Chapter 11
Displacement Current and Maxwell's Equations

11.1 Displacement Current

In a steady state, Ampere's law, Eq. (9.14), holds for a closed line, C, with different surfaces on it, S_1 and S_2, as shown in Fig. 11.1a, b. If the magnetic field $\boldsymbol{H}$ is integrated on C in opposite directions as drawn in Fig. 11.1a, b, the sum of the two integrations is naturally zero. At the same time, the sum of the surface integrals of the current density $\boldsymbol{i}$ on S_1 and S_2 is also zero. This sum becomes the surface integral on closed surface S_{12} composed of S_1 and S_2 (see Fig. 11.1c). Thus, we have

$$\int_{S_{12}} \boldsymbol{i} \cdot \mathrm{d}\boldsymbol{S} = 0. \tag{11.1}$$

This agrees with Eq. (5.8) in a steady state. However, it means that Ampere's law contradicts to Eq. (5.8) in a non-steady state. In such a general case, the law of conservation of electric charge, Eq. (5.8), must be satisfied. Substituting Eq. (4.14) into this equation, we have

$$\int_{S_{12}} \left(\boldsymbol{i} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot \mathrm{d}\boldsymbol{S} = 0. \tag{11.2}$$

This strongly suggests that, for generalizing to a non-steady state, we can assume

$$\oint_C \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = \int_S \left(\boldsymbol{i} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot \mathrm{d}\boldsymbol{S} \tag{11.3}$$

instead of Ampere's law. In the above, S is the surface surrounded by C. The second term on the right side, $\partial \boldsymbol{D}/\partial t$, is called **displacement current**, and has the same unit as electric current density [A/m^2]. In a steady state, Eq. (11.3) reduces to the usual form of Ampere's law, Eq. (9.14), and there is no problem. Equation (11.3) is called the **generalized form of Ampere's law**. The corresponding **generalized differential form of Ampere's law** is
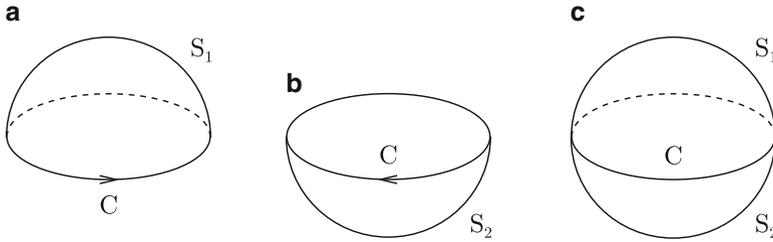
Fig. 11.1 (**a**) Surface S_1 and (**b**) surface S_2 surrounded by closed line C, and (**c**) closed surface composed of S_1 and S_2

$$\nabla \times \boldsymbol{H} = \boldsymbol{i} + \frac{\partial \boldsymbol{D}}{\partial t}. \tag{11.4}$$

Here we show the validity of the displacement current. Suppose that a capacitor is energized using an electric power source. When we apply current I to the capacitor, as shown in Fig. 11.2a, the electric charge Q in the electrode changes. We assume a closed line, C, around a wire through which the current flows and a surface, S_1, as in the figure. We apply Eq. (11.3) to C and S_1. The displacement current is zero on S_1 and the right side is equal to the current I applied to the capacitor. Next, we assume another surface S_2 that does not contain the wire, as shown in Fig. 11.2b. In this case, while the left side does not change, $\boldsymbol{i}$ is zero on S_2 and the right side is

$$\frac{\partial}{\partial t} \int_{S_2} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = \frac{\mathrm{d}Q}{\mathrm{d}t}. \tag{11.5}$$

In the above, we changed the order of the time differential and surface integral, since S_2 does not change with time. Hence,

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} \tag{11.6}$$

and no contradiction results from Eq. (11.4).
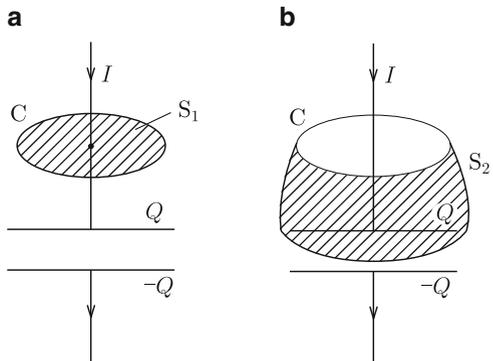
Fig. 11.2 Closed line C around a current-carrying wire and surfaces surrounded by C: (**a**) surface that includes the wire and (**b**) surface that does not include the wire

When current changes with time as in the case of alternating current (AC), electric charges are stored in the electrodes of a capacitor, and hence, current flows through the capacitor. Thus, the time variation in the electric flux in the space between the electrodes generates a magnetic field as well as current.

Example 11.1. We apply AC $I(t) = I_0 \sin \omega t$ to a capacitor with circular electroplates of radius a separated by d, as shown in Fig. 11.3. Determine the displacement current and magnetic field in the space between the electroplates.

Fig. 11.3 AC flowing through circular parallel-plate capacitor
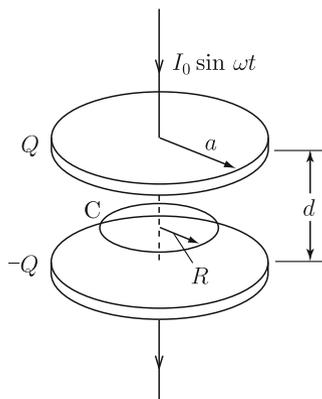
Solution 11.1. The electric charge on the electroplate is $Q(t) = -(I_0/\omega) \cos \omega t$ and the electric flux density is directed downward with magnitude

$$D(t) = \frac{Q(t)}{\pi a^2} = -\frac{I_0}{\pi a^2 \omega} \cos \omega t.$$

Hence, the displacement current is

$$\frac{\partial D(t)}{\partial t} = \frac{I_0}{\pi a^2} \sin \omega t.$$

This is similar to a virtual situation in which the current of the same density flows uniformly in the space between electroplates, suggesting the continuity of current. In fact, this situation is realized if the space is occupied by a material with electric resistivity sufficiently higher than that of the electroplates.

The magnetic field produced by the displacement current is concentric around the central axis of the capacitor. If the magnetic field on a circle of radius R from the axis is $H(R)$, the left side of Eq. (11.3) is $2\pi R H(R)$. The right side is $\pi R^2 \partial D(t)/\partial t = (R^2 I_0/a^2) \sin \omega t$. Thus, we have

$$H(t) = \frac{I_0 R}{2\pi a^2} \sin \omega t.$$

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11.2 Maxwell's Equations

All of the equations that describe electromagnetic phenomena have been introduced. They are summarized as follows:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{11.7}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{i} + \frac{\partial \boldsymbol{D}}{\partial t}, \tag{11.8}$$

$$\nabla \cdot \boldsymbol{D} = \rho, \tag{11.9}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{11.10}$$

These are **Maxwell's equations**. Equation (11.7) describes the law of electromagnetic induction, which directly connects the first fundamental variables in electricity and magnetism. Equation (11.8) is the generalized differential form of Ampere's law that directly connects the second fundamental variables in electricity and magnetism. Equations (11.9) and (11.10) are Gauss' laws on electric flux and magnetic flux, respectively, and represent the conditions of divergence.

When we look back at Eqs. (11.7)–(11.10), a break of formal symmetry may be found among them. However, if there are no sources ($\rho = 0$, $\boldsymbol{i} = 0$), a beautiful symmetry appears. This is the essential feature of electromagnetism. The break of symmetry comes from the difference in the nature of fields produced by different sources, electric charge density ρ (scalar) and current density $\boldsymbol{i}$ (vector). The electric field is an irrotational field with divergence and the magnetic flux density is a solenoidal field with rotation. It should be noted that such a difference comes from the more fundamental nature of mathematics; i.e., the difference between the source being a scalar or vector.

The field (i.e., a distortion vector in space) that a potential due to a scalar source can produce is only a gradient. This field naturally has no rotation and has divergence. In contrast, the field that a potential due to a vector source can produce is only a rotation, which naturally has no divergence. Poisson's equation and its solution clearly show that a scalar or vector source produces a scalar or vector potential, respectively. Therefore, including the effects of these sources in the above equations automatically determines the corresponding equations to be modified. The electric charge density ρ is included in Eq. (11.9), which represents divergence, and the current density $\boldsymbol{i}$ is included in Eq. (11.8), which represents rotation. As a result, the effects are not included in Eqs. (11.7) and (11.10) for $\boldsymbol{E}$ and $\boldsymbol{B}$.

These equations are solved for electromagnetic fields with material relationships:

$$\boldsymbol{D} = \epsilon \boldsymbol{E}, \tag{11.11}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}, \tag{11.12}$$

$$\boldsymbol{i} = \sigma_{\mathrm{c}} \boldsymbol{E}. \tag{11.13}$$

The variables to be obtained are the electric field $\boldsymbol{E}$, magnetic flux density $\boldsymbol{B}$, electric flux density $\boldsymbol{D}$, magnetic field $\boldsymbol{H}$ and current density $\boldsymbol{i}$. These five variables are obtained with five equations, (11.7), (11.8) and (11.11)–(11.13). Equations (11.9) and (11.10) provide supplementary conditions.

We transform the differential Eqs. (11.7)–(11.10) to integral equations:

$$\oint_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}, \tag{11.14}$$

$$\oint_C \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = \int_S \left(\boldsymbol{i} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot \mathrm{d}\boldsymbol{S}, \tag{11.15}$$

$$\int_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = \int_V \rho \, \mathrm{d}V, \tag{11.16}$$

$$\int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0. \tag{11.17}$$

In the above, C is the closed line that surrounds the surface S in Eqs. (11.14) and (11.15), S is the closed surface and V is its internal region in Eqs. (11.16) and (11.17).

Here we show an example for solving Maxwell's equations. In terms of only the electric field and magnetic flux density, Eq. (11.8) is rewritten as

$$\frac{1}{\mu} \nabla \times \boldsymbol{B} = \sigma_c \boldsymbol{E} + \epsilon \frac{\partial \boldsymbol{E}}{\partial t}, \tag{11.18}$$

and Eq. (11.9) gives

$$\epsilon \nabla \cdot \boldsymbol{E} = \rho. \tag{11.19}$$

Substituting Eq. (11.18) into a rotation of Eq. (11.7) gives

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}) = -\mu\epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \mu\sigma_c \frac{\partial \boldsymbol{E}}{\partial t}. \tag{11.20}$$

It is common to solve Eq. (11.20) under the condition of Eq. (11.19). When there is no electric charge ($\rho = 0$) in a material, according to Eq. (A1.46), the left side of Eq. (11.20) is equal to $-\Delta \boldsymbol{E}$, and we have

$$\Delta \boldsymbol{E} - \mu\epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \mu\sigma_c \frac{\partial \boldsymbol{E}}{\partial t} = 0. \tag{11.21}$$

The same equations are obtained for the other four variables including $\boldsymbol{B}$. This equation is the **telegraphic equation**.

The second and third terms on the left side of Eq. (11.21) correspond to the second (displacement current) and first terms (electric current) on the right side

of Eq. (11.18), respectively. Hence, in usual cases where we can neglect the displacement current, the second term in the telegraphic equation disappears and the equation leads to the diffusion equation, Eq. (10.50). Here we discuss the condition in which this approximation holds. Assume AC electromagnetic fields of angular frequency ω. The magnitudes of the second and third terms are $\mu\epsilon\omega^2|E|$ and $\mu\sigma_c\omega|E|$, respectively. The ratio of the second and third terms is

$$\frac{\epsilon\omega}{\sigma_c}. \tag{11.22}$$

This ratio is also directly derived from the displacement current and electric current. In usual metals ϵ and σ_c are of the order of $\epsilon_0 \simeq 1 \times 10^{-11}$ C^2/N m^2 and 1×10^7 S/m, respectively. Hence, this ratio is as small as 6×10^{-8} even for a microwave of 10 GHz ($\omega = 2\pi \times 10^{10}$). In usual metals, we can therefore safely neglect the displacement current even at a very high frequency. In contrast, for insulating materials such as mica, typical material constants are $\epsilon \simeq 7\epsilon_0 \simeq 6 \times 10^{-11}$ C^2/N m^2 and $\sigma_c \simeq 1 \times 10^{-14}$ S/m. Hence, the ratio takes a value as large as 2×10^6 even for a low frequency like a commercial one of 50 Hz. In this case we can neglect the current. It is easily understood that a large difference in the electric conductivity dramatically affects the electromagnetic phenomena. In the latter case, Eq. (11.21) reduces to

$$\Delta E - \mu\epsilon\frac{\partial^2 E}{\partial t^2} = 0. \tag{11.23}$$

This differential equation of the second order is called the **wave equation**. This will be investigated in Chap. 12, in which we learn the electromagnetic wave.

Example 11.2. Show that the equation for the magnetic flux density B is also the telegraphic equation.

Solution 11.2. Substituting Eq. (11.7) into a rotation of Eq. (11.8) gives

$$\nabla \times (\nabla \times B) = \mu\sigma_c \nabla \times E + \mu\epsilon\frac{\partial}{\partial t}\nabla \times E = -\mu\sigma_c\frac{\partial B}{\partial t} - \mu\epsilon\frac{\partial^2 B}{\partial t^2}.$$

Using Eqs. (A1.46) and (11.10), the left side reduces to $-\Delta B$ independently of the existence of electric charges, and we have

$$\Delta B - \mu\epsilon\frac{\partial^2 B}{\partial t^2} - \mu\sigma_c\frac{\partial B}{\partial t} = 0.$$

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11.3 Boundary Conditions

Since the equation for the magnetic field H changes from Eqs. (9.12) to (11.4), we investigate the boundary condition for H here.

We denote the magnetic fields in materials 1 and 2 in the vicinity of an interface by H_1 and H_2, respectively. Consider a plane that is normal to the boundary and contains H_1 and H_2 (see Fig. 9.12a, b). Integrating the magnetic field on the small rectangle ΔC with two sides parallel to the boundary, we have

$$\oint_{\Delta C} H \cdot ds = \int_{\Delta S} \nabla \times H \cdot dS = \int_{\Delta S} i \cdot dS + \frac{\partial}{\partial t} \int_{\Delta S} D \cdot dS, \qquad (11.24)$$

where we have used Stokes' theorem and Eq. (11.4), and ΔS is the surface surrounded by ΔC. If the height Δh is sufficiently small, only the surface current contributes to the first term on the right side and we can neglect the displacement current of a finite density. Hence, the boundary condition to be satisfied is the same as Eq. (9.24); i.e., if the surface density of current flowing on the boundary is τ, Eq. (11.24) gives

$$n \times (H_1 - H_2) = \tau, \qquad (11.25)$$

where n is the unit vector normal to the boundary and is directed from material 1 to material 2. The difference in the parallel component of the magnetic field is equal to the surface density of current flowing on the boundary.

11.4 Electromagnetic Potential

Here we summarize the potentials that describe the electromagnetic fields. The electric field is given by Eq. (10.25) with the electric potential (scalar potential) ϕ and the vector potential A:

$$E = -\nabla\phi - \frac{\partial A}{\partial t}. \qquad (11.26)$$

On the other hand, the magnetic flux density B always satisfies Eq. (11.10) and is given by Eq. (6.29) with the vector potential A:

$$B = \nabla \times A. \qquad (11.27)$$

Thus, the electric field and magnetic flux density are given by ϕ and A, and the set of these potentials are called the **electromagnetic potential**.

There is no change in the magnetic flux density, even if we add a gradient of an arbitrary scalar function ψ to the vector potential A. However, it is necessary to make the change

$$\phi \to \phi - \frac{\partial \psi}{\partial t}, \qquad A \to A + \nabla \psi, \tag{11.28}$$

so that the electric field does not change. This transformation is the **gauge transformation**. Since the electromagnetic fields do not change under this transformation, the electromagnetic potential is arbitrary. Hence, it is necessary to impose a condition to determine the electromagnetic potential uniquely.

When an electric charge of density ρ and a current of density i coexist in space, it is common to use the condition

$$\epsilon\mu \frac{\partial \phi}{\partial t} + \nabla \cdot A = 0. \tag{11.29}$$

This condition is the **Lorentz gauge**. The equations associated with the electric charge and current are Eqs. (11.9) and (11.8), respectively. The left side of Eq. (11.9) is written with the electromagnetic potential as

$$-\epsilon \left(\Delta\phi + \frac{\partial}{\partial t}\, \nabla \cdot A \right) = -\epsilon \left(\Delta\phi - \epsilon\mu\, \frac{\partial^2 \phi}{\partial t^2} \right). \tag{11.30}$$

Hence, the equation that ϕ should satisfy is

$$\Delta\phi - \epsilon\mu\, \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon}. \tag{11.31}$$

Equation (11.8) leads similarly to

$$\Delta A - \epsilon\mu\, \frac{\partial^2 A}{\partial t^2} = -\mu i. \tag{11.32}$$

Under the Lorentz gauge, the scalar function ψ must satisfy

$$\Delta\psi - \epsilon\mu\, \frac{\partial^2 \psi}{\partial t^2} = 0. \tag{11.33}$$

When there is neither electric charge nor current, all these equations reduce to the same form as Eq. (11.23), i.e., the wave equation. This type of equation expresses the electromagnetic wave as will be shown in Chap. 12.

Example 11.3. Derive Eq. (11.32).

Solution 11.3. The left side of Eq. (11.8) is

$$\frac{1}{\mu}\, \nabla \times (\nabla \times A) = \frac{1}{\mu}\, [\nabla(\nabla \cdot A) - \Delta A] = -\epsilon\, \nabla\frac{\partial \phi}{\partial t} - \frac{1}{\mu}\, \Delta A,$$

where Eq. (11.29) is used. The right side is

$$\boldsymbol{i} - \epsilon\,\nabla\frac{\partial \phi}{\partial t} - \epsilon\,\frac{\partial^2 \boldsymbol{A}}{\partial t^2}.$$

Thus, Eq. (11.32) is derived.

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11.5   The Poynting Vector

The total energy density of electromagnetic fields varying with time is given by

$$u = \frac{1}{2}\epsilon E^2 + \frac{1}{2\mu}B^2 + \int \boldsymbol{i}\cdot \boldsymbol{E}\,\mathrm{d}t. \qquad (11.34)$$

The first, second and third terms are the electric energy, magnetic energy and mechanical energy of charged particles, respectively. Hence, the variation in the total energy in space V with time is

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial t}\int_{\mathrm{V}} u\,\mathrm{d}V$$

$$= \int_{\mathrm{V}}\left(\epsilon \boldsymbol{E}\cdot\frac{\partial \boldsymbol{E}}{\partial t} + \frac{\boldsymbol{B}}{\mu}\cdot\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{i}\cdot\boldsymbol{E}\right)\mathrm{d}V. \qquad (11.35)$$

The first and third terms are transformed using Eq. (11.8) and $\partial \boldsymbol{B}/\partial t$ in the second term is eliminated by substituting Eq. (11.7). The right side of the above equation then becomes

$$\int_{\mathrm{V}}[\boldsymbol{E}\cdot(\nabla\times\boldsymbol{H}) - \boldsymbol{H}\cdot(\nabla\times\boldsymbol{E})]\mathrm{d}V = -\int_{\mathrm{V}}\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H})\mathrm{d}V = -\int_{\mathrm{S}}(\boldsymbol{E}\times\boldsymbol{H})\cdot\mathrm{d}\boldsymbol{S}, \qquad (11.36)$$

where S is the surface of V. Here we define

$$\boldsymbol{S}_{\mathrm{P}} = \boldsymbol{E}\times\boldsymbol{H}. \qquad (11.37)$$

We then rewrite Eq. (11.35) as

$$\frac{\partial}{\partial t}\int_{\mathrm{V}} u\,\mathrm{d}V + \int_{\mathrm{S}} \boldsymbol{S}_{\mathrm{P}}\cdot\mathrm{d}\boldsymbol{S} = 0. \qquad (11.38)$$

The vector $\boldsymbol{S}_{\mathrm{P}}$ is called the **Poynting vector**. This equation says that the variation in the energy in space V with time is equal to the Poynting vector that enters V

through the surface S. Hence, we understand that the Poynting vector represents the flow of electromagnetic energy, and Eq. (11.38) represents the law of conservation of energy. When this equation is written in differential form, we have

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{S}_{\mathrm{P}} = 0. \tag{11.39}$$

This gives the **continuity equation of energy**.

However, there is an extraordinary case where it is difficult to understand that the Poynting vector gives a real energy flow. See Column (2) in this chapter.

Example 11.4. The electric field is given by Eq. (10.56) in the case of the skin effect discussed in Sect. 10.5. Prove that the total energy loss $\int \mathrm{d}t \int_{\mathrm{V}} \boldsymbol{E} \cdot \boldsymbol{i} \, \mathrm{d}V$ is equal to $-\int \mathrm{d}t \int_{\mathrm{S}} \boldsymbol{S}_{\mathrm{P}} \cdot \mathrm{d}\boldsymbol{S}$, where the integration with time is carried out over one period of the AC field.

Solution 11.4. The current density corresponding to Eq. (10.56) is

$$i_y(x,t) = \sigma_{\mathrm{c}} E_y(x,t) = B_0 \left(\frac{\sigma_{\mathrm{c}} \omega}{\mu} \right)^{1/2} \mathrm{e}^{-x/\delta} \cos\left(\omega t - \frac{x}{\delta} + \frac{\pi}{4} \right)$$

and the power loss in a unit area of the y-z plane is

$$\int_0^\infty E_y(x,t) i_y(x,t) \mathrm{d}x = \frac{B_0^2 \omega}{\mu} \int_0^\infty \exp\left(-\frac{2x}{\delta} \right) \cos^2\left(\omega t - \frac{x}{\delta} + \frac{\pi}{4} \right) \mathrm{d}x$$

$$= \frac{B_0^2 \omega \delta}{4\mu} \left[1 + \frac{1}{\sqrt{2}} \cos\left(2\omega t + \frac{\pi}{4} \right) \right].$$

Integrating this with time over one period, the total loss energy density is

$$W = \frac{B_0^2 \omega \delta}{4\mu} = \frac{B_0^2}{2\mu} \left(\frac{\omega}{2\mu\sigma_{\mathrm{c}}} \right)^{1/2}.$$

On the other hand, the Poynting vector on the surface is $\boldsymbol{S}_{\mathrm{P}}(x = 0) = i_x(E_y B_z)_{x=0}/\mu$. Noting that $\mathrm{d}\boldsymbol{S} = -i_x \mathrm{d}S$, the surface integral of the Poynting vector is

$$-\int_{\mathrm{S}} \boldsymbol{S}_{\mathrm{P}} \cdot \mathrm{d}\boldsymbol{S} = \left(\frac{E_y B_z}{\mu} \right)_{x=0} = \frac{B_0^2}{\mu} \left(\frac{\omega}{\mu\sigma_{\mathrm{c}}} \right)^{1/2} \cos\omega t \, \cos\left(\omega t + \frac{\pi}{4} \right).$$

Integrating this with time over one period, we have

$$\frac{B_0^2}{2\mu} \left(\frac{\omega}{2\mu\sigma_{\mathrm{c}}} \right)^{1/2},$$

which agrees with the above result. The reason for the same result is that the conditions of electric field and magnetic flux density are the same before and after one period, and hence, the electric and magnetic energies are unchanged. All the energy that flows into the conductor in one period is consumed and converted to Joule heat.

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Example 11.5. We apply current I to a wide parallel-plate transmission line using an electric power source of output voltage V, as shown in Fig. 11.4a. Determine the Poynting vector and discuss the flow of energy. Neglect the electric resistance of the conductor. The width of a plate is w and the plate separation is d. When we cannot neglect the electric resistance, how does the energy flow?
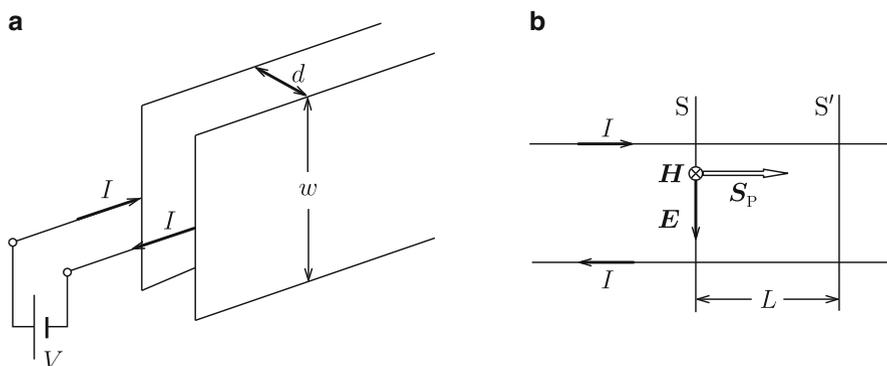
Fig. 11.4 (**a**) Parallel-plate transmission line and (**b**) the Poynting vector in the transmission line

Solution 11.5. First, we consider the case where the electric resistance can be neglected. We determine the Poynting vector on a surface, S, at some distance from the electric power source (see Fig. 11.4b). The electric field of strength $E = V/d$ is directed downward and the magnetic field of strength $H = I/w$ is directed backward. Hence, the magnitude of the Poynting vector is $S_P = VI/(wd)$ and the vector is directed from the electric power source to the terminal of the transmission line. Hence, the energy that flows from the electric power source to the transmission line in unit time is

$$S_P wd = VI$$

and is equal to the electric power as well known. This value is constant and independent of the position of surface S.

Second, we consider the case where the electric resistance cannot be neglected. We use R'_r and V' to denote the electric resistance of the conducting plate in a unit length and the potential difference between the conductors on surface S, respectively. The potential difference on surface S′ at distance L from S is then $V' - 2LR'_r I$. The electric power that enters through surface S is $V'I$ and the electric power that exits through S′ is $(V' - 2LR'_r I)I = V'I - 2LR'_r I^2$. Hence, the difference, $2LR'_r I^2$, is the power consumed as the Joule heat in the region between S and S′. Here we did not discuss the detailed energy flow into the conductor. See Exercise 11.8 for this discussion.

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Column: (1) Polarization Current Density

Using the electric polarization $\boldsymbol{P}$, the displacement current is written as

$$\frac{\partial \boldsymbol{D}}{\partial t} = \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{P}}{\partial t}.$$

The second term is the **polarization current density** due to the movement of the polarization charge of density ρ_p:

$$\frac{\partial \boldsymbol{P}}{\partial t} = \boldsymbol{i}_p.$$

Using this equation and Eq. (4.7), the continuity equation for the polarization charge is obtained as

$$\nabla \cdot \boldsymbol{i}_p + \frac{\partial \rho_p}{\partial t} = 0$$

The first term of the displacement current is independent of the movement of charges and is not a current in the usual sense. The displacement current discussed in Fig. 11.2 and that in the vacuum region of the capacitor in Example 11.1 are examples of this component.

When a capacitor is occupied by a dielectric material, the polarization current given by the second term also flows. For a dielectric material with a larger dielectric constant, most of the displacement current is polarization current.

(2) The Poynting Vector and Flow of Energy

We open the end terminal of the parallel-plate transmission line in Fig. 11.4b and apply voltage V and uniform magnetic flux density B_0 directed normally into the sheet. In this case the electric field applied

between the two plate conductors is $E = V/d$, and a Poynting vector of magnitude $S = VB_0/\mu_0 d$ is directed from the electric power source to the terminal. Does this mean there is a steady energy flow from the electric power source even in this static condition?

It is difficult to consider that the energy flows continuously, since there is no electric power from the source. How should we understand Eq. (11.39), which is considered to represent the energy flow? If we substitute

$$\boldsymbol{S}_{\mathrm{P}} = \boldsymbol{S}_{\mathrm{P0}} + \nabla \times \boldsymbol{K}.$$

into Eq. (11.39), we have

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{S}_{\mathrm{P0}} = 0.$$

If $\boldsymbol{S}_{\mathrm{P0}}$ represents the real energy flow, the Poynting vector differs from the real energy flow by a rotation of an arbitrary vector function.[1] Thus, the Poynting vector does not necessarily give the energy flow, and there is always arbitrariness in determining the energy flow. The above case is one example.

Literature

1. Matsushita T, Funaki K (2004) TEION KOGAKU (J Cryo Soc Jpn) 39:2 [in Japanese]

Exercises

11.1. Derive the equations that the electric potential and vector potential satisfy when we use the Coulomb gauge, Eq. (6.30).

11.2. Suppose we apply an external AC electric field $E_0 \cos \omega t$ along the z-axis parallel to the surface of a semi-infinite dielectric material (dielectric constant: ϵ, magnetic permeability: μ) that occupies $x \geq 0$, as shown in Fig. E11.1. Determine the electric field and magnetic flux density in the dielectric material.

11.3. Discuss the energy flow in Exercise 11.2.

11.4. Suppose that an electric power source supplies electric charges to the electrodes of the circular parallel-plate capacitor in Fig. 11.3. Assume that the distance d between the electroplates is much smaller than a. Determine the Poynting vector that enters the capacitor when the electric charges increase to $\pm q(t)$. Then calculate the energy stored in the capacitor, when the electric charges are $\pm Q$.

11.5. We apply direct current I to a long cylindrical conductor of radius a and electric conductivity σ_c. Determine the Poynting vector and discuss the energy flow.

11.6. We apply current to a cylindrical one-turn coil made of a thin conducting plate shown in Fig. E11.2. Determine the Poynting vector as the current increased from zero to I and discuss the flow of the energy. The height h is sufficiently greater and the width of the gap δ is sufficiently smaller than the radius a. Neglect the electric resistance of the conducting plate.

11.7. Discuss the case where we cannot neglect the electric resistance of the conductor in Exercise 11.6. The electric resistivity of the conductor is ρ_r, and the thickness b of the conductor is sufficiently less than the diameter a so that a slab approximation holds for the conductor.

11.8. Discuss the energy flow into the conductor but including the electric resistance that is not discussed in Example 11.5.
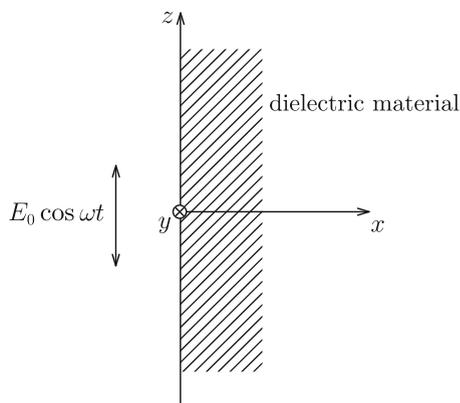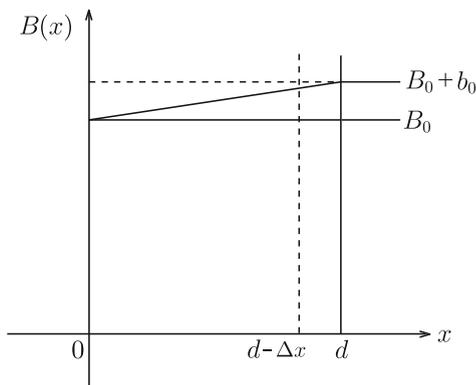

Fig. E11.1 AC electric field applied parallel to the surface of semi-infinite dielectric material
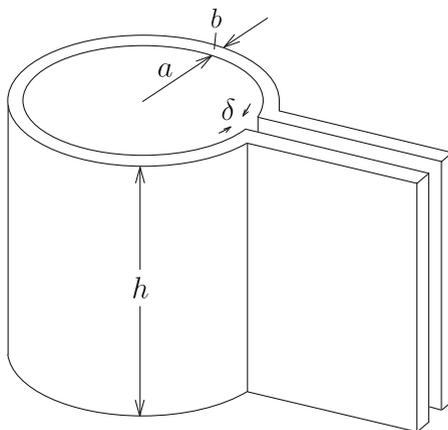
Fig. E11.2 Cylindrical one-turn coil

Fig. E11.3 Magnetic flux
distribution in half a
superconducting slab

11.9. Suppose we apply a uniform magnetic flux density B_0 along the z-axis parallel to a wide superconducting slab occupying $-d \leq x \leq d$ and virtually change the magnetic flux distribution as $B(x) = B_0 + b_0 x/d$ in the region $0 \leq x \leq d$ (see Fig. E11.3) by increasing the external magnetic flux density from B_0 to $B_0 + b_0$. In this case, a current flows along the y-axis, and it is known that the Lorentz force acts on the current. Note that we can presume that this force acts on flux lines whose distortion produces the current. Calculate the energy that flows into the region from $d - \Delta x$ to d through a unit area in the y-z plane as the external magnetic flux density increases. Derive an expression of the Lorentz force from the difference between the energy input and the enhancement of magnetic energy. (Hint: To derive the Lorentz force, it is necessary to determine the displacement of flux lines, u. For this purpose, use the continuity equation of magnetic flux, Eq. (10.20): $\mathrm{d}u/\mathrm{d}x = -\Delta b/B$ with Δb denoting a variation in the magnetic flux density.)