Chapter 8
Current Systems

8.1 Inductance

In Chap. 6 we learned that the magnetic flux is produced around currents. When a current, I, flows in a closed circuit, C, as shown in Fig. 8.1, the magnetic flux penetrating C is proportional to I:

$$\Phi = LI. \tag{8.1}$$

The proportional constant L is called **self-inductance**. The unit of self-inductance is [Wb/A] and is newly defined as [H] (**Henry**). The self-inductance is determined only by the shape of C and is defined as a positive quantity. That is, the directions of the current and magnetic flux follow the right-hand rule.

Second, we suppose that there are two closed circuits and current I_1 flows along circuit C_1. The magnetic flux penetrating itself is expressed as

$$\Phi_1 = L_{11} I_1, \tag{8.2}$$

similarly to Eq. (8.1). The magnetic flux also penetrates the other circuit C_2, as illustrated in Fig. 8.2, and it is expressed as

$$\Phi_2 = L_{21} I_1. \tag{8.3}$$

The constant L_{21} is influenced by the geometrical arrangement of C_1 and C_2, while L_{11} is determined only by the shape of C_1. If current I_2 flows along C_2, the resultant magnetic flux penetrates C_2 itself and C_1. Thus, the magnetic fluxes penetrating C_1 and C_2 are formally given by

$$\Phi_1 = L_{11} I_1 + L_{12} I_2, \tag{8.4a}$$

$$\Phi_2 = L_{21} I_1 + L_{22} I_2. \tag{8.4b}$$

T. Matsushita, *Electricity and Magnetism: New Formulation by Introduction of Superconductivity*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-4-431-54526-2_8, © Springer Japan 2014

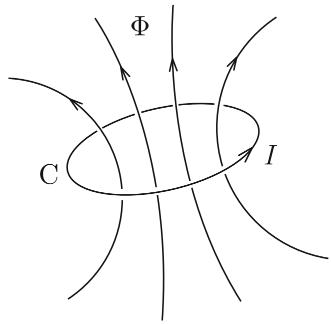


Fig. 8.1 Magnetic flux penetrating closed circuit produced by current flowing along itself

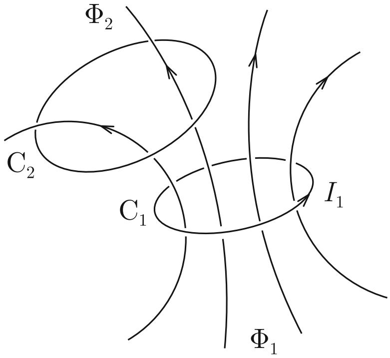


Fig. 8.2 Magnetic flux produced by current flowing along one of two closed circuits

The self-inductances L_{11} and L_{22} are positive as mentioned above. The coefficients L_{12} and L_{21} are called **mutual inductances** and have the following relationship,

$$L_{12} = L_{21}. \tag{8.5}$$

The mutual inductance takes a positive or negative value depending on the directions of the current and magnetic flux.

Extending the above case of two closed electric circuits, we consider a system composed of n electric circuits in Fig. 8.3, where current I_i flows in the i-th circuit C_i $(i = 1, 2, \cdots, n)$. We express the magnetic flux Φ_i penetrating C_i as

$$\Phi_i = \sum_{j=1}^{n} L_{ij} I_j. \tag{8.6}$$

In the above, the L_{ij}'s are **inductance coefficients**. The L_{ii}'s are self-inductances and L_{ij}'s $(i \neq j)$ are mutual inductances. The reciprocity theorem

$$L_{ij} = L_{ji}, \tag{8.7}$$

holds generally.

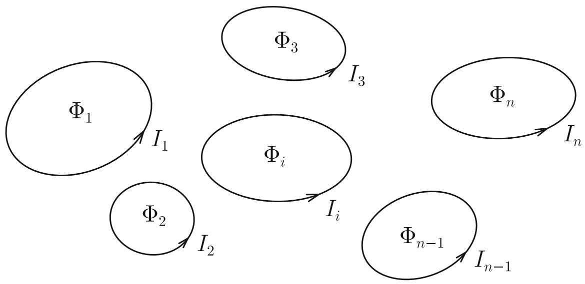


Fig. 8.3 System composed of n closed circuits

Now we prove Eq. (8.7). From Eq. (6.34) the vector potential produced by current I_j flowing in the j-th closed circuit C_j is given by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I_j}{4\pi} \oint_{\mathrm{C}_j} \frac{\mathrm{d}\boldsymbol{r}_j}{|\boldsymbol{r} - \boldsymbol{r}_j|}. \tag{8.8}$$

With the aid of Eq. (6.35), we rewrite the magnetic flux penetrating C_i as

$$\Phi_{ij} = \oint_{\mathrm{C}_i} \boldsymbol{A}(\boldsymbol{r}_i) \cdot \mathrm{d}\boldsymbol{r}_i = \frac{\mu_0 I_j}{4\pi} \oint_{\mathrm{C}_i} \oint_{\mathrm{C}_j} \frac{\mathrm{d}\boldsymbol{r}_i \cdot \mathrm{d}\boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}. \tag{8.9}$$

Hence, the mutual inductance is given by

$$L_{ij} = \frac{\mu_0}{4\pi} \oint_{\mathrm{C}_i} \oint_{\mathrm{C}_j} \frac{\mathrm{d}\boldsymbol{r}_i \cdot \mathrm{d}\boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}. \tag{8.10}$$

This is called **Neumann's formula**. The result is the same even if subscripts i and j are exchanged. Thus, Eq. (8.7) is proved.

Here, we show an example of calculating a mutual inductance. Two circular coils of radius a are separated by distance d with a common central axis, as shown in Fig. 8.4a. The currents flow in the same direction. We define the coordinates as in Fig. 8.4b and position vector $\boldsymbol{r}_1$ for the lower coil is fixed at point P. Then, the contribution from a small region $\mathrm{d}\boldsymbol{r}_2$ in the upper coil to the mutual inductance is written as

$$\frac{\mu_0 \mathrm{d}\boldsymbol{r}_2}{4\pi[2a^2(1 + \sin\theta) + d^2]^{1/2}} = \frac{\mu_0 a \mathrm{d}\theta(-\boldsymbol{i}_x \sin\theta + \boldsymbol{i}_y \cos\theta)}{4\pi[2a^2(1 + \sin\theta) + d^2]^{1/2}}.$$

Integrating this for the upper coil, the y-component reduces to zero because of symmetry, and only the x-component, i.e., the tangential component at point P, remains. Here we put $\theta = 2\psi + \pi/2$. Then, after a simple calculation we write the above integration with respect to $\boldsymbol{r}_2$ as

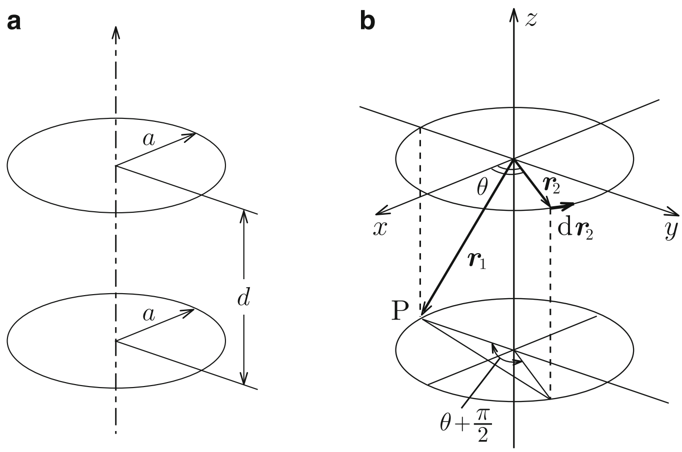


Fig. 8.4 Two circular coils with common axis: (**a**) arrangement and (**b**) coordinates

$$-\frac{\mu_0 k}{2\pi}\int_0^{\pi/2}\frac{1-2\sin^2\psi}{(1-k^2\sin^2\psi)^{1/2}}\mathrm{d}\psi,$$

where $k = 2a/(4a^2+d^2)^{1/2}$. Although the integration is not simplified any more, the above calculated vector is directed along $\mathrm{d}\boldsymbol{r}_1$ and its value is constant. Hence, integrating with respect to $\boldsymbol{r}_1$ gives simply the factor $2\pi a$. Thus, we obtain the mutual inductance as

$$M = \frac{\mu_0 a}{k}[(2-k^2)F(k) - 2E(k)], \tag{8.11}$$

where $F(k)$ and $E(k)$ are complete elliptic integrals of the first and second kind, respectively:

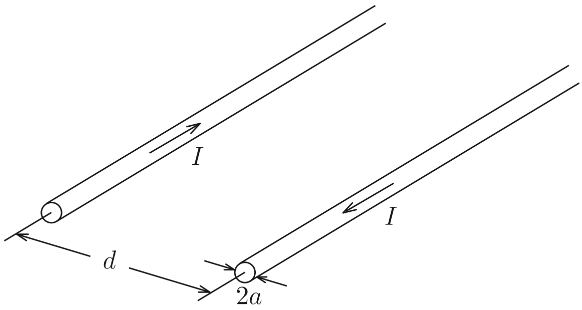
$$F(k) = \int_0^{\pi/2}\frac{1}{(1-k^2\sin^2\psi)^{1/2}}\mathrm{d}\psi, \tag{8.12}$$

$$E(k) = \int_0^{\pi/2}(1-k^2\sin^2\psi)^{1/2}\mathrm{d}\psi. \tag{8.13}$$

When the current path has a finite cross-sectional area, it is not easy to define the magnetic flux penetrating the electric circuit. This may indicate that the inductance cannot be exactly defined. However, the inductance can be exactly determined using the magnetic energy, as will be shown later (see Exercise 8.3).

Example 8.1. Determine the self-inductance of a unit length of the parallel-wire transmission line of radius a separated by distance d in Fig. 8.5. Assume that d is much larger than a and we can neglect the magnetic flux inside the conductors.

Fig. 8.5 Parallel-wire transmission line



Solution 8.1. We suppose that current I flows as shown in the figure. We assume a plane that includes the axes of the two cylindrical conductors and calculate the magnetic flux penetrating the plane between the two conductors. The magnetic flux density produced by current I flowing along the left conductor at distance x from its central axis is

$$B_1 = \frac{\mu_0 I}{2\pi x}$$

and is directed downwards. Hence, the magnetic flux in a unit length produced by this current is

$$\Phi_1' = \int_a^{d-a} \frac{\mu_0 I}{2\pi x} \mathrm{d}x = \frac{\mu_0 I}{2\pi} \log \frac{d-a}{a}.$$

The magnetic flux produced by the current along the right conductor is the same, and we have the self-inductance of a unit length as

$$L' = \frac{2\Phi_1'}{I} = \frac{\mu_0}{\pi} \log \frac{d-a}{a} \simeq \frac{\mu_0}{\pi} \log \frac{d}{a}.$$

◇

Example 8.2. Determine the mutual inductance in a unit length between the two parallel-wire transmission lines in Fig. 8.6. We define the current directions as in the figure.

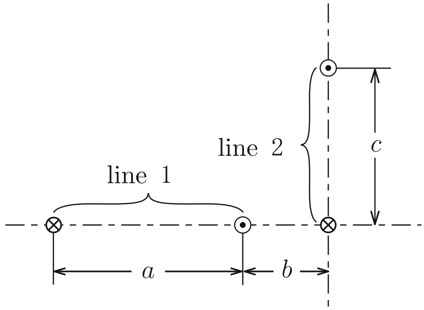


Fig. 8.6 Two parallel-wire transmission lines

Solution 8.2. We apply current I_1 to transmission line 1. We calculate the magnetic flux that penetrates transmission line 2 in a unit length. The magnetic flux produced by the right current of transmission line 1 is

$$\Phi'_{\mathrm{r}} = -\frac{\mu_0 I_1}{2\pi}\int_b^{\sqrt{b^2+c^2}} \frac{\mathrm{d}r}{r} = -\frac{\mu_0 I_1}{4\pi}\log\frac{b^2+c^2}{b^2}.$$

The magnetic flux produced by the left current of transmission line 1 is

$$\Phi'_{\mathrm{l}} = \frac{\mu_0 I_1}{2\pi}\int_{a+b}^{\sqrt{(a+b)^2+c^2}} \frac{\mathrm{d}r}{r} = \frac{\mu_0 I_1}{4\pi}\log\frac{(a+b)^2+c^2}{(a+b)^2}.$$

The total magnetic flux penetrating transmission line 2 is

$$\Phi'_2 = \Phi'_{\mathrm{r}} + \Phi'_{\mathrm{l}} = -\frac{\mu_0 I_1}{4\pi}\log\frac{(b^2+c^2)(a+b)^2}{b^2[(a+b)^2+c^2]}$$

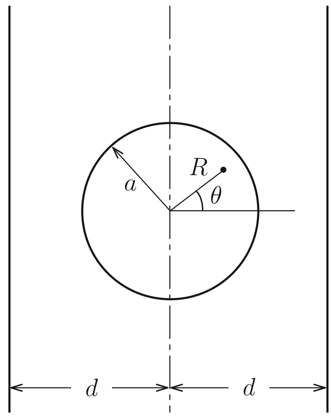
and the mutual inductance is given by

$$L'_{21} = \frac{\Phi'_2}{I_1} = -\frac{\mu_0}{4\pi}\log\frac{(b^2+c^2)(a+b)^2}{b^2[(a+b)^2+c^2]}.$$

Confirm for yourself that the same result is obtained by calculating the magnetic flux produced by transmission line 2 that penetrates transmission line 1. ◇

Example 8.3. A circular coil is placed just between a parallel-wire transmission line separated by d, as shown in Fig. 8.7. These stay on a common plane. Determine the mutual inductance between the transmission line and circular coil.

Fig. 8.7 Circular coil placed at the center of the parallel-wire transmission line



Solution 8.3. We calculate the magnetic flux produced by current I flowing along the left line of the transmission line that penetrates the circular coil. Using two-dimensional polar coordinates with the origin at the center of the coil, the magnetic flux density at (R, θ) is $B = \mu_0 I/[2\pi(d + R\cos\theta)]$. Hence, the magnetic flux is

$$\Phi_1 = \frac{\mu_0 I}{2\pi} \int_0^a \int_0^{2\pi} \frac{R\mathrm{d}R\mathrm{d}\theta}{d + R\cos\theta}.$$

The integral with respect to angle θ is carried out using Eq. (7.26):

$$\Phi_1 = \mu_0 I \int_0^a \frac{R\mathrm{d}R}{(d^2 - R^2)^{1/2}} = \mu_0 I[d - (d^2 - a^2)^{1/2}].$$

Since the magnetic flux due to the current flowing along the right line is the same, we obtain the mutual inductance as

$$M = \frac{2\Phi_1}{I} = 2\mu_0[d - (d^2 - a^2)^{1/2}].$$

◇

8.2 Coils

In Sect. 3.2 we learned about the electric property of capacitors used for storing electric charges in electric circuits. The component used to store the magnetic flux is a **coil**. Coils are also used for other purposes such as producing various magnetic flux densities or generating electric power, that will be covered in Chap. 10. Here, we introduce the magnetic property of coils.

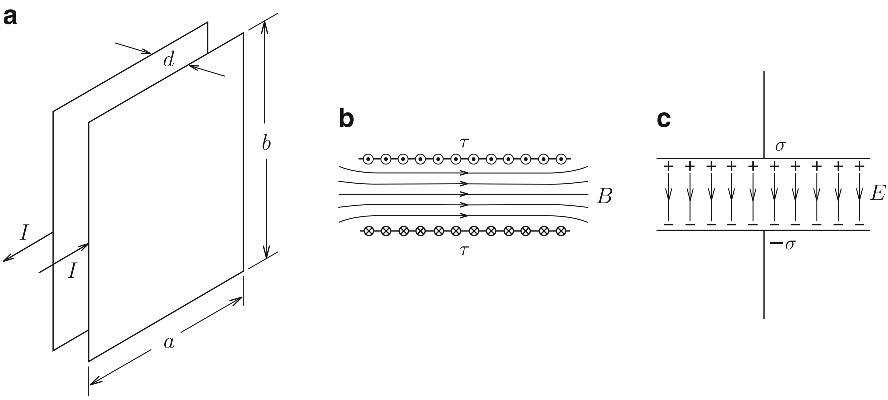


Fig. 8.8 (**a**) Current in parallel-plate transmission line, (**b**) magnetic flux lines and (**c**) electric field lines in a capacitor of the same geometry

When we apply a uniform current, I, to a parallel plate transmission line in Fig. 8.8a, a magnetic flux with a uniform density is produced in the space between the two plates (see Fig. 8.8b). We assume that the distance d between the two plates is sufficiently small and the magnetic flux density on the outside can be neglected. The magnetic flux in the space is directed normally to the currents and its density is

$$B = \frac{\mu_0 I}{b} = \mu_0 \tau, \tag{8.14}$$

where $\tau = I/b$ is the planar current in a unit width. The magnetic flux between the two plates is

$$\Phi = Bad = \frac{\mu_0 Iad}{b}. \tag{8.15}$$

Hence, we obtain the self-inductance as

$$L = \frac{\mu_0 ad}{b}. \tag{8.16}$$

The magnetic flux density produced by parallel planar currents corresponds to the electric field of strength $E = \sigma/\epsilon_0$ produced by planar electric charges in a parallel-plate capacitor (see Fig. 8.8c).

The coil used to produce a uniform magnetic flux density is a **solenoid coil**. For example, when we apply current I to a long solenoid coil with a winding of n turns in a unit length, the interior magnetic flux density is uniform with the value

$$B = \mu_0 nI, \tag{8.17}$$

as shown in Example 6.7. This value does not depend on the radius or length of the coil. For a coil of radius a, the magnetic flux that penetrates one turn of the coil is

$$\phi = \pi\mu_0 n a^2 I. \tag{8.18}$$

Thus, the magnetic flux penetrating the coil of a unit length is

$$\Phi' = n\phi = \pi\mu_0 n^2 a^2 I. \tag{8.19}$$

Hence, the self-inductance in a unit length is given by

$$L' = \frac{\Phi'}{I} = \pi\mu_0 n^2 a^2. \tag{8.20}$$

When the length of the solenoid coil is l, its self-inductance is smaller than $L'l$ and is expressed as

$$L = K\left(\frac{2a}{l}\right) L'l. \tag{8.21}$$

In the above, $K(2a/l)$ is a function only of the ratio $2a/l$ and is called **Nagaoka's coefficient**. Figure 8.9 shows Nagaoka's coefficient.

A **spherical coil** is a special coil for producing a uniform magnetic flux density in a limited space. That is, when current flows on the surface of a sphere as given by Eq. (7.33), the interior magnetic flux density is uniform. Assume a spherical coil of N turns as in Fig. 8.10a. We take the number of turns in a unit zenithal length to be $[N/(2a)]\sin\theta$, where θ is the zenithal angle. When we apply current I, the surface current density on the sphere is

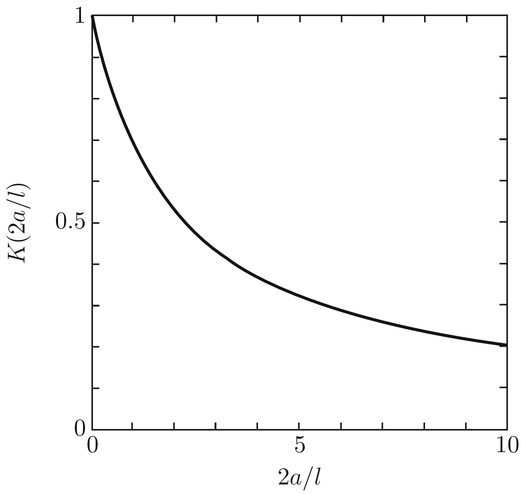


Fig. 8.9 Nagaoka's coefficient

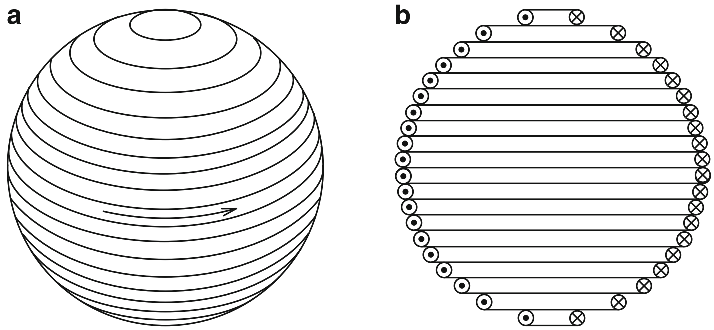


Fig. 8.10 Spherical coil: (**a**) geometry and (**b**) windings

$$\tau = \frac{NI}{2a} \sin\theta \tag{8.22}$$

and from Eq. (7.33) we obtain the interior magnetic flux density as

$$B_0 = \frac{\mu_0 NI}{3a}. \tag{8.23}$$

To realize such a winding, the number of turns in a unit length along the axis is $N/(2a)$.

However, fabricating such a spherical coil is not easy, and the **Helmholtz coil** introduced below is commonly used. This coil consists of a pair of circular coils of the same size, as shown in Fig. 8.11. The radius of the circular coils is a and the distance between the two coils arranged on the common axis is $2z'$. The center is defined to be $z = 0$. We apply current I to the two coils in the same direction. Using the result in Example 6.1, the magnetic flux density on the common axis is given by

$$B(z) = \frac{\mu_0 I a^2}{2} \left\{ \frac{1}{[(z - z')^2 + a^2]^{3/2}} + \frac{1}{[(z + z')^2 + a^2]^{3/2}} \right\}. \tag{8.24}$$

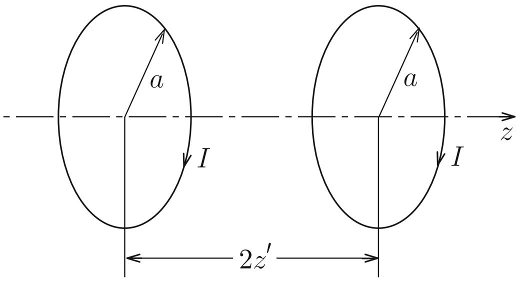


Fig. 8.11 The Helmholtz coil

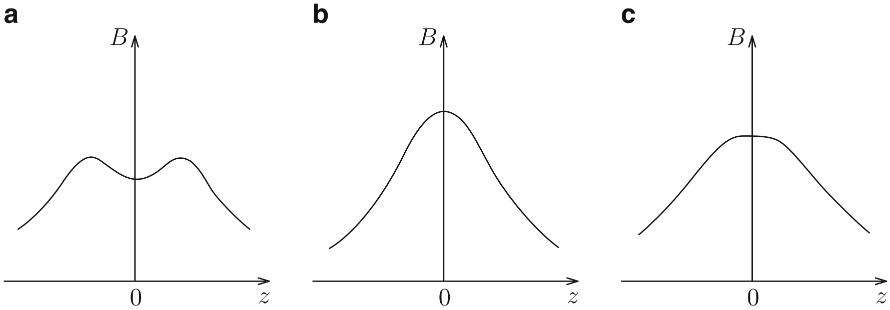


Fig. 8.12 Magnetic flux distribution along the central axis of the Helmholtz coil for the cases where the distance between the two coils is (**a**) too long, (**b**) too short and (**c**) optimum

When the distance between the two coils is too large, the magnetic flux density is locally minimum at the center, as shown in Fig. 8.12a. When this distance is too short, the variation in the magnetic flux density around the center is steep (see Fig. 8.12b). No uniform magnetic flux density is achieved in either of these two cases. We obtain the optimum arrangement under the condition $\mathrm{d}^2 B/\mathrm{d}z^2 = 0$ at the center, $z = 0$, which gives

$$z' = \frac{a}{2}. \tag{8.25}$$

Figure 8.12c shows the magnetic flux distribution for this condition. The magnetic flux density is uniform over a fairly wide area. The magnetic flux density at the center is

$$B(0) = \frac{8\mu_0 I}{5\sqrt{5}a}. \tag{8.26}$$

Example 8.4. We apply current I to a solenoid coil of radius a, length l and N total number of turns. Determine the magnetic flux density on the central axis.

Solution 8.4. We define the z-axis on the central axis, as shown in Fig. 8.13, with the origin at the center of the coil. We regard the windings in the region z to $z + \mathrm{d}z$ as a one-turn coil. The current flowing there is $\mathrm{d}I = (NI/l)\mathrm{d}z$. Using the result in Example 6.1, the magnetic flux density at $z = z_0$ produced by this current is

$$\mathrm{d}B = \frac{\mu_0 \mathrm{d}I a^2}{2[(z - z_0)^2 + a^2]^{3/2}} = \frac{\mu_0 N I a^2}{2l[(z - z_0)^2 + a^2]^{3/2}} \mathrm{d}z.$$

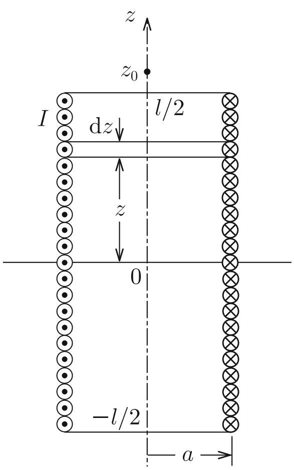


Fig. 8.13 Longitudinal cross-section of solenoid coil

Hence, the total magnetic flux density at the observation point is

$$B(z_0) = \int_{-l/2}^{l/2} \frac{\mu_0 N I a^2}{2l[(z - z_0)^2 + a^2]^{3/2}} \mathrm{d}z.$$

Here we define

$$z - z_0 = a \tan\theta$$

with

$$\tan\theta_1 = -\frac{1}{a}\left(\frac{l}{2} + z_0\right), \qquad \tan\theta_2 = \frac{1}{a}\left(\frac{l}{2} - z_0\right).$$

Then, we calculate the total magnetic flux density to be

$$\begin{aligned} B(z_0) &= \frac{\mu_0 N I}{2l} \int_{\theta_1}^{\theta_2} \cos\theta \mathrm{d}\theta = \frac{\mu_0 N I}{2l}(\sin\theta_2 - \sin\theta_1) \\ &= \frac{\mu_0 N I}{2l} \left\{ \frac{l + 2z_0}{[(l + 2z_0)^2 + 4a^2]^{1/2}} + \frac{l - 2z_0}{[(l - 2z_0)^2 + 4a^2]^{1/2}} \right\}. \end{aligned}$$

◇

8.3 Magnetic Energy

The electric field fills the space between two electrodes in a charged capacitor, and we can regard that the space as filled with electric energy. For a coil that stores the magnetic flux, we can also regard the interior space as filled with **magnetic energy**.

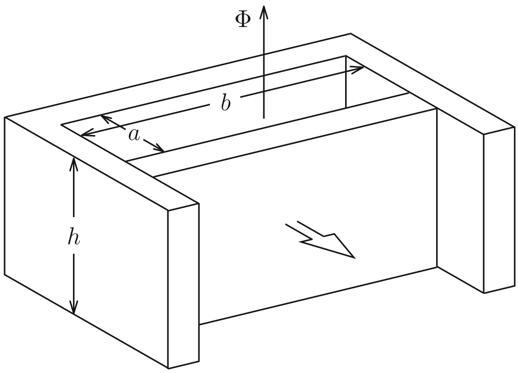


Fig. 8.14 Magnetic flux trapped in closed circuit composed of two superconductors. The superconducting plate is movable as shown by the arrow and in electrical contact with the fixed piece

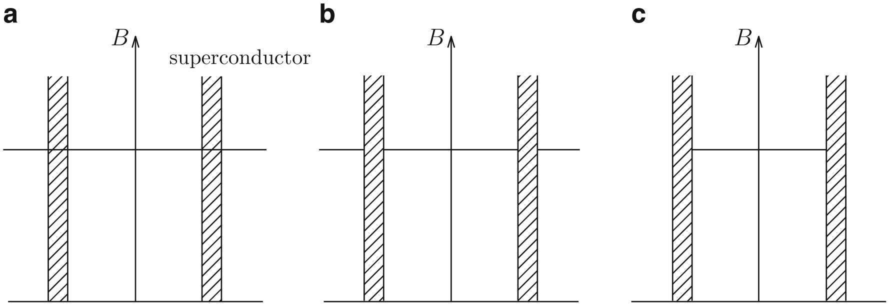


Fig. 8.15 Magnetic flux density in the field-cooled process: (**a**) after applying magnetic flux density above the critical temperature, (**b**) after cooling below the critical temperature and (**c**) after removing external magnetic flux density

Here we suppose two superconductors in electrical contact with each other, as shown in Fig. 8.14. One of them is a movable plate. Assume that magnetic flux Φ is trapped within the space surrounded by the superconductors. We can realize this situation by applying an external magnetic flux density at a temperature above the critical value, reducing the temperature below the critical value to make the superconductors superconducting, and then removing the external magnetic flux density, as shown in Fig. 8.15. A current flows on the inner surface to shield the superconductors from the magnetic flux. Hence, a repulsive force given by Eq. (6.9) is exerted on the movable superconducting plate. If the plate is displaced by distance x, the interior magnetic flux density changes to

$$B = \frac{\Phi}{(a+x)b} \tag{8.27}$$

and the density of current flowing on the inner surface changes to

$$\tau = \frac{B}{\mu_0} = \frac{\Phi}{\mu_0(a+x)b}. \tag{8.28}$$

We assume that a is much smaller than b. Since the magnetic flux density produced by the fixed superconductor is half of the value given by Eq. (8.27), we estimate the force on the movable plate to be

$$F = \frac{1}{2}\tau Bbh = \frac{\Phi^2 h}{2\mu_0 (a+x)^2 b}, \tag{8.29}$$

which is directed along increasing x. This is an isolated system and there is no electromagnetic interaction with the surroundings after the initial condition is established. Thus, this force is attributed to the variation in the magnetic energy U_{m} of this system. From the relationship

$$F = -\frac{\partial U_{\mathrm{m}}}{\partial x}, \tag{8.30}$$

we estimate the magnetic energy as

$$U_{\mathrm{m}} = \frac{\Phi^2 h}{2\mu_0 (a+x)b} = \frac{1}{2\mu_0} B^2 (a+x)bh. \tag{8.31}$$

In the above $(a+x)bh$ is the volume of the space in which the uniform magnetic flux is trapped. Hence, the **magnetic energy density** is given by

$$u_{\mathrm{m}} = \frac{1}{2\mu_0} B^2. \tag{8.32}$$

This is similar to the electric energy density given by Eq. (3.40).

Since the total current flowing in the closed circuit is $I = \tau h$, the self-inductance of the system is

$$L = \frac{\Phi}{I} = \frac{\mu_0 (a+x)b}{h}. \tag{8.33}$$

In terms of the self-inductance we rewrite the magnetic energy, Eq. (8.31), as

$$U_{\mathrm{m}} = \frac{1}{2} L I^2 = \frac{1}{2}\Phi I = \frac{1}{2L}\Phi^2. \tag{8.34}$$

These expressions are similar to those for electric energy in a capacitor, Eq. (3.38).

We consider a system composed of n closed electric circuits. Suppose that current I_i flows in the i-th circuit and magnetic flux Φ_i penetrates it ($i = 1, 2, \cdots, n$). Then, extending the result of Eq. (8.34) to this case, the magnetic energy of this system is given by

$$U_{\mathrm{m}} = \frac{1}{2}\sum_{i=1}^{n}\Phi_i I_i = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} L_{ij} I_i I_j . \tag{8.35}$$

The magnetic energy is usually derived using electromagnetic induction, as will be shown in Chap. 10.

Equation (8.32) is the result when the magnetic flux density is uniform in space. Here we determine the magnetic energy density for a non-uniform magnetic flux density. Substituting Eq. (6.35) into Eq. (8.34), the energy of the system is written as

$$U_{\mathrm{m}} = \frac{1}{2}\oint_{\mathrm{C}} I\boldsymbol{A}\cdot \mathrm{d}\boldsymbol{s}, \tag{8.36}$$

where C is the closed circuit with the current I. When the current is not concentrated but flows widely in space, we extend Eq. (8.36) to

$$U_{\mathrm{m}} = \frac{1}{2}\int_{\mathrm{V}} \boldsymbol{A}\cdot\boldsymbol{i}\ \mathrm{d}V, \tag{8.37}$$

where V is the region in which the current with density $\boldsymbol{i}$ flows. In terms of Eqs. (6.27) and (A1.41) in the Appendix, the magnetic energy becomes

$$U_{\mathrm{m}} = \frac{1}{2\mu_0}\int_{\mathrm{V}} \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})\,\mathrm{d}V = \frac{1}{2\mu_0}\int_{\mathrm{V}}[\boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})]\mathrm{d}V. \tag{8.38}$$

Using Gauss' theorem, the second volume integral is transformed to the surface integral

$$-\int_{\mathrm{S}}(\boldsymbol{A}\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{S}. \tag{8.39}$$

Assuming a sphere of sufficiently large radius r for V, we have $|\boldsymbol{A}| \propto r^{-1}$, $|\boldsymbol{B}| \propto r^{-2}$and $\int \mathrm{d}S \propto r^2$on its surface, and the surface integral is proportional to r^{-1}. Hence, taking the limit $r \to \infty$, the integral reduces to zero. Neglecting this integral, the magnetic energy reduces to

$$U_{\mathrm{m}} = \frac{1}{2\mu_0}\int_{\mathrm{V}} \boldsymbol{B}^2\,\mathrm{d}V, \tag{8.40}$$

where we have used Eq. (6.29). Thus, we can prove that the magnetic energy density is given by Eq. (8.32) even when the magnetic flux density is not uniform in space.

We compare the magnetic energy obtained here and the electric energy obtained in Chap. 3 in Table 8.1. These are quite analogous to each other.

Table 8.1 Comparison of electric energy and magnetic energy

	Electric energy	Magnetic energy
Separated system	$\frac{1}{2}\sum_{i=1}^{n}\phi_i Q_i$	$\frac{1}{2}\sum_{i=1}^{n}\Phi_i I_i$
Continuum system	$\frac{1}{2}\int_V \phi\rho \mathrm{d}V$	$\frac{1}{2}\int_V \boldsymbol{A}\cdot\boldsymbol{i}\,\mathrm{d}V$
Energy density	$\frac{1}{2}\epsilon_0\boldsymbol{E}^2$	$\frac{1}{2\mu_0}\boldsymbol{B}^2$

Example 8.5. We apply current I to the superconducting coaxial transmission line in Fig. 7.5. Determine the magnetic energy stored in a unit length of the transmission line and derive the self-inductance with this result.

Solution 8.5. Currents flow only on the surfaces $R = a$ and $R = b$ so that the magnetic flux does not penetrate the superconductors. The magnetic flux density is $B = \mu_0 I/(2\pi R)$ only in the region $a < R < b$ and is zero in other regions. Hence, the magnetic energy is non-zero only in the region $a < R < b$ and its density is

$$\frac{B^2}{2\mu_0} = \frac{\mu_0 I^2}{8\pi^2 R^2}.$$

Integrating this over the volume in a unit length, we have

$$U'_{\mathrm{m}} = \frac{\mu_0 I^2}{8\pi^2}\int_a^b \frac{1}{R^2}\cdot 2\pi R\mathrm{d}R = \frac{\mu_0 I^2}{4\pi}\log\frac{b}{a}.$$

We can also obtain the magnetic energy from Eq. (8.36). The vector potential has only the axial component A_z similarly to the current. From the relationship $\partial A_z/\partial R = -B$ with $A_z(b) = 0$ we obtain the vector potential as

$$A_z(R) = \frac{\mu_0 I}{2\pi}\log\frac{b}{R}$$

in the region $a < R < b$. Thus, the magnetic energy is

$$U'_{\mathrm{m}} = \frac{1}{2}A_z(a)I = \frac{\mu_0 I^2}{4\pi}\log\frac{b}{a},$$

which agrees with the above result.

Using this result, the self-inductance in a unit length is

$$L' = \frac{2U'_{\mathrm{m}}}{I^2} = \frac{\mu_0}{2\pi}\log\frac{b}{a}.$$

◇

Example 8.6. We apply current I to a sufficiently long solenoid coil of radius a with n turns in a unit length. Calculate the magnetic energy in a unit length using either Eq. (8.37) or (8.40).

Solution 8.6. The magnetic flux density inside the coil is $B = \mu_0 n I$ (see Example 6.7). Using Eq. (8.40) the magnetic energy in a unit length of the coil is

$$U'_{\mathrm{m}} = \frac{1}{2\mu_0}(\mu_0 n I)^2 \pi a^2 = \frac{\pi\mu_0}{2}(naI)^2.$$

On the other hand, the current flows only on the coil surface ($R = a$). The vector potential on this surface is $A_\varphi(a) = \mu_0 n I a/2$ and the surface current density is $\tau = nI$. Thus, using Eq. (8.37), the magnetic energy is

$$U'_{\mathrm{m}} = \frac{1}{2}\int_{\mathrm{S}} A_\varphi(a)\tau \mathrm{d}S = \frac{1}{2}\cdot\frac{\mu_0 n I a}{2} nI \cdot 2\pi a = \frac{\pi\mu_0}{2}(naI)^2.$$

This agrees with the above result. ◇

8.4 Magnetic Force

Magnetic force works between current-carrying conductors. This force is equal to the sum of the Lorentz force on each current. We can expect to derive this force using the magnetic energy and the principle of virtual displacement, similarly to the electrostatic force learned in Sect. 3.4. In fact, we learned the reverse process, namely estimating magnetic energy from the Lorentz force, in Sect. 8.3. However, we need to pay special attention when applying such a method to general cases.

We suppose that current I_1 flows in a straight line and current I_2 flows along a rectangular circuit with two sides parallel to the straight line, as shown in Fig. 8.16. These are placed on a common plane. Now we determine the force on

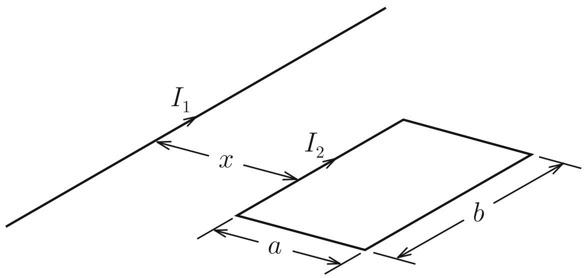


Fig. 8.16 Long straight line and rectangular circuit

the rectangular circuit using the Lorentz force. An attractive force works on the closer side and a repulsive force works on the opposite side. If the distance between the straight line and the closer side is x, the force on the rectangular circuit is

$$F' = -\frac{\mu_0 I_1 I_2}{2\pi x} b + \frac{\mu_0 I_1 I_2}{2\pi (x+a)} b = -\frac{\mu_0 a b I_1 I_2}{2\pi x (x+a)}, \tag{8.41}$$

where we define the force in the direction of increasing x to be positive. This force is negative, i.e., attractive.

Next we calculate the force with the magnetic energy. The magnetic energy due to current I_1 only and that due to current I_2 only are expressed in terms of the self-inductances of each circuit. These energies are independent of the displacement of the rectangular circuit, since the self-inductances do not change under the relative displacement between the two circuits. From Eq. (8.35) for $n = 2$, the associated energy is the interaction energy between the two currents, i.e., the energy from mutual induction. The magnetic flux produced by current I_1 that penetrates the rectangular circuit is

$$\Phi = L_{21} I_1 = \frac{\mu_0 I_1 b}{2\pi} \int_x^{x+a} \frac{\mathrm{d}r}{r} = \frac{\mu_0 I_1 b}{2\pi} \log \frac{x+a}{x}. \tag{8.42}$$

Hence, the associated energy is

$$U_\mathrm{m} = \frac{1}{2}(L_{12} + L_{21}) I_1 I_2 = \Phi I_2 = \frac{\mu_0 b I_1 I_2}{2\pi} \log \frac{x+a}{x}, \tag{8.43}$$

using Eq. (8.5). Thus, the magnetic force seems to be

$$F = -\frac{\partial U_\mathrm{m}}{\partial x} = \frac{\mu_0 a b I_1 I_2}{2\pi x (x+a)}. \tag{8.44}$$

However, this disagrees with F' in Eq. (8.41) which shows that there is a problem with the above procedure.

What does this mean? The above procedure seems to indicate that the magnetic energy decreases by $F\Delta x$ during the displacement of the rectangular circuit from x to $x + \Delta x$. However, the total magnetic energy must additionally increase by $\Delta U_\mathrm{m} = 2F\Delta x$ to reach the correct result. From the reciprocity theorem in Eq. (8.5), half of this increment comes from the increase in the magnetic energy in the rectangular circuit. We expect this to be caused by the **induced electromotive force** V in the circuit. Namely, when the rectangular circuit is displaced by Δx within time Δt, the work done by the induced electromotive force is $V I_2 \Delta t$. We can rewrite this as $-\Delta\Phi I_2$ in terms of the change in the magnetic flux that penetrates the circuit,

$$\Delta\Phi = -\frac{\mu_0 I_1 a b}{2\pi x (x+a)} \Delta x. \tag{8.45}$$

Taking the limit $\Delta t \to 0$, we have

$$V = -\frac{\Delta\Phi}{\Delta t} \to -\frac{d\Phi}{dt}. \tag{8.46}$$

This is the induced electromotive force that will be learned in Chap. 10.

What we can say from the above example is that estimating the magnetic force from the magnetic energy is valid only when the circuit is isolated and the magnetic flux is conserved as treated in Sect. 8.3. In most cases electromagnetic induction is involved and such an estimation is not correct.

Column: (1) Method of Deriving Static Magnetic Energy

This chapter shows that there is a formal similarity between the electric energy and magnetic energy. However, the method of deriving the energy is completely different between the two cases. For example, the electric energy is estimated from the mechanical work needed to carry electric charges from infinity until the final distribution of electric charge is attained, whereas the magnetic energy cannot be estimated from the mechanical work to carry currents from infinity. That is, there is a problem of divergence of the energy. In addition, the more severe problem is that the final energy must be negative, since the work to carry a current is negative because of the attractive force between currents of the same direction. Thus, we conclude that the magnetic energy cannot be derived with this method. This is because of the induced electromotive force mentioned in Sect. 8.4.

For this reason we estimate the magnetic energy in a virtual experiment using a superconducting circuit in this chapter. The merits of this method are that it enables us to construct an isolated system from surroundings such as a power source, and that it is free from electromagnetic induction since it conserves the magnetic flux because of the perfect diamagnetic property (see Exercise 8.9).

Consider possible ways to derive the magnetic energy other than the method introduced here.

(2) Is Magnetic Flux a Magnetic Potential?

There is a famous analogy between the electric energy and magnetic energy, as shown in Table 8.1. The electric energy is given by the product of the electric source, i.e., the electric charge Q_i and the resultant electric potential ϕ_i divided by 2. On the other hand, the magnetic energy is given by the product of the magnetic source, i.e., the current I_i and the resultant magnetic flux Φ_i divided by 2. Does this analogy mean that the magnetic flux is a magnetic potential, i.e., the vector potential? It is clear that the magnetic flux is not the vector potential. How can we explain such a disagreement between electricity and magnetism?

The answer is that this difference is caused by the difference in the dimension of the sources. Originally the electric charge density corresponds to the current density, as can be seen in Table 8.1. For a conductor system, the electric potential is constant in each conductor. In the i-th conductor, the electric energy reduces to

$$\frac{1}{2}\int_{V_i} \phi\rho \mathrm{d}V = \frac{1}{2}\phi_i \int_{V_i} \rho \mathrm{d}V = \frac{1}{2}\phi_i Q_i.$$

On the other hand, we obtain the current by integrating the current density in the cross-sectional area. In the i-th circuit, the magnetic energy reduces to

$$\frac{1}{2}\int_{V_i} \boldsymbol{A}\cdot\boldsymbol{i}\ \mathrm{d}V = \frac{1}{2}\int_{S_i} i\ \mathrm{d}S \oint_{C_i} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s} = \frac{1}{2}\Phi_i I_i.$$

In the above the volume integral was divided into the cross-sectional integral and the integral along the current path as $\mathrm{d}V = \mathrm{d}S\mathrm{d}s$, and we used the relationship $\boldsymbol{i}\,\mathrm{d}s = i\,\mathrm{d}\boldsymbol{s}$. That is, the magnetic flux is not the vector potential but is the vector potential integrated along the circuit. This may be simply understood from dimensions. In this sense the magnetic flux is a kind of magnetic potential.

Exercises

8.1. Determine the mutual inductance between a parallel-wire transmission line and a triangular circuit on the common plane in Fig. E8.1.

8.2. Calculate the self-inductance directly from the penetrating magnetic flux for the superconducting coaxial transmission line in Example 8.5.

8.3. Suppose that the coaxial transmission line in Example 8.6 is not made of a superconductor but of a usual conductor. Determine the self-inductance. (Hint: Determine the self-inductance using the magnetic energy for a current.)

8.4. Determine the mutual inductance between a parallel-wire transmission line and a rectangular circuit placed at distance b from the transmission line (see Fig. E8.2).

8.5. Determine the mutual inductance between two coaxial solenoid coils in Fig. E8.3. The inner and outer coils have n_a and n_b turns in a unit length, respectively.

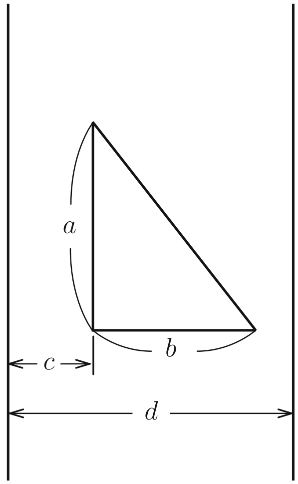


Fig. E8.1 Parallel-wire transmission line and triangular circuit

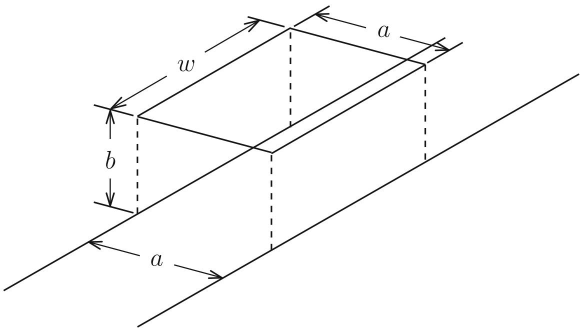


Fig. E8.2 Parallel-wire transmission line and rectangular circuit

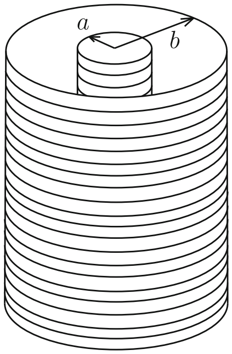


Fig. E8.3 Two coaxial solenoid coils

8.6. Determine the interior magnetic flux density, magnetic energy and self-inductance when we apply current I to the **toroidal coil** in Fig. E8.4. The radius of the central axis is d, the radius of the winding region is a and the total number of turns is N.

Fig. E8.4 Toroidal coil

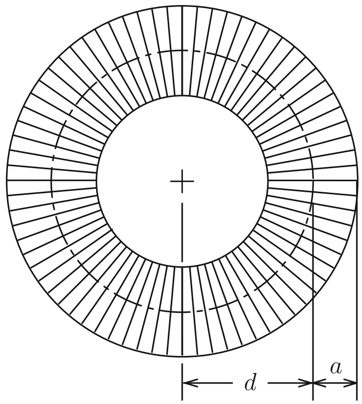
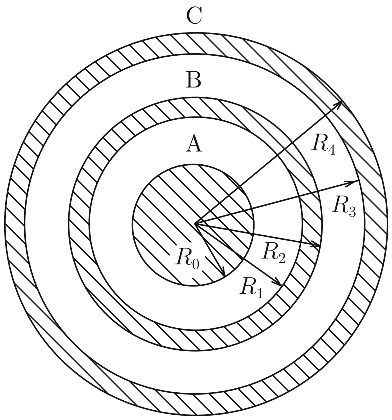


Fig. E8.5 Cross-section of coaxial superconducting cylinders



8.7. We denote the inner, middle and outer long coaxial superconducting cylinders in Fig. E8.5 as superconductors 1–3. (a) Determine the inductance coefficients assuming the reference point for zero vector potential at $R = R_\infty > R_4$, and (b) determine the magnetic energy in a unit length using the inductance coefficients when we apply currents I_1, I_2 and I_3 to superconductors 1, 2 and 3, respectively.

8.8. Suppose that current I' is flowing along the azimuthal direction in a unit length of a long hollow superconducting cylinder with inner diameter $2b$, as shown in Fig. E8.6a. Determine the force on the cylindrical superconducting rod of radius a when we insert the rod into the superconducting cylinder to a depth x from the edge, as shown in Fig. E8.6b. Neglect the disturbance of magnetic flux density around the edge of the rod.

8.9. Currents I_1 and I_2 flow along two long rectangular superconducting circuits, as shown in Fig. E8.7. Calculate the magnetic force using the magnetic energy. (Hint: Note that when the distance x changes to $x + \Delta x$, currents I_1 and I_2 change in a way that keeps the penetrating magnetic flux constant in each circuit.)

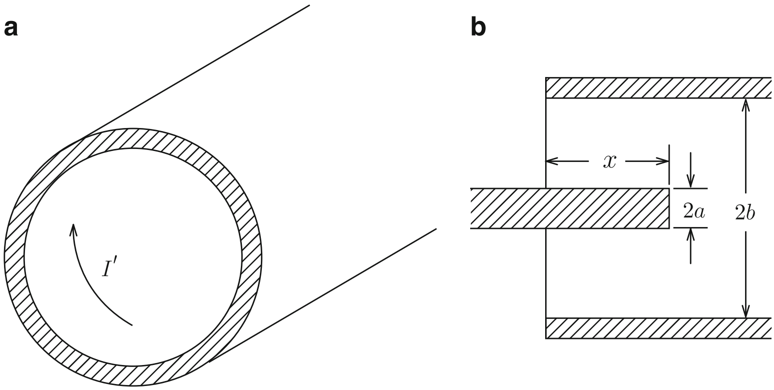


Fig. E8.6 (**a**) Long hollow superconducting cylinder with azimuthal current and (**b**) penetration of superconducting rod into the hollow superconducting cylinder

Fig. E8.7 Two current-carrying long rectangular superconducting circuits placed on a common plane