Chapter 2
Conductors

2.1 Electric Properties of Conductors

In terms of electric properties, materials are roughly classified into **conductors**, which can easily transport electric current, and **insulators**, which can hardly do so. The classification is based on electric conductivity, as shown in Chap. 5. Metals are conductors, and their electric property originates from free electrons that can move freely in the material. On the other hand, electrons in insulators such as mica and glass cannot move because of their bonding to atomic nuclei. Hence, the electric behavior of conductors and insulators is very different. This chapter describes the electric behavior of conductors. Chapter 4 describes that of insulators, which are also called **dielectrics** or **dielectric materials** because of their other electric properties.

The electric behavior of conductors is defined as follows: the electric field and the electric charge density inside the conductor are zero in the static condition after the conductor is put in an external electric field. That is,

$$\boldsymbol{E} = 0 \tag{2.1}$$

and

$$\rho = 0. \tag{2.2}$$

The properties given by the above two equations are not independent of each other. Namely, Eq. (2.2) is derived from Eq. (2.1) with Eq. (1.21). From Eq. (2.1) we have

$$\phi = \text{const.} \tag{2.3}$$

Thus, we can also say that conductors are equipotential.

Here we mention the relationship between electrical conductivity and the above definition of a conductor. If some electric field remains in the conductor,
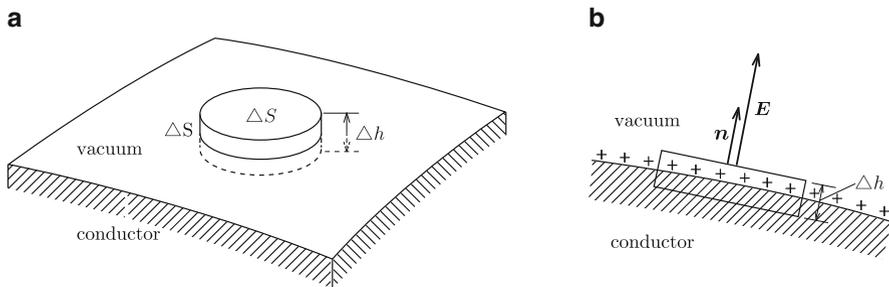
Fig. 2.1 (**a**) Small closed surface that includes part of the conductor surface and (**b**) electric field vector around the closed surface. The vector n is a unit vector normal to the surface

free electrons in the material will be driven by this field, which contradicts the assumption of a static condition. Thus, there is no electric field in a static conductor.

Suppose that an isolated conductor is placed in an electric field. The field forces the free electrons in the conductor to move. These electrons cannot go outside the conductor, and some of them accumulate on the surface of the conductor. The electric field produced by the electric charges on the surface exactly cancels the external electric field, resulting in a zero electric field inside the conductor. This realizes the situation assumed above for a conductor.

The appearance of electric charge on the surface of a conductor placed in an electric field is called **electrostatic induction**. The free electrons that appear on the surface are true charges in electromagnetism. It is possible to make an electric field stay inside the conductor. In this case electric charges move inside the conductor, resulting in electric current, as will be described in Chap. 5. Hence, it is not a static situation. It should be noted that, even if the electric current does not change with time in a steady state, it is different from a static situation. This chapter describes static electric phenomena without movement of electric charges.

Here we investigate the electric field in the vicinity of the conductor surface. Suppose a small closed pellet-shaped surface includes the interface between the conductor and vacuum, as shown in Fig. 2.1a. We denote the height of the pellet and the area of the conductor surface inside the pellet by Δh and ΔS, respectively. Suppose that the density of electric charge on the surface of the conductor is σ. We apply Gauss' law, Eq. (1.19), to the pellet. In this case, the electric field vector, E, is perpendicular to the surface of the conductor because of the orthogonality between the electric field and equipotential surface, since the surface of the conductor is equipotential. Hence, the electric field lines that go out of the side surface of the pellet are negligible if Δh is sufficiently small. Thus, all the electric field lines go out from the outer surface (see Fig. 2.1b). Since E is perpendicular to this surface, we have

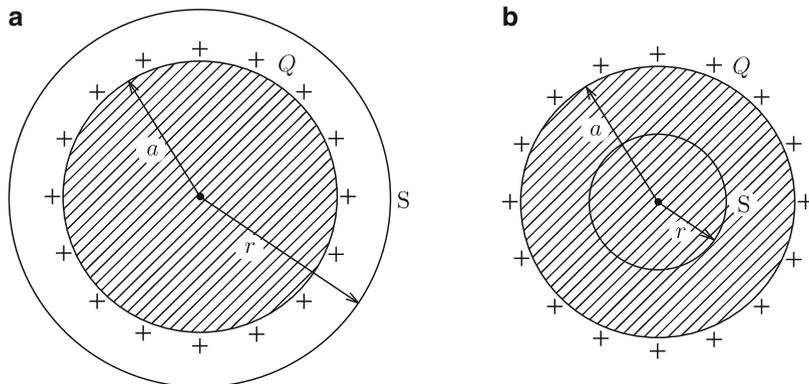$$\int_{\Delta S} E \cdot dS = \int_{outer surface} E \, dS = E \Delta S. \qquad (2.4)$$

Fig. 2.2 Spherical conductor and virtual spherical surface, S: (**a**) case where S is outside the conductor and (**b**) case where S is inside the conductor

The total electric charge inside the pellet is $\sigma \Delta S$, and the left side of Eq. (1.19) is $\sigma \Delta S / \epsilon_0$. This gives

$$E = \frac{\sigma}{\epsilon_0}. \tag{2.5}$$

That is, the electric field strength on the surface of the conductor is equal to the surface electric charge density divided by ϵ_0.

Suppose we apply an electric charge, Q, to a spherical conductor of radius a. This determines the electric field and electric potential inside and outside the conductor. Since the electric charge stays on the surface and charges repel each other, the charge is uniformly distributed on the surface. Hence, the surface electric charge density is $\sigma = Q/(4\pi a^2)$. We apply Gauss' law to a supposed spherical surface, S, of radius r with the same center as that of the conductor. Since the electric charge distribution has spherical symmetry, we can also assume the electric field to have spherical symmetry. Hence, the electric field is directed normally to S, and its strength is uniform on S. If its strength is denoted by E, the surface integral of the electric field strength in Eq. (1.19) is $4\pi r^2 E$. For $r > a$, as shown in Fig. 2.2a, all the electric charge stays inside S, and the right side of Eq. (1.19) is Q/ϵ_0. Thus,

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2}; \qquad r > a. \tag{2.6}$$

The electric field outside the conductor is the same as that when all the electric charge is concentrated on the center. For $r < a$, as shown in Fig. 2.2b, the total electric charge inside S is zero. This gives
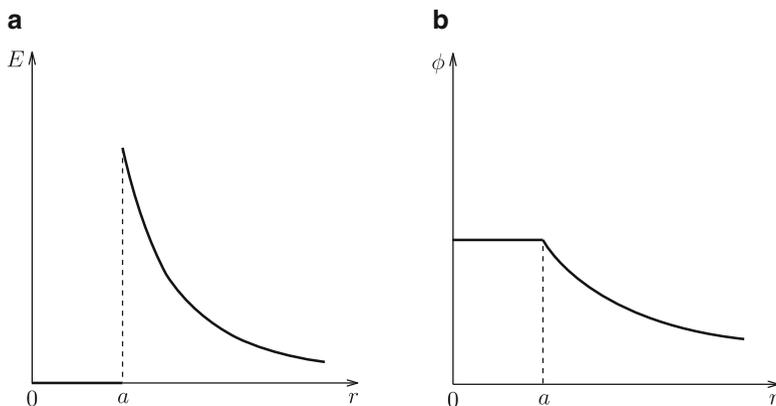
$$E(r) = 0; \qquad 0 \leq r < a. \tag{2.7}$$

a

b

Fig. 2.3 (**a**) Electric field strength and (**b**) electric potential inside and outside the charged spherical conductor

Thus, Eq. (2.1) is fulfilled inside the conductor. It can also be shown that Eq. (2.6) satisfies Eq. (2.5) on the surface of the conductor ($r = a$) with the surface electric charge density determined above.

We determine the electric potential from

$$\phi(r) = -\int_{\infty}^{r} E(r)\mathrm{d}r \tag{2.8}$$

with Eqs. (2.6) and (2.7) and the condition that the electric potential is zero at infinity. This gives

$$\phi(r) = \frac{Q}{4\pi\epsilon_0 r}; \qquad r > a, \tag{2.9a}$$

$$= \frac{Q}{4\pi\epsilon_0 a}; \qquad 0 \leq r < a. \tag{2.9b}$$

Figure 2.3a, b shows the determined electric field strength and electric potential, respectively.

Example 2.1. Suppose a pair of concentric spherical conductors, as shown in Fig. 2.4. Determine the electric field strength and electric potential in all regions when the electric charge, Q, is given on the inner conductor.

Fig. 2.4 Isolated concentric
spherical conductors
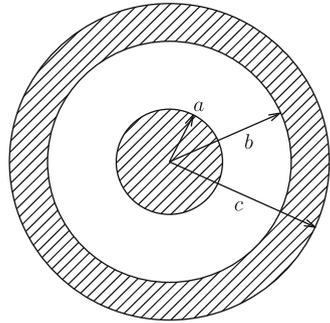
Solution 2.1. We can assume that Q is uniformly distributed on the surface ($r = a$)
of the inner conductor because of the spherical symmetry. This distribution makes
the electric field zero inside the inner conductor ($r < a$). The electric charge appears
on the inner surface ($r = b$) of the outer conductor because of the electrostatic
induction. This electric charge is denoted by Q_b. We apply Gauss' law to a spherical
surface, S, of radius r ($b < r < c$) with the same center as that of the conductors:

$$\int_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{Q + Q_b}{\epsilon_0}.$$

Since $\boldsymbol{E} = 0$ on S, we obtain $Q_b = -Q$. Since no electric charge is given to the
outer conductor, the electric charge that appears on the outermost surface ($r = c$) is
$-Q_b = Q$.

If the total electric charge inside the virtual sphere, S, of radius r is denoted by
Q_r, Gauss' law gives

$$E(r) = \frac{Q_r}{4\pi\epsilon_0 r^2}.$$

Since Q_r is equal to $Q, 0$ and Q for $a < r < b, b < r < c$ and $r > c$, respectively,
we determine the electric field strength to be

$$
\begin{aligned}
E &= 0; & 0 < r < a, \\
&= \frac{Q}{4\pi\epsilon_0 r^2}; & a < r < b, \\
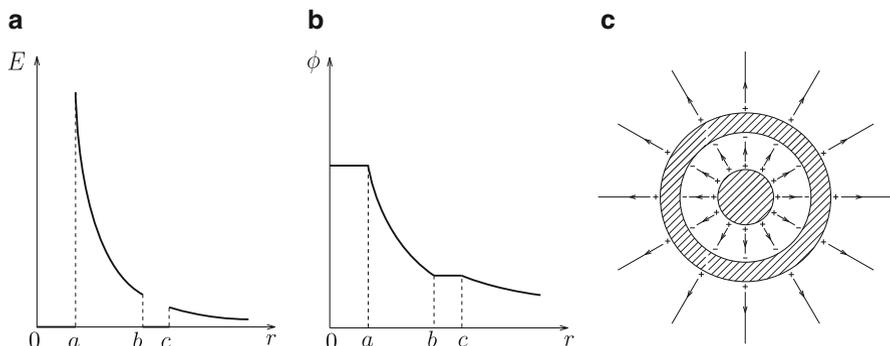&= 0; & b < r < c, \\
&= \frac{Q}{4\pi\epsilon_0 r^2}; & r > c.
\end{aligned}
$$

Fig. 2.5 (**a**) Electric field strength, (**b**) electrical potential and (**c**) electric field lines when electric charge is given to the inner conductor of a set of concentric spherical conductors

Then, we obtain the electric potential as

$$
\begin{aligned}
\phi(r) = -\int_{\infty}^{r} E(r)\mathrm{d}r &= \frac{Q}{4\pi\epsilon_0 r}; & r > c, \\
&= \frac{Q}{4\pi\epsilon_0 c}; & b < r < c, \\
&= \phi(b) - \int_{b}^{r} E(r)\mathrm{d}r = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b} + \frac{1}{c}\right); & a < r < b, \\
&= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right); & 0 \le r < a.
\end{aligned}
$$

Figure 2.5a–c shows the obtained electric field strength, electrical potential and electric field lines, respectively.

$\diamond$

Here, we suppose that the outer conductor in Example 2.1 is grounded. **Grounding** is a method to make the electric potential of a conductor zero by connecting it to the ground. It sometimes accompanies transfer of electric charge. In the above case, the electric charge on the outer surface ($r = c$) of the outer conductor transfers to the ground through the grounding. This occurs because of the repulsive Coulomb interaction between electric charges on the outer surface. This can also be understood from the fact that the free electric charge transfers from the position of higher electric potential, $\phi = Q/(4\pi\epsilon_0 c)$, to the position of lower electric potential, $\phi = 0$. The electric charge on the inner surface ($r = b$) of the outer conductor does not transfer to the ground. This is because it is attracted by

Fig. 2.6 Electric field lines when electric charge is given to the inside of a set of concentric spherical conductors and the outside is grounded
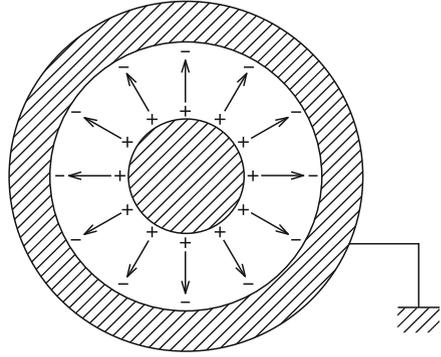
the electric charge on the surface ($r = a$) of the inner conductor (see Fig. 2.6). When some area is surrounded by a grounded conductor, changes in the outside do not influence the electric field inside the grounded conductor. Such shielding from outside influence is called **electrostatic shielding**.

In this case, Q_r is Q and 0 for $a < r < b$ and $r > b$, respectively. This gives

$$E = 0; \qquad 0 \leq r < a,$$

$$= \frac{Q}{4\pi\epsilon_0 r^2}; \quad a < r < b,$$

$$= 0; \qquad r > b.$$

Thus, we determine the electric potential to be

$$\phi(r) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right); \qquad 0 \leq r < a,$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right); \qquad a < r < b,$$

$$= 0; \qquad\qquad\qquad r > b.$$

Figure 2.7a, b shows the obtained electric field strength and electric potential, respectively.
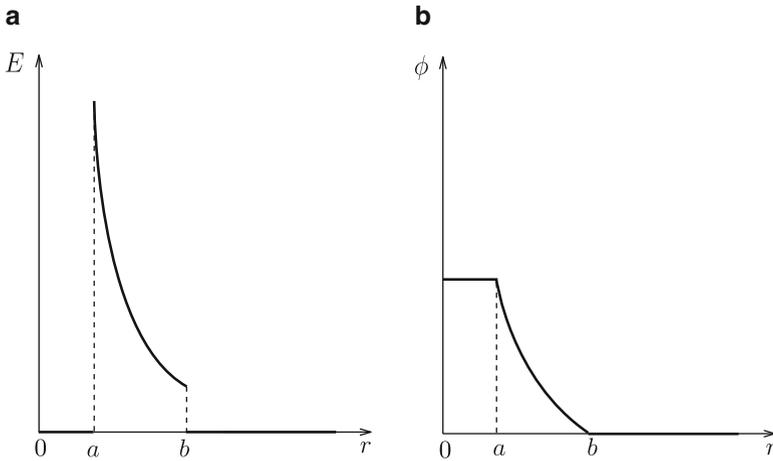
a

b

Fig. 2.7 (**a**) Electric field strength and (**b**) electric potential when electric charge is given to the inner conductor of a set of concentric spherical conductors and the outer conductor is grounded

Example 2.2. Suppose a pair of long coaxial conductors, as shown in Fig. 2.8. Determine the electric field strength and electric potential in all regions when an electric charge, λ, is given to the inner conductor of unit length. The electric potential is defined to be zero at a point at distance $R_0(> c)$ from the central axis.

Fig. 2.8 Isolated long coaxial conductors
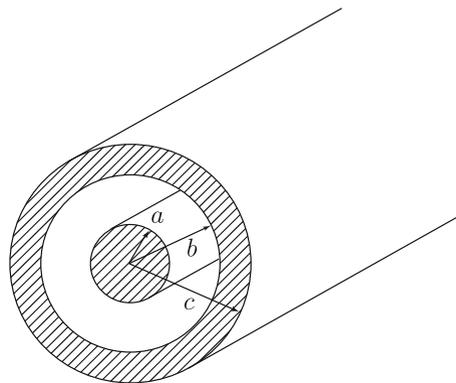
Solution 2.2. The electric charge is uniformly distributed on the surface $(R = a)$ of the inner conductor with value λ in unit length. The induced electric charges on the inner $(R = b)$ and outer $(R = c)$ surfaces of the outer conductor are $-\lambda$ and λ

in unit length, respectively. We determine the electric field strength to be

$$E(R) = 0; \qquad 0 \leq R < a,$$

$$= \frac{\lambda}{2\pi\epsilon_0 R}; \qquad a < R < b,$$

$$= 0; \qquad b < R < c,$$

$$= \frac{\lambda}{2\pi\epsilon_0 R}; \qquad R > c.$$

From the definition the electric potential is given by

$$\phi(R) = -\int_{R_0}^{R} E(R)\mathrm{d}R = \frac{\lambda}{2\pi\epsilon_0} \log \frac{R_0}{R}; \qquad c < R < R_0,$$

$$= \frac{\lambda}{2\pi\epsilon_0} \log \frac{R_0}{c}; \qquad b < R < c,$$

$$= \frac{\lambda}{2\pi\epsilon_0} \log \frac{bR_0}{cR}; \qquad a < R < b,$$

$$= \frac{\lambda}{2\pi\epsilon_0} \log \frac{bR_0}{ac}; \qquad 0 \leq R < a.$$

The reason why infinity is not defined as the reference point of zero electric potential is that the total electric charge is infinite because of the infinite length, as mentioned in Example 1.7.

$$\diamond$$

2.2   Special Solution Method for Electrostatic Field

Suppose we need to determine the density of electric charge on the surface of a conductor or the electric field strength around the conductor when the conductor is in an external electric field. The electric potential in the conductor is constant in space, as shown in Eq. (2.3). Outside the conductor, there is no electric charge and the electric potential ϕ satisfies Laplace's equation (1.38).

When we are given the boundary condition on the surface of a treated area, such as the value of ϕ above or a value of its derivative along the direction normal to the surface, Laplace's equation can be solved uniquely. Hence, there is only one solution of ϕ in the space outside the conductor, which becomes a constant value on the surface of the conductor. This means that, if some function satisfies the boundary condition, it is a solution, even though it may be obtained by intuition. In the case of conductors we know some methods to solve problems. These will be introduced in this section. When we obtain a solution for ϕ, we obtain the electric field, E, using Eq. (1.24) and determine the surface electric charge density from the value of E with Eq. (2.5).
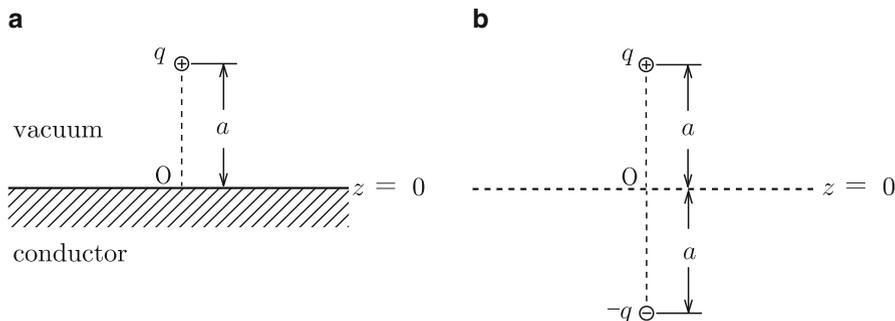
Fig. 2.9 (**a**) Point charge, q, at distance a from a wide flat conductor surface and (**b**) image charge, $-q$, put at the symmetric point of the given charge with respect to the conductor surface

Here, suppose that an electric point charge, q, is put at a position at distance a from a flat infinite conductor surface, as shown in Fig. 2.9a. Electric charge of different signs appears on the conductor surface because of the electrostatic induction and exerts an attractive force on q. The x-y plane is defined on the conductor surface with the origin, O, at the foot of a perpendicular line from the electric charge. The electric potential is constant on the conductor surface ($z = 0$), as discussed in Sect. 2.1. Figure 1.16b shows that such an electric potential can be realized in the following way: the conductor is virtually removed, and then an electric charge, $-q$, is put at the point $(0, 0, -a)$, the point symmetric to the location of q with respect to the conductor surface, as shown in Fig. 2.9b. We now check the validity of this speculation. The electric potential that the two electric charges produce outside the conductor ($z > 0$) is

$$\phi(x, y, z) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q}{[x^2 + y^2 + (z - a)^2]^{1/2}} - \frac{q}{[x^2 + y^2 + (z + a)^2]^{1/2}} \right\}.$$

$$(2.10)$$

It is easily found that this satisfies the condition, $\phi = 0$, on the conductor surface ($z = 0$). Since this satisfies Laplace's equation outside the conductor and the boundary condition of Eq. (2.3) on the conductor surface, this is the solution. This shows that the above intuitive method is useful. In the conductor ($z < 0$) the electric potential is not given by Eq. (2.10) but by $\phi = 0$. This solution method is called the **method of images** and the virtual electric charge is called an **image charge**.

From Eq. (2.10) we obtain the electric field strength outside the conductor as

$$E_x = -\frac{\partial\phi}{\partial x} = \frac{q}{4\pi\epsilon_0} \left\{ \frac{x}{[x^2+y^2+(z-a)^2]^{3/2}} - \frac{x}{[x^2+y^2+(z+a)^2]^{3/2}} \right\},$$

$$E_y = -\frac{\partial\phi}{\partial y} = \frac{q}{4\pi\epsilon_0} \left\{ \frac{y}{[x^2+y^2+(z-a)^2]^{3/2}} - \frac{y}{[x^2+y^2+(z+a)^2]^{3/2}} \right\}, \quad (2.11)$$
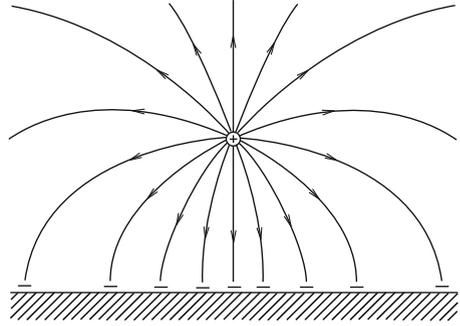
$$E_z = -\frac{\partial \phi}{\partial x} = \frac{q}{4\pi\epsilon_0} \left\{ \frac{z-a}{[x^2+y^2+(z-a)^2]^{3/2}} - \frac{z+a}{[x^2+y^2+(z+a)^2]^{3/2}} \right\}.$$

On the conductor surface this reduces to

$$E_x(x, y, 0) = E_y(x, y, 0) = 0, \qquad E_z(x, y, 0) = -\frac{qa}{2\pi\epsilon_0(x^2 + y^2 + a^2)^{3/2}}$$

$$(2.12)$$

Figure 2.10 shows the electric field lines. Then, from Eq. (2.5) we obtain the density of electric charge induced on the conductor surface as

$$\sigma = -\frac{qa}{2\pi(x^2 + y^2 + a^2)^{3/2}}. \tag{2.13}$$

Now we determine the total electric charge. Using the two-dimensional polar coordinates $(x = r\cos\varphi, \ y = r\sin\varphi)$, we have

$$\int dx \int dy \ \sigma = -\frac{qa}{2\pi} \int_0^{2\pi} d\varphi \int_0^\infty \frac{r}{(r^2+a^2)^{3/2}} dr = -qa \left[-\frac{1}{(r^2+a^2)^{1/2}} \right]_0^\infty = -q. \tag{2.14}$$

That is, the total electric charge is equal to the amount of the image charge. The Coulomb force exerted on the electric charge q by the electric charge induced on the conductor surface is equal to that exerted by the image charge:

$$F = -\frac{q^2}{4\pi\epsilon_0(2a)^2} = -\frac{q^2}{16\pi\epsilon_0 a^2}. \tag{2.15}$$

This force is attractive $(F < 0)$. This force is called **image force**.

   The electric field strength inside the conductor $(z < 0)$ produced by the electric charge on the conductor surface is equal to that produced by the electric charge $-q$ placed at the position of q, $(0, 0, a)$. Since the latter electric field absolutely cancels out the electric field produced by q, the electric field in the conductor can be shown to be zero.
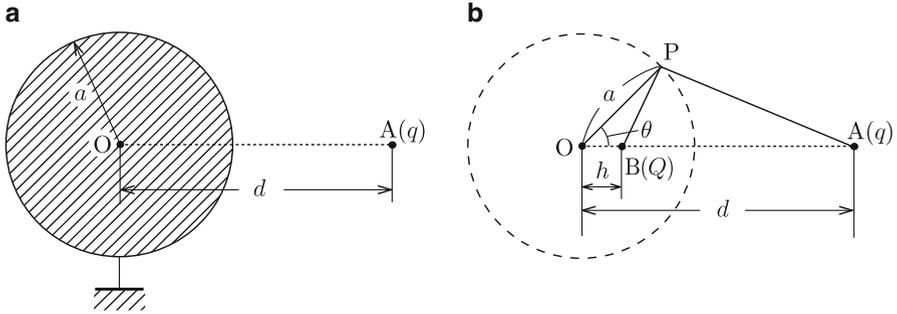
Fig. 2.11 (**a**) Grounded spherical conductor and point charge at point A and (**b**) image charge at point B after removal of the conductor

Suppose that a point charge, q, is placed at point A at distance d from the center, O, of a grounded spherical conductor of radius a $(d > a)$, as illustrated in Fig. 2.11a. Now we determine the electric potential outside the spherical conductor. Assume that the conductor is removed and an image charge, Q, is placed at point B at distance h from the center, as shown in Fig. 2.11b. The quantities Q and h are unknown and need to be determined. Then, the electric potential on the conductor surface is

$$\phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(a^2+d^2-2ad\cos\theta)^{1/2}}+\frac{Q}{(a^2+h^2-2ah\cos\theta)^{1/2}}\right],$$

$$= \frac{1}{4\pi\epsilon_0}\left\{\frac{q/\sqrt{a^2+d^2}}{[1-2ad\cos\theta/(a^2+d^2)]^{1/2}}+\frac{Q/\sqrt{a^2+h^2}}{[1-2ah\cos\theta/(a^2+h^2)]^{1/2}}\right\}, \quad (2.16)$$

where angle $\angle$POA is represented by θ. Hence, $\phi = 0$ is realized at any point on the conductor surface $(r = a)$ and the boundary condition is satisfied, if the following conditions are fulfilled:

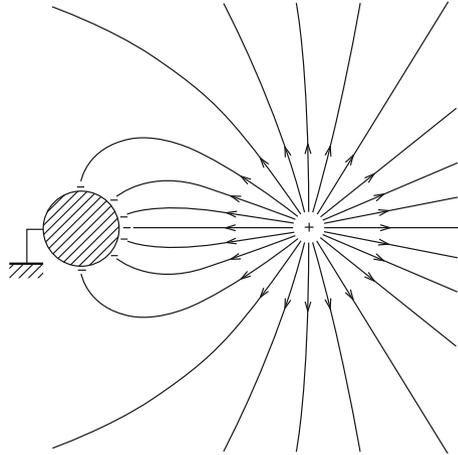$$\frac{q}{\sqrt{a^2+d^2}}+\frac{Q}{\sqrt{a^2+h^2}} = 0, \tag{2.17}$$

and

$$\frac{2ad}{a^2+d^2} = \frac{2ah}{a^2+h^2}, \tag{2.18}$$

which reduce to

$$h = \frac{a^2}{d}, \qquad Q = -\frac{aq}{d}. \tag{2.19}$$

Fig. 2.12 Electric field lines between the point electric charge and grounded spherical conductor

Thus, the electric potential at point (r, θ) outside the conductor is given by

$$\phi(r, \theta) = \frac{q}{4\pi\epsilon_0} \left\{ \frac{1}{(r^2 + d^2 - 2rd\cos\theta)^{1/2}} - \frac{a}{d[r^2 + (a^2/d)^2 - 2(a^2r/d)\cos\theta]^{1/2}} \right\}.$$
(2.20)

The electric field strength can be calculated with this electric potential (see Exercise 2.7). Figure 2.12 shows the electric field lines.

We obtain the density of electric charge on the conductor surface as

$$\sigma(\theta) = \epsilon_0 E_r(r = a) = -\epsilon_0 \left(\frac{\partial \phi}{\partial r} \right)_{r=a} = -\frac{q(d^2 - a^2)}{4\pi a(a^2 + d^2 - 2ad\cos\theta)^{3/2}}.$$
(2.21)

The total electric charge is

$$\int_0^\pi \sigma(\theta) \cdot 2\pi a^2 \sin\theta \, d\theta = -\frac{qa(d^2-a^2)}{2} \int_0^\pi \frac{\sin\theta \, d\theta}{(a^2+d^2-2ad\cos\theta)^{3/2}}$$

$$= \frac{q(d^2-a^2)}{2d} \left[(a^2+d^2-2ad\cos\theta)^{-1/2} \right]_0^\pi = -\frac{a}{d}q, (2.22)$$

which is equal to the image charge, Q. This charge is transferred from the ground to the conductor because of attraction by the point charge q. The reason it is smaller by factor a/d than in the case shown in Fig. 2.9 is that the size of the conductor is finite. For an infinitely long cylindrical conductor and line charge, the electric charge induced in a conductor of unit length is equal to the density of the given line charge (see Exercise 2.8).

Example 2.3. Suppose that the conductor is not grounded in the problem shown in Fig. 2.11a. Determine the electric potential outside the conductor.

Solution 2.3. In this case the total electric charge on the conductor surface is zero. This problem is solved using the method of superposition. That is, this situation is obtained by a superposition of the electric charge $-aq/d$, which is distributed according to Eq. (2.21), and the charge aq/d, which is uniformly distributed on the surface. The distributed charge $-aq/d$ and point charge q give the zero electric potential of the conductor, and the distributed charge aq/d makes the conductor equipotential, $q/(4\pi\epsilon_0 d)$. Hence, this situation satisfies the conductor condition. If the electric potential given by Eq. (2.20) is denoted by $\phi_1(r, \theta)$, the electric potential outside the conductor is

$$\phi(r, \theta) = \phi_1(r, \theta) + \frac{aq}{4\pi\epsilon_0 dr}.$$

$\diamondsuit$

2.3   Electrostatic Induction

Suppose that a spherical conductor of radius a is put in a uniform electric field of strength $\boldsymbol{E}_0$ (see Fig. 2.13). An electric charge appears on the conductor surface and cancels out the electric field in the conductor. This phenomenon is the electrostatic induction. Here we determine the surface electric charge density and the electric field around the conductor. We use cylindrical coordinates and define the z-axis as the line through the center of the conductor along the direction of the applied electric field.
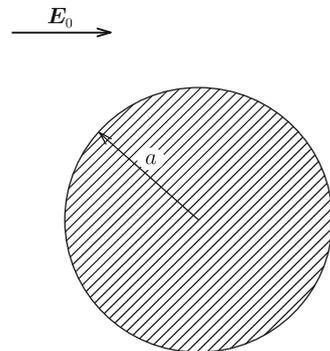

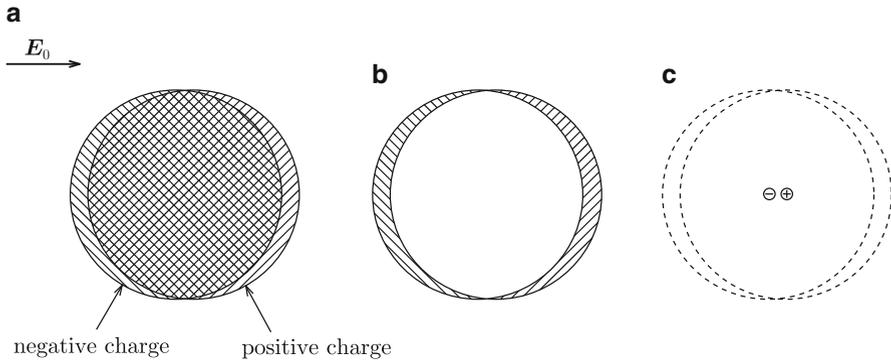**Fig. 2.13** Spherical conductor put in a uniform electric field

Fig. 2.14 Electrostatic induction in spherical conductor in uniform electric field: (**a**) displacement of positive and negative electric charges driven by the electric field, (**b**) electric charge that appears on the surface and (**c**) electric dipole at the center

Before the electric field is applied, positive and negative electric charges are uniformly distributed inside the conductor, and the conductor is electrically neutral. When the electric field is applied, the positive and negative electric charges are displaced in and against the direction of the electric field, respectively, as illustrated in Fig. 2.14a. This leads to a surface distribution of electric charge that keeps the inside electrically neutral (see Fig. 2.14b). Hence, this seems to realize the proper condition of a spherical conductor. We determine the displacement of the electric charges. In this case we find the electric field produced by the positive charge to be the same as that produced when all the positive charge is concentrated at the center, as predicted by Gauss' law. The electric field produced by the negative charge is also the same as that produced by all the negative charge if concentrated at the center. As a result an electric dipole appears at the center of the conductor (see Fig. 2.14c). The electric dipole moment, p, that satisfies the electric potential is to be determined.

The electric potential outside the conductor is composed of the electric potential, ϕ_f, due to the applied electric field, E_0, and the electric potential, ϕ_d, due to the electric dipole placed at the center after virtually removing the spherical conductor. These are given by

$$\phi_f = -E_0 r \cos \theta, \tag{2.23}$$

$$\phi_d = \frac{p \cos \theta}{4\pi\epsilon_0 r^2}, \tag{2.24}$$

where θ is the zenithal angle measured from the direction of applied electric field. These potentials are independent of the azimuthal angle, φ. Now prove for yourself that ϕ_f satisfies the requirements

$$-\frac{\partial \phi_f}{\partial r} = E_0 \cos \theta, \qquad -\frac{1}{r} \cdot \frac{\partial \phi_f}{\partial \theta} = -E_0 \sin \theta.$$

The electric potential is given by

$$\phi = \phi_f + \phi_d = \left(-E_0 r + \frac{p}{4\pi\epsilon_0 r^2}\right)\cos\theta. \tag{2.25}$$

We determine the electric dipole moment, p, to be

$$p = 4\pi\epsilon_0 a^3 E_0 \tag{2.26}$$

so that the condition, $\phi(r = a) = 0$, is satisfied independently of the angle θ. Thus, we have

$$\phi = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta. \tag{2.27}$$

Since this satisfies the boundary condition on the conductor surface ($r = a$) and satisfies Laplace's equation (note that each component of ϕ satisfies it), this is the unique solution. Thus, we can say the above speculation is valid. The electric potential inside the conductor is $\phi = 0$.

   The electric field strength outside the conductor is given by

$$E_r = -\frac{\partial\phi}{\partial r} = E_0\left(1 + \frac{2a^3}{r^3}\right)\cos\theta, \tag{2.28a}$$

$$E_\theta = -\frac{1}{r}\cdot\frac{\partial\phi}{\partial\theta} = -E_0\left(1 - \frac{a^3}{r^3}\right)\sin\theta, \tag{2.28b}$$

$$E_\varphi = -\frac{1}{r\sin\theta}\cdot\frac{\partial\phi}{\partial\varphi} = 0. \tag{2.28c}$$

Figure 2.15 shows electric field lines on the plane that includes the z-axis. We can see that $E_\theta(r = a) = 0$ from Eq. (2.28b). This shows that the electric field vector is normal to the conductor surface. Equation (2.28a) shows that the electric field strength has the maximum value, $3E_0$, at both poles ($\theta = 0, \pi$). We determine the surface electric charg density to be
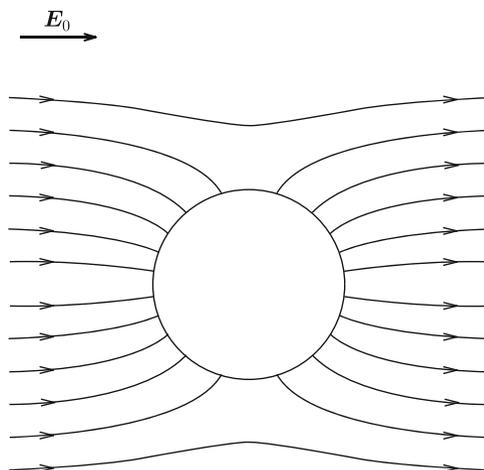
$$\sigma = \epsilon_0 E_r(r = a) = 3\epsilon_0 E_0\cos\theta. \tag{2.29}$$

The electric dipole moment in a unit volume of the spherical conductor is

$$P = 3\epsilon_0 E_0, \tag{2.30}$$

which corresponds to the electric polarization in dielectric materials.

Fig. 2.15 Electric field lines outside the spherical conductor

E_0

Example 2.4. A long cylindrical conductor is placed in a uniform normal electric field of strength E_0. Determine the electric potential and electric field outside the conductor and the density of electric charge on the conductor surface.

Solution 2.4. We use cylindrical coordinates and define the z-axis as the central axis of the conductor, and measure the azimuthal angle, φ, from the direction of the applied electric field. The electric potential outside the conductor can be determined by putting the electric dipole line, as shown in Example 1.8, on the central axis after removing the conductor similarly to what we did in the above analysis. We denote by $\hat{p}$ the moment of the electric dipole line in a unit length along the z-axis. Then, the electric potential outside the conductor is given by

$$\phi(R, \varphi) = \left(-E_0 R + \frac{\hat{p}}{2\pi\epsilon_0 R} \right) \cos\varphi$$

with the aid of Eq. (1.53). The first and second terms are the electric potential due to the applied electric field and the electric dipole line, respectively. Hence, from the requirement that $\phi = 0$ at $R = a$, we have

$$\hat{p} = 2\pi\epsilon_0 a^2 E_0,$$

which gives

$$\phi(R, \varphi) = -E_0 \left(R - \frac{a^2}{R} \right) \cos\varphi.$$

Thus, we obtain the electric field strength as

$$E_R = -\frac{\partial \phi}{\partial R} = E_0 \left(1 + \frac{a^2}{R^2}\right) \cos \varphi,$$

$$E_\varphi = -\frac{1}{R} \cdot \frac{\partial \phi}{\partial \varphi} = -E_0 \left(1 - \frac{a^2}{R^2}\right) \sin \varphi,$$

$$E_z = 0.$$

We can see that the electric field is normal to the conductor surface from $E_\varphi(R = a) = 0$. The surface electric charge density is

$$\sigma = \epsilon_0 E_R(R = a) = 2\epsilon_0 E_0 \cos \varphi.$$

The electric dipole moment in a unit volume of the conductor is

$$P = 2\epsilon_0 E_0.$$

◇

Column: Applicability of Method of Images

The method of images is useful for solving problems when a conductor is put in an electric field, as shown in the Examples and Exercises. Now, suppose that an electric charge is given on a spherical conductor placed at some distance from a wide flat conductor surface. Can we also use the method of images in this case?

This problem can be compared with Exercise 2.9. Following the solution for that exercise, we first remove the spherical conductor and then place a point charge equal to the given charge at a point at some distance from the center. Next, we remove the wide flat conductor and put the same electric charge at the point symmetric to the location of the former charge with respect to the flat conductor surface. At first this may seem useful in determining the electric potential outside the two conductors.

However, the electric potential cannot be determined with this method. How can we prove it? To satisfy the boundary condition on the infinitely wide conductor surface, all electric charges distributed on the surface must be connected to the electric charges on the spherical conductor surface through the electric field lines. In fact, from a superposition of point electric charges we can show that the total amount of electric charge induced on the flat conductor surface is equal to the electric charge given to the spherical conductor. However, to satisfy the boundary condition on the spherical conductor surface after virtually concentrating all the electric charge at the image point, the absolute value of the electric charge must be smaller than the point charge assumed inside the infinite flat conductor [see Eq. (2.22)].

The method shown in Example 2.3 seems useful for making the spherical conductor surface equipotential with the same electric charge. However, the boundary condition on the infinitely wide conductor surface is not satisfied between one point charge and two separate point charges. For this reason we cannot obtain an analytic solution. To get a solution it is necessary to distribute the image charge in such way that the boundary condition is satisfied with the given electric charge.

For an infinitely long cylindrical conductor as in Exercise 2.9, even if the electric charge is concentrated on an infinitely thin line, the two conductors have equal total amounts of electric charge. Hence, we can obtain an analytic solution that simultaneously satisfies the two boundary conditions using the method of images. The method of images is useful also for dielectric materials, and even for magnetic phenomena in superconductors and magnetic materials. Consider the possibility of solving other problems using this method.

Exercises

2.1. Determine the electric field strength and electric potential when electric charges Q_1 and Q_2 are given to the inner and outer conductors, respectively, of the concentric spherical conductors in Fig. 2.4.

2.2. In a pair of concentric spherical conductors, an electric charge, Q, is given to the outer conductor and the inner conductor is grounded, as shown in Fig. E2.1. Determine the electric charge induced on the inner conductor surface. (Hint: Use the condition that the electric potential is also zero at infinity).

Fig. E2.1 Concentric spherical conductors with grounded inner conductor
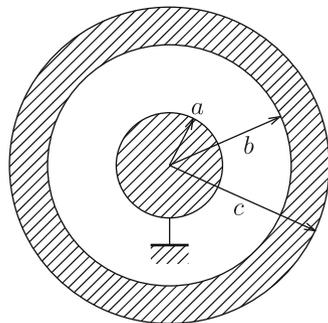
2.3. Two wide slab conductors are parallel to each other, as shown in Fig. E2.2, and an electric charge, Q, is given to the left conductor. The area of each flat surface is S.

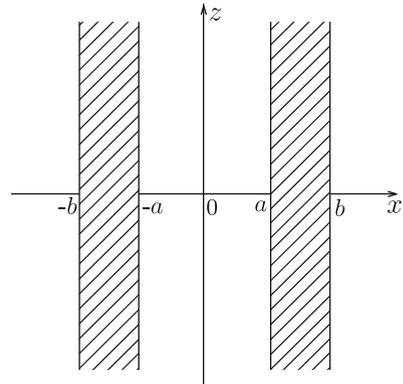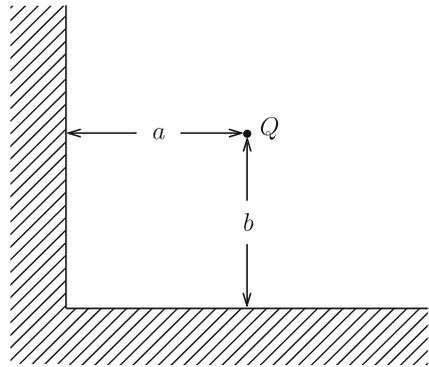**Fig. E2.2** Two parallel slab
conductors

Fig. E2.3 Two perpendicular
flat conductor surfaces and
point charge

Determine the electric charge that appears on each conductor surface, the electric
field strength and electric potential inside and outside the conductors.

2.4. When an electric charge is uniformly distributed with a surface density σ on
a thin flat plane, the electric field strength near the plane is given by Eq. (1.22).
However, Eq. (2.5) yields double this electric field strength near the conductor
surface with the same charge density. Discuss the reason for the difference.

2.5. When an electric charge, q, is put at a point at distance a from a wide conductor
surface, the electric charge induced on the conductor surface is given by Eq. (2.13).
Prove that the Coulomb force exerted on q by the induced electric charge is given
by Eq. (2.15).

2.6. Point charge Q is placed at a point at distances a and b from two flat conductor
surfaces that are perpendicular to each other, as shown in Fig. E2.3. Determine the
electric potential and electric field strength in the vacuum.

2.7. Determine the electric field strength in the space around a spherical conductor
using the electric potential given by Eq. (2.20).

Fig. E2.4 Cylindrical
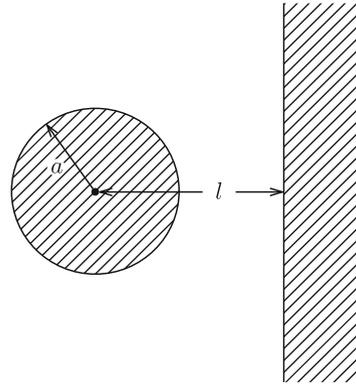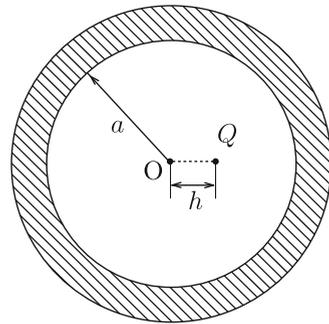conductor parallel to infinite
flat conductor surface

Fig. E2.5 Hollow spherical
conductor and electric charge
at a point inside the conductor

2.8. A long line of electric charge of uniform linear density λ is placed at distance d from the central axis of a grounded parallel long cylindrical conductor of radius $a(< d)$. Determine the electric charge induced on the conductor surface.

2.9. A long cylindrical conductor of radius a is placed at distance $l(> a)$ from an infinite flat conductor surface, as shown in Fig. E2.4, and an electric charge of linear density λ is given to the cylindrical conductor. Determine the density of electric charge on the surfaces of the two conductors.

2.10. Electric charge Q is placed at a point at distance h from the center, O, of a hollow spherical conductor, as shown in Fig. E2.5. Determine the electric potential in the vacuum and the electric charge density on the inner surface of the conductor.