Chapter 10
Work

How can you find the motion of an atom moving near a surface according to a complicated position-dependent force without solving Newton's equation?

Up to now, you have learned to use Newton's laws of motion to determine the motion of an object based on the forces acting on it. The methods you have learned are completely general and can always be applied to solve a problem. Unfortunately, in many cases we cannot find an exact solution to the equations of motion we get from Newton's second law.

Here we introduce a commonly used technique that allows us to find the velocity as a function of position without finding the position as a function of time—an alternate form of Newton's second law. The method is based on a simple principle: Instead of solving the equations of motion directly, we integrate the equations of motion. Such a method is called an *integration method*. You will learn two integration methods: In this chapter we integrate Newton's second law in space using the work-energy theorem to find the speed as a function of position; in Chap. 12 we integrate Newton's second law in time to get conservation of momentum. While these methods are simple from a mathematical point of view, they introduce very important physical concepts that you will rely on throughout your career. You should therefore pay more attention to the use of these methods than to their derivation.

In this chapter, we introduce the work-energy theorem as a method to find the velocity as a function of position for an object even in cases when we cannot solve the equations of motion. This introduces us to the concept of work and kinetic energy—an energy related to the motion of an object. Finally we also address the rate of work done by a force—the power.

10.1 Integration Methods

In principle, we can determine the motion of any object if we know the net force, $\mathbf{F}^{net}$ acting on the object, by applying Newton's second law:

A. Malthe-Sørenssen, *Elementary Mechanics Using Matlab*,
Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-19587-2_10

$$m\mathbf{a} = \mathbf{F}^{\text{net}}(\mathbf{r}, \mathbf{v}, t). \tag{10.1}$$

and solve to the position at a time t, $\mathbf{r}(t)$, if we start from $\mathbf{r}(t_0)$. Since (10.1) is true, the integral of this equation must also be valid for the motion. Integral, you ask, what integral? Both the integral over time and the integral over the actual motion—the curve integral along the motion. The following two integrals of (10.1) also describe the motion:

$$\int_{t_0}^{t} m\mathbf{a}\, dt = \int_{t_0}^{t} \mathbf{F}^{\text{net}}(\mathbf{r}, \mathbf{v}, t)\, dt, \tag{10.2}$$

$$\int_C \mathbf{a} \cdot d\mathbf{r} = \int_C \mathbf{F}^{\text{net}}(\mathbf{r}, \mathbf{v}, t) \cdot d\mathbf{r}. \tag{10.3}$$

Ok. This may be true, but it seems entirely unmotivated. Why would we want to do this? Bear with us. It turns out that both these integrals are very useful and introduce powerful new concepts. In this chapter, we will focus on the integral in (10.3), while in the next chapter we focus on (10.2).

Path Integral

To understand the motivation, let us look at the integral in (10.3) in detail and calculate the left-hand side. What does the integral in (10.3) mean? It is the path integral along the curve, $\mathbf{r}(t)$. However, this integral may depend not only on the path, but also on how we move along the path—it may depend on the velocity $\mathbf{v}(t)$ along the curve. We should therefore replace the $d\mathbf{r}$ with $(d\mathbf{r}/dt)dt$ on both sides of the equation. The equation is still true since it is simply an integral of Newton's second law:

$$\int_{t_0}^{t} m\mathbf{a} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_{t_0}^{t} \mathbf{F}^{\text{net}}(\mathbf{r}, \mathbf{v}, t) \cdot \frac{d\mathbf{r}}{dt}\, dt. \tag{10.4}$$

Again, you may ask why this is useful. The short answer is that it is useful because we always can find the analytical solution to the left-hand side and we sometimes can find the solution to the right-hand side, even if we cannot find the analytical solution to the acceleration from Newton's second law.

What is the left-hand side of (10.4)? It can be solved using integration by parts:

$$\int_{t_0}^{t} m\mathbf{a} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_{t_0}^{t} m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\, dt = \mathbf{v} \cdot \mathbf{v} - \int_{t_0}^{t} \mathbf{v} \cdot \frac{d\mathbf{v}}{dt}\, dt, \tag{10.5}$$

which gives

$$\int_{t_0}^{t} m\, \mathbf{a} \cdot \frac{d\mathbf{r}}{dt}\, dt = \frac{1}{2} m\mathbf{v} \cdot \mathbf{v}. \tag{10.6}$$

The left-hand side of (10.4) therefore only depends on the magnitude of the velocity! We can therefore find the change in velocity, if we only can calculate the integral on the right-hand side of (10.4). The resulting integral equation is called the Work-energy theorem:

Work-energy theorem: For any motion $\mathbf{r}(t)$, we can find the change in velocities from the integral $W_{0,1}$:

$$W_{0,1} = \int_{t_0}^{t_1} \mathbf{F}^{\text{net}}(\mathbf{r}, \mathbf{v}, t) \cdot \frac{d\mathbf{r}}{dt}\, dt = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2. \tag{10.7}$$

Application of the Work-Energy Theorem

The power of the work-energy theorem is best demonstrated by an example. For an atom moving along a surface, as shown in Fig. 10.1, the force from the surface on the atom can ba approximated as:

$$F(x) = -F_0 \sin \frac{2\pi x}{b}, \tag{10.8}$$

where x is the position of the atom and b is the distance between the atoms on the surface. If we apply Newton's second law to find the motion of the atom, we get

$$\sum F_x = F(x) = -F_0\, sin \frac{2\pi x}{b} = ma \Rightarrow a = -\frac{F_0}{m} sin \frac{2\pi x}{b}, \tag{10.9}$$

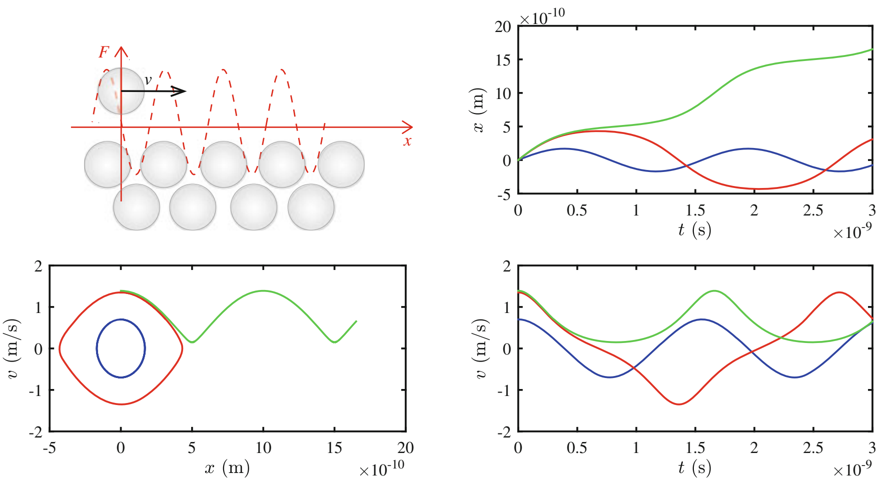

Fig. 10.1 Illustration of an atom moving along a period lattice of atoms, giving rise to a periodic force, $F(x) = -F_0 \sin kx$

which we can solve numerically, but not analytically. However, we can use the Work-energy integral to find the velocity as a function of position for this motion. We calculate the work integral

$$\int_{t_0}^{t} F(x(t))\,\frac{dx}{dt}\,dt = \int_{x(t_0)}^{x(t)} F(x)\,dx = \int_{x_0}^{x} -F_0 \sin\frac{2\pi x}{b}\,dx \tag{10.10}$$

$$= \frac{F_0 b}{2\pi}\left(\cos\frac{2\pi x}{b} - \cos\frac{2\pi x_0}{b}\right). \tag{10.11}$$

If the motion starts from $x_0 = 0$ with $v = v_0$, we get

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{t_0}^{t} F(x(t))\,\frac{dx}{dt}\,dt = \frac{F_0 b}{2\pi}\left(\cos\frac{2\pi x}{b} - 1\right). \tag{10.12}$$

and for the velocity, we find

$$v(x) = \pm\sqrt{v_0^2 + \frac{F_0 b}{m\pi}\left(\cos\frac{2\pi x}{b} - 1\right)}. \tag{10.13}$$

where the sign depends on what direction the atom is moving in. This expression is a complete solution of the motion. Figure 10.1 illustrates the numerical solution for $x(t)$ and $v(t)$ for various initial velocities and positions, and the corresponding plot of $v(x)$. We have plotted the analytical solution on top using circles. Notice the interesting pattern in this figure. We will spend more time developing our understanding of this model further on.

10.2 Work-Energy Theorem

The work-energy theorem is an alternative form of Newton's second law and therefore has the same range of applicability. We call the path integral along the curve the **work of the net force**:

$$\boxed{W_{0,1}^{\text{net}} = \int_{t_0}^{t_1} \mathbf{F}^{\text{net}}(\mathbf{r}, \mathbf{v}, t) \cdot \mathbf{v}\,dt.} \tag{10.14}$$

But this definitions seems to require that we know the both $\mathbf{r}(t)$ and $\mathbf{v}(t)$ in order to solve the integral. Hmmmm. Was not the whole point that we did need to find the analytical solution?

The usefulness of the formulation first comes to its right when the net force depends on the position *only*—that is, when the net force does *not* depend on the velocity:

$$W_{0,1}^{\text{net}} = \int_{t_0}^{t_1} \mathbf{F}^{\text{net}}(\mathbf{r}) \cdot \mathbf{v}\, dt = \int_C \mathbf{F}^{\text{net}}(\mathbf{r}) \cdot d\mathbf{r}. \tag{10.15}$$

In this case, we may be able to solve the integral, as we saw above, even if we cannot solve to find the motion. This gives the **work-energy theorem** for a position-dependent force:

$$W_{0,1}^{\text{net}} = \int_C \mathbf{F}^{\text{net}}(\mathbf{r}) \cdot d\mathbf{r} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2. \tag{10.16}$$

This equation has an even simpler form in one dimension when $\mathbf{F} = F\mathbf{i}$ and $d\mathbf{r} = dx$, giving

$$\boxed{W_{0,1}^{\text{net}} = \int_{x_0}^{x} F(x)\, dx = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.} \tag{10.17}$$

It is usual to introduce the term **kinetic energy**, K, for the right-hand side in the work-energy theorem

$$\boxed{K = \frac{1}{2}mv^2.} \tag{10.18}$$

This is the reason why we call the theorem the **work-energy theorem**. And we can now formulate it very compactly:

$$W_{0,1}^{\text{net}} = K_1 - K_0. \tag{10.19}$$

where it is usual to drop the subindex 0, 1 for the work.

Unit of work: The unit for work is *Joule* (J), which is defined as: $1\,\text{J}\,(\text{Joule}) = 1\,\text{Nm} = 1\,\text{kgm}^2/\text{s}^2$.

Comments on the Work-Energy Theorem

The work-energy theorem has several important features:

- The work-energy theorem is an alternative formulation of Newton's second law of motion, and is therefore valid as long as Newton's laws are valid. For example, it is only valid in an inertial system. It is not valid in an accelerated coordinate system.
- The work-energy is only true if you find the work of the *net force*. Do not forget or leave out any of the forces acting on the object.
- Notice that most microscopic laws of motion, including all interatomic interactions, only depend on position. The same is true for gravitational forces between astronomical objects. There is a large span of processes where the net force is only

position dependent. The special formulation in (10.17) is therefore a very useful law.

For example, if you take a block and pull it back and forth a few times on the floor, you cannot use (10.17) to find the work because both the friction force from the floor on the block and the driving force (you pulling or pushing the block) varies not only with position, but also with time and velocity. After pushing the block back and forth you end up at the same place. If you used (10.17), the work would therefore be zero, which is incorrect. You have performed work on the block even if the block ends in the same place it started.

Superposition of Work

The work-energy theorem is only valid for the work done by the net force. If there are several forces acting on an object, we may number the forces $\mathbf{F}_j$ from $j = 1$ to $j = n$. The net force is the sum of the forces:

$$\mathbf{F}^{\text{net}} = \sum_j \mathbf{F}_j , \tag{10.20}$$

The work done by the net force can therefore be written as:

$$W^{\text{net}} = \int_{t_0}^{t_1} \mathbf{F}^{\text{net}} \cdot \mathbf{v}\, dt = \int_{t_0}^{t_1} \sum_j \mathbf{F}_j \cdot \mathbf{v}\, dt = \sum_j \int_{t_0}^{t_1} \mathbf{F}_j \cdot \mathbf{v}\, dt = \sum_j W_j . \tag{10.21}$$

where W_j is the work done by force j. The work done by the net force is therefore the sum of the work done by each of the forces acting. For a one-dimensional motion with a position-dependent force, $\mathbf{F} = F(x)\,\mathbf{i}$ this simplifies to:

$$W_j = \int_{t_0}^{t_1} \mathbf{F}_j \cdot \mathbf{v}\, dt = \int_{x_0}^{x} F(x)\, dx . \tag{10.22}$$

The Concept of Work

I am sure you have an intuition of what physical work is, but does that correspond to our definition of mechanical work? Intuitively, it requires work to push a box along the floor. The longer we push, the more work it requires. The heavier the box (and hence the larger the friction force), the more work is done. Here, our intuition is consistent with our definition.

However, if you lift a box from the floor, it requires work. Both in the ordinary use of the word and in our precise definition. But I am sure you know that just holding a heavy box in your arms requires effort, although it requires no work according to our definition. Trying to push a very heavy box without succeeding also requires no mechanical work, but still requires effort on your behalf. The reasons for this discrepancy are related to how we perceive and experience trying to move something, to how our muscles work inside our bodies, and to how we perceive movement: your body may still move somewhat while the box is kept approximately at a constant height.

However, a mechanical analysis of the work done when you move your body is useful, and does result in real effects that you can feel. It is, for example, possible to design a backpack that requires less effort to carry long distances based on our understanding of work (and energy conservation).

Most importantly, it is important to try to separate the very precise definition of mechanical work from the looser concept from everyday speech.

10.3 Work Done by One-Dimensional Force Models

We apply the work-energy theorem to various force models, such as a constant force, a spring force, and a given position-dependent force—all in one dimension.

Work of a Constant Force

The work done by a constant force, $F = F_0$, as a car accelerates/decelerates from x_0 to x_1 is

$$W = \int_{t_0}^{t_1} F_0 \, v \, dt = \int_{x_0}^{x_1} F_0 \, dx = F_0 \, (x_1 - x_0) = F_x \Delta x. \tag{10.23}$$

If the force F_0 is the only force (or the net force) on the car, the work W corresponds to the change in kinetic energy.

- Notice that if the force and the displacement, Δx, are in the same direction, the **work is positive**. If this is the net force, it means that the kinetic energy increases, and that the speed increases.
- If the force and the displacement are in opposite direction, for example if the car is moving in the positive x-direction while it is breaking with a constant force in the negative x-direction, the **work is negative**. If this is the net force on the car, the kinetic energy decreases and the speed decreases.

Work of a Spring Force

One of the most commonly used models for a contact force is the spring model. What is the work done by a spring force? Figure 10.2 illustrates the motion of a block on a frictionless horizontal surface. The block is attached to a spring with spring constant k. The other end of the spring is attached at the origin, $x = 0$, and the equilibrium length of the spring is L_0. The force, F_x, from the spring on the block is then:

$$F_x = -k\,(x - L_0)\,. \tag{10.24}$$

The position, x, where the spring force is zero is called the *equilibrium position*. Here, the equilibrium position is $x = L_0$.

The work done by the spring force on the block as the block moves from $x(t_0) = x_0$ to $x(t_1) = x_1$ is:

$$W_{0,1} = \int_{t_0}^{t_1} F_x v_x dt = \int_{x_0}^{x_1} F(x) dx = \int_{x_0}^{x_1} -k\,(x - L_0)\,dx = . \tag{10.25}$$

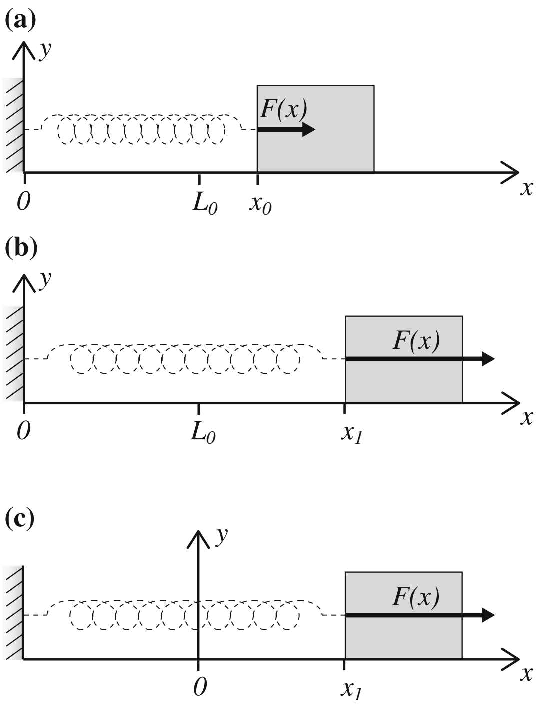

Fig. 10.2 A block attached to a spring on a frictionless surface. **a**, **b** The friction force F is illustrated at two different positions x_0 and x_1 of the block. **c** The origin is moved to the equilibrium position for the spring

We change integration variable to $u = x - L_0$, $du = dx$, getting:

$$W = \int_{x_0-L_0}^{x_1-L_0} -ku\,du = \frac{1}{2}k(x_0 - L_0)^2 - \frac{1}{2}k(x_1 - L_0)^2. \quad (10.26)$$

This result becomes simpler if we move the origin to the equilibrium position of the spring, as shown in Fig. 10.2c, so that $F(x) = -kx$. The work from x_0 to x_1 is then

$$W = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_0^2. \quad (10.27)$$

Work of a Position-Dependent Force

The work of a position-dependent force $F(x)$ is found through the integral

$$W = \int_{x_0}^{x} F(x)\,dx. \quad (10.28)$$

This force $F(x)$ may be a simple function, such as $F(x) = -kx$ or $F(x) = F_0 \sin 2\pi x/b$. In that case you can simply solve the integral analytically. But what if you cannot solve the integral analytically or the function $F(x)$ is not know exactly, but instead is measured in a discrete number of points, x_i. How can you then find the work?

Symbolic Integration of a Function $F(x)$

Even if *you* cannot (or you are too lazy to) solve the integral analytically, you can always check if matlab can the indefinite integral symbolically. We demonstrate this for a force $F(x) = 1/(a + x^2)$ for $x > 0$. We integrate this function using the symbolic package in matlab as follows:

```
>> syms x a
>> int(1/(a+x^2),x)
ans = atan(x/a^(1/2))/a^(1/2)
```

Numerical Integration of a Function $F(x)$

However, if we cannot find an analytical solution to the integral of $F(x)$, we can always calculate the definite integral numerically. The integral from x_0 to x_1 of $F(x)$ in Fig. 10.3 corresponds to the area under the curve from x_0 to x_1. We can calculate the area by first dividing the interval into smaller pieces, finding an approximate value

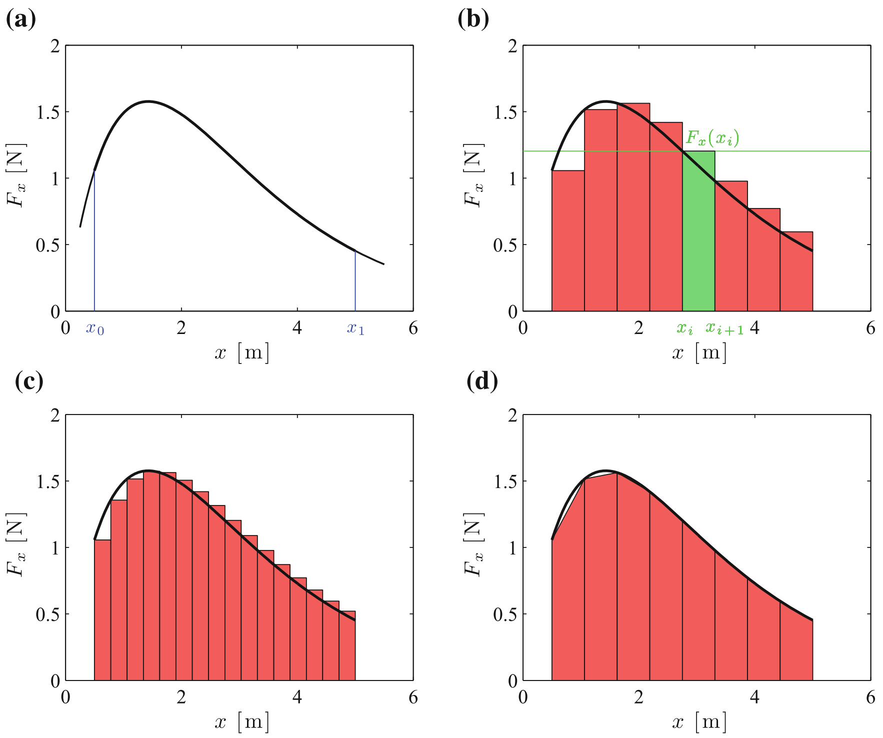

Fig. 10.3 Plot of the net force $F_x(x)$ on an object moving from x_0 to x_1. **a** Plot of $F_x(x)$. **b** Illustration of the numerical integral of $F_x(x)$ found as a sum of small rectangles of size Δx. **c** The numerical integral with half the box width, $\Delta x/2$. **d** The numerical integral as a sum of trapezoids

for the area of each such piece, and summing the areas to find total area corresponding to the integral.

We divide the interval from x_0 to x_1 into n intervals of length $\Delta x = (x_1 - x_0)/n$ so that interval i spans from x_i to $x_{i+1} = x_i + \Delta$ (see Fig. 10.3). The area under the curve $F(x)$ over the interval from x_i to x_{i+1} is the integral:

$$W_{i,i+1} = \int_{x_i}^{x_{i+1}} F(x)\, dx. \tag{10.29}$$

When Δx is small, the area under the curve is approximately equal to the area of a rectangle of width Δx and a height given by the value of $F(x)$ at x_i, as illustrated in Fig. 10.3b. The area of this rectangle is $W_{i,i+1} = A_i \simeq \Delta x\ F(x_i)$, and the total area is the sum of all the areas A_i, $i = 1, \ldots, n$ as illustrated in Fig. 10.3b:

$$W_{0,1} = \sum_{i=1}^{n} W_{i,i+1} = \sum_{i=1}^{n} \Delta x\ F(x_i). \tag{10.30}$$

This approach is identical to Euler's method from Chap. 4. As we increase the number of intervals n, the numerical approximation becomes better and better. Figure 10.3c shows how the area of the rectangles becomes a better approximation to the area under the curve when n is doubled. We expect that our approximation approaches the exact value for the integral as n increases.[1]

While Euler's method is simple to explain, we can obtain a better approximation by using trapeziods with corners $F(x_i)$ and $F(x_{i+1})$. The area of one such trapezoid is the baseline Δx multiplied by the average height h_i:

$$W_{i,i+1} = A_i \simeq \Delta x \, h_i = \Delta x \, \frac{1}{2} \left[F(x_i) + F(x_{i+1}) \right] , \tag{10.31}$$

as illustrated by the green trapezoid in Fig. 10.3d. The total integral is then:

$$W_{0,1} = \sum_{i=1}^{n} \Delta x \frac{1}{2} \left[F_x(x_i) + F_x(x_{i+1}) \right] . \tag{10.32}$$

This method is called the *trapezoidal rule* for numerical integration. The numerical implementation of this method is just as simple as Euler's method.

The trapezoidal rule is a standard numerical integration method that is built into matlab through the function `trapz`. For example, we can integrate the work done by the function $F(x)$ shown in Fig. 10.3a:

$$F_x(x) = 3xe^{-0.7\,x} , \tag{10.33}$$

from $x_0 = 0.5$ m to $x_1 = 5$ m in $n = 1000$ steps by: first generating a set of n x_i values; then generating the corresponding set of $F(x_i)$ values; and finally calculating the integral using `trapz`:

```
x = linspace(0.5,5,1000);
y = 3*x.*exp(-0.7*x);
W = trapz(x,y);
```

(Notice that the integral of this particular function $F(x)$ is analytically solvable. We have used it here to illustrate the principle of how to solve an integral numerically).

Numerical Integration of a Measured $F(x_i)$

In some case, the force $F(x)$ may be the result of a more complicated calculation or it may be an experimentally measured value. For example, you may calculate the net force on a basketball bouncing on the floor using an advanced model for the deformation of the surface of the ball; you may calculate the forces between two atoms as a function of their position based on an underlying microscopic picture such as quantum mechanics; or you may measure the force as a function of position

[1] In the limit when n becomes infinitely large, this is indeed the definition of the integral.

as you pull on the string of your bow. In all these cases you know the force $F(x_i)$ for some values x_i, but you do not know the general function $F(x)$.

How can we estimate the work integral based on a few points? One approach would be to try to find a smooth function $F(x)$ that goes through all the points, and then use this function to calculate the work numerically. This is a powerful method, but beyond the scope of this text. Another method is to apply the trapezoidal rule on the discrete data as an approximation to the integral, as illustrated in Fig. 10.4. Fortunately, the trapezoidal rule is so robust that we may apply it also in cases when the data is not uniformly spaced—that is, when the intervals $\Delta x_i = x_{i+1} - x_i$ vary. In this case, we approximate the integral from x_0 to x_1 by the sum:

$$W_{0,1} = \int_{x_0}^{x_1} F(x)\,dx \simeq \sum_{i=1}^{n} \Delta x_i \, \frac{1}{2}\left[F(x_i) + F(x_{i+1})\right]. \tag{10.34}$$

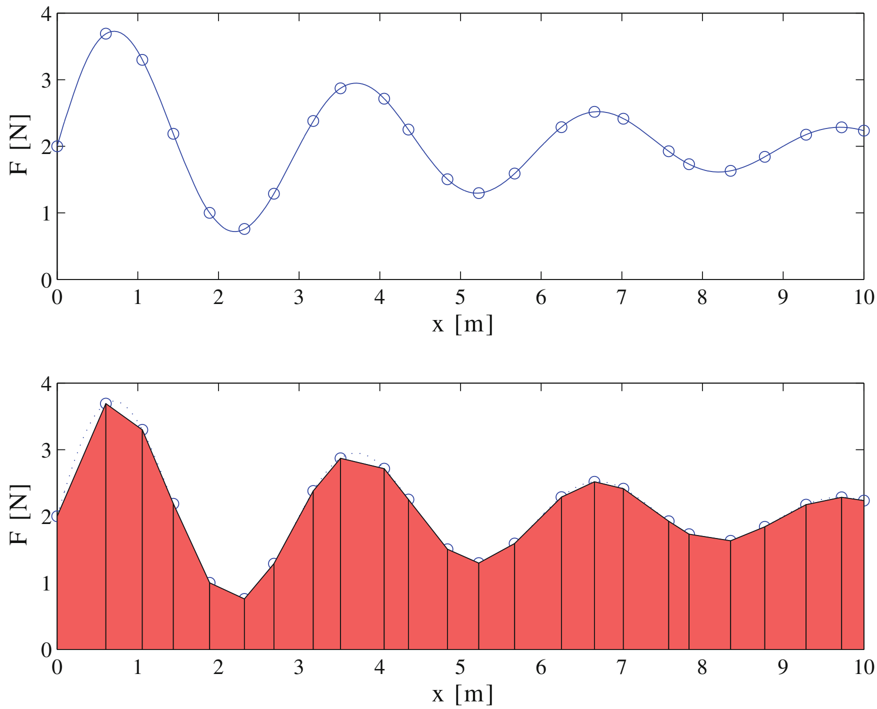

Fig. 10.4 **a** Illustration of a position-dependent force $F(x)$ showing a real underlying function, corresponding to the unknown $F(x)$ (which we here have supposed that we do not really know), and the values x_i where we have measured or calculated the value of the force, $F_i = F(x_i)$. **b** Illustration of the area under the curve, corresponding to the work integral, calculated using the trapezoidal rule using the discrete dataset only

This method works for both calculated and measured values of $F(x_i)$, and has exactly the same implementation as above. For example, given a file forcedata.d[2] consisting of lines with values of x_i and $F(x_i)$, we can read the data and calculate the work integral by:

```
load forcedata.d
x = forcedata(:,1);
F = forcedata(:,2);
W = trapz(x,F);
```

The dataset is shown in Fig. 10.4 with an illustration of the trapezoidal approximation to the integral.

10.3.1 Example: Jumping from the Roof

In this example you will be introduced to how to apply the work-energy problem to solve actual problems, applying it to a constant force and a spring force.

You are standing on top of your house and are wondering how to jump down without getting hurt: You can jump into a thick snow cover, which exhibits a constant force, or onto a trampoline, which exhibits a spring force. What alternative would exert the smallest force?

Specify the problem: This problem is formulated loosely on purpose. You should be able to address such problems by adding the necessary components yourself. Let us make the problem more specific by adding details and assumptions: You have a mass m and your roof is a height h above the ground. You stop after a distance d. Also, we neglect air resistance.

Sketch: The problem addresses the motion of a person falling through air and then into various materials. We use $y(t)$ for the vertical position and place the origin at the top of the cushion with positive direction upward as shown in Fig. 10.5.

Model: We divide the motion into two phases, as illustrated in Fig. 10.5. Just like when we solve problems using Newton's second law, we start by analyzing the forces acting on the object.

Phase 1: The person falls through air from $y_0 = h$ to $y_1 = 0$. The only force acting is gravity, $G_y = -mg$.

Phase 2: The person is in contact with the surface from $y_1 = 0$ to $y_2 = -d$. There is a contact force from the surface, F acting upward in the positive y-direction and gravity, $G_y = -mg$.

Applying the work-energy theorem: We could now use Newton's second law to find the position $y(t)$ as a function of time. We know this would work, but it is a lot of work.

[2] http://folk.uio.no/malthe/mechbook/forcedata.d.

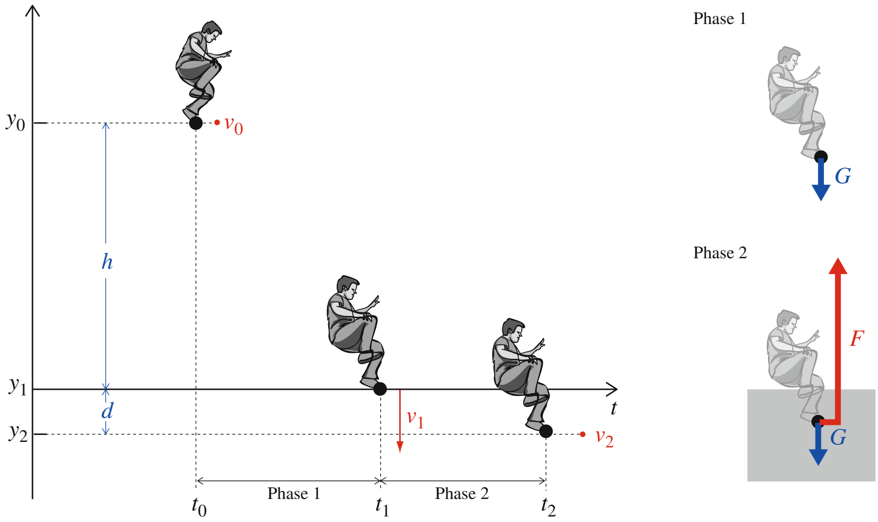

Fig. 10.5 A person jumping from the roof into various braking “devices”

However, in this case, we do not care about the time it takes for the person to fall from the house and then brake during contact with the surface. We only care about the velocity immediately before contact with the surface, and the distance he needs to stop while in contact with the surface: We are only interested in questions relating the velocity v_y to the position y: First, what is the velocity when $y = 0$? Second, what is the position y when the velocity is zero, that is, when you have stopped? (Ok, this is not really the question. We actually want to find the force F necessary to ensure that the velocity is zero after a length d into the surface, but this is essentially the same as finding the distance d it moves before the velocity is zero when affected by a force F).

Our plan is to use the work-energy theorem for the first phase to find the velocity immediately before you hit the surface, expressed as a function of the height, h, you jumped from. We will then use the work-energy theorem for the second phase to find the distance d you move before you stop, expressed in terms of the velocity you had when you hit the surface. Finally, we will relate the stopping distance d to the initial height h, and then find the maximum force during phase two.

Finding the velocity before impact: In *Phase 1* the net force is constant: $F_y^{\text{net}} = -mg$. We find the change in velocity from y_0 to y_1 using the work-energy theorem: $K_1 - K_0 = W_{0,1}$, where the work of the net force from y_0 to y_1 is:

$$W_{0,1} = \int_{t_0}^{t_1} F_y^{\text{net}} v dt = \int_{y_0}^{y_1} (-mg)\, dy = -mg\,(y_1 - y_0) = mgh, \qquad (10.35)$$

where we have used that $y_1 = 0$ and $y_0 = h$. This is equal to the change in kinetic energy:

$$W_{0,1} = mgh = K_1 - K_0 = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 = K_1. \tag{10.36}$$

where we have used that the person starts from rest, $v_0 = 0\,\text{m/s}$. This gives that $K_1 = mgh$, which is what we will need to determine the stop length, d.

Contact with the surface: In *Phase 2* the net force is:

$$F_y^{\text{net}} = F - mg, \tag{10.37}$$

where the force F from the surface depends on the nature of the surface. We will therefore treat the various surfaces independently.

Falling into snow—Constant force model: For snow, we assume that F is a constant force. We use the work-energy theorem $W_{1,2} = K_2 - K_1$ to relate the stopping distance d to the kinetic energy K_1 when the contact started. The work of the net force over the distance $\Delta y = y_2 - y_1 = 0 - d = -d$ is:

$$W_{1,2} = \int_{y_1}^{y_2} (F - mg)\; dy = (F - mg)\; (-d) = K_2 - K_1. \tag{10.38}$$

where $K_2 = 0$ sice the person stops at y_2. We insert K_1 from (10.36), getting:

$$-(F - mg)\, d = -K_1 = -mgh \;\Rightarrow\; F - mg = mg\frac{h}{d} \;\Rightarrow\; F = mg\left(1 + \frac{h}{d}\right). \tag{10.39}$$

Falling into snow—Discussion: Notice the simplicity of this approach. We do not even have to calculate the velocity v_1 after the person has fallen a height h.

For a typical house of 6 m height and for a typical person of height 2 m and a stopping distance of 1 m, the force from the snow is $F = mg\,(1 + 6) = 7\,mg$.

The work integral during the free fall is illustrated as the blue area in Fig. 10.6, and the work integral during contact is illustrated as the red area. After the free fall, the person has a kinetic energy corresponding to the blue area. Similarly, the red area corresponds to the change in kinetic energy during contact. The person stops when the red area is equal to the blue area. Notice that the fall starts on the right hand side of Fig. 10.6, which corresponds to high y-values, and then progresses toward the left. This graph can be a useful tool to discuss the motion.

Test your understanding: Based on Fig. 10.6 and that you stop over a distance d, what do you think is the force model that gives the lowest maximum force during the brake?

Falling onto a trampoline—Spring force model: For a trampoline, the force F is a spring force $F(y) = -ky$ when $y < 0\,\text{m}$. The net force on the person in *Phase 2* is therefore

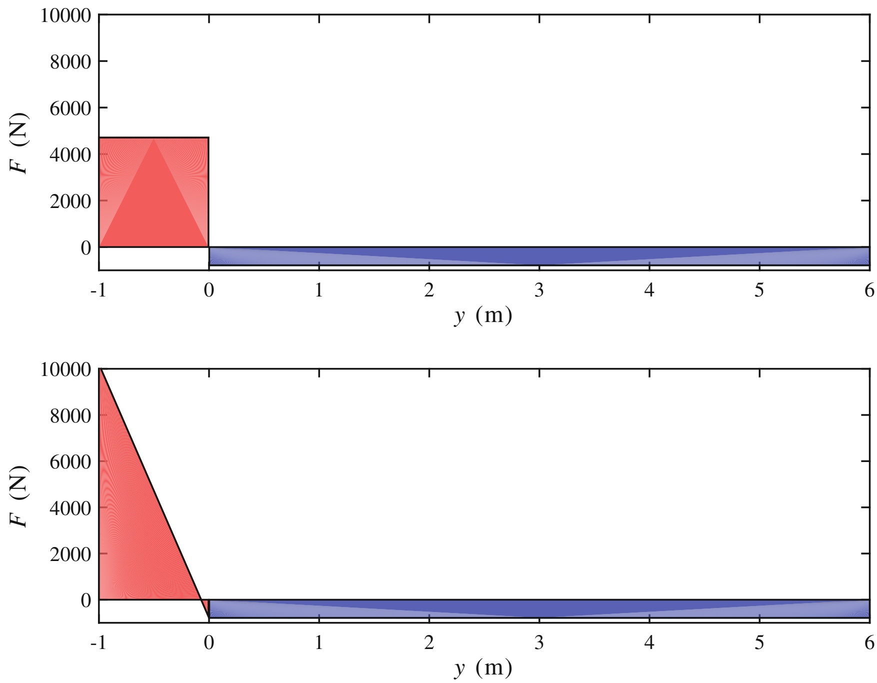

Fig. 10.6 Illustration of the work integrals. The *red area* corresponds to phase 1, when the person is falling through the air, and the blue area corresponds to phase 2, when the person is in contact with the surface. (*Top*) Constant forces. (*Bottom*) Spring forces

$$F_y^{\text{net}} = F - mg = -ky - mg, \tag{10.40}$$

which is a force that depends *only* on the position. The work of the net force from y_1 to y_2 is therefore:

$$\begin{aligned} W_{1,2} &= \int_{t_1}^{t_2} F_y^{\text{net}} v dt = \int_{y_1}^{y_2} (-ky - mg)\, dy = \int_{y_1}^{y_2} -ky dy + \int_{y_1}^{y_2} -mg dy \\ &= -k\left(\frac{1}{2}y_2^2 - \frac{1}{2}y_1^2\right) - mg\,(y_2 - y_1) = -\frac{1}{2}kd^2 + mgd, \end{aligned} \tag{10.41}$$

where we have used that $y_2 = -d$ and $y_1 = 0\,\text{m}$. We insert this result in the work-energy theorem:

$$W_{1,2} = -\frac{1}{2}kd^2 + mgd = K_2 - K_1 = -K_1 = -mgh = \tag{10.42}$$

where we used that $K_2 = 0$ and that $K_1 = mgh$ from (10.36).

Falling onto a trampoline—Discussion: This result allows us to find k if we know d and h, or, alternatively, to find d if we know k and h. Here, we know d and h and are interested in finding k, since this allows us to calculate the force from the trampoline. From (10.42) we get:

$$\frac{1}{2}kd^2 = mg\,(d+h)\,. \tag{10.43}$$

The force from the trampoline on the person jumping increases with the deformation of the trampoline, and is at its maximum when the trampoline is maximally deformed, which occurs when $y = -d$. The force is then:

$$F_{\max} = kd = \frac{2}{d}\frac{1}{2}kd^2 = \frac{2}{d}mg\,(d+h)\,, \tag{10.44}$$

where we have used the result from (10.43). Again, we assume that reasonable values are $h = 6\,\text{m}$ and $d = 1\text{m}$, giving:

$$F_{\max} = kd = 2\,mg\,(1 + (6\,\text{m}/(1\,\text{m})) = 14\,mg, \tag{10.45}$$

which is double the value of the constant force $F = 7\,mg$ we found for the constant force above.

Again, the simplicity of the approach is striking. We do not have to calculate the velocity v_1 after the person has fallen a height h, we only need to know the kinetic energy at this point in order to carry out the rest of the calculation.

The work integrals are illustrated in Fig. 10.6. The blue area corresponds to the net work during free fall, and the red area corresponds to the net work during braking. Here, we have essentially selected the spring constant k, which is the slope of the curve, so that the red area over a length d corresponds to the blue area over the length h.

Test your understanding: How will d and F change if you double the spring constant k of the trampoline?

10.3.2 Example: Stopping in a Cushion

You have installed a new soft cushion from SoftCush technology to prevent falling damage. According to the producers, the cushion has been developed to produce a normal force that depends on the compression of the cushion only (and not on the speed of deformation). You do not know the functional form of the force response, $F(y)$, of the cushion, but you have measured the response in a controlled experiment where you pushed the cushion down to a position y and measured the correspond-

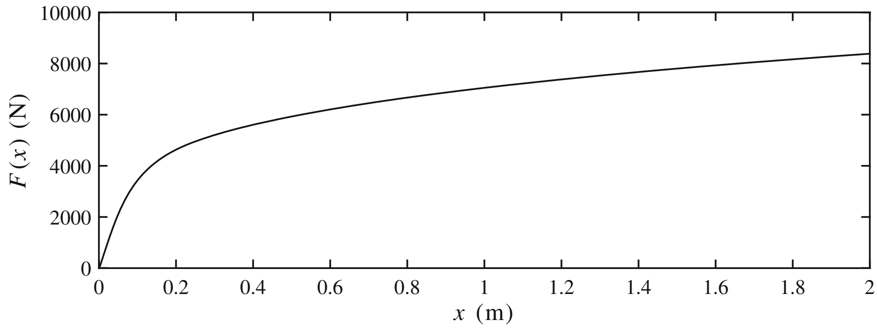

Fig. 10.7 The measured force $F(y)$ as a function of the position y of the *top* of the cushion. Notice that y is negative when the cushion is compressed

ing force $F(y)$. The results are provided in the file cushionforce.d,[3] and shown in Fig. 10.7.

Problem: If you jump from a height h and onto the cushion, how far will it compress before you stop? What is the maximum of the force from the cushion on you?

Approach: The cushion reaches its maximum compression when you stop, that is, when your kinetic energy is zero. We will first use the work-energy theorem to find the kinetic energy you have after falling a distance h, and then we will use the work-energy theorem during the contact with the cushion, from you touch the cushion and until you stop.

Sketch and Identify: Figure 10.8 shows an illustration of the forces acting on the person during the jump and a free-body diagram of the person.

Model: We divide the motion into two phases, phase 1 from $x_0 = h$ to $x_1 = 0$, where the person is only affected by gravity, $G = mg$, and phase 2 from $x_1 = 0$ to $x_2 = -d$, where the person is affected by gravity, G, and the force from the cushion, F.

Phase 1: Free fall: In phase 1 the person starts from rest, $v_0 = 0$ and $K_0 = 0$, at a height $x_0 = h$, and reaches the height, $x_1 = 0$ with a velocity, v_1, and a kinetic energy, K_1.

We find K_1 by applying the work-energy theorem from 0 to 1:

$$W_{0,1} = \int_{x_0}^{x_1} G\,dx = \int_{h}^{0} -mg\,dx = mgh = K_1 - K_0 = K_1, \tag{10.46}$$

where we have used that $K_0 = 0$. We see that $K_1 = (1/2)mv_1^2 = mgh$, so that we can determine v_1 if needed.

[3] http://folk.uio.no/malthe/mechbook/cushionforce.d.

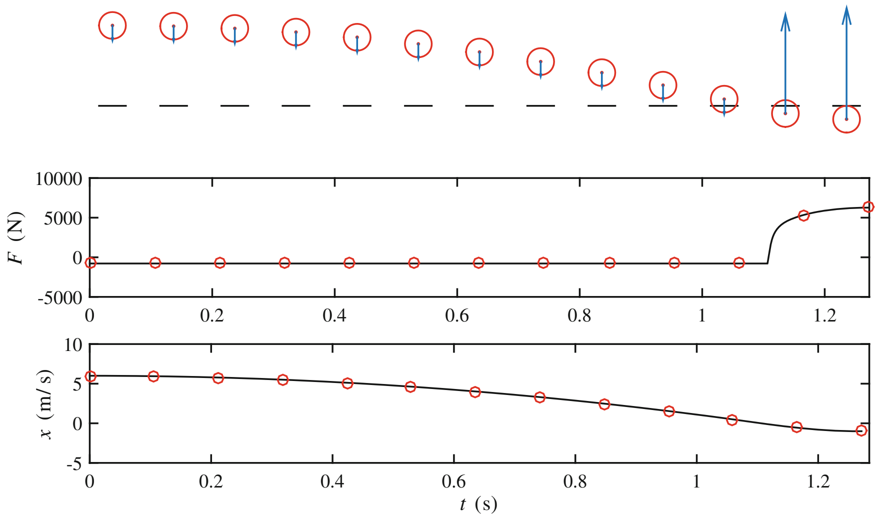

Fig. 10.8 The measured force $F(y)$ as a function of the position y of the *top* of the cushion. Notice that y is negative when the cushion is compressed

Phase 2: Contact with cushion: In phase 2 the person starts with velocity v_1 and kinetic energy K_1 at the height $y_1 = 0$ and stops with velocity $v_2 = 0$ and kinetic energy $K_2 = 0$ at height $y_2 = -d$. You are in contact with the SoftCusion, and the net force affecting you is therefore:

$$F^{\text{net}} = F(x) - mg, \tag{10.47}$$

We apply the work-energy theorem from 1 to 2 to determine d, the stopping distance:

$$W_{1,2} = \int_{x_1}^{x_2} F^{\text{net}}\, dx = \int_0^{-d} (F(x) - mg)\, dx = K_2 - K_1 = -mgh, \tag{10.48}$$

where we have used that $K_2 = 0$ and $K_1 = mgh$ from (10.46).

The idea is that the value for d that satisfies (10.48) gives us the maximum compression of the cushion. Unfortunately, (10.48) is not an explicit equation in d that we can solve since the unknown d is the upper limit in the integral. If we knew how to solve the integral analytically, we might have been able to solve the equation, but in this case we only know how to solve the integral numerically. How can we solve such an equation and find the d that satisfies (10.48)?

What we need to do, is to calculate the work:

$$W_{1,2} = \int_0^{x^*} F(x)dx - mgx^*, \tag{10.49}$$

as a function of the position x^* of the jumper. Our plan is to vary x^* until we find a value for x^* that satisfies the equation, which corresponds to $-d$. We need to vary x^* systematically. We start from $x^* = 0$, where the jumper comes in contact with the cushion, and then gradually decrease x^* (remember that the jumper is moving down during the contact with the cushion) until the equation is satisfied, that is, until we find the value for x^* for which $W_{1,2}$ is closest to $-mgh$. More precisely, we make a sequence of x^* values starting from 0, decreasing in small steps, Δx: $x_0^* = 0$, $x_1^* = x_0^* + \Delta x$, and we calculate the work for each of these numbers until the work is equal to $-mgh$.

Can we use any step size Δx in this sequence of x_i^* values? If we knew the force $F(x)$ for any position x, we could calculate the work at any resolution Δx. But in this case we only know the forces $F(x_i)$ for the values x_i where it has been measured. Therefore, the best possible resolution we can get is to use the measured x_i values as our sequence of x_i^*-values. This means that we start by calculating the work for $x_1^* = x_1$. Then we calculate the work for $x_2^* = x_2$, then for $x_3^* = y_3$ and so on, until, for some value i, the work is approximately equal to $-mgh$.

How do we calculate the work for $x^* = x_k$, where k is a number in the sequence of x_i values in the datafile? The integral in (10.49) can be calculated using the trapezoidal rule even if we do not know the underlying function—it works well also for a measured dataset:

$$\int_0^{x_k} F(x)\,dx \simeq \sum_{i=1}^{k} (x_{i+1} - x_i)\,\frac{1}{2}\left[F(x_i) + F(x_{i+1})\right]. \tag{10.50}$$

Numerically, this is done using the function `trapz`. Here, we only want to do the sum over the first k values of x_i, that is, we only want to do the sum from x_1 to x_k in the list of n such x-values. This corresponds to doing the sum over the first `(1:k)` elements in the position array `x`:

```
load cushionforce.d
x = cushionforce(:,1);
F = cushionforce(:,2);
k = 10;
I = trapz(x(1:k),F(1:k));
```

We calculate the work for all possible values of k, and store the results in an array `work`, which gives us the work as a function of k:

```
m = 80.0; % kg
g = 9.8;  % m/s^2
h = 6.0;  % m
load cushionforce.d
x = -cushionforce(:,1);
F = cushionforce(:,2);
n = length(x);
work = zeros(n,1);
for k = 2:n
    I = trapz(x(1:k),F(1:k));
    work(k) = I - m*g*x(k);
end
plot(x,work,'-b')
xlabel('x (m)');
ylabel('W(x) (J)');
```

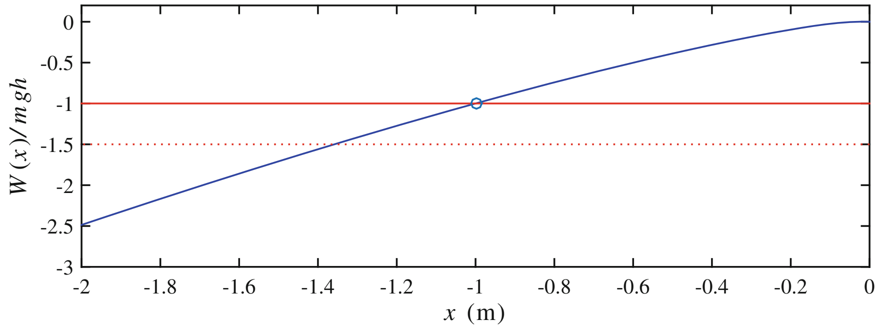

Fig. 10.9 Plot of the work $W_{1,2}$ as a function of the position x of the jumper

Figure 10.9 shows the work $W_{1,2}$ as a function of the position x calculated using this program. In the same plot we have also shown the value for $-mgh$. We see that the work starts at zero and decreases gradually as the deformation increases and x becomes a large negative number because the work done is negative: The net work acts to reduce the kinetic energy. When the work reaches the value $-mgh$ the jumper stops: the kinetic energy is now zero. How can we find the x-value this occurs at from our numerical data? Based on the plot, we realize that the x value when $W_{1,2} = -mgh$ can be found as the first value in the sequence of x_i's for which $W_{1,2} < -mgh$. This value of x will correspond to a work that is slightly smaller than $-mgh$, but this is a good first approximation. How can we find this value numerically? We use the function `find`:

```
i2 = min(find(work<mgh));
x2 = x(i2)
```

The function `find(work<-mgh)` returns a list of all the indexes of `work` that satisfies the condition `work<-mgh`. We need to find the first of the indexes in this sequence, therefore we take the minimum index in this list, and find the corresponding y-value. We have found the maximum compression of the cushion!

What is the cushion force at this value? This is the corresponding cushion force: $F(y_2) = 7050$ N found by:

```
>>F2 = F(i2)
F2 =      7050
```

What would happen if the jumper started from $h = 9$ m instead? We plot the corresponding value for $-mgh$ in Fig. 10.9. You can read the maximum compression directly from this graph, or determine it numerically as we did above.

10.4 Work Done in Two- and Three-Dimensional Motions

The work-energy theorem was demonstrated for a three-dimensional motion. We have so far studied one-dimensional motions only. How does the application of the theorem change for three-dimensional motion?

Work of Tangential and Normal Forces

Figure 10.10 illustrates a general three-dimensional motion from 0 to 1. For this motion, the net work is

$$W = \int_{t_0}^{t_1} \mathbf{F}^{\text{net}} \cdot \mathbf{v}\, dt = K_1 - K_0. \tag{10.51}$$

Because $\mathbf{v}$ points in the tangential direction, we notice that *it is only the tangential component of the force* that does any work! We can demonstrate this by introducing a local coordinate system along the path of motion, with unit vectors $\hat{u}_T$ in the tangential direction and $\hat{u}_N$ in the normal direction. The velocity vector points along the tangential unit vector:

$$\mathbf{v} = v\, \hat{u}_T(t). \tag{10.52}$$

If we decompose a force, $\mathbf{F}_j$, in the tangential and normal directions along the path of motion:

$$\mathbf{F}_j = F_{j,T}(t)\, \hat{u}_T(t) + F_{j,N}(t)\, \hat{u}_N(t), \tag{10.53}$$

we see that the work done by the force $\mathbf{F}_j$ is:

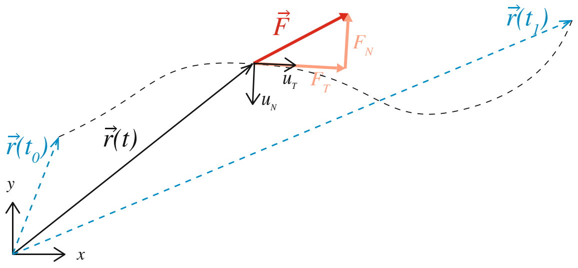

Fig. 10.10 Illustration of the path followed by an object moving from position 0 to position 1

$$W_j = \int_{t_0}^{t_1} \mathbf{F}_j \cdot \frac{d\mathbf{r}}{dt}\, dt \tag{10.54}$$

$$= \int_{t_0}^{t_1} \left(F_{j,T}(t)\hat{u}_T(t) + F_{j,N}(t)\hat{u}_N(t)\right) \cdot \frac{d\mathbf{r}}{dt}\, dt \tag{10.55}$$

$$= \int_{t_0}^{t_1} F_{j,T}(t)\hat{u}_T(t) \cdot \frac{d\mathbf{r}}{dt}\, dt + \int_{t_0}^{t_1} F_{j,N}(t)\, \underbrace{\hat{u}_N(t) \cdot \frac{d\mathbf{r}}{dt}}_{=0}\, dt \tag{10.56}$$

$$= \int_{t_0}^{t_1} F_{j,T}(t)\hat{u}_T(t) \cdot \frac{d\mathbf{r}}{dt}\, dt. \tag{10.57}$$

where we have used that the normal unit vector, $\hat{u}_N$, is normal to the velocity vector, $\mathbf{v}$. Consequently, it is only the tangential component of the force that contributes to the work.

This is also intuitive, since we know that normal forces only contribute to a change in the direction of the velocity vector and not to a change in the speed—it is only the tangential force that can cause a tangential acceleration which causes a change in speed.

Work of a Constant Force in Two and Three Dimensions

The work done by a constant force has a particularly simple solution. Let us address the work done on an object by a constant force $\mathbf{F}$ as the object is moved from $\mathbf{r}(t_0)$ to $\mathbf{r}(t_1)$ as illustrated in Fig. 10.11. The work done on the object is

$$W_F = \int_{t_0}^{t_1} \mathbf{F} \cdot \mathbf{v} dt = \int_0^1 \mathbf{F} \cdot d\mathbf{r}, \tag{10.58}$$

Since the force is constant, we move it outside the integration:

$$W_F = \mathbf{F} \cdot \underbrace{\int_0^1 d\mathbf{r}}_{=(\mathbf{r}(t_1)-\mathbf{r}(t_0))} = \mathbf{F} \cdot (\mathbf{r}(t_1) - \mathbf{r}(t_0))\,. \tag{10.59}$$

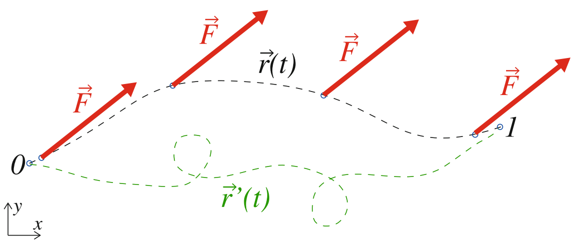

Fig. 10.11 Work done by a constant force $\mathbf{F}$ as the object is moved from point 0 at $\mathbf{r}(t_0)$ to point 1 at $\mathbf{r}(t_1)$

We call $\mathbf{s} = \Delta\mathbf{r} = \mathbf{r}(t_1) - \mathbf{r}(t_0)$ the displacement. The **work of a constant force** is therefore:

$$\boxed{W = \mathbf{F} \cdot \Delta\mathbf{r}.} \tag{10.60}$$

Notice that the work done by a constant force depends only on the displacement $\Delta\mathbf{r}$ and not on the path taken! The work done along the two paths $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ in Fig. 10.11 are therefore the same for a constant force. This is a very nice property of a force, since it makes it convenient to calculate the work done. A force with this property is called a conservative force, and we will see that such forces play a prominent role in our studies of mechanical energy.

10.4.1 Example: Work of Gravity

Problem: A projectile is moving through the air starting with an initial velocity $\mathbf{v}_0$ at the height y_0. Find the speed of the projectile at the height y_1. You can neglect air resistance.

Identify: In this problem we address the motion of the projectile, described by the position $\mathbf{r}(t)$. The projectile is at y_0 at t_0 with a velocity $\mathbf{v}_0$, and at a height h_1 at t_1 with a speed v_1.

Model: The projectile is only affected by gravity, which is constant, $\mathbf{G} = -mg\,\mathbf{j}$.

Solve: We can therefore use the work-energy theorem to find the kinetic energy of the projectile:

$$W = \int_{t_0}^{t_1} \mathbf{G} \cdot \mathbf{v}\, dt = \int_0^1 \mathbf{G} \cdot d\mathbf{r} = K_1 - K_0. \tag{10.61}$$

Since the force is constant, the work only depends on the displacement:

$$W = \mathbf{G} \cdot \mathbf{s} = (-mg\,\mathbf{j}) \cdot (\Delta x\,\mathbf{i} + \Delta y\,\mathbf{j}) = -mg\Delta y. \tag{10.62}$$

The motion in the x-direction has no impact on the work. It is only the displacement in the y-direction that contributes to a change in kinetic energy:

$$K_1 - K_0 = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = -mg\Delta y, \tag{10.63}$$

The speed v_1 is therefore:

$$v_1^2 = v_0^2 - 2g\,(y_1 - y_0)\,, \tag{10.64}$$

The right-hand side must be positive for this equation to have a meaningful solution.

10.4.2 Example: Roller-Coaster Motion

Problem: A roller-coaster cart is rolling from the height h to the height 0 along a curving roller-coaster track. It starts with the speed v_0 at the top of the track. Find the speed v_1 of the roller-coaster cart at the bottom of the track. You can ignore air resistance and friction.

Identify: The cart follows the path $\mathbf{r}(t)$ from $\mathbf{r}(t_0) = h\,\mathbf{j}$ at $t = t_0$ to $\mathbf{r}(t_1) = 0\,\mathbf{j}$ at $t = t_1$, as illustrated in Fig. 10.12.

Model: The cart is affected by the normal force, $\mathbf{N}$, the friction force, $\mathbf{f}$, and gravity, $\mathbf{G} = -mg\,\mathbf{j}$. We assume that friction is negligible, $\mathbf{f} = 0$ throughout the motion. The normal force $\mathbf{N}$ varies throughout the motion. It is therefore not that simple to apply Newton's second law directly to determine the motion of the cart. However, because the normal force always is normal to the path of motion, it performs no work. We can therefore apply the work-energy theorem to determine the velocity of the cart at the end of the track.

Solve: The work energy-theorem relates the work of the net force to the change in kinetic energy:

$$W_{\text{net}} = \Delta K = K_1 - K_0 = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2. \tag{10.65}$$

We can therefore find the speed, v_1, if we know the net work:

$$W_{\text{net}} = \int_{t_0}^{t_1} \mathbf{F}_{\text{net}} \cdot \mathbf{v}\,dt = \int_{t_0}^{t_1} (\mathbf{N} + \mathbf{G}) \cdot \mathbf{v}\,dt = \int_{t_0}^{t_1} \underbrace{\mathbf{N} \cdot \mathbf{v}}_{=0}\,dt + \int_{t_0}^{t_1} \mathbf{G} \cdot \mathbf{v}\,dt = W_G \tag{10.66}$$

As we argued above, the normal force does no work. The work done by gravity is the same as we have found previously:

$$W_G = \int_0^1 \mathbf{G} \cdot d\mathbf{r} = \mathbf{G} \cdot \int_0^1 d\mathbf{r} = \mathbf{G} \cdot (\mathbf{r}(t_1) - \mathbf{r}(t_0))\,. \tag{10.67}$$

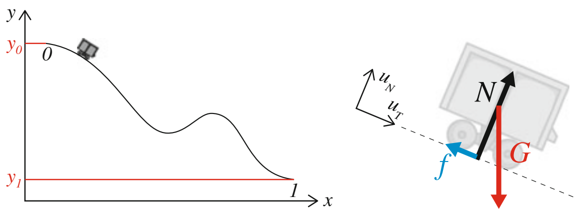

Fig. 10.12 A roller-coaster cart moving along a roller-coaster track

Since $\mathbf{G} = -mg\,\mathbf{j}$, only the vertical component of the displacement is included in the work:

$$W_G = -mg\mathbf{j}\cdot(\mathbf{r}(t_1) - \mathbf{r}(t_0)) = -mg(y_1 - y_0) = -mg(0 - h) = mgh. \tag{10.68}$$

We use this result in the work-energy theorem to find v_1:

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = mgh \;\Rightarrow v_1^2 = v_0^2 + 2gh. \tag{10.69}$$

Analyze: We have learned that for motion with a normal force, the normal force does no work and we can therefore use the same analysis as for an object falling due to gravity. However, this is only true as long as there are no additional forces that depend on the normal force, such as a friction force, or a force that depends on for example the velocity of the cart, such as the air resistance. In these cases, we need to make a more detailed analysis.

10.4.3 Example: Work on a Block Sliding Down a Plane

Problem: A classical problem in mechanics is the motion of a block sliding down an inclined plane. If the block starts from rest, what is the velocity of the block when it has moved a vertical distance h? The plane forms an angle α with the horizontal, the mass of the block is m, the acceleration of gravity is g, and μ is the dynamic coefficient of friction between the block and the plane.

Approach: We use the work-energy theorem to find the change in velocity from the change in kinetic energy, which only depends on the vertical displacement.

Identify: We use $\mathbf{r}(t)$ to describe the motion of the block from $\mathbf{r}(t_0)$ at $t = t_0$ to $\mathbf{r}(t_1)$ at $t = t_1$, as illustrated in Fig. 10.13.

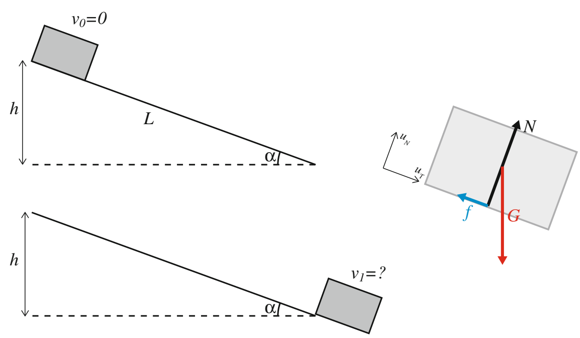

Fig. 10.13 A block sliding down along an inclined plane

Model: The block is affected by the normal force from the plane, $\mathbf{N}$, the friction force, $\mathbf{f}$, and gravity, $\mathbf{G} = -mg\,\mathbf{j}$. Since the block is sliding relative to the surface, the we use a dynamic friction model:

$$\mathbf{f} = -\mu N \frac{\mathbf{v}}{v}. \tag{10.70}$$

where the normal force must be determined from the motion in the direction normal to the plane. Since we assume that there is no motion normal to the plane, Newton's second law gives:

$$\sum F_N = N + G_N = ma_N = 0 \;\Rightarrow\; N = -G_N. \tag{10.71}$$

where G_N is the component of gravity in the normal direction, given by the unit vector, $\hat{u}_N$. From Fig. 10.13 we see that:

$$\hat{u}_N = \sin\alpha\,\mathbf{i} + \cos\alpha\,\mathbf{j}. \tag{10.72}$$

We find G_N by projecting $\mathbf{G}$ on $\hat{u}_N$:

$$G_N = \mathbf{G}\cdot\hat{u}_N = -mg\,\mathbf{j}\cdot(\sin\alpha\,\mathbf{i} + \cos\alpha\,\mathbf{j}) = -mg\cos\alpha, \tag{10.73}$$

which inserted in (10.71) gives: $N = mg\cos\alpha$.
Solve: The work-energy theorem for the motion of the block relates the work done by the net force on the block to the change in kinetic energy, $W_{\text{net}} = K_1 - K_0$, where

$$W_{\text{net}} = \sum_j W_j = W_N + W_f + W_G. \tag{10.74}$$

The work done by the normal force is zero, since it is always normal to the direction of motion: $W_N = 0$. Since gravity is a constant, the work done by gravity only depends on the displacement, $\Delta\mathbf{r}$:

$$W_G = \mathbf{G}\cdot\Delta\mathbf{r} = -mg\,(y_0 - y_1) = mgh. \tag{10.75}$$

In this particular case, we can find the work done by friction, because the friction force is constant during the motion:

$$W_f = \int_{t_0}^{t_1} \mathbf{f}\cdot\mathbf{v}\,dt = \mathbf{f}\cdot\Delta\mathbf{r}. \tag{10.76}$$

where both $\mathbf{f}$ and $\Delta\mathbf{r}$ are directed along the slope. The work done by friction is therefore

$$W_f = -f\,s, \tag{10.77}$$

where s is the distance along the slope. For a slope with an inclination α, the distance s is related to the height h by:

$$\frac{h}{s} = \sin\alpha \;\Rightarrow\; s = \frac{h}{\sin\alpha}. \tag{10.78}$$

This gives:

$$W_f = -fs = -\mu N\,\frac{h}{\sin\alpha} = -\mu\,mg\,h\,\frac{\cos\alpha}{\sin\alpha}. \tag{10.79}$$

We apply the work-energy theory to find the kinetic energy and the speed at the end of the slope:

$$W_{\text{net}} = W_N + W_f + W_G = mgh - \mu mgh\cot\alpha = K_1 - K_0 = \frac{1}{2}mv_1^2, \tag{10.80}$$

where we have used that $K_0 = 0$. This gives

$$v_1^2 = 2gh\,\left(1 - \mu\cot\alpha\right). \tag{10.81}$$

10.5 Power

You are pulling a crate along the floor with a force $\mathbf{F}$. In a small time interval Δt you have pulled the crate from the position $\mathbf{r}(t)$ to $\mathbf{r}(t+\Delta t)$, and the force $\mathbf{F}$ has performed the work

$$\Delta W = \int_t^{t+\Delta t} \mathbf{F}\cdot\mathbf{v}\,dt. \tag{10.82}$$

For a small time interval we can assume that the force $\mathbf{F}$ is constant, hence:

$$\Delta W = \mathbf{F}\cdot\int_t^{t+\Delta t}\mathbf{v}\,dt = \mathbf{F}\cdot(\mathbf{r}(t+\Delta t) - \mathbf{r}(t)) = \mathbf{F}\cdot\Delta\mathbf{r}. \tag{10.83}$$

We divide by Δt on each side and find that:

$$\frac{\Delta W}{\Delta t} = \mathbf{F}\cdot\frac{\Delta r}{\Delta t} \tag{10.84}$$

and in the limit when $\Delta t \rightarrow 0$ we have the definition of

Power: Power is the amount of work performed per unit time, that is, the rate at which we perform work:

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \mathbf{F} \cdot \mathbf{v}. \qquad (10.85)$$

The unit of power is *Watt* (W), which is defined as:

$$1\,\mathrm{W} = 1\,\mathrm{J/s} = 1\,\mathrm{Nm/s}. \qquad (10.86)$$

We also use the unit of horsepower, hP or hK. The metric horsepower is defined as:

$$1\,\mathrm{hP} = 735.5\,\mathrm{W}. \qquad (10.87)$$

10.5.1 Example: Power Exerted When Climbing the Stairs

Problem: You are climbing the stairs, moving upward at a constant vertical velocity v_0. What is the power exerted?

Solution: Since you are moving upward with a constant velocity, v_0, your acceleration upward is zero. Therefore, there is no net force in the vertical direction. The normal force N exerted from the ground on you therefore equals the gravitational force, $W = mg$, if we assume that other forces, such as air resistance, are negligible. The rate at which the normal force does work on you—the power of the normal force—is:

$$P = N v_0 = m g v_0. \qquad (10.88)$$

10.5.2 Example: Power of Small Bacterium

Problem: A small bacterium with approximately spherical shape is moving with a constant velocity **v** through water. Find the power exerted by the bacterium.

Model: The motion of the bacterium is determined by the forces acting on the bacterium. The bacterium is affected by two main forces: The fluid drag force, **D**, and the propulsion force, **F**, which is a force from the fluid on the bacterium due to the swirling motion of the bacterium's tail.

From Newton's second law we see that:

$$\mathbf{F} + \mathbf{D} = m\mathbf{a} = 0, \qquad (10.89)$$

where $\mathbf{a} = 0$ since the bacterium is moving with constant velocity. Consequently,

$$\mathbf{F} = -\mathbf{D}. \tag{10.90}$$

We have a good force model for the fluid drag. Because the bacterium is small and is moving at a tiny velocity, the fluid drag force is well approximated by the viscous force model:

$$\mathbf{D} = -k_v \mathbf{v}. \tag{10.91}$$

The power exerted by the bacterium corresponds to the power exerted by the force $\mathbf{F}$:

$$P = \mathbf{F} \cdot \mathbf{v} = k_v \mathbf{v} \cdot \mathbf{v} = k_v v^2. \tag{10.92}$$

Summary

Work: The work done on an object by the force $\mathbf{F}$ moving the object along the path $\mathbf{r}(t)$ is:

$$W = \int_{t_0}^{t_1} \mathbf{F} \cdot \mathbf{v} dt.$$

We can write this as a line integral along the path C given by $\mathbf{r}(t)$:

$$W = \int_0^1 \mathbf{F} \cdot d\mathbf{r}.$$

Kinetic energy: The kinetic energy of an object is defined as:

$$K = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} = \frac{1}{2} m v^2.$$

Work-energy theorem: The work done by the *net force* on an object along the path from 0 to 1 corresponds to the change in kinetic energy of the object:

$$W_{\text{net}} = \int_{t_0}^{t_1} \mathbf{F}_{\text{net}} \cdot \mathbf{v} dt = K_1 - K_0$$

Work of a constant force: The work of a constant force is $W = \mathbf{F} \cdot \mathbf{s}$.

Work of a position-dependent force in one dimension: If *a force only depends on the position*, $F_x = F_x(x)$, the work is an integral over x only:

$$W_F = \int_{x_0}^{x_1} F_x(x) dx.$$

It is often easier to solve the work integral for a complicated position-dependent force than to find the motion of the object. The work-energy theorem is therefore a useful tool to relate the velocity and position of an object without finding the complete motion $x(t)$.

Power: The power P exerted by a force $\mathbf{F}$ is the rate of work done by the force:

$$P = \mathbf{F} \cdot \mathbf{v}$$

Exercises

Discussion Questions

10.1 Accelerating a car. Compare the work required to accelerate a car from 50 to 70 km/h and from 70 to 90 km/h.

10.2 Leverage. Why is it easier to lift a car using a long lever than with your hands?

10.3 Friction. Can a frictional force increase the kinetic energy of a system?

10.4 Diving board. Divers often double-jump on a diving board to get higher. How does this work?

10.5 Trampoline. While you jump on a trampoline you are able to control your jump by either jumping higher or to almost stop at a single "jump". How?

10.6 Work in an elevator. You carry a heavy crate first from your car into your house, then into an elevator, and then you hold the crate in the elevator until you reach the top. What is the work done by your arm on the crate in the different cases?

Problems

10.7 Dragging a cart. You push a small cart 10 m along a flat floor with a force $F = 90$ N. The friction force on the cart is $f = 30$ N. The mass of the cart is 10 kg, and the cart starts at rest.

(a) What is the work done by you on the cart?
(b) What is the work done by the friction force?
(c) What is the velocity of the cart after 10m?

10.8 Toboggan slide. A child is sliding down a hill in a toboggan. He starts from rest, and when he reaches the end of the slope, he has moved a vertical distance of 10 m and he has a speed of 13.5 m/s. The mass of the child is 40 kg.

(a) What is the work done by friction?

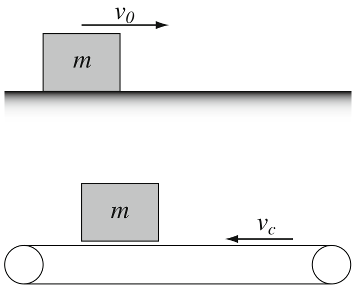

Fig. 10.14 (*Top*) Illustration of a crate sliding along the floor. (*Bottom*) Illustration of a crate released down onto a conveyor belt

10.9 Crate on conveyor belt. A crate is sliding along the floor with a horizontal velocity v_0, as illustrated in Fig. 10.14. The mass of the crate is m and the coefficient of dynamic friction between the floor and the crate is μ_d.
(a) Draw a free-body diagram for the crate.
(b) Find the distance, s, the crate slides before stopping.

Let us now study a slightly different situation. A conveyor belt is moving with a constant velocity, v_c, as illustrated in Fig. 10.14. A crate is released onto the conveyor belt, starting at rest relative to the ground. After a while the crate attains the same velocity as the belt. The dynamic coefficient of friction between the belt and the crate is μ_d. You can assume that the belt is long compared to the motion of the crate.
(c) Draw a free-body diagram for the crate on the belt.
(d) Find the work, W, done by the friction force.
(e) What distance, s, is the crate transported relative to the ground before it get the same velocity as the belt?

10.10 Volleyball smash. A volleyball player is smashing a ball. He gets the ball from behind with a velocity of 2 m/s towards the net. During the smash, he hits the ball with an approximately constant force of 300 N in the direction of the initial velocity. The mass of a volleyball is 270 g.
(a) Over how long distance must he be in contact with the ball in order to give the ball a velocity of 10 m/s?

10.11 A bouncing ball. A ball is lifted to a height h and released. When the ball hits the ground, the force from the ground on the ball has the form $F = k|\Delta y|^{3/2}$, where Δy is the indentation of the ball into the ground as illustrated in Fig. 10.15.

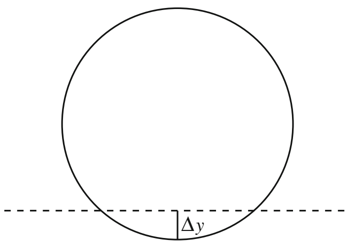

Fig. 10.15 Illustration of the deformation, Δy, of a ball in contact with the ground

This force-law is called the Hertz law for the deformation of a solid sphere. In this exercise we will find the maximum indentation of the ball—how long the surface of the ball has been deformed when the ball stops.
(a) Find the velocity v_1 of the ball as it just touches the ground.
(b) Draw a free-body diagram for the ball when it is in contact with the ground.
(c) Find the work done by gravity, W_G, and the deformation force, W_k, as a function of the vertical position y of the ball, when the ball is in contact with the ground.
(d) Find an equation for the maximum indentation, y_2, of the ball. (You do not have to solve the equation).
(e) What is the velocity of the ball when it loses contact with the ground?

10.12 Power of the heart. 7500 L of blood is pumped through a typical human heart each day. Let us assume that the work done to each liter of blood by the heart corresponds to the work used to lift the blood to 1/2 of the height of a typical male. The average male height is 180 cm and the density of blood is 1065 kg/m^3.
(a) How much work is done by the heart every day?
(b) What is the (average) power exerted by the heart?

10.13 Power station. At a small hydro-electrical facility operated on a small stream, water is released down a 10 m drop at a rate of 100 L/s.
(a) If you were able to convert all the work done by gravity on the water into energy, what would the power generated by the station be?

10.14 Accelerating car. A car engine produces a power of 250 hP and has a mass of 1200 kg.
(a) If we ignore the effects of air resistance and friction, how long time does the car need to accelerate to 100 km/h?

10.15 An accelerating motorbike. The engine of a motorbike produces a constant power P. The bike starts at rest and drives in a straight line. We neglect effects of friction and air resistance.
(a) Find the velocity of the bike as a function of time?
(b) Find the acceleration and show that it is not a constant.
(c) Find the position, $x(t)$, for the bike as a function of time.

Projects

10.16 Driving efficiently. In this project we address what energy dissipation mechanisms are dominating when driving a car: braking or air resistance. You will learn how to use work and energy arguments, and hopefully also realize how to design energy-efficient cars. We start by studying a car driving along a horizontal surface.
(a) Identify the forces acting on the car, and draw a free-body diagram for the car.
(b) If the car is driving at constant velocity, in what direction does the friction force from the ground on the car act?

Let us first assume that the engine delivers a constant power, P_0, so that the force on the car acting in the direction of motion, F, is $F\,v = P_0 = \text{const.}$, where v is the velocity of the car. You may assume a square-law for the air resistance on the car $F_D = -Dv^2$.

(c) Find an expression for the acceleration of the car as a function of the velocity.

(d) Show that if the car drives with constant velocity, the velocity is $v = (P_0/D)^{1/3}$. What is the physical interpretation of this velocity?

The air resistance coefficient is $D = (1/2)\,\rho\,C_D\,A$ where $\rho = 1.208\,\text{kg/m}^3$ is the mass density of air (at 20°C), C_D is the drag coefficient, and A is the cross-sectional area of the car. A few examples are given in the following table.

Car	Power (hp)	$C_D\,A$ (m^2)	m (kg)
Audi A2 2001 1.4 TDI	89	0.616	1030
Honda Civic 2001 1.4	115	0.682	1091
Hummer H2 2001 V8	316	2.46	3000

(e) Assume that 60 % of the engine power is used to propel the car forwards. Find the maximum velocities for each of the cars in the table.

Now, we want to study the motion of a car as it starts from rest, accelerates to a velocity v over a distance b, then travels at a constant velocity v, and finally brakes and comes to rest over a distance b. In total, the car has driven a length L.

(f) Sketch the velocity as a function of position for the car.

(g) Assume that we can ignore the effect of air resistance as the car accelerates. Find the work W_E done by the driving force on the car from the car starts at rest and until the car reaches the constant velocity v after a distance b.

(h) We introduce the effect of air resistance and assume that the car moves the whole distance L at a constant velocity v. (We ignore the brief acceleration and deceleration periods). Find the work W_D done by the driving force over the distance L.

(i) Compare the work W_E done in order to accelerate the car to the velocity v and the work W_D done in order to move the car a distance L through the air at constant velocity v. What term is the largest? At what distance L^* does the air resistance term become dominating? Calculate the distance L^* for the cars in the table above. Comment on your results.

(j) Based on your results, can you make recommendations for how to design cars for city traffic and for long distance travel?