Chapter 6
Current and Magnetic Flux Density

6.1 Magnetic Flux Density by Current

We know that there exists a force between currents. This phenomenon is similar to the Coulomb force between electric charges. Hence, we can presume that currents also make some field in space similar to the electric field made by electric charges. This field is called the **magnetic field**.

Magnets also make a magnetic field. However, the magnetic field source that we can quantitatively define is current. In fact, the amount of current is defined based on the force between currents. In the case of magnets the magnetic field strength depends on the substance of the magnet and its condition of magnetization, which cannot be controlled exactly. We discuss the relationship between the current and magnetic field in this chapter. It should be noted, however, that we do not define the magnetic field itself but the **magnetic flux density** to express the magnetic field strength. The relationship between the magnetic field and magnetic flux density will be described in Chap. 9.

We suppose that two parallel straight wires separated by distance d carry currents I_1 and I_2, as shown in Fig. 6.1a. In this case a force of strength

$$F' = -\frac{\mu_0 I_1 I_2}{2\pi d} \tag{6.1}$$

works on each wire of a unit length. It is attractive ($F' < 0$) for currents in the same direction ($I_1 I_2 > 0$) and repulsive ($F' > 0$) for currents in opposite directions ($I_1 I_2 < 0$). The constant

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2} \tag{6.2}$$

is called the **magnetic permeability of vacuum**. The unit of current, [A], is defined using Eq. (6.1). The magnitude of this force corresponds to the magnitude of the Coulomb force on electric charges. That is, when electric charges of linear densities

T. Matsushita, *Electricity and Magnetism: New Formulation by Introduction of Superconductivity*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-4-431-54526-2_6, © Springer Japan 2014

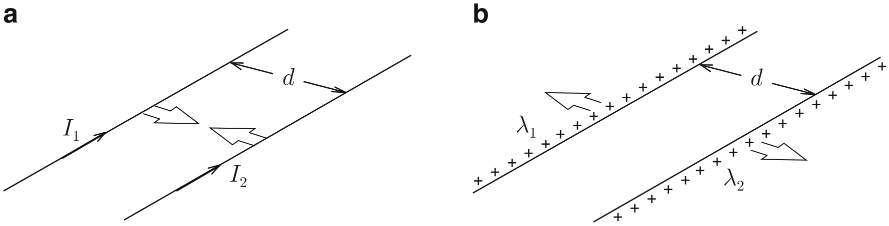

Fig. 6.1 (**a**) Force between two parallel currents and (**b**) force between two parallel line charges

λ_1 and λ_2 are uniformly distributed on two parallel straight lines separated by d, as shown in Fig. 6.1b, the Coulomb force on each line of a unit length is given by

$$F' = \frac{\lambda_1 \lambda_2}{2\pi\epsilon_0 d}, \tag{6.3}$$

which is of the same form as the magnetic force of Eq. (6.1). Namely, the magnetic force is proportional to the product of two currents and is inversely proportional to their distance. The unique difference is that the magnetic force is attractive for currents in the same direction.

The magnetic flux produced by currents is largely different from the electric field produced by electric charges. The magnetic flux can be visualized using magnetic particles such as iron sand. For example, the magnetic flux lies in a plane normal to a straight current and forms vortices around it, as shown in Fig. 6.2. On the other hand, the electric field radiates from a line charge.

Below are the main differences between the electric interaction between electric charges and the magnetic interaction between currents:

- Electric charges are scalars and currents are vectors.
- While the electric field is directed parallel to the straight line connecting the electric charge and observation point, the magnetic flux is directed normally to

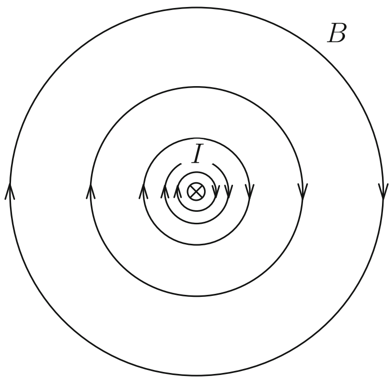

Fig. 6.2 Magnetic flux produced by straight current

both the current and the straight line connecting a part of the current and the observation point.

- While the electric force between electric charges of the same kind is repulsive, the magnetic force between currents in the same direction is attractive.

6.2 The Biot–Savart Law

The law that expresses the magnetic flux density produced by a current is the **Biot–Savart law**. Suppose that current I flows along line C, as shown in Fig. 6.3. The law states that the magnetic flux density at point P produced by an elementary current $I\,\mathrm{d}\boldsymbol{s}$ flowing in a small segment $\mathrm{d}\boldsymbol{s}$ is given by

$$\mathrm{d}\boldsymbol{B} = \frac{\mu_0}{4\pi} \cdot \frac{I\,\mathrm{d}\boldsymbol{s} \times \boldsymbol{i}_r}{r^2} = \frac{\mu_0}{4\pi} \cdot \frac{I\,\mathrm{d}\boldsymbol{s} \times \boldsymbol{r}}{r^3}. \tag{6.4}$$

In the above $\boldsymbol{r}$ is the position vector from the small segment to point P and $|\boldsymbol{r}| = r$ with $\boldsymbol{i}_r = \boldsymbol{r}/r$. The unit of the magnetic flux density is [T] (**tesla**). If the angle between $\mathrm{d}\boldsymbol{s}$ and $\boldsymbol{r}$ is θ, the magnitude of magnetic flux density is

$$\mathrm{d}B = \frac{\mu_0 I\,\mathrm{d}s}{4\pi r^2} \sin\theta \tag{6.5}$$

and the vector points along the motion of a screw when a screw driver is rotated from $\mathrm{d}\boldsymbol{s}$ to $\boldsymbol{r}$.

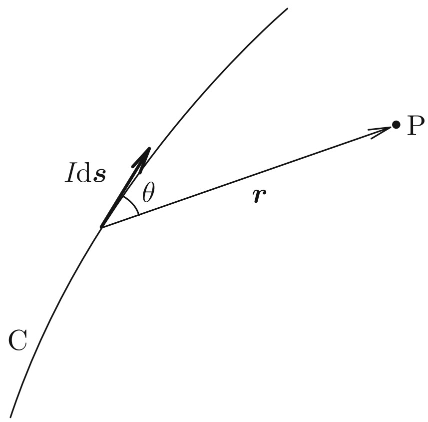

Fig. 6.3 Elementary current along C and observation point P

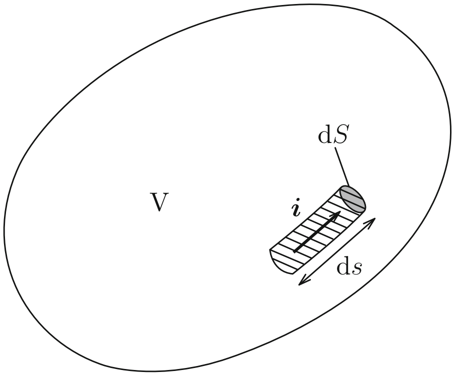

Fig. 6.4 Elementary current in space V

Thus, the magnetic flux density produced at $\boldsymbol{r}$ by current I flowing through line C is given by

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_C \frac{I\,\mathrm{d}\boldsymbol{r}' \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}. \tag{6.6}$$

When there are many currents, the total magnetic flux density is the superposition of all the individual magnetic flux densities they produce.

Here, we consider a current flowing with density $\boldsymbol{i}$ in space V, as shown in Fig. 6.4. The elementary current that flows in a small region of length $\mathrm{d}s$ and cross-sectional area $\mathrm{d}S$ is $\boldsymbol{i}\,\mathrm{d}S\mathrm{d}s = \boldsymbol{i}\,\mathrm{d}V$ with $\mathrm{d}V$ denoting the volume of this region. Thus, the magnetic flux density is given by

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\boldsymbol{i}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \mathrm{d}V'. \tag{6.7}$$

One can see that this equation corresponds to Eq. (1.12) for the electric field produced by electric charges. The electric charge density ρ corresponds to the current density $\boldsymbol{i}$, and the vector product of current density and position is necessary for yielding a vector for the magnetic flux density. This explains why the magnetic flux density is perpendicular to the current.

Example 6.1. Current I flows in a circle of radius a, as shown in Fig. 6.5a. Determine the magnetic flux density at point P located at distance z in the normal direction from the center O of the circle.

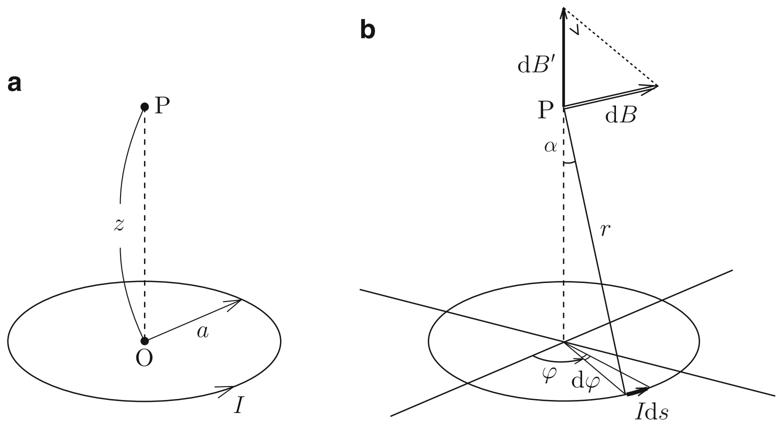

Fig. 6.5 (**a**) Point P on axis of circular current and (**b**) magnetic flux density at P produced by elementary current

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Solution 6.1. We define angle φ as shown in Fig. 6.5b. The elementary current in the part φ to $\varphi + \mathrm{d}\varphi$ has magnitude $I\mathrm{d}s = Ia\mathrm{d}\varphi$ and is directed normally to the position vector from this segment to point P [$\theta = \pi/2$ in Eq. (6.5)], as shown in the figure. The magnetic flux density at P produced by this elementary current is

$$\mathrm{d}B = \frac{\mu_0 I a \mathrm{d}\varphi}{4\pi r^2}$$

with $r = (z^2 + a^2)^{1/2}$. From symmetry only the component along the z-axis remains:

$$\mathrm{d}B' = \frac{\mu_0 I a \mathrm{d}\varphi}{4\pi r^2} \sin\alpha = \frac{\mu_0 I a^2 \mathrm{d}\varphi}{4\pi r^3}.$$

This will be understood by considering a contribution from the opposite side of the circle at angle $\varphi + \pi$. Integrating with respect to angle φ, we have

$$B = \int_0^{2\pi} \frac{\mu_0 I a^2 \mathrm{d}\varphi}{4\pi r^3} = \frac{\mu_0 I a^2}{2(z^2 + a^2)^{3/2}}.$$

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Example 6.2. Current I flows along a long straight line (see Fig. 6.6). Determine the magnetic flux density at point P at distance a from the line.

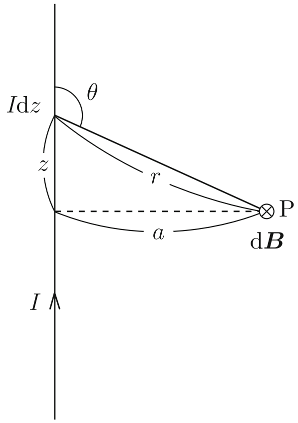

Fig. 6.6 Long straight line with current and observation point P

Solution 6.2. We define coordinates as shown in Fig. 6.6. The magnetic flux density produced at P by an elementary current $I\,\mathrm{d}z$ in a small region z to $z + \mathrm{d}z$ is

$$\mathrm{d}B = \frac{\mu_0 I\,\mathrm{d}z}{4\pi r^2}\sin\theta,$$

where angle θ is defined as in the figure and $r = (z^2 + a^2)^{1/2} = a/\sin\theta$. The relationship $z = -a\cot\theta$ gives $\mathrm{d}z = (a/\sin^2\theta)\mathrm{d}\theta$. Thus, the elementary magnetic flux density is transformed to be

$$\mathrm{d}B = \frac{\mu_0 I}{4\pi a}\sin\theta\ \mathrm{d}\theta.$$

Since this vector is directed normally to this sheet, a simple superposition holds for summing the contribution from each small region. We obtain the magnetic flux density as

$$B = \int_0^{\pi}\frac{\mu_0 I}{4\pi a}\sin\theta\ \mathrm{d}\theta = \frac{\mu_0 I}{2\pi a}. \tag{6.8}$$

◇

6.3 Force on Current

The force on an elementary part $\mathrm{d}\boldsymbol{s}$ of a current I in a magnetic flux density $\boldsymbol{B}$ is given by

$$\mathrm{d}\boldsymbol{F} = I\,\mathrm{d}\boldsymbol{s} \times \boldsymbol{B} \tag{6.9}$$

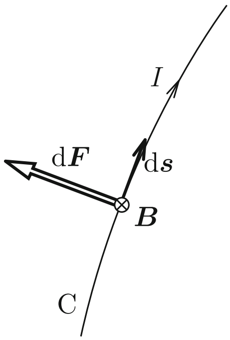

Fig. 6.7 Current I in magnetic flux density $\boldsymbol{B}$

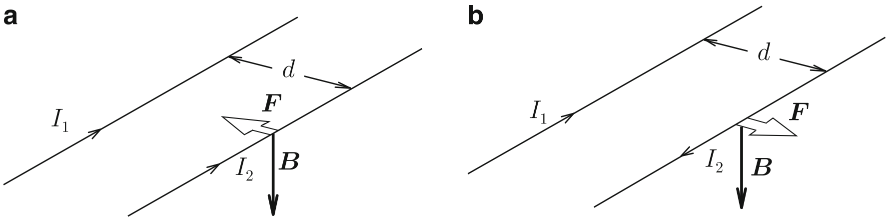

Fig. 6.8 Magnetic flux density produced by current I_1 and resultant force on current I_2 when I_2 has (**a**) the same and (**b**) opposite directions to I_1

(see Fig. 6.7). This is called the **Lorentz force** in a narrow sense. From the mathematical requirement that the force vector results from the product of two vectors, the vector product appears again. The force that line C with current I experiences in the magnetic flux density $\boldsymbol{B}$ is

$$\boldsymbol{F} = I \int_{\mathrm{C}} \mathrm{d}\boldsymbol{s} \times \boldsymbol{B}. \tag{6.10}$$

We apply this to the case of a force on two parallel currents in Fig. 6.1a. The magnetic flux density that current I_1 produces at the position of current I_2 is $B = \mu_0 I_1/(2\pi d)$ using the result of Example 6.2. It is directed as shown in Fig. 6.8a. Hence, if I_2 is in the same direction as I_1, the force on I_2 is attractive. If I_2 is directed opposite to I_1 as in Fig. 6.8b, the force is repulsive. This results in Eq. (6.1).

Since the current is composed of flowing electric charges, we can consider that the force on the current to be a force on the electric charges. The force on the region of length ds along the direction of the current I and cross-sectional area dS in the magnetic flux density $\boldsymbol{B}$ is

$$\mathrm{d}\boldsymbol{F} = I\mathrm{d}\boldsymbol{s} \times \boldsymbol{B} = \boldsymbol{i}\,\mathrm{d}S\mathrm{d}s \times \boldsymbol{B}. \tag{6.11}$$

Since $\mathrm{d}S\mathrm{d}s$ is the volume of this region, $\mathrm{d}V$, we rewrite this force as

$$\mathrm{d}\boldsymbol{F} = (q\boldsymbol{v} \times \boldsymbol{B})n\mathrm{d}V \tag{6.12}$$

using Eq. (5.4). Since $n\mathrm{d}V$ is the number of electric charges in this volume, the force on one electric charge is given by

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}. \tag{6.13}$$

For the special case where the electric field $\boldsymbol{E}$ and the magnetic flux density $\boldsymbol{B}$ coexist, the force on the electric charge is

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \tag{6.14}$$

This is called the **Lorentz force** in a broad sense. This equation shows that $\boldsymbol{E}$ and $\boldsymbol{B}$ are important variables that connect electromagnetism with dynamics. The **$\boldsymbol{E}$–$\boldsymbol{B}$ analogy**is the standpoint from which electromagnetism is described using these variables. Chapter 9 will include a description based on another standpoint.

We can easily show that the total force on a closed circuit, C, along which current I flows in a uniform magnetic flux density is zero:

$$\boldsymbol{F} = I\left(\oint_{\mathrm{C}} \mathrm{d}\boldsymbol{s}\right) \times \boldsymbol{B} = 0. \tag{6.15}$$

However, it should be noted that the torque, i.e., the moment of force on a closed circuit is not necessarily zero.

For example, we suppose rectangular circuit PQRS in a uniform magnetic flux density, $\boldsymbol{B}$, as shown in Fig. 6.9a. We assume sides PS and QR are normal to $\boldsymbol{B}$. When current I flows in this circuit, the force $aIB\cos\theta$ works on side PQ, where θ is the angle between the unit vector $\boldsymbol{n}$ normal to the circuit and the magnetic flux density $\boldsymbol{B}$. The strength of this force is the same as that on side RS, and these forces are on the same line and directed opposite to each other (see Fig. 6.9b). Hence, these forces completely cancel and do not contribute to the torque. On the other hand, forces of strength bIB work on sides PS and QR in opposite directions to each other, but these forces do not lie on the same line. Hence, a torque appears and rotates the circuit (see Fig. 6.9c). Its magnitude is

$$N = bIBa\sin\theta = BIS\sin\theta, \tag{6.16}$$

where $S = ab$ is the area of the circuit.

The unit vector $\boldsymbol{n}$ specifies the direction of movement of a right thumb when it is rotated along the direction of the current, and the surface vector of the closed circuit is defined by $\boldsymbol{S} = \boldsymbol{n}S$. Then, the torque is expressed in the form of a vector: $\boldsymbol{N} = I\boldsymbol{S} \times \boldsymbol{B}$, where $\boldsymbol{N}$ is a vector with the same magnitude as the torque and is

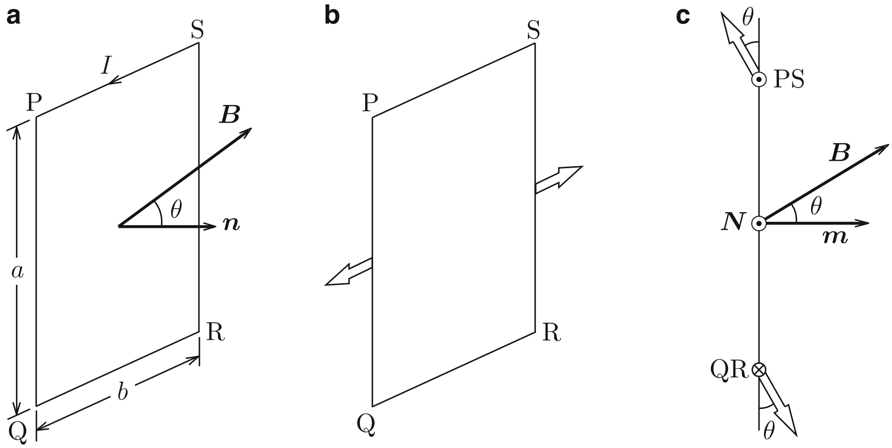

Fig. 6.9 (**a**) Rectangular circuit with current I in uniform magnetic flux density $\boldsymbol{B}$, (**b**) two forces that cancel each other and (**c**) two forces that cause torque

directed along the motion of a screw when a screw driver is rotated along the torque. If we define the **magnetic moment** of the closed circuit as

$$\boldsymbol{m} = I\boldsymbol{S}, \tag{6.17}$$

the torque is given by

$$\boldsymbol{N} = \boldsymbol{m} \times \boldsymbol{B}. \tag{6.18}$$

Equation (6.18) holds for closed current I with an arbitrary shape on a plane in a uniform magnetic flux density. We can explain this as follows. Any given current-carrying closed circuit is expressed as a superposition of small rectangular closed circuits with the same current I, as shown in Fig. 6.10, since two currents in opposite directions in adjacent closed circuits completely cancel out. The above discussion holds for each small rectangular circuit. The magnitude of the total magnetic moment is equal to the product of current I and the total area of the closed circuit. Thus, we can prove that Eqs. (6.17) and (6.18) hold for any closed circuit on a plane.

Example 6.3. Electric charge q of mass m is ejected with velocity v along of the x-axis in magnetic flux density B along the z-axis. Discuss the motion of the electric charge after ejection.

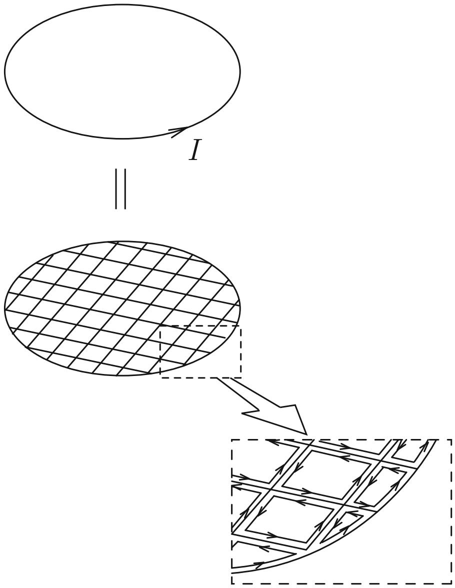

Fig. 6.10 Division of closed current of arbitrary shape on a plane

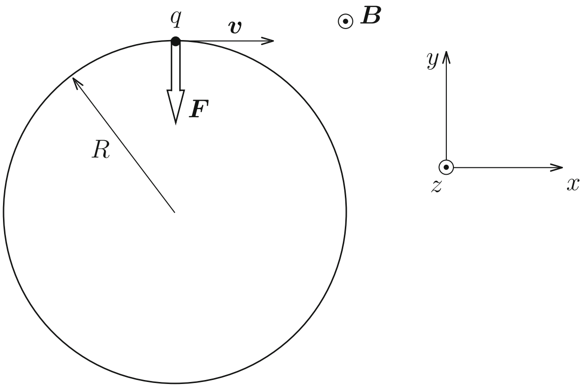

Fig. 6.11 Motion of electric charge in magnetic flux density

Solution 6.3. The Lorentz force given by Eq. (6.13) works on the electric charge along the y-axis just after it is ejected. Since the force is directed perpendicularly to the motion, a circular motion of the electric charge occurs (see Fig. 6.11). We denote the radius of this circular motion as R. The magnitude of the Lorentz force on the electric charge is qvB and the centrifugal force is mv^2/R. From the condition of balance between these forces, we obtain the radius of the circular motion as

$$R = \frac{mv}{qB}.$$

This circular motion of a charge is called **cyclotron motion**. The angular frequency of the motion, which is called **cyclotron angular frequency**, is given by

$$\omega = \frac{v}{R} = \frac{qB}{m}.$$

Since the Lorentz force is always perpendicular to the direction of motion, the work done by the Lorentz force is zero. ◇

6.4 Magnetic Flux Lines

The electric field produced by electric charges can be visualized by electric field lines. We can similarly see the magnetic flux density with **magnetic flux lines**. The magnetic flux line is defined as follows: the direction of a tangential line at any point on the magnetic flux line is the same as the direction of $\boldsymbol{B}$, and its line density is defined as equal to the magnitude of $\boldsymbol{B}$. Figure 6.12 shows examples of magnetic flux lines. We define the **magnetic flux** that passes through arbitrary surface S as

$$\Phi = \int_{\mathrm{S}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}. \tag{6.19}$$

The unit of magnetic flux is [T m^2], which is newly defined as [Wb] (**weber**).

From the examples in Fig. 6.12 it seems that magnetic flux lines are closed lines. This is different from electric field lines that start from positive electric charges and terminate at negative electric charges. If this speculation is valid,

$$\int_{\mathrm{S}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0 \tag{6.20}$$

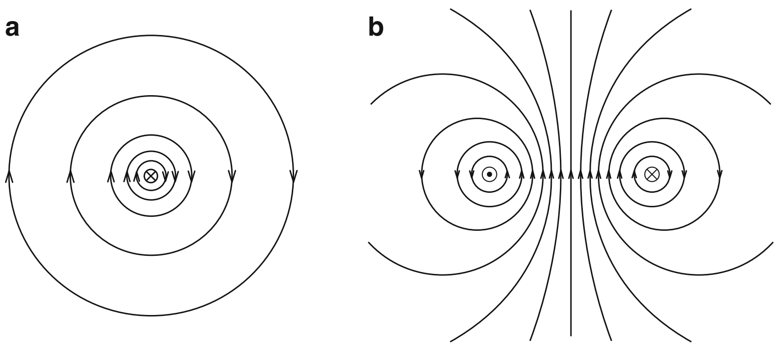

Fig. 6.12 Examples of magnetic flux lines for (**a**) a straight current and (**b**) a circular current

holds for an arbitrary closed surface, S. Using Gauss' theorem on the left side, Eq. (6.20) gives

$$\int_{\mathrm{V}} \nabla \cdot \boldsymbol{B} \, \mathrm{d}V = 0,$$

where V is the interior of S. Since this holds for arbitrary V, we have

$$\nabla \cdot \boldsymbol{B} = 0. \tag{6.21}$$

In fact, we can prove Eq. (6.21) as described in Sect. A2.2 in the Appendix. That is, the magnetic flux lines are closed lines. Equations (6.20) and (6.21) are called **Gauss' law for magnetic flux** and **Gauss' divergence law for magnetic flux**, respectively.

6.5 Ampere's Law

Consider the circular integral of magnetic flux density $\boldsymbol{B}$ along a closed line, C:

$$\oint_{\mathrm{C}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s}.$$

When current I flows in a straight line and C is a circle of radius R from the current in a plane normal to the current, $\boldsymbol{B}$ is parallel to the line element d$\boldsymbol{s}$ (see Fig. 6.13) and its magnitude B is a constant, $\mu_0 I/(2\pi R)$, as discussed in Example 6.2. Hence, the above integral gives

$$\oint_{\mathrm{C}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = \frac{\mu_0 I}{2\pi R} \oint_{\mathrm{C}} \mathrm{d}s = \mu_0 I \tag{6.22}$$

and the value is independent of R.

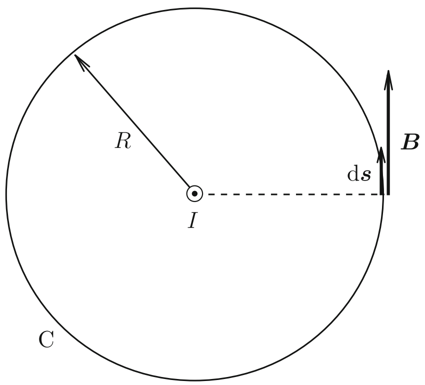

Fig. 6.13 Circle C around straight current

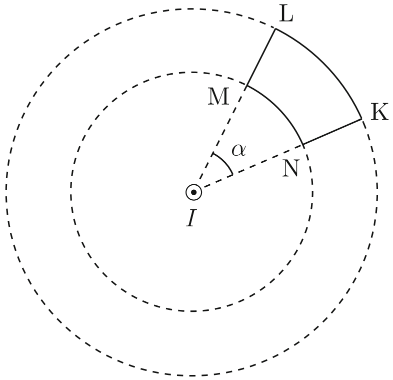

Fig. 6.14 Closed line KLMN on a plane normal to straight current that is composed of two arcs and two straight segments

Suppose a closed line, C, on a plane normal to the straight current I that does not encircle the current. For simplicity we assume that C is composed of two arcs with different radii and two straight segments extending from the current, as shown in Fig. 6.14. Here, we calculate the circular integral of $\boldsymbol{B}$ on C. We denote the angle of arcs KL and MN as α. The integrals along these arcs are:

$$\int_{\mathrm{K}}^{\mathrm{L}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = \int_{\mathrm{N}}^{\mathrm{M}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = \frac{\mu_0 I \alpha}{2\pi}.$$

On the two straight segments NK and LM, $\boldsymbol{B}$ is perpendicular to d$\boldsymbol{s}$ and we have

$$\int_{\mathrm{N}}^{\mathrm{K}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = \int_{\mathrm{L}}^{\mathrm{M}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = 0.$$

Thus, the circular integral gives

$$\oint_{\mathrm{C}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = 0. \tag{6.23}$$

Using the above results, we can show that Eq. (6.22) holds when straight current I penetrates arbitrary closed line C on the normal plane, and Eq. (6.23) holds when this is not so. In the latter case, for example, we can realize an arbitrary closed line with a set of small closed loops composed of two arcs with the center on the current and two straight segments extending from the current, as shown in Fig. 6.15, and Eq. (6.23) holds for each closed loop. Hence, we can show that Eq. (6.23) holds for any closed line.

If I is defined as the current that penetrates closed line C in Eq. (6.22), it includes Eq. (6.23). This equation holds also when the closed line is not on a plane normal to the straight current and/or when the current is not straight. Since it is time-consuming to prove each of them, this will be proved indirectly using another method.

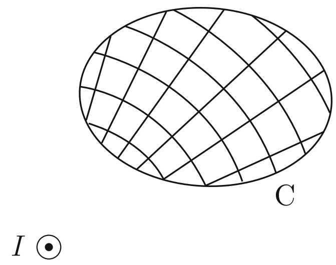

Fig. 6.15 Closed line divided into a set of closed loops composed of two arcs and two straight segments when straight current does not penetrate the closed line

We suppose that currents I_1, I_2, $\cdots$ flow separately in space. In this case Eq. (6.22) is extended to

$$\oint_{\mathrm{C}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = \mu_0 \sum_{m=1}^{n} I_m . \tag{6.24}$$

In the above, the right side is the sum of the currents that penetrate C. When current flows with the density $\boldsymbol{i}$, the corresponding equation is

$$\oint_{\mathrm{C}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = \mu_0 \int_{\mathrm{S}} \boldsymbol{i} \cdot \mathrm{d}\boldsymbol{S} , \tag{6.25}$$

where S is the surface surrounded by C. Equations (6.22), (6.24) and (6.25) are called **Ampere's law**.

Applying Stokes' theorem to the left side of Eq. (6.25) gives

$$\int_{\mathrm{S}} \nabla \times \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \mu_0 \int_{\mathrm{S}} \boldsymbol{i} \cdot \mathrm{d}\boldsymbol{S} . \tag{6.26}$$

Since this holds for arbitrary S, we have

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{i} . \tag{6.27}$$

This is called the **differential form of Ampere's law**. Hence, if Eq. (6.27) holds, then Eq. (6.22) must hold. The proof of Eq. (6.27) is given in Sect. A2.3 in the Appendix.

From Eq. (A1.44) in the Appendix, we can see that Eq. (6.27) satisfies Eq. (5.11). That is, the magnetic flux density $\boldsymbol{B}$ is produced by a steady current. Equation (6.27) shows that the current produces rotation of the magnetic flux density. This is in contrast with Eq. (1.21) that shows that an electric charge produces divergence of the electric field.

Example 6.4. Current I flows uniformly in a long cylinder of radius a. Determine the magnetic flux density inside and outside the cylinder.

Solution 6.4. We can determine the magnetic flux density using the Biot–Savart law. However, it is not easy. On the other hand, this can be easily done using Ampere's law. We apply this law to a circle, C, of radius R from the central axis of the cylinder, as shown in Fig. 6.16. From symmetry the magnetic flux density $\boldsymbol{B}$ is parallel to a line element, d$\boldsymbol{s}$, and its magnitude B is constant on C. Hence, the left side of Eq. (6.25) gives

$$\oint_{\mathrm{C}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = 2\pi R B.$$

For $R > a$, the total current that flows inside C is I and the right side of Eq. (6.25) is equal to $\mu_0 I$. Thus, we have

$$B = \frac{\mu_0 I}{2\pi R}.$$

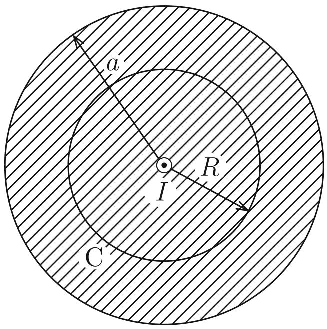

Fig. 6.16 Circle C with the same central axis as cylinder for $R < a$

This result is the same as for the case where the total current flows along the central axis. For $R < a$, the current inside C is $(R/a)^2 I$, and we have

$$B = \frac{\mu_0 I R}{2\pi a^2}.$$

◇

Example 6.5. Current flows uniformly with surface density τ on a wide plane. Determine the magnetic flux density at a position at distance h from the plane.

Solution 6.5. We can easily obtain the answer using Ampere's law also for this case. Suppose a rectangle, KLMN, normal to the direction of current with two sides (KL and MN) of length w parallel to the plane, as shown in Fig. 6.17. The other two sides (LM and NK) of length $2h$ are normal to the plane. We apply Ampere's law to this rectangle. From symmetry the magnetic flux density has the same value on sides KL and MN at the same distance from the plane, and its vectors are parallel to these sides but opposite to each other. Hence, the contribution from these sides to the circular integral of the magnetic flux density gives $2wB$. On the other hand, the contribution from other two sides is zero. As a result we have

$$\oint_{\mathrm{C}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = 2wB.$$

The total current inside the rectangle is $w\tau$. Thus, we obtain the magnetic flux density as

$$B = \frac{\mu_0 \tau}{2}. \tag{6.28}$$

It should be noted that this value is independent of the distance h from the plane. Figure 6.18 shows the magnetic flux lines produced by the sheet current. This is similar to the electric field produced by an electric charge distributed uniformly on a wide plane (see Example 1.4). ◇

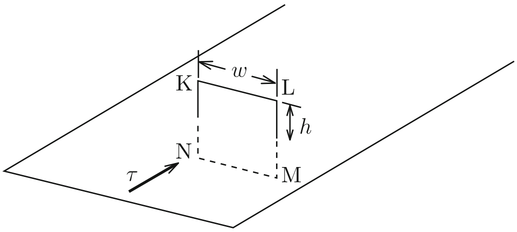

Fig. 6.17 Rectangle normal to the direction of current flowing uniformly on a plane

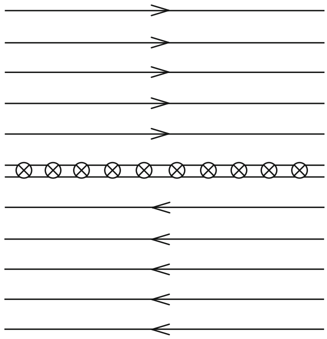

Fig. 6.18 Magnetic flux lines produced by uniform sheet current

Table 6.1 Correspondence of laws describing electric and magnetic phenomena

	Electricity	Magnetism
Local	$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\boldsymbol{r}')(\boldsymbol{r}-\boldsymbol{r}')}{\lvert\boldsymbol{r}-\boldsymbol{r}'\rvert^3}\mathrm{d}V'$ (Coulomb's law)	$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\boldsymbol{i}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{\lvert\boldsymbol{r}-\boldsymbol{r}'\rvert^3}\mathrm{d}V'$ (The Biot–Savart law)
Global	$\int_S \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S} = \frac{1}{\epsilon_0}\int_V \rho\,\mathrm{d}V$ (Gauss' law)	$\oint_C \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{s} = \mu_0\int_S \boldsymbol{i}\cdot\mathrm{d}\boldsymbol{S}$ (Ampere's law)
Differential	$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\epsilon_0}$	$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{i}$

As shown above we can say that the Biot–Savart law for the local magnetic flux density produced by a current and Ampere's law for the global relationship between the magnetic flux density and current describe the same magnetic phenomenon from opposite viewpoints. This is similar to the relation between Coulomb's law and Gauss' law describing electric phenomena in Chap. 1. Table 6.1 shows the correspondence between electric and magnetic phenomena.

Although there is a difference between the scalar source (electric charge) and vector source (current) that produce the fields, the correspondence between electric and magnetic phenomena is clear. These sources cause a divergence of the electric field and rotation of the magnetic flux density. As for the physical constants, the magnetic permeability of vacuum, μ_0, corresponds to the inverse of the permittivity of vacuum, ϵ_0^{-1}.

6.6 Vector Potential

The electric field $\boldsymbol{E}$ is given by Eq. (1.24) in terms of the electric potential, i.e., a kind of scalar potential. This originates from the irrotational nature of the electric field as given by Eq. (1.28) and from the mathematical property that the gradient of a scalar is irrotational.

Magnetic flux density is a solenoidal field with no divergence, as shown by Eq. (6.21). Mathematically rotation of a vector has no divergence, as shown by Eq. (A1.44) in the Appendix. Hence, the magnetic flux density can be mathematically expressed as a rotation of some vector:

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}. \tag{6.29}$$

The quantity $\boldsymbol{A}$ is called the **vector potential**. As can be seen from Eq. (A1.45), we could add the gradient of any scalar function to the vector potential, and the vector potential would still correspond to the same magnetic flux density. This arbitrary gradient of some scalar function means the vector potential cannot be uniquely determined without specifying some condition. In the case of static magnetic phenomenon that does not change with time, the condition

$$\nabla \cdot \boldsymbol{A} = 0 \tag{6.30}$$

is usually used. This is called the **Coulomb gauge**. We assume a new vector potential,

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\alpha, \tag{6.31}$$

under this condition. It yields the same magnetic flux density. Using Eq. (6.30), α satisfies Laplace's equation:

$$\nabla \cdot \nabla\, \alpha = \Delta\alpha = 0. \tag{6.32}$$

Since α is uniquely determined under a given boundary condition as discussed in Sect. 2.2, the vector potential is uniquely determined using the Coulomb gauge.

The solution of the vector potential is given by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\mathrm{V}} \frac{\boldsymbol{i}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V' \tag{6.33}$$

One can show that this proves the Biot–Savart law for the magnetic flux density $\boldsymbol{B}$ (see Sect. A2.4 in the Appendix). In this case the Coulomb gauge is fulfilled (see Exercise 6.8). This equation corresponds to Eq. (1.27) describing the electric potential produced by electric charge. These are compared in Table 6.2. The result that the potential is scalar or vector depends on whether the source of the field is scalar or vector. The similarity in Table 6.1 is found again in this table. When current I flows along a line circuit, C, Eq. (6.33) reduces to

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\int_{\mathrm{C}} \frac{\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{6.34}$$

Table 6.2 Comparison between electric potential due to electric charge and vector potential due to current

	Electric potential by charge	Vector potential by current
Potential	$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathrm{V}} \frac{\rho(\boldsymbol{r}')}{\lvert\boldsymbol{r}-\boldsymbol{r}'\rvert}\mathrm{d}V'$	$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\mathrm{V}} \frac{\boldsymbol{i}(\boldsymbol{r}')}{\lvert\boldsymbol{r}-\boldsymbol{r}'\rvert}\mathrm{d}V'$
Source	ρ	$\boldsymbol{i}$
Constant	ϵ_0	μ_0^{-1}
Equation	$\Delta\phi = -\frac{\rho}{\epsilon_0}$	$\Delta\boldsymbol{A} = -\mu_0\boldsymbol{i}$

Integrating the vector potential along C gives

$$\oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{s} = \int_S \nabla \times \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} = \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \Phi, \tag{6.35}$$

where S is a surface surrounded by C and Φ is a magnetic flux that penetrates C. In the above we have used Stokes' theorem and Eqs. (6.19) and (6.29).

Substituting Eq. (6.29) into Eq. (6.27) gives

$$\nabla \times (\nabla \times \boldsymbol{A}) = \mu_0 \boldsymbol{i}\,. \tag{6.36}$$

With Eqs. (A1.46) in the Appendix and (6.30), Eq. (6.36) becomes

$$\Delta \boldsymbol{A} = -\mu_0 \boldsymbol{i}\,. \tag{6.37}$$

That is, each component of the vector potential satisfies Poisson's equation. In the region where current does not flow ($\boldsymbol{i} = 0$), this reduces to Laplace's equation,

$$\Delta \boldsymbol{A} = 0. \tag{6.38}$$

This is also similar to the electric potential in electric phenomena. When a boundary condition is given, $\boldsymbol{A}$ in Eqs. (6.37) or (6.38) is uniquely determined. One can directly prove that Eq. (6.36) holds for the vector potential $\boldsymbol{A}$ given by Eq. (6.33). This will be apparent from the proof of Eq. (6.33) in Sect. A2.4 and that of Eq. (6.27) in Sect. A2.3 in the Appendix.

Example 6.6. Determine the vector potential for the case discussed in Example 6.4.

Solution 6.6. We use cylindrical coordinates. Since the current flows only along the z-axis, the vector potential has only the z-component A_z, as indicated by Eq. (6.33). In addition, from symmetry it does not depend on z or the azimuthal angle φ. The magnetic flux density has only the azimuthal component B_φ. Thus, we have

$$B_\varphi = -\frac{\partial A_z}{\partial R}.$$

The vector potential is given by

$$A_z = -\int_{R_0}^{R} B_\varphi \mathrm{d}R,$$

where $R_0 (> a)$ is the distance from the central axis to the reference point at which $A_z = 0$. The reason why infinity is not chosen as the reference point is that the vector potential diverges because of the requirement that the current flows over an infinitely long distance. In fact, we find that the vector potential directly estimated from Eq. (6.34) diverges. This corresponds to the divergence of the electric potential for an infinitely long line charge (see Example 1.7).

We determine the vector potential as

$$A_z = \frac{\mu_0 I}{2\pi} \log \frac{R_0}{R}$$

from $B_\varphi = \mu_0 I/(2\pi R)$ for $R > a$ and as

$$A_z = \frac{\mu_0 I}{4\pi a^2}(a^2 - R^2) + \frac{\mu_0 I}{2\pi} \log \frac{R_0}{a}$$

from $B_\varphi = \mu_0 I R/(2\pi a^2)$ for $0 \leq R < a$. In the above the vector potential takes a constant value on the cylindrical surface with the radius R. Thus, we can define an **equivector-potential surface**, which is similar to the equipotential surface. The vector potential is parallel to the equivector-potential surface. ◇

Example 6.7. Current I is applied to an infinitely long solenoid coil of radius a with n turns in a unit length. Determine the vector potential.

Solution 6.7. Firstly, we determine the magnetic flux density. Suppose rectangles C_1 and C_2 as shown in Fig. 6.19. Applying Ampere's law to these rectangles, the circular integral of the magnetic flux density is zero in each case. It shows that the magnetic flux density is constant inside and outside the coil. Since the magnetic flux density outside the coil must be uniform up to infinity and the total magnetic flux must be finite, we can show that the magnetic flux density must be zero outside the coil. Then, we apply Ampere's law to rectangle C_3 to determine the magnetic flux density B inside the coil. The left side of Eq. (6.25) is Bl with l denoting the axial length of C_3. Since the total current inside C_3 is nIl, we have

$$B = \mu_0 n I.$$

Then, we could determine the vector potential with Eq. (6.29), but we use Eq. (6.35) here. We apply this equation to a circle, C, of radius R from the central axis of the coil in Fig. 6.20. Since the current flows only in the azimuthal direction, the vector potential has only the azimuthal component A_φ. Hence, this is parallel to C and its magnitude is constant. We have

$$\oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{s} = 2\pi R A_\varphi(R).$$

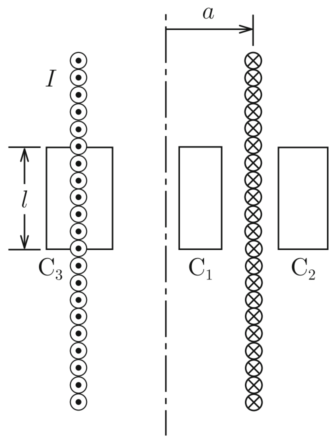

Fig. 6.19 Longitudinal cross-section of solenoid coil and rectangles C_1–C_3

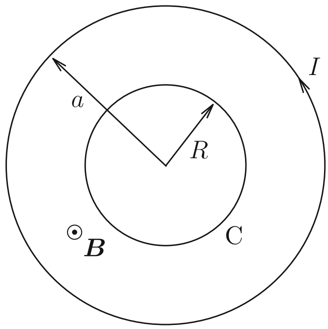

Fig. 6.20 Circle C of radius R from the central axis of the coil (for $R < a$)

On the other hand, the magnetic flux penetrating C is

$$\int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = B\pi R^2 = \pi\mu_0 n I R^2; \qquad 0 \le R < a,$$
$$= B\pi a^2 = \pi\mu_0 n I a^2; \qquad R > a.$$

Thus, we determine the vector potential as

$$A_\varphi(R) = \frac{\mu_0 n I R}{2}; \qquad 0 \le R < a,$$
$$= \frac{\mu_0 n I a^2}{2R}; \qquad R > a.$$

Determine the vector potential with Eq. (6.29) and confirm that it agrees with the above result (see Exercise 6.9). ◇

6.7 Small Closed Current

Suppose that current I flows around a small square of side length d. The vector potential produced by this current is determined at a point, P, sufficiently far from this square. We denote the direction vector from the center of the square to point P as $\boldsymbol{r}$. The assumption allows $|\boldsymbol{r}| = r \gg d$. We define the origin of the coordinates at the center of the square placed on the x-y plane. Their sides are parallel to the x- or y-axis, as shown in Fig. 6.21a. We also assume that P is on the y-z plane, and θ is the angle between $\boldsymbol{r}$ and the z-axis. Hence, the position of P is $\boldsymbol{r} = (0, r\sin\theta, r\cos\theta)$ in polar coordinates. We suppose that a point, Q, with the position vector $\boldsymbol{r}'$ moves on square KLMN (see Fig. 6.21b). When Q is on side KL, $\boldsymbol{r}' = (x, d/2, 0)$ with $-d/2 \le x \le d/2$. A simple calculation gives

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{[r^2 - rd\sin\theta + (d^2/4) + x^2]^{1/2}} \simeq \frac{1}{r}\left(1 + \frac{d}{2r}\sin\theta\right).$$

Integrating this from K to L (along the negative x-axis), we have

$$\int_{\mathrm{K}}^{\mathrm{L}} \frac{\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\frac{d}{r}\left(1 + \frac{d}{2r}\sin\theta\right)\boldsymbol{i}_x. \tag{6.39a}$$

When Q is on MN, we similarly have $\boldsymbol{r}' = (x, -d/2, 0)$ and

$$\int_{\mathrm{M}}^{\mathrm{N}} \frac{\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{d}{r}\left(1 - \frac{d}{2r}\sin\theta\right)\boldsymbol{i}_x. \tag{6.39b}$$

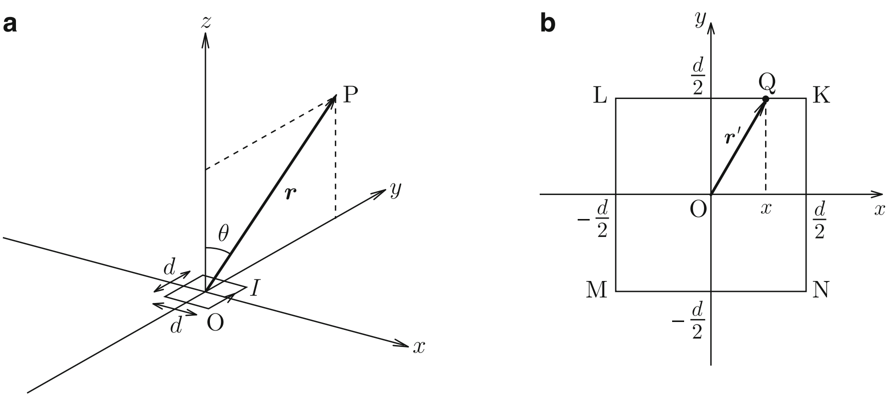

Fig. 6.21 (**a**) Closed current flowing along a small square with the center at the origin and observation point P and (**b**) position of Q on the square

When Q is on LM, substituting $\boldsymbol{r}' = (-d/2, y, 0)$ gives

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{[r^2 - 2ry\sin\theta + (d^2/4) + y^2]^{1/2}} \simeq \frac{1}{r}\left(1 + \frac{y}{r}\sin\theta\right)$$

and

$$\int_{\mathrm{L}}^{\mathrm{M}} \frac{\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\frac{d}{r}\boldsymbol{i}_y. \tag{6.39c}$$

Similarly we have $\boldsymbol{r}' = (d/2, y, 0)$ and

$$\int_{\mathrm{N}}^{\mathrm{K}} \frac{\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{d}{r}\boldsymbol{i}_y, \tag{6.39d}$$

when Q is on NK.

The vector potential is determined with Eqs. (6.39a)–(6.39d) as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\oint_{\mathrm{C}} \frac{\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\frac{\mu_0 I d^2}{4\pi r^2}\sin\theta\ \boldsymbol{i}_x. \tag{6.40}$$

We rewrite this as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\cdot\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3} \tag{6.41}$$

in terms of the magnetic moment,

$$\boldsymbol{m} = I\boldsymbol{S}, \tag{6.42}$$

with $\boldsymbol{S} = d^2\boldsymbol{i}_z$ denoting the surface vector. Equation (6.41) corresponds to the electric potential, Eq. (1.47), produced by an electric dipole. Practically Eq. (6.41) reduces to

$$A_\varphi = \frac{\mu_0 m}{4\pi r^2}\sin\theta \tag{6.43}$$

in our polar coordinates. In the above, $m = |\boldsymbol{m}|$ is the magnitude of the magnetic moment.

For simplicity we treated the closed current flowing on the small square in the above. This result is valid for small closed current of arbitrary shape. One can easily prove this, since any closed current can be realized by a superposition of small square currents.

The magnetic flux density produced by the closed current is determined with Eqs. (6.29) and (6.43) as

$$B_r = \frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\theta}(\sin\theta A_\varphi) = \frac{\mu_0 m\cos\theta}{2\pi r^3}, \tag{6.44a}$$

$$B_\theta = -\frac{1}{r} \cdot \frac{\partial}{\partial r}(rA_\varphi) = \frac{\mu_0 m \sin\theta}{4\pi r^3}, \tag{6.44b}$$

$$B_\varphi = 0. \tag{6.44c}$$

It is found that the obtained magnetic flux density has the same form as the electric field produced by the electric dipole given by Eqs. (1.48a)–(1.48c) under the correspondence of $m \to p$ and $\mu_0 \to \epsilon_0^{-1}$. This is reasonable, since the magnetic flux density is expressed as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = -\frac{\mu_0}{4\pi}\nabla\left(\frac{\boldsymbol{m}\cdot\boldsymbol{r}}{r^3}\right) \tag{6.45}$$

as is shown in Sect. A2.5 in the Appendix.

6.8 Magnetic Charge

Magnets are materials that cause magnetic interaction similarly to currents. Magnets have north (N) and south (S) poles. The force between poles of the same kind is repulsive and the force between poles of the different kinds is attractive. This property is similar to that of electric charges. Hence, one can compare the magnetic interaction between magnetic poles to Coulomb's law for electric charges. A **magnetic charge** is an imaginary source that causes magnetic interaction corresponding to the magnetic pole. In fact, it was assumed in the past that N and S poles had magnetic charges, q_m and $-q_\mathrm{m}$, respectively, and that a force similar to the Coulomb force worked on magnetic charges. The force exerted by q'_m on q_m would then be given by

$$\boldsymbol{F}_\mathrm{m} = \frac{\mu_0 q_\mathrm{m} q'_\mathrm{m} \boldsymbol{r}}{4\pi r^3} \tag{6.46}$$

similarly to Eq. (1.3), where $\boldsymbol{r}$ is the position vector from q'_m to q_m and $r = |\boldsymbol{r}|$.

Since the magnetic force, Eq. (6.46), comes from some magnetic distortion in space, we can define the magnetic flux density $\boldsymbol{B}$ as the magnetic field that quantitatively expresses the strength of magnetic distortion. When we express the magnetic force as

$$\boldsymbol{F}_\mathrm{m} = q_\mathrm{m}\boldsymbol{B}, \tag{6.47}$$

Table 6.3 Formal correspondence between electricity and magnetism

	Electricity	Magnetism
Source	Electric charge (q)	Magnetic charge (q_{m})
Potential	Electric potential (ϕ)	Magnetic potential (ϕ_{m})
Field	Electric field ($\boldsymbol{E} = -\nabla\phi$)	Magnetic flux density ($\boldsymbol{B} = -\nabla\phi_{\mathrm{m}}$)

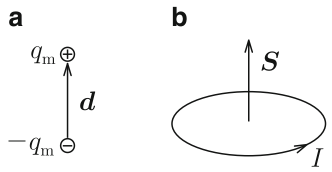

Fig. 6.22 Magnetic moment produced by (**a**) magnetic dipole and (**b**) small closed current

the magnetic flux density is given by

$$\boldsymbol{B} = \frac{\mu_0 q'_{\mathrm{m}} \boldsymbol{r}}{4\pi r^3}. \tag{6.48}$$

It should be noted that this definition of magnetic charge is different by a factor of μ_0^{-1} from that used in other books, in which the magnetic field $\boldsymbol{H}$ defined in Chap. 9 was used for the magnetic interaction instead of the magnetic flux density $\boldsymbol{B}$. One can also define a scalar potential, ϕ_{m}, called **magnetic potential** similarly to the electric potential. Its relation to $\boldsymbol{B}$ is

$$\boldsymbol{B} = -\nabla\phi_{\mathrm{m}}. \tag{6.49}$$

Table 6.3 compares the charges, scalar potentials and fields between electricity and magnetism. However, we cannot apply this scheme in a space in which current flows.

Suppose that imaginary magnetic charges $\pm q_{\mathrm{m}}$ are separated by a small distance, $\boldsymbol{d}$, like an electric dipole. This pair of magnetic charges is called a **magnetic dipole**. We define the **magnetic dipole moment** as

$$\boldsymbol{m} = q_{\mathrm{m}} \boldsymbol{d}. \tag{6.50}$$

The magnetic flux density produced by the magnetic dipole is given by Eq. (1.48) with $\boldsymbol{p}$ replaced by $\boldsymbol{m}$. It was shown above that the magnetic flux density has the same form as Eq. (6.44) produced by a small closed current. Hence, the magnetic dipole and the small closed current are equivalent to each other (see Fig. 6.23). This also supports the formal correspondence between electric and magnetic charges. It is empirically known that even when a magnet is divided into small pieces, it is impossible for any piece to pick up only one type of magnetic pole, as shown in the

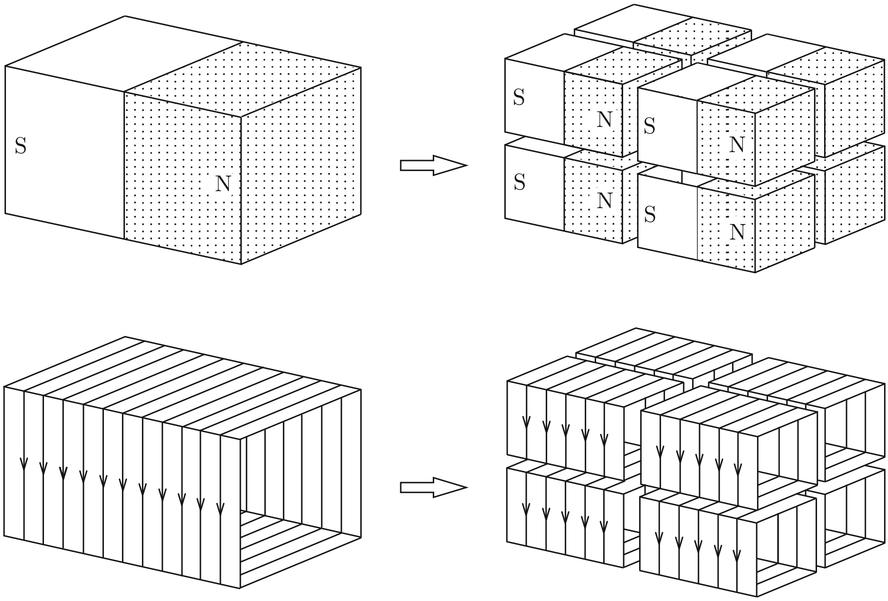

Fig. 6.23 Division of permanent magnet (*upper half*) and corresponding division of closed current (*lower half*)

upper part of Fig. 6.23. This can be explained assuming equivalent closed currents, as shown in the lower figure.

Thus, it may be advantageous in some cases to assume magnetic virtual charges and compare them to electric charges, since we can directly use all knowledge of electric phenomena. However, the obvious problem is that magnetic charge has never been observed. That is, the magnetic flux density empirically satisfies Eq. (6.21), including the flux for permanent magnets. On the other hand, the right side should be equal to **magnetic charge density** ρ_m multiplied by μ_0. Hence, this results in $\rho_m = 0$. For this reason textbooks now base their treatment of magnetic phenomena on current.

Example 6.8. Currents I and $-I$ flow along lines at $y = d/2$ and $y = -d/2$, respectively, on the y-z plane, as shown in Fig. 6.24. Determine the vector potential at point P sufficiently far from the z-axis. This pair of parallel opposite currents is equivalent to a pair of virtual line magnetic charges and is called a **magnetic dipole line**.

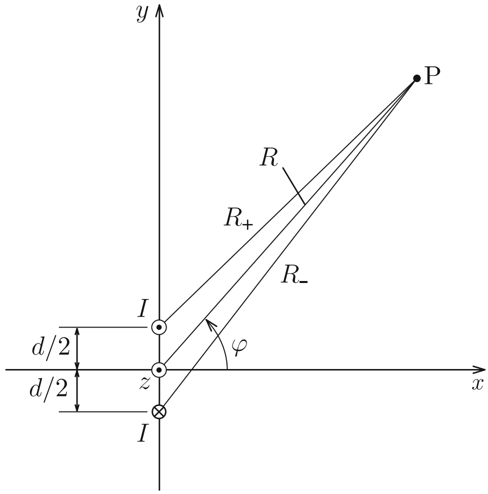

Fig. 6.24 Magnetic dipole line composed of pair of parallel opposite currents

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Solution 6.8. We define cylindrical coordinates, as shown in the figure, and measure the azimuthal angle from the x-axis. We denote the distance between observation point P and the line at $y = d/2$ as R_+. With the solution of Example 6.6 we obtain the contribution of this current to the vector potential as

$$A_{z+} = \frac{\mu_0 I}{2\pi} \log \frac{R_0}{R_+},$$

where R_0 is the distance to the reference point at which the vector potential is zero. The vector potential produced by the current at $y = -d/2$ is

$$A_{z-} = -\frac{\mu_0 I}{2\pi} \log \frac{R_0}{R_-}$$

using the distance R_- between the current and point P. Hence, the vector potential is given by

$$A_z = A_{z+} + A_{z-} = \frac{\mu_0 I}{2\pi} \log \frac{R_-}{R_+}.$$

For $R \gg d$, R_+ reduces approximately to

$$R_+ = \left[R^2 + \left(\frac{d}{2}\right)^2 - Rd \sin\varphi \right]^{1/2} \simeq R \left(1 - \frac{d}{2R} \sin\varphi\right).$$

Similarly we have $R_- \simeq R\{1 + [d/(2R)] \sin\varphi\}$. Thus, the vector potential reduces to

$$A_z(R,\varphi) \simeq \frac{\mu_0 I}{2\pi} \log \frac{1 + [d/(2R)] \sin\varphi}{1 - [d/(2R)] \sin\varphi} \simeq \frac{\mu_0 I d}{2\pi R} \sin\varphi.$$

If we define the **moment of a magnetic dipole line** in a unit length by

$$\hat{m} = Id, \tag{6.51}$$

the vector potential is written as

$$A_z(R,\varphi) \simeq \frac{\mu_0 \hat{m} \sin\varphi}{2\pi R}. \tag{6.52}$$

At sufficient distance ($R \gg d$) the equality holds. In this case the total vector sum of the current is zero, and hence, the vector potential goes to zero at infinity. Thus, there is no problem of divergence.

If we carry out the same calculation using a virtual magnetic charge, we obtain the magnetic potential $\phi_{\mathrm{m}}(R,\varphi)$ that corresponds to the electric potential in Example 1.8. Assume that the magnetic charges of linear densities $\pm\lambda_{\mathrm{m}}$ stay at $x = \pm d/2$, respectively, and define the moment of the magnetic dipole line by

$$\hat{m} = \lambda_{\mathrm{m}} d. \tag{6.53}$$

Then, the magnetic potential is given by

$$\phi_{\mathrm{m}}(R,\varphi) = \frac{\mu_0 \hat{m} \cos\varphi}{2\pi R}. \tag{6.54}$$

This magnetic potential has the same form as the electric potential, Eq. (1.53).

From the above result for vector potential or magnetic potential we determine the magnetic flux density as

$$B_R = \frac{\mu_0 \hat{m}}{2\pi R^2} \cos\varphi, \tag{6.55a}$$

$$B_\varphi = \frac{\mu_0 \hat{m}}{2\pi R^2} \sin\varphi, \tag{6.55b}$$

$$B_z = 0. \tag{6.55c}$$

This corresponds to the electric field in Eq. (1.54). ◇

Column: (1) Forces Between Electric Charges and Between Currents

We have said that, since the source of magnetic phenomena is the current, a vector, the resultant vector field of magnetic flux density must be given by the vector product of the current and position vector. As a result, the magnetic flux density is perpendicular to both the current and position vector. This is in contrast to the electric field, which is simply given by the product of the electric charge, a scalar, and the position vector and is in the same direction as the position vector. Thus, these results follow the mathematical requirements and the correspondence holds in this sense.

Here, we compare the forces arising from electric charges and currents. To obtain a force vector for electric charges, we need the direct product of the electric charge, a scalar, and the electric field, a vector. Thus, the force between electric charges of the same kind is repulsive. For currents the vector product must appear again to yield a force vector from the current vector and the magnetic flux density vector. Thus, the magnetic force is perpendicular to both the current and magnetic flux density. This explains why the force is attractive between currents in the same direction. We again show the correspondence between the forces based on the mathematical requirements.

(2) Coulomb Magnetic Field

In a space with no current, as in vacuum, the magnetic flux density obeys

$$\nabla \times \boldsymbol{B} = 0$$

from Eq. (6.27). Hence, we can describe the magnetic flux density in terms of the magnetic potential ϕ_{m} as in Eq. (6.49) from the mathematical viewpoint. Such a magnetic field is called a **Coulomb magnetic field**.

In the past the magnetic phenomena were described using virtual magnetic charges and the Coulomb magnetic field. This is the magnetic field $\boldsymbol{H}$ defined in Chap. 9 and is equal to $\boldsymbol{B}/\mu_0$ in the present description. This method is beneficial because various descriptions of electric phenomena can be used to explain some magnetic phenomena. For example, the static magnetic energy can be simply determined as the work needed to carry magnetic charges in the magnetic field. This is not possible for currents because of electromagnetic induction (see Column (1) in Chap. 8). In addition, from similarity to the electric field, one can easily show that the parallel component of the magnetic field is continuous at an interface using Eq. (4.22). In fact, this coincides with the boundary condition of Eq. (9.24) when there is of no surface current on the interface.

Thus, the magnetic field in the absence of current behaves similarly to the electric field in the absence of a true electric charge. We can see such an analogy between the electric field in a dielectric material in Example 4.4 and the magnetic flux density in a magnetic material in Example 9.4.

Exercises

6.1. Current I flows along a line composed of a semicircle and two straight lines on a common plane, as shown in Fig. E6.1. Determine the magnetic flux density at the center O of curvature of the semicircle.

6.2. Current I flows on a square closed line of side length a, as shown in Fig. E6.2. Determine the magnetic flux density at point P at distance b from the center O of the square.

6.3. Current of density $\boldsymbol{i}$ flows in a slab conductor along the x-axis in the normal magnetic flux density $\boldsymbol{B}$ parallel to the z-axis (see Fig. E6.3). Determine the steady electric field produced in the direction normal to both the current and magnetic flux density. The electric charge and number density of particles that carry current are q and n.

6.4. Prove that Eq. (6.22) holds for an arbitrary closed line C that straight current I penetrates (see Fig. E6.4).

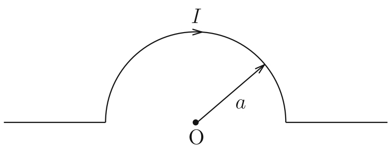

Fig. E6.1 Current on line composed of semicircle and two straight lines

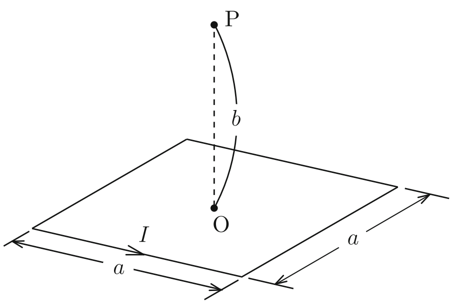

Fig. E6.2 Square current and observation point P

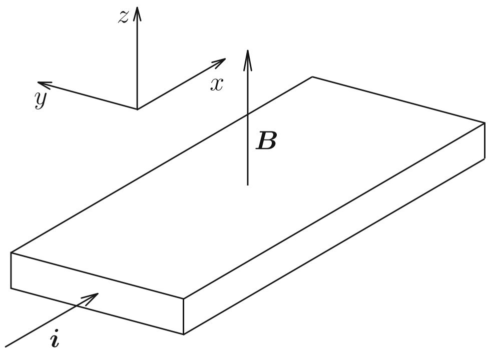

Fig. E6.3 Slab conductor carrying current in normal magnetic flux density

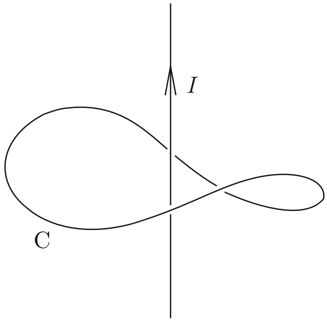

Fig. E6.4 Straight current and surrounding closed line of arbitrary shape

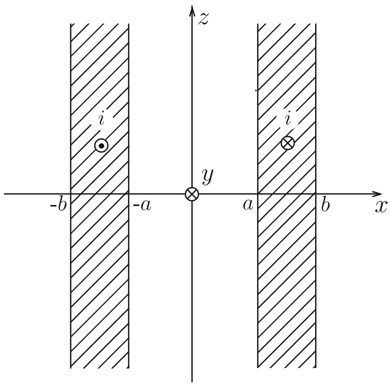

Fig. E6.5 Two parallel slab conductors with uniform currents flowing in opposite directions

6.5. Currents of density i flow uniformly in two parallel wide slab conductors in opposite directions along the y-axis (see Fig. E6.5). Determine the magnetic flux density and vector potential inside and outside of the conductors.

6.6. Current I flows uniformly in a thin planar conductor of width w. Determine the magnetic flux density and vector potential at point P at distance $d(> w/2)$ from the center of the conductor (see Fig. E6.6). The conductor and P are on a common plane.

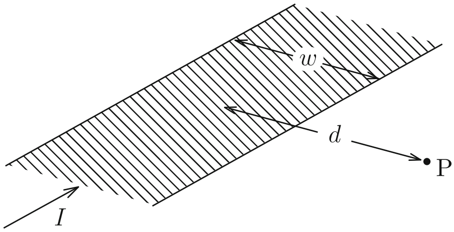

Fig. E6.6 Thin planar conductor with uniform current and observation point P

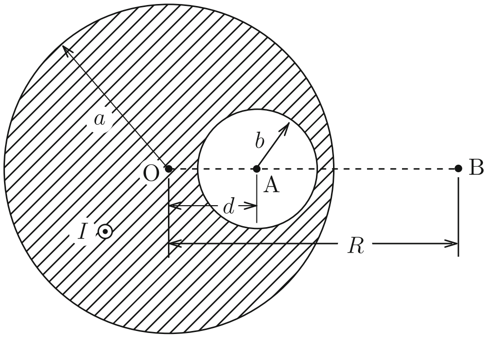

Fig. E6.7 Cross-section of long cylindrical conductor with cylindrical hollow

6.7. Current I flows uniformly in a long cylindrical conductor of radius a that contains a cylindrical hollow of radius b, as shown in Fig. E6.7. The center of the hollow is located at distance d from the center of the conductor, where $a > b + d$. Determine the magnetic flux density at the center of the hollow (point A) and at point B outside the conductor. The distance of point B from the center is $R(> a)$. The central axis O and points A and B are on a common plane.

6.8. Prove that the vector potential given by Eq. (6.33) satisfies the Coulomb gauge, Eq. (6.30).

6.9. Use Eq. (6.29) to determine the vector potential in the solenoid in Example 6.7.

6.10. Determine the magnetic potential produced by the small closed current in Fig. 6.21b in Sect. 6.7.