For the sphere, we see that the position of point P is related to the center of mass of the sphere, $p_y = Y - R$, where Y is the vertical position of the center of mass, and R is the radius of the sphere.

This result is not only valid for a sphere, but for another body affected by a spring force. The potential energy of a spring force is related to the elongation of the spring. We therefore need to relate the motion of the rigid body to the elongation of the spring by relating for example the position of the center of mass of the body or the rotation angle θ of the body to the elongation of the spring. This is illustration by the rod on the right in Fig. 15.10: The rod can rotate freely around the point O, but is affected by a spring attached to the rod at the point P and to the wall at the point Q, so that when the rod is parallel to the wall, corresponding to $\theta = 0$, the spring is in its equilibrium position. The potential energy in the spring then depends on the distance between P and Q, which for small rotation angles θ can be approximated as $\Delta QP \simeq R\theta$, where R is the distance from P to O. The potential energy due to the spring force affecting the rotating rod is therefore:

$$U_s(\theta) = \frac{1}{2}k\,(\Delta QP)^2 = \frac{1}{2}kR^2\theta^2. \tag{15.38}$$

If the rod also is subject to gravity, we must also add the potential energy due to gravity:

$$U_g(\theta) = -MgD\sin\theta, \tag{15.39}$$

where D is the distance from O to the center of mass of the rod. The total potential energy of the rod is then $U = U_s + U_g$.

If you want to use a spring force model for a rigid body, you must therefore consider how to describe it in detail. We will introduce several ways to model such interactions in the following, but generally you are left to your own ingenuity.

Conservation of Energy for a Rigid Body

How can we now put the pieces together and use conservation of energy of a rigid body relate its velocity (angular and translational) to its position (rotational and translational)? As long as the system is only subject to conservative forces, the total energy of a rigid body is:

$$E = K + U = \frac{1}{2}MV^2 + \frac{1}{2}I_{cm}\omega^2 + U. \tag{15.40}$$

For a *rigid* body the potential energy can only depend on the translational and rotational motion of the body and not on the internal deformation. We have therefore only included one term for the potential energy: The potential energy due to external forces.

If the object is rotating around a fixed axis—an axis that does not move—we can simplify the kinetic energy, getting

$$E = \frac{1}{2}I_O\omega^2 + U. \tag{15.41}$$

These two equations ((15.40) and (15.41)) forms the basis for using energy conservation to solve problems with rigid bodies.

15.4.1 Example: Rotating Rod

Problem: Find the angular velocity for a rod that rotates without friction about an attachment point at one of its ends. The rod starts in a horizontal position. You can neglect air resistance.

Approach: We use an energy argument to relate the angular velocity of the rod to its position based on the kinetic energy of a rotating rigid body and the potential energy of a solid body affected by gravity.

Solution: How can we address this problem with what we have learned so far? Can we use Newton's laws of motion? Not really, because we do not know the forces acting in the attachment point. Normal forces will be present in this point—but we do not have a model to determine their magnitude.

What about energy conservation? We know how the kinetic energy of the rod depends on the angular velocity, and we know the potential energy of the rod in

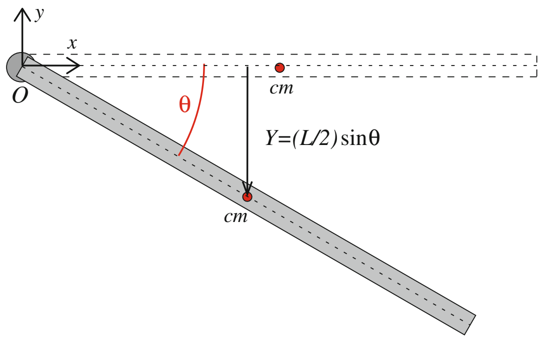

Fig. 15.11 A thin rod of length L and mass M attached at the point O. The orientation of the rod is given by the angle θ. The position of the center of mass of the rod is found from the geometry to be $Y = -(L/2)\sin\theta$

the gravity field. But is the mechanical energy conserved in this case? There are no frictional forces and no air resistance. The only external forces acting are the normal force in the attachment point and gravity. The force acting on the attachment point does no work, because this point is not moving! And we know that the work done by gravity can be expressed as a potential energy. We can therefore use energy conservation to solve this problem!

Identify: We start from a sketch of the system in Fig. 15.11. The rod has a length L. Its position is given by the angle θ, and the rod starts at $\theta = 0$. Our task is to find the angular velocity as a function of θ. We choose a positive rotational direction, which means that the angle θ in Fig. 15.11 is negative!

Kinetic energy: The kinetic energy of a rigid rod rotating around the z-axis through O is given as:

$$K = \frac{1}{2} I_z \omega^2 , \tag{15.42}$$

where I_z is the moment of inertia for the rod around the axis z through O. For a rod rotating around a *fixed* axis, this is the only term for the kinetic energy.

Caution: Notice that for an object rotating around a fixed axis you use the term $K = (1/2) I\omega^2$ for the kinetic energy, where ω is the angular velocity around the fixed axis. If you instead want to separate the kinetic energy into the motion of the center of mass and the rotation around the center of mass, you have to figure out what the angular velocity around the center of mass is, which may not be the same as the angular velocity around O. A frequent mistake is the confuse the two cases of rotation around a fixed axis and rotation around the center of mass.

Potential energy: From (15.35) we found that the potential energy of a solid body due to gravity is

$$U = M\, g\, Y , \tag{15.43}$$

where Y is the vertical position of the center of mass of the object. For a homogenous rod, the center of mass is located at its geometric center, that is at a distance $L/2$

from the end. What is the vertical coordinate of this position? From Fig. 15.11 we see that of the center of mass is:

$$Y = \frac{L}{2} \sin\theta. \tag{15.44}$$

This is a negative number as long as θ is negative (and larger than $-\pi$). The potential energy of the rod is therefore:

$$U = -Mg\frac{L}{2} \sin\theta. \tag{15.45}$$

Solve: The total energy of the rod is conserved since all the forces are conservative (or not doing any work on the body), the total energy, $E = K + U$, is conserved. The total energy is therefore the same in the initial position, 0, when $\theta = \theta_0 = 0$, and in the position 1, when the angle is θ:

$$E_0 = K_0 + U_0 = E_1 = K_1 + U_1, \tag{15.46}$$

where

$$E_0 = K_0 + U_0 = \frac{1}{2}I_z\omega_0^2 + MgY_0 = 0, \tag{15.47}$$

and

$$E_1 = K_1 + U_1 = \frac{1}{2}I_z\omega_1^2 + MgY_1 = \frac{1}{2}I_z\omega^2 - \frac{L}{2}Mg\sin\theta. \tag{15.48}$$

Energy conservation, $E_0 = E_1$, gives:

$$\frac{1}{2}I_z\omega^2 - \frac{L}{2}Mg\sin\theta = 0 \ \Rightarrow\ \frac{1}{2}I_z\omega^2 = \frac{L}{2}Mg\sin\theta \ \Rightarrow\ \omega = \pm\sqrt{\frac{MgL}{I_z}\sin\theta}, \tag{15.49}$$

where we get two solutions, a positive and a negative solution, depending which way the rod is swinging. Both directions are possible solutions: For every angle θ the rod will first swing one way, then swing the other way. Since there are no damping it will continue swinging back and forth indefinitely.

Analyze: We have found a general solution, valid for the rotation of any type of homogeneous staff of length L. The solution depends on the moment of inertia for the staff. If the staff is slim, we know that the moment of inertia around the center of mass is:

$$I_{\text{cm}} = \frac{1}{12}ML^2, \tag{15.50}$$

but the rod is not rotating around the center of mass, instead we need to find the moment of inertia around the end-point of the rod, that is, around the axis z. We use the parallel-axis theorem to find this, since the rod is rotating around an axis parallel to an axis through the center of mass of the rod. The distance s from the center of mass to the axis z in point O is $L/2$. The parallel-axis theorem therefore gives:

$$I_z = I_{\text{cm}} + Ms^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2. \tag{15.51}$$

For this rod, the angular velocity is:

$$\omega = \sqrt{\frac{MgL}{I_z}\sin\theta} = \sqrt{\frac{MgL}{\frac{1}{3}ML^2}\sin\theta} = \sqrt{3\frac{g}{L}\sin\theta}. \tag{15.52}$$

Test your understanding: How would your results change if the rod was not homogeneous, but instead had all its mass located in a point at a distance L from the attachment point?

15.5 Relating Rotational and Translational Motion

For a wheel rolling along a surface or for a rope running over a spinning wheel, the rotational and translational motion of the objects are related: If the wheel is to roll without slipping the center of the wheel must move a distance equal to the path "rolled out" by the wheel, and for the rope to run over the spinning wheel without slipping the length of rope pulled from the wheel must correspond to the length a point on the wheel has moved (see Fig. 15.14). How do we introduce such relations between the rotation and the motion of the object?

Rolling Motion

The left part of Fig. 15.12 illustrates the motion of a wheel that is rolling without slipping. What does this mean? That it is rolling without slipping? It means that the distance moved by the center of the wheel during a time interval Δt must correspond to the distance "rolled out"—the distance moved by a point on the surface of the wheel in the same time interval. Alternatively, we could say that the point P on the wheel that is in contact with the floor, does not move relative to the floor (see Fig. 15.13). This point must therefore have zero velocity relative to the floor. Let us find an expression for the point P. We do this by first finding the velocity of the point relative to the center of mass, and then adding the velocity of the center of mass to find the velocity relative to the ground. In the center of mass system (a system with an origin that moves with the center of mass), the wheel is rotating around the origin. The velocity of P relative to the center of mass is therefore the velocity of a point moving in a circle around the origing with angular velocity $\boldsymbol{\omega}$:

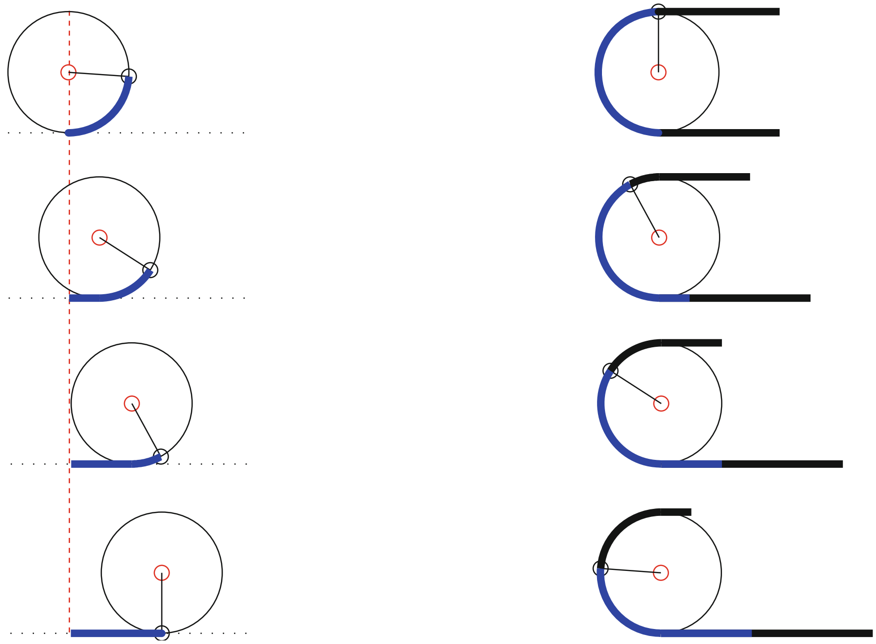

Fig. 15.12 Illustration of a wheel rolling without slipping (*left*) and a rope running over a spinning wheel without slipping (*right*)

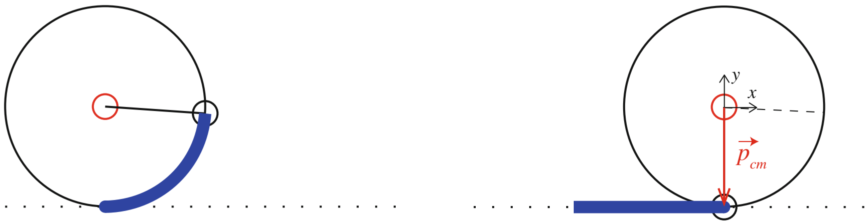

Fig. 15.13 Illustration of rolling condition: The point P is at a position $\mathbf{p}_{cm} = -R\,\mathbf{j}$ relative to the center of mass

$$\mathbf{v}P, cm = \boldsymbol{\omega} \times \mathbf{p}_{cm}, \tag{15.53}$$

where $\mathbf{p}_{cm}$ is the position of point P relative to the center of mass. If we place the origin at the center of mass, we find that

$$\mathbf{p}_{cm} = -R\,\mathbf{j}, \tag{15.54}$$

as illustrated in Fig. 15.13. The angular velocity is:

$$\boldsymbol{\omega} = \omega\,\mathbf{k}, \tag{15.55}$$

and therefore

$$\mathbf{v}_{P,cm} = \omega\,\mathbf{k} \times (-R)\,\mathbf{j} = \omega R\,\mathbf{i}. \quad (15.56)$$

The velocity of point P relative to the ground is found by adding the velocity of the center of mass relative to the ground (using the Galilei-transformations):

$$\mathbf{v}_P = \mathbf{v}_{cm} + \mathbf{P, cm} = \mathbf{V} + \omega R\,\mathbf{i}. \quad (15.57)$$

The wheel is rolling if this is zero:

$$\mathbf{v}_P = \mathbf{V} + \omega R\mathbf{i} = 0 \Rightarrow \mathbf{V} = -\omega R\,\mathbf{i}. \quad (15.58)$$

Are the signs in this equation correct? Yes. If V_x is positive we see that ω must be negative, as expected.

Rolling condition: If the condition $V_x = -\omega R$ is satisfied, the wheel is rolling without slipping. This relation is called the *rolling condition*. If this condition is not satisfied, the contact between the wheel and the ground is moving relative to the ground, and we say that the wheel is *sliding*. This condition is often used to determine when a wheel starts rolling without sliding.

Caution: A common mistake is to use the rolling condition also in the case when the wheel is slipping. The rolling condition is only valid in the case of rolling without slipping.

15.5.1 Example: Weight and Spinning Wheel

Problem: A weight of mass m is hanging from a massless rope that is wound around a spinning wheel of mass M and radius R. The spinning wheel can rotate without friction around its attachment point at the center of the wheel. The weight is released from rest. Find the velocity v of the weight as a function of its height h. You may neglect air resistance.

Approach: We plan to use energy conservation for the (wheel+weight) system. Both the wheel and the weight has a kinetic energy, but only the potential energy of the weight changes as it descends. The velocity of the weight and the angular velocity of the wheel is related because they are connected by the rope. We solve to find how the velocities depend on the position of the weight.

Solution: This problem may be addressed in several ways: We could apply Newton's laws of motion, but then we need a law for rotational motion. We learn that in Chap. 16, and we return to this example then. The problem can also be addressed

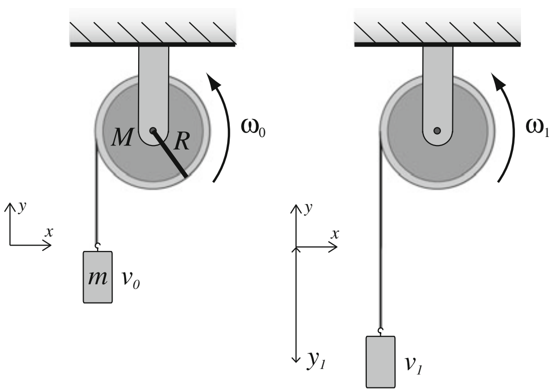

Fig. 15.14 A weight of mass m is attached to a massless rope wrapped around a spinning wheel of radius R and mass M

using energy conservation—energy is conserved for the complete system consisting of the spinning wheel, the rope, and the weight since the only external forces are gravity, which is a conservative force, and the force acting at the attachment point of the spinning wheel—and this force does no work since the spinning wheel does not move.

Identify: The system consists of the weight, the rope, and the spinning wheel (see Fig. 15.14). The weight of mass m has the position y and velocity v, with the positive direction of y chosed upwards. The spinning wheel rotates with an angular velocity ω around its attachment point at its center (of mass). Positive rotational direction is chosen according to the right hand rule, and is shown with the arrow indicating the angular velocity ω. The weight starts at the position $y_0 = 0$ with the velocity $v_0 = 0$.

Model: Because all the external forces acting on the system are either conservative (gravity) or not performing any work (the normal force on the axis of the spinning wheel), we can use energy conservation. The total energy is $E = K_m + K_W + U_m + U_w$, where $K_m = (1/2)mv^2$ is the kinetic energy of the weight, $K_w = (1/2)I\omega^2$ is the kinetic energy of the spinning wheel, $U_m = mgy$ is the potential energy of the weight (where $y < 0$ since we assume the weight to start at $y = 0$), and $U_w = Mgy_w$ is the potential energy of the spinning wheel (which is constant since center of mass of the spinning wheel is not moving). Since the rope does not have any mass and does not stretch, we do not need to include the kinetic or potential energy of the rope.

Solve: The total energy is calculated for two configurations: the initial configuration 0 where the velocity of the weight is $v_0 = 0$ and $\omega_0 = 0$, and the position 1 where the angular velocity of the spinning wheel is ω and the velocity of the weight is v:

$$
\begin{aligned}
E_0 &= K_{m,0} + U_{m,0} + K_{w,0} + U_{w,0} \\
&= \frac{1}{2}mv_0^2 + mgy_0 + \frac{1}{2}I\omega_0^2 + Mgy_w \\
&= 0 + 0 + 0 + Mgy_w ,
\end{aligned}
\tag{15.59}
$$

and

$$E_1 = K_{m,1} + U_{m,1} + K_{M,1} + U_{M,1} \\ = \frac{1}{2}mv^2 + mgy + \frac{1}{2}I\omega^2 + Mgy_w. \tag{15.60}$$

These equations involve both v and ω. However, the motions of the weight and the spinning wheel are related because they are connected by a massless rope. In order for the rope not to slip along the wheel, the rope must follow the motion of the wheel at point P. This means that the velocity of point P on the wheel must be the same as the velocity of the rope at this point:

$$\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r} = \omega\,\mathbf{k} \times (-\mathbf{R}\,\mathbf{i}) = -\omega R\,\mathbf{j}. \tag{15.61}$$

In order for the rope to remain tight and not stretch, the velocity of the rope at point P must be the same as the velocity of the weight. We therefore have:

$$\mathbf{v}_P = -\omega R\,\mathbf{j} = \mathbf{v}, \tag{15.62}$$

which is the velocity of the weight. We insert $v = -\omega R$ into (15.60):

$$E_1 = \frac{1}{2}mv^2 + mgy + \frac{1}{2}I\left(-\frac{v}{R}\right)^2 + Mgy_w = \frac{1}{2}\left(m + \frac{I}{R^2}\right)v^2 + mgy + Mgy_w. \tag{15.63}$$

What is the moment of inertial I of the spinning wheel around its center of mass? If the mass is homogeneously distributed, the wheel is a cylinder with mass M and radius R, and the moment of inertia of a cylinder around its center of mass is $I = (1/2)MR^2$, which we insert into (15.63):

$$E_1 = \frac{1}{2}\left(m + \frac{MR^2}{2R^2}\right)v^2 + mgy + Mgy_w = \frac{1}{2}\left(m + \frac{M}{2}\right)v^2 + mgy + Mgy_w. \tag{15.64}$$

Applying energy conservation $E_0 = E_1$ we find:

$$Mgy_w = \frac{1}{2}\left(m + \frac{M}{2}\right)v^2 + mgy + Mgy_w \;\Rightarrow\; -\frac{1}{2}\left(m + \frac{M}{2}\right)v^2 = mgy, \tag{15.65}$$

$$v = \sqrt{2mg(-y)/\left(m + (M/2)\right)}, \tag{15.66}$$

where we have chosen the negative solution for v, since we know that the weight is moving downwards. We recall that y is negative since the weight is falling down.

Analyze: Let us test this result by checking what happens when the spinning wheel is massless, that is when $M \ll m$. In this limit we find $v = \sqrt{2g(-y)}$, which is the result we expected. Since the velocity of the weight for finite masses of the spinning wheel is smaller than this, the effect of the spinning wheel is to slow down the acceleration of the weight.

15.5.2 Example: Rolling Down a Hill

Problem: You are arranging a rolling competition. Various symmetrical, round objects of mass M, radius R, and moment of inertia I about an axis through the center of mass, are rolled down a hill of vertical height h. Find the velocity v of the object at the end of the hill. You may neglect the effects of air resistance.

Approach: We plan to use energy conservation: The initial potential energy is converted to translational and rotational energy of the object. The rolling condition relates translational and rotational motion.

Solution: We may use the conservation of total mechanical energy if all the forces are conservative. A rolling object is affected by gravity, which is conservative, by the normal force, which does no work, and by a friction force from the surface. What, you say, friction? How can we then use energy conservation? It turns out that for an object that is rolling without slipping, the friction force acts in the point of contact between the rolling object and the surface, and the velocity of this point is zero. The work done by the friction force on the object is therefore zero:

$$W = \int_{t_0}^{t_1} \mathbf{F} \cdot \mathbf{v} \, dt = 0. \tag{15.67}$$

In this particular case we can therefore use energy conservation. Notice that you cannot use energy conservation if the object is slipping!

Identify: The rolling object has mass M, radius R, and moment of inertia I as illustrated in Fig. 15.15. The object starts at Y_0 with the velocity $V_0 = 0$ and the angular velocity $\omega_0 = 0$. What is the velocity (of the center of mass) at $Y = Y_1 = Y_0 - h$?

Model: The total energy of the object is $E = K_T + K_R + U$, where $K_T = (1/2)MV^2$ is the translational kinetic energy, related to the motion of the center of mass, and $K_R = (1/2)I\omega^2$ is the rotational kinetic energy, related to the motion relative to the center of mass. The potential energy is due to gravity $U = MgY$.

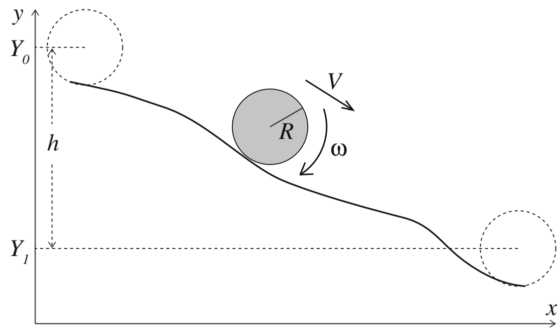

Fig. 15.15 A round object rolling down a hill

Solve: Conservation of energy gives:

$$
\begin{aligned}
E_0 &= E_1 \\
\frac{1}{2}MV_0^2 + \frac{1}{2}I\omega_0^2 + MgY_0 &= \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 + MgY_1 \\
0 + 0 + MgY_0 &= \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 + MgY_1 \\
Mg\underbrace{(Y_0 - Y_1)}_{=h} &= \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 .
\end{aligned}
\tag{15.68}
$$

We relate the translational and rotational motion using the rolling condition. As long as the object is rolling without slipping:

$$
V = R\omega \;\Rightarrow\; \omega = V/R, \tag{15.69}
$$

which gives

$$
Mgh = \frac{1}{2}MV^2 + \frac{1}{2}I\left(\frac{V}{R}\right)^2 = \frac{1}{2}M\left(1 + \frac{I}{MR^2}\right)V^2 . \tag{15.70}
$$

Analyze: The result depends on the ratio I/MR^2. Let us introduce the number c for this:

$$
c = \frac{I}{MR^2}, \tag{15.71}
$$

The number c characterizes how the mass is distributed around the center of mass. Large values of c means that the mass is placed far from the center of mass. As more mass is pulled towards the center of mass, c is reduced. We can find c for various ordinary shapes. For a cylinder shell $c = 1$, for a cylinder $c = 1/2$, and for a sphere $c = 2/5$. We interpret $c = 0$ as the case where a block is sliding without rolling on a frictionless surface. We can find the velocity V expressed using c:

$$gh = \frac{1}{2} V^2 (1 + c) , \tag{15.72}$$

$$V = \sqrt{\frac{2gh}{1 + c}} . \tag{15.73}$$

We see that smaller c gives larger velocities, and the object with the largest velocity wins the race. Hence the object with the smallest value of c would win. Notice that the results do not depend on the mass, only on how the mass is distributed around the axis of rotation.

Test your understanding: How would you construct a rolling object with a small value of c?

Summary

A rigid body: A rigid body can be **translated** or **rotated**, but the distance between any two points in the body does not change.

Moment of inertia:

- The moment of inertia of a multiparticle system around an axis O is: $I_O = \sum_i m_i \rho_i^2$, where m_i is the mass of particle i, and ρ_i is the distance from particle i to the rotation axis O.
- The moment of inertia of a solid body with mass density ρ_M around an axis O is: $I_O = \iiint \rho_M \, \rho^2 \, dV$, where ρ is the distance from the element dV to the axis O.
- Moments of inertia are added according to the **superposition principle**: The moment of inertial of two systems A and B around the axis O is the sum of the moments of inertia for each of the systems: $I_{O,AB} = I_{O,A} + I_{O,B}$
- The **Parallel-axis theorem**: The moment of inertia I_O of an object around an axis O can be found from the moment of inertia of the object around an *parallel* axis through the center of mass of the object, I_{cm}: $I_O = I_{\text{cm}} + Ms^2$, where s is the distance between the axis O and the center of mass.

Energy of a rotating body:

- The **kinetic energy** of a rigid body rotating around the axis O is: $K = (1/2) I_O \omega^2$, where ω is the angular velocity around the axis O.
- The **potential energy** of a rigid body in a **homogeneous gravity field** is: $U = MgY$, where Y is the vertical position of the center of mass of the system.

Relating translational and rotational motion:

- For two objects connected by a thin, non-elastic rope, the point of contact with the rope must have the same speeds for both objects.
- An object is **rolling without slipping** along a surface if the point in contact with the surface has zero velocity relative to the surface.

Exercises

Discussion Questions

15.1 Moment of inertia. Can you find an object where the moment of inertia is larger around the center of mass than around an axis directed in the same direction but going through a different point?

15.2 Symmetries. Are there any objects for which the moment of inertia around an axis through the center of mass is the same for all possible axes? For all possible axes in a plane?

15.3 Dumbbell. A dumbbell consists of two spheres connected by a thin rod. Around which axis is the moment of inertia minimal for the dumbbell?

Problems

15.4 Three-particle system. Three particles of mass m are placed at $(-a, -a)$, $(a, -a)$, and $(0, a)$.
(a) Find the center of the mass of this system.
(b) Find the moment of inertia I_{cm} around the center of mass of the system for an axis along the z-axis.
(c) Find the moment of inertia $I_{0,z}$ for an axis along the z-axis through the origin.
(d) Find the moment of inertia $I_{0,x}$ for an axis along the x-axis through the origin.
(e) Find the moment of inertia $I_{0,z}$ for an axis along the y-axis through the origin.

15.5 Compound system. A dumbbell consists of two spheres of radius R and mass M connected by a rigid rod of mass m and length L, as illustrated in Fig. 15.16.
(a) Find the moment of inertia for the system around an axis normal to the direction of the rod and through the center of mass of the system.
(b) Find the moment of inertia for the system around an axis along the direction of the rod and through the center of mass of the system.
(c) Find the moment of inertia for the system around an axis normal to the direction of the rod and through the center of one of the spheres.
(d) Find the moment of inertia for the system around an axis along the direction of the rod and through the center of one of the spheres.

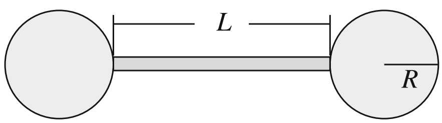

Fig. 15.16 A dumbbell system

15.6 Water molecule. A water molecule consists of an oxygen atom of mass 16u and two hydrogen atoms of mass 1u each. The two hydrogen atoms are placed at a distance a from the center of the oxygen atom, and the angle between the lines from the center of the oxygen atom to each of the hydrogen atoms is 105°. You can assume that each atom is a point particle with all its mass at its center.
(a) Find the moment of inertia for the molecule around an axis normal to the plane with all three atoms and through the center of mass of the molecule.
(b) Find the moment of inertia for the molecule around an axis normal to the plane with all three atoms and through the oxygen atom.

15.7 Compound system. A dumbbell consists of two spheres of radius R and mass M attached together so that the spheres are just touching. The dumbbell can rotate freely without friction about an attachment point through one of the spheres. The system starts at rest in a horizontal position and is released.
(a) Find the moment of inertia for the dumbbell around an axis normal to the plane of the dumbbell through one of the spheres.
(b) Find the angular velocity ω of the system as a function of its rotation angle θ.

15.8 Atwood's fall machine. Atwood's fall machine consists of two weight of mass m_1 and m_2 attached with a massless rope running around a spinning wheel of mass M and radius R without slipping. The spinning wheel is attached at its center and rotates around an axis through its center without friction.
(a) Find velocity of each of the weights as a function of their vertical positions.
(b) Find angular velocity of the spinning wheel as a function of the vertical positions of the weights.

15.9 Triangular pendulum. In this exercise we study a pendulum consisting of two point masses, each of mass, m attached to the ends of a massless rod of length, L as illustrated in Fig. 15.17. Two massless strings of length, L, are attached to each of the mass points and to the point, O, so that the pendulum forms an equilateral triangle. The pendulum is lifted to the position 0 and released. You may ignore air resistance.
(a) Find the moment of inertia, I_O about the point O for the pendulum.
(b) Show that the angular acceleration of the pendulum about the point O when it is at the position θ is:

$$\alpha = -\left(\sqrt{3}\right)(2)\ (g/L)\ \sin\theta. \tag{15.74}$$

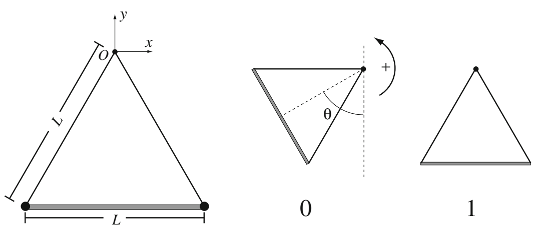

Fig. 15.17 A triangular pendulum

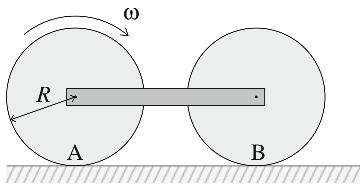

Fig. 15.18 Illustration of a model toy car

(c) Find the angular velocity of the pendulum about the point, O, when the pendulum is in its lowest position.
(d) When the pendulum reaches its lowest position, both strings break. Describe the subsequent motion of the rod. Justify your answer.

15.10 Spinning toy car. In this exercise we study a simplified model for a toy car. The car consists of two identical wheels with mass m and radius R connected by a massless rod as illustrated in Fig. 15.18. The wheels rotate without friction around their attachment points on the rod. Initially, the trailing wheel (wheel A) starts with an angular velocity ω_0, with positive direction as illustrated in the figure. The leading wheel (wheel B) starts from rest. You put the car down on a flat, horizontal floor, and release it. We assume that the trailing wheel (wheel A) initially slides on the floor, whereas the leading wheel (wheel B) is rolling without slipping. You can also assume that the case remains horizontal throughout the motion.

The dynamic coefficient of friction between wheen A and the floor is μ, the acceleration of gravity is g, and the moment of inertia for each of the wheel around their respective center of masses are I.
(a) Draw a free-body diagram for each of the three bodies (wheel A, wheel B, and the rod), showing the forces acting on each of the objects.
(b) Show that the acceleration of the car immediately after it is put down on the floor is $a = g\ (\mu)\ /\ (2 + c)$ where $c = I/\left(mR^2\right)$.
(c) Find the angular velocity for wheel A as a function of time. (The expression is only valid until the trailing wheel starts rolling without sliding).
(d) How long time does it take before both wheels roll without sliding? Describe the motion after this.

Projects

15.11 Micro-electromechanical system. In this project you will learn about the moment of inertia, and the potential and kinetic energy of a rotating object, and we will use this to study a simple electromechanical system, similar to a micro-mirror used in most modern projectors.

Modern production techniques for microscopic systems allows us to construct small mechanical elements made of silicon. For example, we can construct small

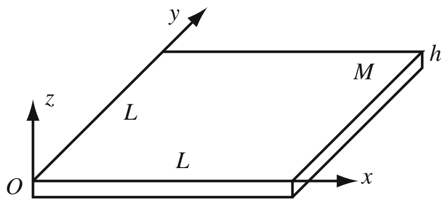

Fig. 15.19 Illustration of a square mechanical element. Each side has a length L, and the mass of the square is M. The thickness h is small compared to L

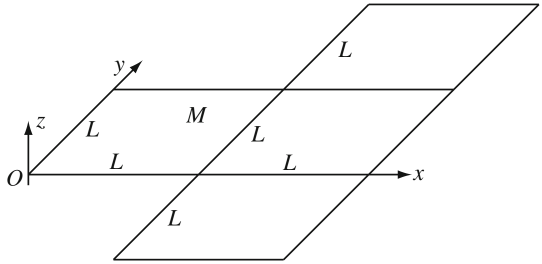

Fig. 15.20 Illustration of a cantilever constructed from four squares

silicon cantilevers with dimensions down to a few micrometers. In this project we will address the motion of a thin, microscopic beam using energy techniques.

First, let us consider a small, square mechanical element of dimensions $L \times L \times h$ and mass M. We assume that the thickness h is so small that we can neglect the finite thickness of the square. The square is attached with a hinge at one of the ends of length L. The hinge follows the y-axis as show in Fig. 15.19.
(a) Find the position $\mathbf{R} = X\,\mathbf{i} + Y\,\mathbf{j}$ of the center of mass of the square. The origin is in one of the corners of the square.
(b) Show by integration that the moment of inertia, $I_{cm,y}$, for rotations around an axis parallel with the y-axis going through the center of mass is $I_{cm,y} = ML^2/12$.
(c) Find the moment of inertia, I_y, for rotations around the y-axis.

The micromechanical cantilever we want to study is a bit more complicated than a single square. We can construct the cantilever from four identical squares, each with dimensions $L \times L$ and masses M, as illustrated in Fig. 15.20. The squares are rigidly attached to each other, so that they move as a single body. At the edge, along the y-axis, the cantilever is attached with a hinge so that the cantilever can rotate freely about this axis.
(d) Show that the center of mass of the cantilever is $\mathbf{R} = X\,\mathbf{i} + Y\,\mathbf{j} = (5/4)\,L\,\mathbf{i} + (L/2)\,\mathbf{j}$.
(e) Show that the moment of inertia, I_y for the whole object is $I_y = (22/3)\,ML^2$. Hint 1: You can calculate the moment of inertia for each part of the object independently and sum the results. (This is called the superposition principle). Hint 2: Use the parallel-axis theorem to find the moment of inertial around the y-axis for each part of the object.

Even though we are considering a microscopic system, where the effect of gravity typically will be negligible, let us first consider the motion of the cantilever when it is

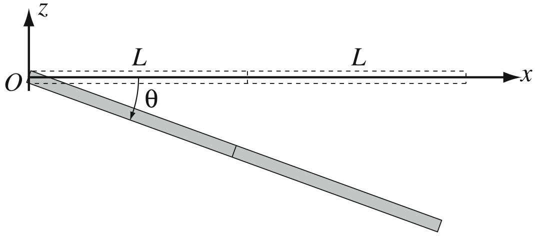

Fig. 15.21 Illustration of a cantilever in the *xz*-plane. The cantilever rotates around the *y*-axis to an angle θ

affected by gravity. For a real microscopic cantilever the effects of electrostatic forces is typically more important. Often such interactions results in a constant electrostatic force on the cantilever. The results from studying the behavior of the cantilever when affected by gravity as we do here can therefore easily be translated into the behavior of a cantilever affected by electrostatic forces.

The gravitational force acts in the negative *z*-direction. As a result of gravity, the cantilever rotates and angle θ around the *y*-axis as illustrated in Fig. 15.21.
(f) Show that the potential energy of the cantilever due to the gravitational force is $U_G = -5MLg \sin\theta$, where the potential energy is zero when the cantilever is horizontal.
(g) Assume that the cantilever starts with an initial angular velocity $\omega_0 = 0$ when $\theta = 0$. Find the angular velocity of the cantilever, $\omega(\theta)$, when it has reached the angle θ.

In the following, we will no longer assume that the cantilever rotates freely around the *y*-axis. Instead, we will assume that it bends around a hinge along the *y*-axis, and that the bending is like the bending of an elastic body. This means that there is an potential energy associated with the bending. The potential energy of the cantilever when it is bent an angle θ due to the stiffness of the hinge is $U_h = (1/2)\,\kappa\,\theta^2$ where κ is a constant that depends on the material properties (and the size) of the hinge.
(h) Again, assume that the cantilever starts with an initial angular velocity $\omega_0 = 0$ when $\theta = 0$. Find the angular velocity of the cantilever, $\omega(\theta)$, when it has reached the angle θ.
(i) Describe the motion of the cantilever.

For small θ we can approximate $\sin\theta \simeq \theta$. We will use this approximation in the following.
(j) Find the maximum angle θ of the cantilever when it is released as described above.
(k) Draw an energy diagram in the form of the total potential energy of the cantilever as a function of θ. Can you find any equilibrium points for the cantilever?

Small cantilevers are used for many technological applications. For example, projectors using the DLP technology consists of a vast number of micromirrors, small cantilevers that reflect light. When an electrical field is applied to the cantilever, the cantilever is affected by an electrostatic force. We can describe this in the same way as we described gravity above, but the electrical field can be turned on or off. As a result, the cantilever can be bent, and the light is reflected in a different direction. You can look at other interesting applications by searching for MEMS in your favourite search engine.