Chapter 10
Electromagnetic Induction

10.1 Induction Law

Magnetic flux density is produced by current as described in Chap. 6. Faraday conducted experiments based on the idea that current might be produced by magnetic flux density. Although he could not produce a steady current by using a static magnetic flux density, he produced a non-steady current by varying a magnetic flux density with time. The results are summarized as follows. Suppose two coils. When a current is applied to coil 1 as shown in Fig. 10.1, a magnetic flux density is produced around it. A current flows in coil 2 in the following cases:

1. the current in coil 1 changes as shown in Fig. 10.1a;
2. coil 1 is moved as shown in Fig. 10.1b.

This phenomenon is called **electromagnetic induction**, and the electromotive force that induces the current in coil 2 is called **induced electromotive force**. One can observe that the current flows in coil 2 in such a way as to reduce any change in the magnetic flux penetrating coil 2. This shows a conservative property in nature similarly to the law of inertia for the matter.

From the above results, if the magnetic flux that penetrates the coil is Φ, the induced electromotive force in the coil is given by

$$V_{\mathrm{em}} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \tag{10.1}$$

where the directions of magnetic flux and electromotive force follow the right-hand rule. This is exactly the result predicted in Sect. 8.4 and is called **Faraday's law**. This shows that the variation in magnetic flux with time causes the electromotive force. In this sense this is also called the **magnetic flux law** or **transformer law**. If the number of turns of the coil is n and the magnetic flux that penetrates one turn of the coil is Φ, the electromotive force is

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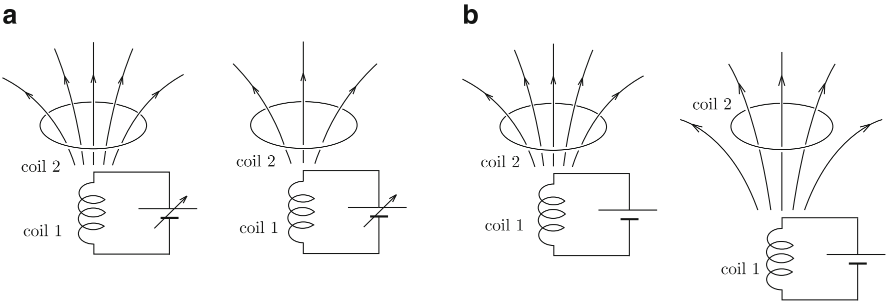
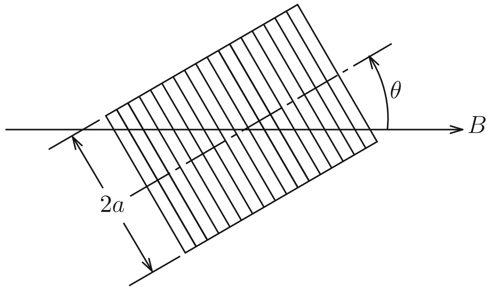


Fig. 10.1 Current changes in coil 2, when (**a**) current in coil 1 changes or (**b**) coil 1 moves

Fig. 10.2 Solenoid coil rotating in uniform magnetic flux density



$$V_{\mathrm{em}} = -N\frac{\mathrm{d}\Phi}{\mathrm{d}t}. \tag{10.2}$$

Here we calculate the electromotive force induced in a solenoid coil of radius a and number of turns N that is rotating with angular frequency ω in a uniform magnetic flux density, B, as shown in Fig. 10.2. If the angle between the coil axis and magnetic flux density is $\theta = \omega t$, the magnetic flux that penetrates one turn of the coil is $\Phi = \pi a^2 B \cos\omega t$, and the electromotive force is

$$V_{\mathrm{em}} = \pi N a^2 B\omega \sin\omega t. \tag{10.3}$$

Here we derive the differential expression of Faraday's law for the above case (1). Using the electric field, the electromotive force induced in closed coil 2 is written as

$$V_{\mathrm{em}} = \oint_{\mathrm{C}} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{s} = \int_{\mathrm{S}} \nabla\times\boldsymbol{E}\cdot \mathrm{d}\boldsymbol{S}, \tag{10.4}$$

where S is the surface surrounded by C and we have used Stokes' theorem. On the other hand, from Eq. (6.19) the magnetic flux is

$$\Phi = \int_{\mathrm{S}} \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S}. \tag{10.5}$$

Thus, Eq. (10.1) gives

$$\int_S \nabla \times \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}. \tag{10.6}$$

In the above we changed the order of the surface integral and differentiation with respect to time, since surface S does not change with time. In this process we change the total differentiation with time to partial differentiation, since we are treating a stationary system. The relationship, Eq. (10.6), holds for arbitrary S, and we have

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{10.7}$$

This is the **differential form of the induction law**. When the magnetic flux density does not change with time, Eq. (10.7) reduces to the equation for electrostatic field, Eq. (1.28). Hence, we can conclude that the electric field given by Eq. (10.7) is a general electric field that includes the induced and static components.

Second, we consider case (2) where coil 2 of closed loop C moves with the velocity $\boldsymbol{v}$ in a magnetic flux density $\boldsymbol{B}$ that does not change with time. The area of the hatched region in Fig. 10.3 that a small segment of the coil, $\mathrm{d}\boldsymbol{s}$, sweeps in short period Δt is $|\boldsymbol{v}\Delta t \times \mathrm{d}\boldsymbol{s}|$, and the magnetic flux that enters the coil through this region is

$$(\boldsymbol{v}\Delta t \times \mathrm{d}\boldsymbol{s}) \cdot \boldsymbol{B} = (\boldsymbol{B} \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{s}\,\Delta t. \tag{10.8}$$

It should be noted that the directions of $\mathrm{d}\boldsymbol{s}$ and $\boldsymbol{B}$ follow the right hand rule. Hence, the total magnetic flux that enters the coil during Δt is

$$\Delta\Phi = \Delta t \oint_C (\boldsymbol{B} \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{s}. \tag{10.9}$$

In the limit $\Delta t \to 0$, we have

$$\frac{\Delta\Phi}{\Delta t} \to \frac{\mathrm{d}\Phi}{\mathrm{d}t} = \oint_C (\boldsymbol{B} \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{s}. \tag{10.10}$$

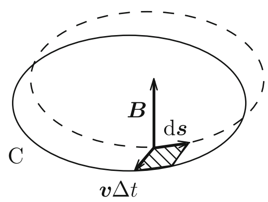


Fig. 10.3 Coil C moving in magnetic flux density $\boldsymbol{B}$. It moves from the position shown by the *dotted line* to that by the *solid line* during short period Δt

Hence, from Eqs. (10.4) and (10.10) we obtain the relationship describing the induced electric field,

$$\boldsymbol{E} = \boldsymbol{v} \times \boldsymbol{B} . \tag{10.11}$$

This is called the **motional law**. Mathematically, the induced electric field should be given by $\boldsymbol{E} = \boldsymbol{v} \times \boldsymbol{B} - \nabla\phi$ with ϕ being an arbitrary scalar function. However, it is empirically known that $\nabla\phi$ is zero in usual cases. Electrons in a conductor of a coil suffer the Lorentz force, $-e\boldsymbol{v} \times \boldsymbol{B}$, when the coil moves with the velocity $\boldsymbol{v}$. We can think of this force as a force caused by the electric field, Eq. (10.11), induced in the coil.

Thus, we can conclude that the electromotive force induced in the coil is caused by the change in the magnetic flux that penetrates the coil in both cases (1) and (2). The induced electromotive force can be summarized as

$$V_{\mathrm{em}} = \oint_{\mathrm{C}} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{s} = -\int_{\mathrm{S}} \frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{S} . \tag{10.12}$$

Using Eq. (A1.43), the condition $\nabla \cdot \boldsymbol{B} = 0$ and the condition that the coil is not deformed during the movement, $\nabla \cdot \boldsymbol{v} = 0$, we have

$$\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} . \tag{10.13}$$

When the velocity is constant, the first term on the right side is zero. The total differentiation with respect to time is written as

$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = \frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} . \tag{10.14}$$

Hence, rewriting Eq. (10.12) with Stokes' theorem, we have

$$\int_{\mathrm{S}} \nabla \times \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = -\int_{\mathrm{S}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S} . \tag{10.15}$$

Thus, we have derived Eq. (10.7), indicating that the above conclusion is valid.

When a conductor carries current $\boldsymbol{I}$ in magnetic flux density $\boldsymbol{B}$, the Lorentz force acts on the conductor. If this force forces the conductor to move with velocity $\boldsymbol{v}$, the power in a unit length of the conductor given by the Lorentz force is

$$\boldsymbol{F}' \cdot \boldsymbol{v} = (\boldsymbol{I} \times \boldsymbol{B}) \cdot \boldsymbol{v} . \tag{10.16}$$

This seems to contradict the fact that the Lorentz force does not do any work on electric charges (see Example 6.3). In a practical case, the induced electric field directed opposite to the current works to reduce the current. To have the same current continue to flow, the electric power source must supply additional electric power,

$$-\boldsymbol{I} \cdot (\boldsymbol{v} \times \boldsymbol{B}) = (\boldsymbol{I} \times \boldsymbol{B}) \cdot \boldsymbol{v} , \tag{10.17}$$

which is equal to the power given by the Lorentz force. That is, the power by the Lorentz force is nothing other than the electric power by the electric source (see Exercise 10.8).

Example 10.1. A triangular closed circuit and a straight current, I, are placed on a common plane. The closed circuit is moving away with velocity v from the current, as shown in Fig. 10.4. The distance between the closed circuit and the current is $r = r_0$ in the initial condition ($t = 0$). Determine the electromotive force induced in the closed circuit with the magnetic flux law. We define the electromotive force to be positive along the direction of ABC.

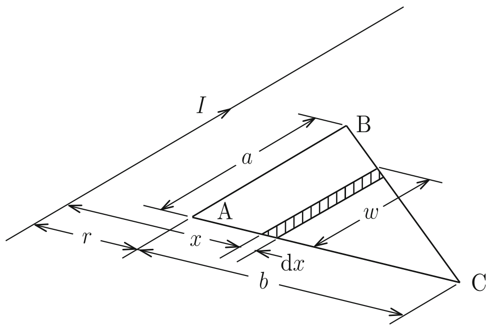


Fig. 10.4 Straight current and triangular closed circuit moving away with constant velocity

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Solution 10.1. At time t the distance between the circuit and current is $r(t) = r_0 + vt$. The width of the triangle at distance x ($r \leq x \leq r + b$) from the current is

$$w(x) = a - \frac{a(x-r)}{b}.$$

The direction of the magnetic flux produced by current I inside the circuit is the same as that of the magnetic flux produced by the current flowing along ABC. Hence, the magnetic flux produced by current I is positive. The magnetic flux density at distance x from the current is $B(x) = \mu_0 I/(2\pi x)$. The magnetic flux penetrating the narrow region x to $x + \mathrm{d}x$ in the circuit is

$$\mathrm{d}\Phi = B(x)w(x)\mathrm{d}x = \frac{\mu_0 a I}{2\pi b}\left(\frac{r+b}{x} - 1\right)\mathrm{d}x.$$

Thus, the total magnetic flux penetrating the circuit is

$$\Phi = \frac{\mu_0 I a}{2\pi b}\int_r^{r+b}\left(\frac{r+b}{x} - 1\right)\mathrm{d}x = \frac{\mu_0 I a}{2\pi b}\left[(r+b)\log\frac{r+b}{r} - b\right].$$

The electromotive force induced in the circuit is

$$V_{\rm em} = -\frac{{\rm d}\Phi}{{\rm d}t} = -\frac{\partial \Phi}{\partial r}\cdot\frac{\partial r}{\partial t} = \frac{\mu_0 I a v}{2\pi b}\left[\frac{b}{r_0 + vt} - \log\left(1 + \frac{b}{r_0 + vt}\right)\right].$$

The penetrating magnetic flux decreases with time and hence, the electromotive force is induced to increase it. Thus, $V_{\rm em}$ is positive.

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Example 10.2. Determine the electromotive force in Example 10.1 with the motional law.

Solution 10.2. Figure 10.5a shows the direction of the induced electric field on each side. We determine the electromotive force induced on each side. On side AB, the magnetic flux density is $B = \mu_0 I/(2\pi r)$, and $\boldsymbol{v} \times \boldsymbol{B}$ is directed from A to B and its magnitude is $\mu_0 I v/(2\pi r)$. Hence, the contribution from this side to the electromotive force is

$$\int_{\rm A}^{\rm B} (\boldsymbol{v} \times \boldsymbol{B}) \cdot {\rm d}\boldsymbol{s} = \frac{\mu_0 I v}{2\pi r}\, a.$$

Next, the magnetic flux density at point P at distance s from B is $\mu_0 I/[2\pi(r + s\sin\theta)]$ with θ denoting the angle of B (see Fig. 10.5b). The induced electric field has magnitude $\mu_0 I v/[2\pi(r + s\sin\theta)]$ and its direction is tilted by $\pi - \theta$ from the direction of integration, d$\boldsymbol{s}$. Thus, we have

$$(\boldsymbol{v} \times \boldsymbol{B}) \cdot {\rm d}\boldsymbol{s} = \frac{\mu_0 I v}{2\pi(r + s\sin\theta)} \cos(\pi - \theta)\, {\rm d}s = -\frac{\mu_0 I v \cos\theta}{2\pi(r + s\sin\theta)}\, {\rm d}s,$$

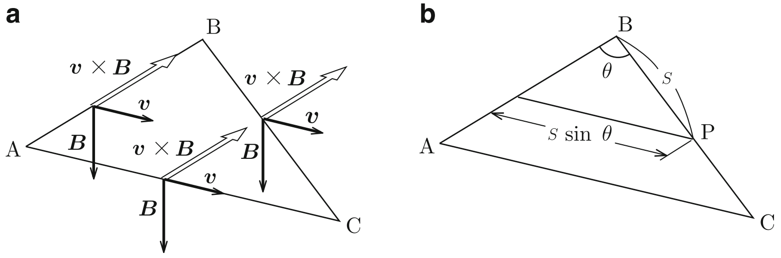

Fig. 10.5 (**a**) Direction of induced electric field on each side and (**b**) a point on side BC

and the contribution from side BC is

$$\int_{\mathrm{B}}^{\mathrm{C}} (\boldsymbol{v} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{s} = -\frac{\mu_0 I v \cos\theta}{2\pi} \int_0^{\overline{\mathrm{BC}}} \frac{\mathrm{d}s}{r + s \sin\theta} = -\frac{\mu_0 I v \cot\theta}{2\pi} \log \frac{r + a\tan\theta}{r}$$
$$= -\frac{\mu_0 I a v}{2\pi b} \log \frac{r+b}{r}.$$

Finally, the induced electric field is perpendicular to the direction of integration on side CA. Hence, there is no contribution from this side. As a result, the induced electromotive force is

$$V_{\mathrm{em}} = \oint (\boldsymbol{v} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{s} = \frac{\mu_0 I a v}{2\pi b} \left[\frac{b}{r_0 + vt} - \log\left(1 + \frac{b}{r_0 + vt}\right) \right],$$

which agrees with the result obtained in Example 10.1.

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Here we discuss a phenomenon that is usually explained using only the motional law. Suppose that a conducting circular plate of radius a is rotated with angular frequency ω around its axis in uniform magnetic flux density B, as shown in Fig. 10.6a. We determine the electromotive force induced between the center O of the plate and point P on the edge. Since the magnetic flux penetrating the closed loop composed of the straight line connecting O and P and the line C outside the plate does not change with time, it seems that no electromotive force is induced in it. However, an electromotive force is induced in reality. This is called **unipolar induction**.

According to the motional law, since the magnetic flux crosses line OP, the electromotive force is induced there. In the arrangement in Fig. 10.6a the induced electric field is directed from O to P. At a point at distance R from the central axis the velocity of rotation, v, is equal to $R\omega$. Hence, the induced electric field is $E = R\omega B$. Integrating this from O to P, the electromotive force is

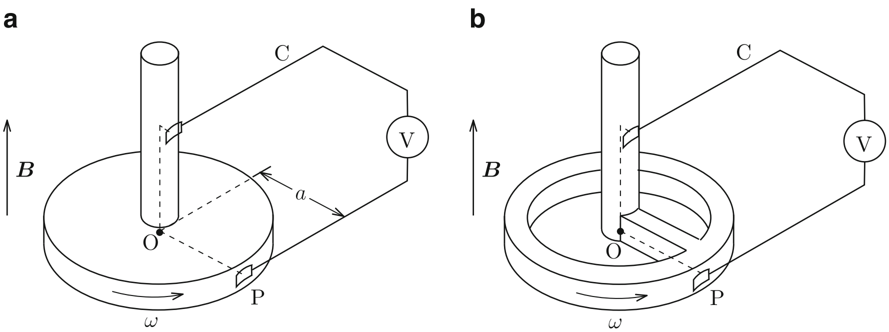

Fig. 10.6 Unipolar induction: (**a**) usual system and (**b**) new system

$$V_{\rm em} = \int_0^a R\omega B \, \mathrm{d}R = \frac{1}{2}\omega B a^2. \tag{10.18}$$

The unipolar induction is really known. However, since the electromotive force is not stable, it is not practically used. If the circular plate is magnetized, the same thing occurs even if no magnetic field is applied. In addition, when a hollow dielectric plate is rotated in a magnetic flux density, an electric polarization occurs because of the electromotive force.

The same unipolar induction can be observed even for the system shown in Fig. 10.6b, in which bar OP rotates. In this case the magnetic flux penetrating the circuit changes with time, and the result can also be explained with the magnetic flux law. To explain the result for the system in Fig. 10.6a with the magnetic flux law, we can suppose that part of the circuit rotates with the plate as the bar in Fig. 10.6b. Another part of the circuit on the edge of the plate is equipotential. Hence, it is difficult to distinguish the two mechanisms.

In the above we learned the magnetic flux law and motional law to describe the induced electromotive force for simple cases. If we discuss the case where a conductor is forced to move in a magnetic flux density varying with time, we have two contributions to the electromotive force, and it is necessary to calculate each contribution using the two laws. Here we propose a general law that combines the two laws.

We assume that the external magnetic field is increasing with time. In this case the magnetic flux density inside a material also increases because of the penetrating magnetic flux. Thus, we can define the velocity of the magnetic flux lines and denote it as $\boldsymbol{V}$. If the coil in Fig. 10.3 stays stationary but the magnetic flux lines move with velocity $\boldsymbol{V} = -\boldsymbol{v}$, the same amount of magnetic flux penetrates into the coil through a segment $\mathrm{d}\boldsymbol{s}$ within period Δt. Hence, repeating a similar argument up to Eq. (10.10), the time-variation in magnetic flux Φ that penetrates the coil is given by

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\oint_{\rm C} (\boldsymbol{B} \times \boldsymbol{V}) \cdot \mathrm{d}\boldsymbol{s}. \tag{10.19}$$

Thus, we obtain the local relationship,

$$\nabla \times (\boldsymbol{B} \times \boldsymbol{V}) = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{10.20}$$

This is called the **continuity equation of magnetic flux** and is frequently used for analyzing electromagnetic phenomena in superconductors. Comparing this equation with Eq. (10.7), we have

$$\boldsymbol{E} = \boldsymbol{B} \times \boldsymbol{V}. \tag{10.21}$$

This is called **Josephson's relation**. It should be noted that this relation expresses the magnetic flux law. Using the above two velocities, the relative velocity of magnetic flux lines from coil C is

$$\boldsymbol{V}' = \boldsymbol{V} - \boldsymbol{v}, \tag{10.22}$$

and combining the above equation with Eq. (10.11) gives the **general law for the induced electric field**,

$$\boldsymbol{E} = \boldsymbol{B} \times \boldsymbol{V}'. \tag{10.23}$$

The Column in this chapter shows an example of calculating induced electromotive force.

Example 10.3. Suppose that we increase by ΔB the magnetic flux density B_0 applied parallel to a long cylindrical conductor of radius a during a short period, Δt. Determine with Eq. (10.21) the electromotive force measured with potential leads with the different arrangements shown in Fig. 10.7a, b.

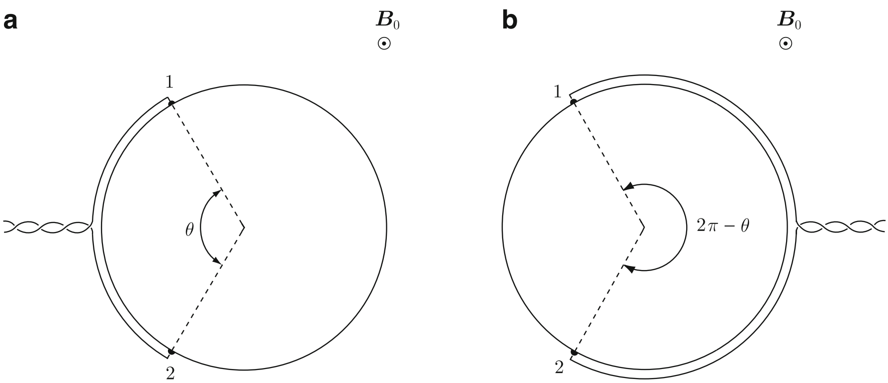

Fig. 10.7 Different arrangements of potential leads for measuring electromotive force induced in cylindrical conductor

Solution 10.3. We denote the velocity of the magnetic flux as

$$\boldsymbol{V} = -\boldsymbol{i}_R V.$$

Then, the continuity equation of magnetic flux shown earlier reduces to

$$-\frac{1}{R} \cdot \frac{\partial}{\partial R}(R B_0 V) = -\frac{\Delta B}{\Delta t}$$

and we have

$$E = B_0 V = \frac{1}{2} \cdot \frac{\Delta B}{\Delta t} a$$

on the surface ($R = a$). The induced electric field is directed counterclockwise. One can easily show that the induced electric field integrated along the circumference in this direction, $2\pi a E$, is equal to the total electromotive force, $\Delta\Phi/\Delta t = \pi a^2 \Delta B/\Delta t$.

Here we consider the case shown in Fig. 10.7a. The azimuthal angle between potential terminals 1 and 2 is θ, and we place the potential leads on the surface of the conductor and twist to eliminate the electromotive force outside the conductor. The measured electromotive force is

$$V_{\mathrm{em}} = a\theta B_0 V = \frac{1}{2} \cdot \frac{\Delta B}{\Delta t} a^2 \theta,$$

where we set a reference point on terminal 1.

Second, we determine the electromotive force for the arrangement shown in Fig. 10.7b. We assume an integral path of the induced electric field on the right conductor surface. In this case we can neglect the magnetic flux penetrating the closed loop composed of this path and the potential leads, and similarly determine the electromotive force only by integrating Eq. (10.21) along the path. Thus, we have

$$V'_{\mathrm{em}} = -(2\pi - \theta) B_0 V = -\frac{1}{2} \cdot \frac{\Delta B}{\Delta t} a^2 (2\pi - \theta)$$

for the same reference point.

On the other hand, it is possible to choose the left conductor surface for the integral path. The integral gives $a^2\theta(\Delta B/\Delta t)/2$. In this case the electromotive force due to the magnetic flux penetrating the integral path should be taken into account. This additional component is $-\pi a^2(\Delta B/\Delta t)$, and we obtain the same result by adding it.

Thus, there is a freedom in choosing the integral path. For example, it is also possible to choose the path shown by the dotted line in Fig. 10.7b. In this case the induced electric field is perpendicular to the path, resulting in a zero line integral. From the magnetic flux penetrating the area surrounded by the path and the potential leads (denoted by the azimuthal angle $2\pi - \theta$), we directly obtain the same result.

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10.2 Potential

As described in Sect. 10.1, the electric field $\boldsymbol{E}$ contains not only the electrostatic field but also the induced electric field. Hence, such a general electric field cannot be described only by the electric potential. Here we note that the right side of Eq. (10.7) is written in terms of the vector potential $\boldsymbol{A}$ as

$$-\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \left(\frac{\partial \boldsymbol{A}}{\partial t} \right), \tag{10.24}$$

where we have changed the order of the time differentiation and spatial differentiation. Comparing this with the left side of Eq. (10.7), it is obvious that the induced electric field is given by $-\partial \boldsymbol{A}/\partial t$. Hence, with the electrostatic field, the general electric field is given by

$$\boldsymbol{E} = -\nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t}. \tag{10.25}$$

This satisfies Eq. (10.7) and reduces to Eq. (1.24) in the static condition. The vector potential satisfies Eq. (9.21) under the Coulomb gauge even in this case.

To discuss the general electromagnetic fields, it is necessary to extend Ampere's law, Eq. (9.12), for the static magnetic field produced by a steady current to a general law including the magnetic field by a non-steady current. This will be covered in Chap. 11 in which we complete the set of Maxwell's equations.

10.3 Boundary Conditions

Here we investigate whether the boundary condition for the electric field $\boldsymbol{E}$ changes in going from Eqs. (1.28) to (10.7).

We denote the electric fields in materials 1 and 2 near the boundary as $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$, respectively. Consider a plane normal to the boundary that includes vectors $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ (see Fig. 4.15a, b). Integrating the electric field around a small rectangle, ΔC, with two sides parallel to the boundary, with Eq. (10.7) and Stokes' theorem we have

$$\oint_{\Delta \mathrm{C}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = \int_{\Delta \mathrm{S}} \nabla \times \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = -\frac{\partial}{\partial t} \int_{\Delta \mathrm{S}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}, \tag{10.26}$$

where ΔS is the surface surrounded by ΔC, and we have changed the order of the spatial integration and time differentiation. When the height of the small rectangle, Δh, is sufficiently small, the amount of magnetic flux that penetrates ΔS is negligible. Thus, the circular integral of the electric field reduces to zero similarly to the case in Sect. 4.3, and we obtain the same result as Eq. (4.22),

$$\boldsymbol{n} \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0. \tag{10.27}$$

That is, the parallel component of the electric field is continuous across the boundary even when electromagnetic induction occurs.

10.4 Magnetic Energy

We already learned the magnetic energy in Sect. 8.3. In the system shown in Fig. 8.14 the magnetic flux that links the circuit made of a superconductor does not change even when the plate moves. This means that there is no electromagnetic induction. This is why the magnetic energy can be determined from the mechanical work needed against the magnetic force for this system. Most textbooks on electromagnetism explain magnetic energy after electromagnetic induction. Hence, this textbook gives the same explanation. An explanation from a different viewpoint is helpful for a deep understanding of the phenomenon. Since there is no new phenomenon in this section, however, readers who do not feel a need for this explanation can skip it.

Suppose a coil of self-inductance L. When we apply current I' to this coil, the magnetic flux penetrating this coil is

$$\Phi' = LI'. \tag{10.28}$$

When the current is increased by a small amount $\mathrm{d}I'$ in a short period $\mathrm{d}t$, the induced electromotive force is

$$V_{\mathrm{em}} = -L\frac{\mathrm{d}I'}{\mathrm{d}t}. \tag{10.29}$$

This acts to restrict the increase in the current. This phenomenon is called **self-induction**. Thus, the electric power source must work against this electromotive force to increase the current, and the electric power in this period is

$$P = -V_{\mathrm{em}}I' = LI'\frac{\mathrm{d}I'}{\mathrm{d}t}. \tag{10.30}$$

The energy stored in the coil when we apply the current I to the coil is equal to the energy supplied by the electric power source until the current increases from 0 to I, and is given by

$$U_{\mathrm{m}} = W = \int LI'\frac{\mathrm{d}I'}{\mathrm{d}t}\,\mathrm{d}t = \int_0^I LI'\mathrm{d}I' = \frac{1}{2}LI^2. \tag{10.31}$$

Using the magnetic flux $\Phi = LI$, this is also written as

$$U_{\mathrm{m}} = \frac{1}{2}LI^2 = \frac{1}{2}\Phi I = \frac{1}{2L}\Phi^2. \tag{10.32}$$

This result agrees with Eq. (8.34), and we can understand that this energy is exactly the magnetic energy.

Following a similar discussion, we can show that the magnetic energy of a system composed of n coils is given by Eq. (8.35). In this case the electromotive force in the i-th coil induced by current I_j flowing in the j-th coil is given by

$$V_{\mathrm{em}i} = -L_{ij}\frac{\mathrm{d}I_j}{\mathrm{d}t}, \tag{10.33}$$

using the mutual inductance. This is called **mutual induction**.

Example 10.4. Prove Eq. (8.35) for a system composed of two coils and prove the reciprocity, $L_{12} = L_{21}$.

Solution 10.4. Currents I_1 and I_2 flow in coils 1 and 2, and the resultant penetrating magnetic fluxes in these coils are Φ_1 and Φ_2, respectively. Assume that this final situation is reached after we apply the currents to coil 1 and then to coil 2. We suppose an intermediate situation where coil 2 has no current and coil 1 has current I_1' (see Fig. 10.8a). With the inductance coefficient the magnetic flux that penetrates coil 1 is $L_{11}I_1'$. Hence, the work done to apply current I_1 to coil 1 is

$$W_1 = \int_0^{I_1} L_{11}I_1'\mathrm{d}I_1' = \frac{1}{2}L_{11}I_1^2.$$

Next, we consider the situation where coils 1 and 2 have currents I_1 and I_2', respectively (see Fig. 10.8b). The magnetic flux penetrating coil 2 is $L_{21}I_1 + L_{22}I_2'$. Hence, the work done to apply current I_2 to coil 2 is

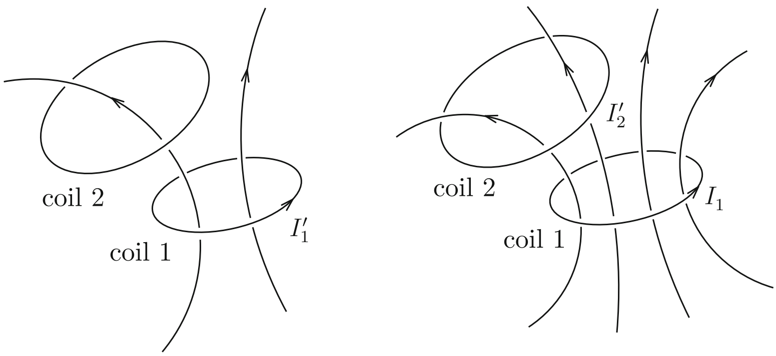

Fig. 10.8 (**a**) Situation in which current I_1' is flowing in coil 1 but no current in coil 2 and (**b**) that in which currents I_1 and I_2' are flowing in coils 1 and 2, respectively

$$W_2 = \int_0^{I_2} (L_{21} I_1 + L_{22} I_2')\mathrm{d}I_2' = L_{21} I_1 I_2 + \frac{1}{2} L_{22} I_2^2.$$

Thus, the magnetic energy of this system is

$$U_\mathrm{m} = W_1 + W_2 = \frac{1}{2} L_{11} I_1^2 + L_{21} I_1 I_2 + \frac{1}{2} L_{22} I_2^2. \tag{10.34}$$

If we apply the currents in reversed order, the magnetic energy is

$$U_\mathrm{m} = \frac{1}{2} L_{11} I_1^2 + L_{12} I_1 I_2 + \frac{1}{2} L_{22} I_2^2. \tag{10.35}$$

Since Eqs. (10.34) and (10.35) must be the same, we can prove the reciprocity, Eq. (8.5),

$$L_{12} = L_{21}.$$

Thus, if we write $L_{21} = (L_{12} + L_{21})/2$ in Eq. (10.34), the magnetic energy is

$$U_\mathrm{m} = \frac{1}{2} I_1 (L_{11} I_1 + L_{12} I_2) + \frac{1}{2} I_2 (L_{21} I_1 + L_{22} I_2) = \frac{1}{2} (I_1 \Phi_1 + I_2 \Phi_2).$$

Thus, Eq. (8.35) holds for $n = 2$.

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Suppose we change magnetic flux densities in coils with electric power sources. We denote the induced electric field by $\boldsymbol{E}$. From Eq. (10.30) the input power into the system is given by

$$P = -\int_\mathrm{V} \boldsymbol{E} \cdot \boldsymbol{i} \, \mathrm{d}V. \tag{10.36}$$

Using Eqs. (6.27) and (A1.41) in the Appendix, this leads to

$$\begin{aligned} P &= -\frac{1}{\mu_0} \int_\mathrm{V} \boldsymbol{E} \cdot (\nabla \times \boldsymbol{B}) \mathrm{d}V \\ &= \frac{1}{\mu_0} \int_\mathrm{V} [\nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) - \boldsymbol{B} \cdot (\nabla \times \boldsymbol{E})] \mathrm{d}V. \end{aligned} \tag{10.37}$$

Using Gauss' theorem we rewrite the first integral as the surface integral

$$\frac{1}{\mu_0} \int_\mathrm{S} (\boldsymbol{E} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{S},$$

where S is the surface of V. We assume a sphere of sufficiently large radius r for V. Since $|\boldsymbol{B}| \propto r^{-2}$, $|\boldsymbol{E}| \propto r^{-1}$ and $\int \mathrm{d}S \propto r^2$, the integral is proportional to r^{-1}. Hence, if we assume a very large sphere, the integral approaches zero and can be disregarded. Substituting Eq. (10.7) for the second integral, we have

$$P = \frac{1}{2\mu_0} \int_{\mathrm{V}} \frac{\partial \boldsymbol{B}^2}{\partial t} \, \mathrm{d}V. \tag{10.38}$$

From the above result we can generally show that the magnetic power density is given by $[1/(2\mu_0)]\partial \boldsymbol{B}^2/\partial t$ and the magnetic energy density by Eq. (8.32).

10.5 Skin Effect

The fundamental equations that we have learned up to now are

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{10.39}$$

$$\boldsymbol{i} = \nabla \times \boldsymbol{H}, \tag{10.40}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}, \tag{10.41}$$

$$\boldsymbol{i} = \sigma_{\mathrm{c}} \boldsymbol{E}. \tag{10.42}$$

The unknown variables, $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{H}$ and $\boldsymbol{i}$ can be obtained by solving the set of these equations. Here we rewrite the above equations in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. Then, Eqs. (10.40) to (10.42) are summarized as

$$\nabla \times \boldsymbol{B} = \mu \sigma_{\mathrm{c}} \boldsymbol{E}. \tag{10.43}$$

We can solve this equation with Eq. (10.39).

Now we learn the **skin effect** for the example of dynamic phenomena. Supposed that we apply an AC magnetic flux density of amplitude B_0 and angular frequency ω along the z-axis parallel to the surface of a semi-infinite conductor that occupies $x \geq 0$ (see Fig. 10.9). We can assume that no physical variable changes along the y- and z-axes:

$$\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z} \to 0. \tag{10.44}$$

The variation with time can be expressed with the factor $\mathrm{e}^{\mathrm{i}\omega t}$, and we replace the time differentiation with

$$\frac{\partial}{\partial t} \to \mathrm{i}\omega. \tag{10.45}$$

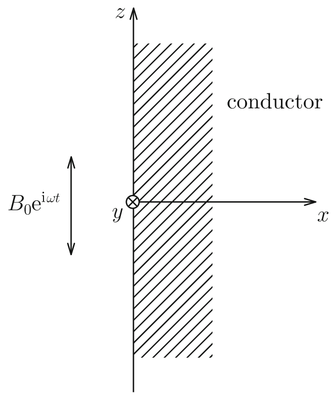

Fig. 10.9 Magnetic flux density applied parallel to the surface of semi-infinite conductor

In addition, since the external magnetic flux density is directed along the z-axis, we can assume that the internal magnetic flux density has only a z-component. Hence, we have

$$\nabla \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{i}_x & \boldsymbol{i}_y & \boldsymbol{i}_z \\ \mathrm{d}/\mathrm{d}x & 0 & 0 \\ 0 & 0 & B_z \end{vmatrix} = -\boldsymbol{i}_y \frac{\mathrm{d}B_z}{\mathrm{d}x}.$$

Thus, we find that the electric field has only a y-component, E_y, and Eq. (10.43) reduces to

$$\frac{\mathrm{d}B_z}{\mathrm{d}x} = -\mu\sigma_\mathrm{c} E_y. \tag{10.46}$$

The left side of Eq. (10.39) is

$$\nabla \times \boldsymbol{E} = \begin{vmatrix} \boldsymbol{i}_x & \boldsymbol{i}_y & \boldsymbol{i}_z \\ \mathrm{d}/\mathrm{d}x & 0 & 0 \\ 0 & E_y & 0 \end{vmatrix} = \boldsymbol{i}_z \frac{\mathrm{d}E_y}{\mathrm{d}x},$$

leading to

$$\frac{\mathrm{d}E_y}{\mathrm{d}x} = -\mathrm{i}\omega B_z. \tag{10.47}$$

This is consistent with the initial assumption that the magnetic flux density has only a z-component. Eliminating E_y in Eqs. (10.46) and (10.47), we have

$$\frac{\mathrm{d}^2 B_z}{\mathrm{d}x^2} - \mathrm{i}\omega\mu\sigma_\mathrm{c} B_z = 0. \tag{10.48}$$

The equation for E_y has the same form as this.

We can derive this equation generally. Taking a rotation of Eq. (10.43) and substituting Eq. (10.39), we have

$$\nabla \times (\nabla \times \boldsymbol{B}) = -\mu\sigma_c \frac{\partial \boldsymbol{B}}{\partial t}. \tag{10.49}$$

Using Eqs. (A1.46) in the Appendix and (6.21), this equation becomes

$$\Delta \boldsymbol{B} - \mu\sigma_c \frac{\partial \boldsymbol{B}}{\partial t} = 0. \tag{10.50}$$

This is a differential equation of the second order called a diffusion equation. Substituting the spatial symmetry, Eq. (10.44), and the time variation, Eq. (10.45), derives Eq. (10.48).

Assume a solution of type $B_z(x) \sim \mathrm{e}^{\alpha x}$. Substituting this into Eq. (10.48), we have $\alpha^2 = \mathrm{i}\omega\mu\sigma_c$. That is,

$$\alpha = \pm(1+\mathrm{i})\left(\frac{\omega\mu\sigma_c}{2}\right)^{1/2}. \tag{10.51}$$

From the condition that the magnetic flux density must be finite in the limit $x \to \infty$, α with the negative real part is the solution. The boundary condition is

$$B_z(x=0) = B_0. \tag{10.52}$$

Thus, we obtain the solution of the magnetic flux density as

$$B_z(x,t) = B_0\mathrm{e}^{-x/\delta}\exp\left[\mathrm{i}\left(\omega t - \frac{x}{\delta}\right)\right] \to B_0\mathrm{e}^{-x/\delta}\cos\left(\omega t - \frac{x}{\delta}\right). \tag{10.53}$$

In the above

$$\delta = \left(\frac{2}{\omega\mu\sigma_c}\right)^{1/2} \tag{10.54}$$

is a quantity with the dimension of length and is called the **skin depth**. In Eq. (10.53) we have adopted the real part for the solution. Figure 10.10 shows the spatial variation in the magnetic flux density given by Eq. (10.53). The magnetic flux density propagates along the x-axis while decaying. The depth of penetration is roughly equal to δ, which is the reason for the name. For larger ω and/or larger σ_c the shielding current density is higher, resulting in shorter δ. The position of a plane on which the phase of the propagating wave is constant is given by the condition

$$\omega t - \frac{x}{\delta} = c, \tag{10.55}$$

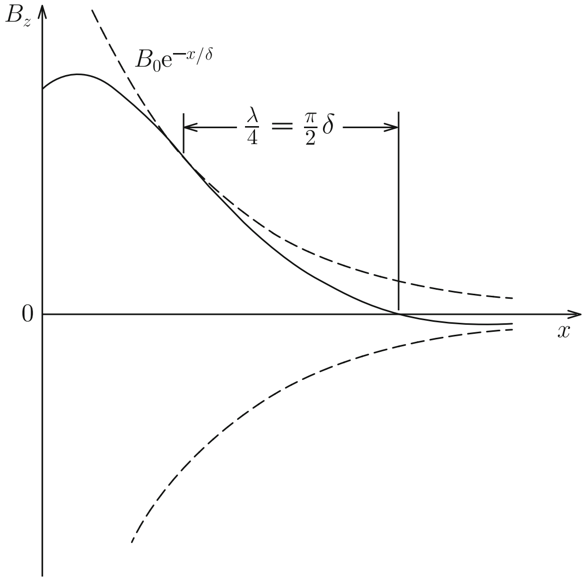

Fig. 10.10 Spatial variation in magnetic flux density

where c is a constant. Thus, the velocity of propagation is $\mathrm{d}x/\mathrm{d}t = \omega\delta$, and the wave length λ is $2\pi\delta$. Substituting this solution into Eq. (10.46) gives the solution of the electric field,

$$E_y(x,t) = B_0 \left(\frac{\omega}{\mu\sigma_c}\right)^{1/2} e^{-x/\delta} \exp\left[i\left(\omega t - \frac{x}{\delta} + \frac{\pi}{4}\right)\right]$$
$$\to B_0 \left(\frac{\omega}{\mu\sigma_c}\right)^{1/2} e^{-x/\delta} \cos\left(\omega t - \frac{x}{\delta} + \frac{\pi}{4}\right). \qquad (10.56)$$

We find that, although this solution is similar to that of the magnetic flux density, the phase is ahead by $\pi/4$. The magnetic field and current density are obtained from $H_z(x,t) = B_z(x,t)/\mu$ and $i_y(x,t) = \sigma_c E_y(x,t)$ with the above results.

We estimate the skin depth of copper at 60 Hz at room temperature. Substituting typical values, $\sigma_c = 0.58 \times 10^8$ S/m and $\mu \simeq \mu_0 = 4\pi \times 10^{-7}$ N/A^2 into Eq. (10.54) gives $\delta = 0.85 \times 10^{-2}$ m.

Example 10.5. We apply an AC electric field of amplitude E_0 and angular frequency ω along the z-axis parallel to an infinitely wide slab conductor that occupies $-d \le x \le d$. Determine the electric field in the conductor.

Solution 10.5. We can assume that $\partial/\partial y$ and $\partial/\partial z$ are zero as in Eq. (10.44) and the electric field has only a z-component. The time differentiation is replaced by the operator to multiply $\mathrm{i}\omega$. Hence, Eq. (10.39) reduces to

$$\frac{\mathrm{d}E_z}{\mathrm{d}x} = \mathrm{i}\omega B_y,$$

showing that the magnetic flux density has only a y-component. Thus, Eq. (10.43) becomes

$$\frac{\mathrm{d}B_y}{\mathrm{d}x} = \mu\sigma_\mathrm{c} E_z.$$

From these equations we have

$$\frac{\mathrm{d}^2 E_z}{\mathrm{d}x^2} - \mathrm{i}\omega\mu\sigma_\mathrm{c} E_z = 0,$$

which has the same form as Eq. (10.48). Using Eqs. (10.51) and (10.54), we obtain the general solution for the electric field as

$$E_z(x) = K_1 \exp\left[(1+\mathrm{i})\frac{x}{\delta}\right] + K_2 \exp\left[-(1+\mathrm{i})\frac{x}{\delta}\right].$$

The coefficients K_1 and K_2 are determined by the boundary conditions:

$$E_z(x=-d) = E_z(x=d) = E_0.$$

Thus, the electric field is given by

$$E_z(x,t) = E_0 \frac{\cosh[(1+\mathrm{i})x/\delta]}{\cosh[(1+\mathrm{i})d/\delta]} \mathrm{e}^{\mathrm{i}\omega t}$$

and its real part is

$$\begin{aligned} E_z(x,t) = {} & \frac{E_0}{\cosh(2d/\delta) + \cos(2d/\delta)} \\ & \times \left\{ \left[\cosh\left(\frac{x+d}{\delta}\right) \cos\left(\frac{x-d}{\delta}\right) + \cosh\left(\frac{x-d}{\delta}\right) \cos\left(\frac{x+d}{\delta}\right) \right] \cos\omega t \right. \\ & \left. - \left[\sinh\left(\frac{x-d}{\delta}\right) \sin\left(\frac{x+d}{\delta}\right) + \sinh\left(\frac{x+d}{\delta}\right) \sin\left(\frac{x-d}{\delta}\right) \right] \sin\omega t \right\}. \end{aligned}$$

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Column: General Law of Electromagnetic Induction

Suppose that the plate in Fig. 10.6a is rotating with angular frequency ω in magnetic flux density, B, that increases with time as $B = B_0 + \gamma t$. For simplicity we assume that the potential leads are arranged as in Fig. 10.11 to eliminate the magnetic flux that interlinks the circuit outside the plate. That is, one potential lead is aligned on the plate edge between P and A′ and twisted with another one up to a voltmeter. Now we determine the induced electromotive force in the direction of OPA′AO. In this case the electromotive force appears on line OP from the motional law and appears also from the magnetic flux law because of the penetrating magnetic flux. We now use Eq. (10.23) for the determination.

The velocity of the plate is

$$\boldsymbol{v} = R\omega \boldsymbol{i}_\theta$$

and from the answer to the problem in Example 10.3, the velocity of the magnetic flux lines is

$$\boldsymbol{V} = -\frac{\gamma R}{2B}\boldsymbol{i}_R.$$

The contribution from line OP to the electromotive force is

$$\int_0^a [\boldsymbol{B} \times (\boldsymbol{V} - \boldsymbol{v})] \cdot \mathrm{d}\boldsymbol{R} = \int_0^a vB\mathrm{d}R = \frac{1}{2}\omega B a^2.$$

That from arc PA′ is

$$\int_{\mathrm{P}}^{\mathrm{A}'} [\boldsymbol{B} \times (\boldsymbol{V} - \boldsymbol{v})] \cdot \mathrm{d}\boldsymbol{s} = -\int_0^\theta BV(R = a)a\mathrm{d}\theta = -\frac{1}{2}\gamma a^2\theta.$$

On line AO outside the plate, $\boldsymbol{v} = 0$ and the induced electric field, $\boldsymbol{B} \times \boldsymbol{V}$, is perpendicular to the integration path. Thus, there is no contribution from this line to the electromotive force. Finally we have

$$V_{\mathrm{em}} = \frac{1}{2}(\omega B - \gamma\theta)a^2.$$

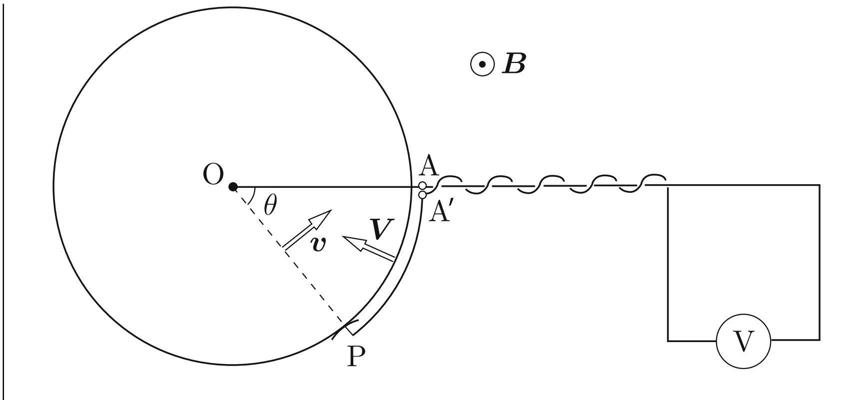


Fig. 10.11 Arrangement of plate and potential leads for measurement

Exercises

10.1. AC current $I(t) = I_{\mathrm{m}} \sin \omega t$ flows along a straight line. Calculate the electromotive force induced in a rectangular coil separated by d from the current (see Fig. E10.1).

10.2. A conducting bar that is in contact with two parallel lines shunted at the terminal is moving with constant velocity v, as shown in Fig. E10.2. A static magnetic flux density, B, is applied normal to the rectangular circuit. Determine the electromotive force induced in the rectangular circuit. The electromotive force is defined to be positive along the direction of PQRS, and the distance between PQ and SR is $b + vt$.

10.3. Determine the electromotive force induced in the rectangular circuit in Exercise 10.2 when the magnetic flux density changes with time as $B(t) = B_0 + \alpha t$. Use the general law, Eq. (10.23).

10.4. A rectangular coil is moving with constant velocity v on a horizontal plane of distance R_0 from a straight line carrying constant current I, as shown in Fig. E10.3. Calculate the electromotive force induced in this coil with the motional law. The electromotive force is defined to be positive along the direction of PQRS and $d = d_0 + vt$.

10.5. A constant current, I, is applied to a straight line and a rectangular coil is rotating with angular frequency ω ($\theta = \omega t$) around side RS parallel to the straight line, as shown in Fig. E10.4. Calculate the electromotive force induced in the coil with the magnetic flux law. The distance d is larger than a and the electromotive force is defined to be positive along the direction of PQRS.

10.6. Solve Exercise 10.5 with the motional law.

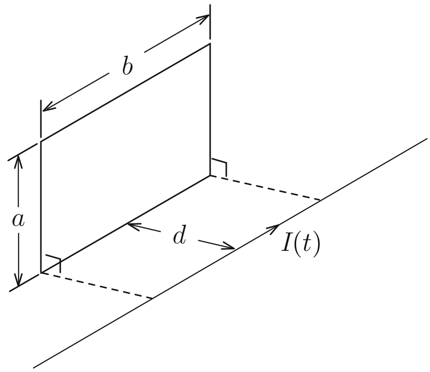


Fig. E10.1 Straight line carrying AC current and rectangular coil

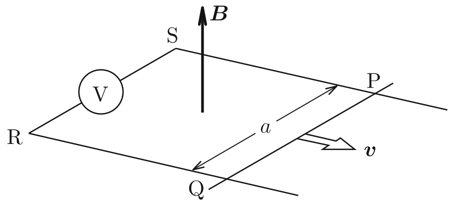


Fig. E10.2 Conducting bar moving with constant velocity on two parallel lines shunted at the terminal

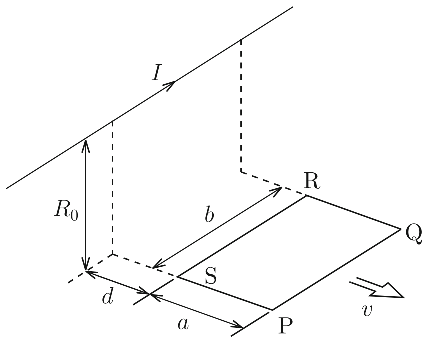


Fig. E10.3 Straight line carrying constant current and rectangular coil moving with constant velocity

10.7. Voltage V is applied to an electric circuit composed of a resistor of electric resistance R_r and a coil of inductance L at $t \geq 0$, as shown in Fig. E10.5. Derive the equation for the circuit and determine the current.

10.8. Suppose that a conductor carrying current $\boldsymbol{I}$ in magnetic flux density $\boldsymbol{B}$ is forced to move with velocity $\boldsymbol{v}$ by the Lorentz force. Thus, we may say that the Lorentz force does mechanical work. In this case the work in unit time done on electric charge by the induced electric field $\boldsymbol{v} \times \boldsymbol{B}$ is given by

$$\boldsymbol{I} \cdot (\boldsymbol{v} \times \boldsymbol{B}) = -(\boldsymbol{I} \times \boldsymbol{B}) \cdot \boldsymbol{v}.$$

Fig. E10.4 Straight line carrying constant current and rectangular coil rotating around side RS

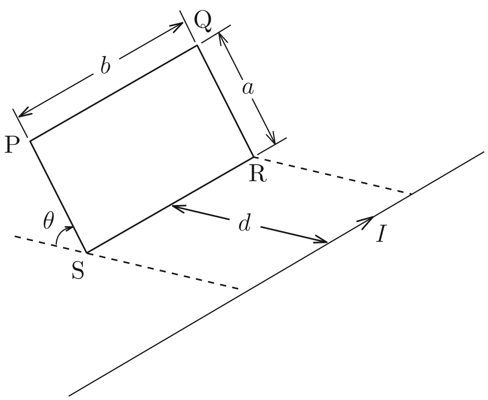
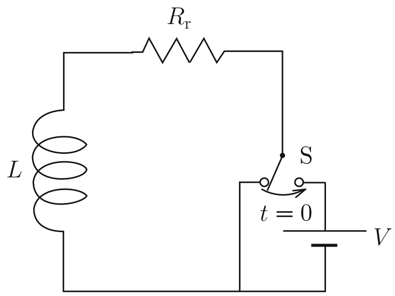


Fig. E10.5 Electric circuit composed of resistor and coil



This is equal to the negative of the work done by the Lorentz force in unit time, Eq. (10.16). Explain what it means. The fact that the Lorentz force does mechanical work seems to contradict the statement in Example 6.3. Discuss whether these are really contradictory.

10.9. Suppose that we apply an AC electric field of amplitude E_0 and angular frequency ω along the z-axis parallel to the surface of a semi-infinite conductor of electric conductivity σ_c that occupies $x \geq 0$. Determine the electric field and magnetic flux density in the conductor.

10.10. Suppose that we apply current I to an infinitely long cylindrical thin conductor of radius a. Calculate the magnetic energy in this condition from the work necessary to carry the current from the position at $R = R_\infty$ sufficiently far from the conductor. We assume that the return current flows uniformly at $R = R_\infty$.