Chapter 12
Electromagnetic Wave

12.1 Planar Electromagnetic Wave

Electromagnetic fields in a dielectric material follow the wave equation, Eq. (11.23), since there is no electric charge in the material. This chapter covers the property of electromagnetic fields described by this equation. For simplicity, we focus only on the electric field and assume that it has only a y-component varying only along the x-axis. Thus, Eq. (11.23) reduces to

$$\frac{\partial^2 E_y}{\partial x^2} - \mu\epsilon \frac{\partial^2 E_y}{\partial t^2} = 0. \tag{12.1}$$

We assume that E_y varies with time as $\mathrm{e}^{\mathrm{i}\omega t}$ with ω denoting the angular frequency. Equation (12.1) then leads to

$$\frac{\partial^2 E_y}{\partial x^2} + \epsilon\mu\omega^2 E_y = 0. \tag{12.2}$$

This equation is easily solved as

$$E_y = E_1 \mathrm{e}^{\mathrm{i}(\omega t + kx)} + E_2 \mathrm{e}^{\mathrm{i}(\omega t - kx)}, \tag{12.3}$$

where E_1 and E_2 are constants determined by initial and boundary conditions, and

$$k = (\epsilon\mu)^{1/2}\omega \equiv \frac{\omega}{c} \tag{12.4}$$

is the wave number. As will be shown later,

$$c = \left(\frac{1}{\epsilon\mu}\right)^{1/2} \tag{12.5}$$

T. Matsushita, *Electricity and Magnetism: New Formulation by Introduction of Superconductivity*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-4-431-54526-2_12,

is the speed of electromagnetic waves or the **light speed**, in the dielectric material. The wavelength is given by

$$\lambda = \frac{2\pi}{k}. \tag{12.6}$$

The practical electric field is given by the real part of the complex solution.

In the first term of Eq. (12.3),

$$\omega t + kx = \mathrm{const} \tag{12.7}$$

gives the position at which the phase of the wave is constant. Hence,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\omega}{k} = -c \tag{12.8}$$

shows that the first term in Eq. (12.3) represents a wave that propagates with the velocity c along the negative x-axis. Namely, this wave is an **electromagnetic wave**. Similarly, the second term in Eq. (12.3) gives the electromagnetic wave propagating along the positive x-axis. Such an electromagnetic wave, whose same phase is on a plane as in Eq. (12.3), is generally called a **plane wave**.

Since the left side of Eq. (11.7) is given by $\boldsymbol{i}_z \partial E_y / \partial x$, the magnetic flux density has only a z-component. Assuming the same time-dependent factor, the magnetic flux density is given by

$$\begin{aligned} B_z &= -B_1 \mathrm{e}^{\mathrm{i}(\omega t + kx)} + B_2 \mathrm{e}^{\mathrm{i}(\omega t - kx)}, \\ &= -\frac{1}{c} E_1 \mathrm{e}^{\mathrm{i}(\omega t + kx)} + \frac{1}{c} E_2 \mathrm{e}^{\mathrm{i}(\omega t - kx)}. \end{aligned} \tag{12.9}$$

The first and second terms in this equation correspond to the first and second terms in Eq. (12.3), and represent electromagnetic waves propagating along the negative and positive x-axis, respectively. Thus, the electric field and magnetic flux density coexist in electromagnetic waves. That is, the time variation in a magnetic flux density induces an electric field and the time variation in the produced electric field induces again a magnetic flux density; the process is repeated and the variation propagates as a wave. Since current does not flow, there is no energy dissipation and the electromagnetic wave does not decay with time.

We can also easily show that, if the electric field has only a z-component, the magnetic flux density has only a y-component, similarly to the above case. Hence, the planar electromagnetic wave is a **transverse wave** in which the electric field and magnetic flux density are perpendicular to each other and directed perpendicularly to the propagation direction. The ratio of these amplitudes is

$$\frac{E_1}{B_1} = \frac{E_2}{B_2} = c. \tag{12.10}$$

The magnetic field has been commonly used instead of the magnetic flux density to describe electromagnetic waves. In this case, the following equation is used:

$$H_z = -H_1 e^{i(\omega t + kx)} + H_2 e^{i(\omega t - kx)}. \tag{12.11}$$

The ratio of the amplitudes,

$$\frac{E_1}{H_1} = \frac{E_2}{H_2} = \mu c = \left(\frac{\mu}{\epsilon}\right)^{1/2} \equiv Z, \tag{12.12}$$

is the **characteristic impedance** or **wave impedance**. This value is inherent to each medium and its unit is [Ω].

The planar electromagnetic wave is generally expressed in the form

$$\exp[i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})]. \tag{12.13}$$

Using the unit vector along the propagation direction (the wave number vector $\boldsymbol{i}_k = \boldsymbol{k}/|\boldsymbol{k}|$), the relationship between the electric field and magnetic field is expressed as

$$\boldsymbol{E} = Z(\boldsymbol{H} \times \boldsymbol{i}_k), \qquad \boldsymbol{H} = Z^{-1}(\boldsymbol{i}_k \times \boldsymbol{E}). \tag{12.14}$$

In other words, the propagation direction of electromagnetic waves coincides with that of the Poynting vector representing the direction of the energy flow. For example, the first terms in Eqs. (12.3) and (12.11) correspond to each other, and since the electric field and magnetic field are directed along the positive y- and negative z-axes, respectively, the Poynting vector is directed along the negative x-axis. Thus, the above relation holds. For the second terms in these equations, a similar relation holds.

In vacuum the speed of the electromagnetic wave is

$$c_0 = \frac{1}{(\epsilon_0 \mu_0)^{1/2}} = 2.997925 \times 10^8 \text{ m/s}, \tag{12.15}$$

and the characteristic impedance is

$$Z_0 = \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2} = 376.730\,\Omega. \tag{12.16}$$

The electromagnetic wave is generally classified depending on its wavelength. Table 12.1 shows the classifications, although the classification ranges somewhat overlap.

Table 12.1 Classification of electromagnetic waves

Name	Wavelength
Electromagnetic wave	Above 10^{-1} mm
Infrared ray	1 mm to 0.76 μm
Visible ray	0.76–0.38 μm
Ultraviolet ray	0.38 μm to 1 nm
X ray	Several 10–10^{-3} nm
γ ray	Below 10^{-1} nm

Example 12.1. Prove that there are no components of the electric field or magnetic flux density parallel to the propagation direction of a planar electromagnetic wave.

Solution 12.1. Assume a planar electromagnetic wave propagating along the positive x-axis. The functional form of variations with respect to time and space for the electric field and magnetic flux density is $e^{i(\omega t - kx)}$. Hence, from Eq. (11.7) the x-component of the magnetic flux density is

$$-i\omega B_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0,$$

and from Eq. (11.8), the x-component of the electric field is

$$i\omega\epsilon E_x = \frac{1}{\mu}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) = 0.$$

Thus, we prove that there are no components along the propagation direction for the electric field or magnetic flux density.

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Equations (12.3) and (12.9) deal with the case where each of the electric field and magnetic flux density remains in its own direction perpendicular to the other. Such a direction is called the direction of polarization. A polarization whose direction is fixed is referred to as **linear polarization**. In general, a superposition of components is possible, such as

$$E_y = E_1 \cos(\omega t - kx), \qquad E_z = E_2 \cos(\omega t - kx + \delta). \tag{12.17}$$

The corresponding components of the magnetic flux density are

$$B_y = -\frac{E_2}{c}\cos(\omega t - kx + \delta) = -\frac{1}{c}E_z, \qquad B_z = \frac{E_1}{c}\cos(\omega t - kx) = \frac{1}{c}E_y. \tag{12.18}$$

Usually, the direction of polarization is designated as that of the electric field, $\boldsymbol{E}$. Eliminating t in the above equations, we obtain the relationship between E_y and E_z as

$$\left(\frac{E_y}{E_1}\right)^2 - 2\cos\delta\,\frac{E_y E_z}{E_1 E_2} + \left(\frac{E_z}{E_2}\right)^2 = \sin^2\delta. \tag{12.19}$$

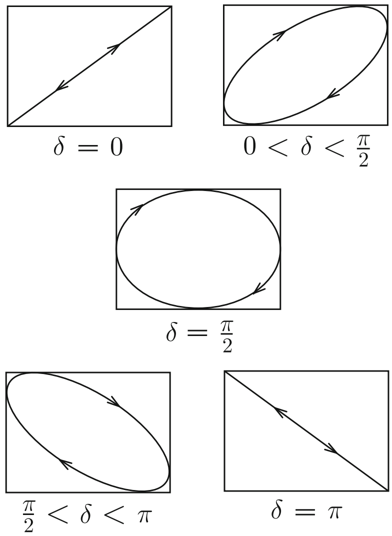

Fig. 12.1 Linear polarization and right-hand elliptical polarization: the abscissa and ordinate are respectively E_y and E_z

In the range $0 < \delta < \pi$, the direction of polarization turns to the right (clockwise) from the view of an observer directed along the propagation of the electromagnetic wave. This is called right-hand polarization (see Fig. 12.1). In the range $-\pi < \delta < 0$, the direction of polarization turns to the left (counter-clockwise). When $\delta = 0$ or π, the electric field $\boldsymbol{E}$ is fixed in one direction and is linearly polarized. When $\delta = \pm\pi/2$ and $E_1 = E_2$, the trace of $\boldsymbol{E}$ is a circle and this polarization is **circular polarization**. In other cases, the trace is an ellipse, and the polarization is **elliptical polarization**. Such phenomena that the directions of $\boldsymbol{E}$ and $\boldsymbol{B}$ in the electromagnetic wave are not uniform but biased are referred to as the **polarization of a wave**, and such a wave is called a **polarized wave**.

12.2 Reflection and Refraction of the Planar Electromagnetic Wave

Electromagnetic wave is a family of waves that includes visible light. Reflection and refraction are well known processes for light in optics. Here, we investigate reflection and refraction using electromagnetism.

Media 1 and 2 with dielectric constants ϵ_1 and ϵ_2 and magnetic permeabilities μ_1 and μ_2 face each other on the plane $z = 0$, as shown in Fig. 12.2a. Suppose that a planar electromagnetic wave propagates from medium 1 to the boundary. The plane formed by the propagation direction and the direction (z-axis) normal to the boundary is called a plane of incidence. We define the x-axis on the line on which the plane of incidence and the boundary meet and the y-axis on the boundary in such a way that it is normal to both the x- and z-axes.

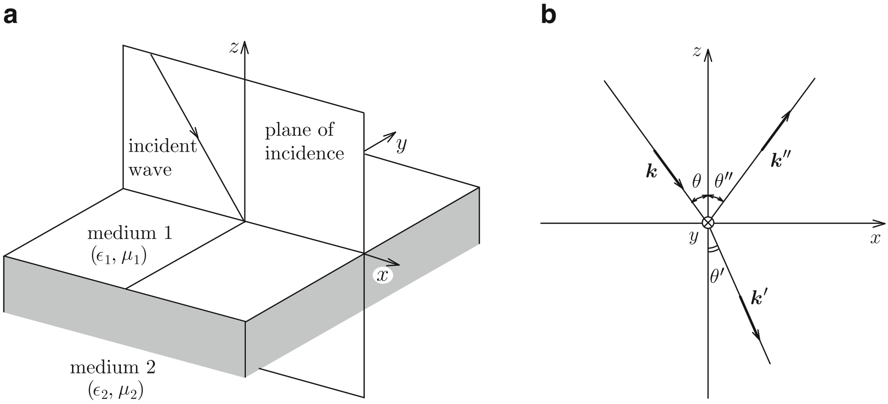

Fig. 12.2 Definition of (**a**) axes on the boundary and (**b**) angles of waves

In this case, the incident wave and reflected wave remain in medium 1, and the transmitted wave remains in medium 2. We use $\boldsymbol{k}$, $\boldsymbol{k}''$ and $\boldsymbol{k}'$ to denote the wave number vectors of the incident, reflected and transmitted waves, respectively. Each wave propagates along the wave number vector. These vectors lie on the plane of incidence, the x-z plane. We denote the angles of the incident, reflected and transmitted waves from the z-axis by θ, θ'' and θ', respectively (see Fig. 12.2b). The factor that represents the variation with time is commonly given by $e^{i\omega t}$. The incident, reflected and transmitted waves are then expressed as

$$\exp[i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})], \qquad \exp[i(\omega t - \boldsymbol{k}''\cdot\boldsymbol{r})], \qquad \exp[i(\omega t - \boldsymbol{k}'\cdot\boldsymbol{r})],$$

with $\boldsymbol{r}$ representing the position vector.

The electric field $\boldsymbol{E}$ and magnetic flux density $\boldsymbol{B}$ are perpendicular to each other and lie on the plane normal to the propagation direction, as shown in the last section. For example, $\boldsymbol{E}$ and $\boldsymbol{B}$ for the incident wave are normal to $\boldsymbol{k}$. Here we consider the boundary conditions for $\boldsymbol{E}$ and $\boldsymbol{B}$: Eqs. (4.19), (10.27), (9.22) and (11.25). When there is neither electric charge nor current on the boundary, the parallel components of the electric field and magnetic field are continuous, and the normal components of the electric flux density and magnetic flux density are continuous across the boundary. That is,

$$\boldsymbol{n} \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0, \tag{12.20}$$

$$\boldsymbol{n} \cdot (\epsilon_1 \boldsymbol{E}_1 - \epsilon_2 \boldsymbol{E}_2) = 0, \tag{12.21}$$

$$\boldsymbol{n} \times \left(\frac{\boldsymbol{B}_1}{\mu_1} - \frac{\boldsymbol{B}_2}{\mu_2} \right) = 0, \tag{12.22}$$

$$\boldsymbol{n} \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0, \tag{12.23}$$

with $\boldsymbol{n}$ denoting the vector normal to the boundary. In the above, the subscripts 1 and 2 represent the variables in media 1 and 2, respectively.

Considering the orthogonality between $\boldsymbol{E}$ and $\boldsymbol{B}$, the incident wave is given by

$$\boldsymbol{E} = \boldsymbol{E}_0 \exp[\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})], \tag{12.24a}$$

$$\boldsymbol{B} = \frac{\boldsymbol{k}}{k} \times \frac{\boldsymbol{E}_0}{c_1} \exp[\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})], \tag{12.24b}$$

where $k = |\boldsymbol{k}|$ and c_1 is the light speed in medium 1. The magnetic flux density $\boldsymbol{B}$ is normal to both $\boldsymbol{E}$ and $\boldsymbol{k}$, and its magnitude is equal to the magnitude of $\boldsymbol{E}$ divided by the corresponding light speed. The reflected wave is similarly given by

$$\boldsymbol{E}'' = \boldsymbol{E}_0'' \exp[\mathrm{i}(\omega t - \boldsymbol{k}''\cdot\boldsymbol{r})], \tag{12.25a}$$

$$\boldsymbol{B}'' = \frac{\boldsymbol{k}''}{k''} \times \frac{\boldsymbol{E}_0''}{c_1} \exp[\mathrm{i}(\omega t - \boldsymbol{k}''\cdot\boldsymbol{r})], \tag{12.25b}$$

and the transmitted wave is given by

$$\boldsymbol{E}' = \boldsymbol{E}_0' \exp[\mathrm{i}(\omega t - \boldsymbol{k}'\cdot\boldsymbol{r})], \tag{12.26a}$$

$$\boldsymbol{B}' = \frac{\boldsymbol{k}'}{k'} \times \frac{\boldsymbol{E}_0'}{c_2} \exp[\mathrm{i}(\omega t - \boldsymbol{k}'\cdot\boldsymbol{r})]. \tag{12.26b}$$

In the above, $k'' = |\boldsymbol{k}''|$, $k' = |\boldsymbol{k}'|$ and c_2 is the light speed in medium 2. Thus, the electric field and magnetic flux density in medium 1 are

$$\boldsymbol{E}_1 = \boldsymbol{E} + \boldsymbol{E}'', \qquad \boldsymbol{B}_1 = \boldsymbol{B} + \boldsymbol{B}'', \tag{12.27}$$

and those in medium 2 are

$$\boldsymbol{E}_2 = \boldsymbol{E}', \qquad \boldsymbol{B}_2 = \boldsymbol{B}'. \tag{12.28}$$

Since Eqs. (12.20)–(12.23) should be satisfied at the boundary ($z = 0$) at any time, the phase must be the same for the three waves. This condition is given by

$$\boldsymbol{k}\cdot\boldsymbol{r}|_{z=0} = \boldsymbol{k}''\cdot\boldsymbol{r}|_{z=0} = \boldsymbol{k}'\cdot\boldsymbol{r}|_{z=0}. \tag{12.29}$$

Equation (12.29) is expressed as

$$\boldsymbol{k}\cdot\boldsymbol{r}_0 = \boldsymbol{k}''\cdot\boldsymbol{r}_0 = \boldsymbol{k}'\cdot\boldsymbol{r}_0 \tag{12.30}$$

in terms of an arbitrary position vector $\boldsymbol{r}_0$ on the boundary. If $\boldsymbol{r}_0$ is given by

$$\boldsymbol{r}_0 = x\boldsymbol{i}_x + y\boldsymbol{i}_y, \tag{12.31}$$

we have

$$k \sin\theta = k'' \sin\theta'' = k' \sin\theta', \tag{12.32}$$

since the wave number vectors are perpendicular to the y-axis. The speeds of the incident and reflected waves in the same medium are the same and the wave numbers of these waves are also the same $k = k''$. Hence, we have

$$\theta = \theta''. \tag{12.33}$$

That is, the incident and reflection angles are the same, and this is called the **law of reflection**. We also have the relationship between the incident and transmission angles as

$$\frac{\sin\theta}{\sin\theta'} = \frac{k'}{k} = \frac{c_1}{c_2} = \left(\frac{\epsilon_2\mu_2}{\epsilon_1\mu_1}\right)^{1/2}. \tag{12.34}$$

This is called **Snell's law** for refraction.

Using the above results, Eqs. (12.20)–(12.23) are rewritten as

$$\boldsymbol{n} \times (\boldsymbol{E}_0 + \boldsymbol{E}''_0 - \boldsymbol{E}'_0) = 0, \tag{12.35}$$

$$\boldsymbol{n} \cdot [\epsilon_1(\boldsymbol{E}_0 + \boldsymbol{E}''_0) - \epsilon_2\boldsymbol{E}'_0] = 0, \tag{12.36}$$

$$\boldsymbol{n} \times \left[\frac{1}{\mu_1}\left(\frac{\boldsymbol{k}\times\boldsymbol{E}_0}{kc_1} + \frac{\boldsymbol{k}''\times\boldsymbol{E}''_0}{k''c_1}\right) - \frac{1}{\mu_2}\cdot\frac{\boldsymbol{k}'\times\boldsymbol{E}'_0}{k'c_2}\right] = 0, \tag{12.37}$$

$$\boldsymbol{n} \cdot \left(\frac{\boldsymbol{k}\times\boldsymbol{E}_0}{kc_1} + \frac{\boldsymbol{k}''\times\boldsymbol{E}''_0}{k''c_1} - \frac{\boldsymbol{k}'\times\boldsymbol{E}'_0}{k'c_2}\right) = 0. \tag{12.38}$$

Although the electric field of the incident wave is directed in various directions, we focus for simplicity on the case where the electric field is directed parallel to the y-axis (i.e., normal to the plane of incidence) as shown in Fig. 12.3. In this case, Eq. (12.35) reduces to

$$E_0 + E''_0 - E'_0 = 0. \tag{12.39}$$

Since the electric field is perpendicular to the normal vector, $\boldsymbol{n}$, Eq. (12.36) is already satisfied. Equation (12.37) becomes

$$\left(\frac{\epsilon_1}{\mu_1}\right)^{1/2}(E_0 - E''_0)\cos\theta - \left(\frac{\epsilon_2}{\mu_2}\right)^{1/2} E'_0\cos\theta' = 0. \tag{12.40}$$

Equation (12.38) is written as

$$\frac{E_0}{c_1}\sin\theta + \frac{E''_0}{c_1}\sin\theta - \frac{E'_0}{c_2}\sin\theta' = 0, \tag{12.41}$$

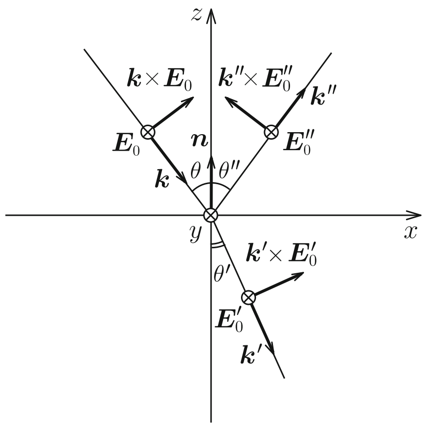

Fig. 12.3 Case where the electric field of the incident wave is normal to the plane of incidence, i.e., parallel to the y-axis

which is found to reduce to Eq. (12.39) using Eq. (12.34). Thus, from Eqs. (12.39) and (12.40), we obtain the amplitudes of the electric fields of the refracted and reflected waves as

$$E_0' = \frac{2(\epsilon_1/\mu_1)^{1/2}\cos\theta}{(\epsilon_1/\mu_1)^{1/2}\cos\theta + (\epsilon_2/\mu_2)^{1/2}\cos\theta'} E_0, \tag{12.42a}$$

$$E_0'' = \frac{(\epsilon_1/\mu_1)^{1/2}\cos\theta - (\epsilon_2/\mu_2)^{1/2}\cos\theta'}{(\epsilon_1/\mu_1)^{1/2}\cos\theta + (\epsilon_2/\mu_2)^{1/2}\cos\theta'} E_0. \tag{12.42b}$$

The amplitudes of magnetic flux densities are

$$B_0' = \frac{E_0'}{c_2} = (\epsilon_2\mu_2)^{1/2} E_0', \tag{12.43a}$$

$$B_0'' = \frac{E_0''}{c_1} = (\epsilon_1\mu_1)^{1/2} E_0''. \tag{12.43b}$$

Example 12.2. Solve Eqs. (12.35)–(12.38) when the electric field of the incident wave is parallel to the plane of incidence, i.e., the magnetic flux density is parallel to the y-axis.

Solution 12.2. Under this condition, $(\boldsymbol{k} \times \boldsymbol{E}_0)$, $(\boldsymbol{k}' \times \boldsymbol{E}_0')$ and $(\boldsymbol{k}'' \times \boldsymbol{E}_0'')$ are directed along the y-axis, as shown in Fig. 12.4. Hence, Eqs. (12.35) and (12.36) become

$$(E_0 - E_0'')\cos\theta - E_0'\cos\theta' = 0,$$
$$\epsilon_1(E_0 + E_0'')\sin\theta - \epsilon_2 E_0'\sin\theta' = 0,$$

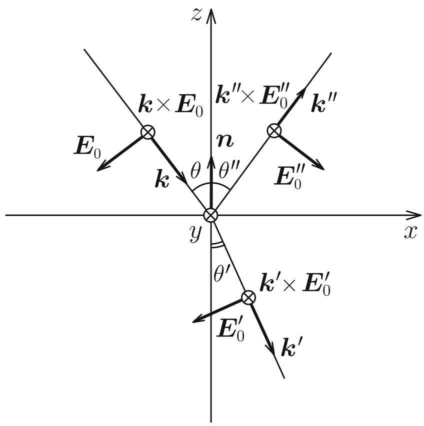

Fig. 12.4 Case where the electric field of the incident wave is parallel to the plane of incidence, i.e., normal to the y-axis

respectively. The latter equation is rewritten as

$$\left(\frac{\epsilon_1}{\mu_1}\right)^{1/2}(E_0 + E_0'') - \left(\frac{\epsilon_2}{\mu_2}\right)^{1/2} E_0' = 0.$$

In Eq. (12.37), $\boldsymbol{n} \times (\boldsymbol{k} \times \boldsymbol{E})$ is parallel to the x-axis, and this equation agrees with the above result from Eq. (12.36). The condition given by Eq. (12.38) is satisfied. From the above two equations, we obtain the electric fields of the refracted and reflected waves as

$$E_0' = \frac{2(\epsilon_1/\mu_1)^{1/2}\cos\theta}{(\epsilon_2/\mu_2)^{1/2}\cos\theta + (\epsilon_1/\mu_1)^{1/2}\cos\theta'} E_0,$$

$$E_0'' = \frac{(\epsilon_2/\mu_2)^{1/2}\cos\theta - (\epsilon_1/\mu_1)^{1/2}\cos\theta'}{(\epsilon_2/\mu_2)^{1/2}\cos\theta + (\epsilon_1/\mu_1)^{1/2}\cos\theta'} E_0.$$

The corresponding magnetic flux densities are derived by substituting these results into Eqs. (12.43a) and (12.43b).

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12.3 Energy of the Electromagnetic Wave

Here we discuss the energy of the planar electromagnetic wave described in Sect. 12.1. For simplicity we treat the second terms in Eqs. (12.3) and (12.9). In this case, the electric field and magnetic flux density are given by

$$\boldsymbol{E} = E_2 \cos(\omega t - kx)\boldsymbol{i}_y, \tag{12.44}$$

$$\boldsymbol{B} = \frac{E_2}{c}\cos(\omega t - kx)\boldsymbol{i}_z. \tag{12.45}$$

Hence, the electric energy density and magnetic energy density are equal to each other and given by

$$\frac{1}{2}\epsilon \boldsymbol{E}^2 = \frac{1}{2\mu}\boldsymbol{B}^2 = \frac{1}{2}\epsilon E_2^2 \cos^2(\omega t - kx). \tag{12.46}$$

Since there is no current, from Eq. (11.34) the total energy density is

$$u = \epsilon E_2^2 \cos^2(\omega t - kx). \tag{12.47}$$

On the other hand, from Eq. (11.37) the Poynting vector is

$$\boldsymbol{S}_{\mathrm{P}} = \boldsymbol{E} \times \frac{\boldsymbol{B}}{\mu} = \left(\frac{\epsilon}{\mu}\right)^{1/2} E_2^2 \cos^2(\omega t - kx)\boldsymbol{i}_x = cu\boldsymbol{i}_x. \tag{12.48}$$

Thus, we find that the Poynting vector has a magnitude equal to the total energy density multiplied by the light speed and is directed along the x-axis, i.e., the propagation direction of the electromagnetic wave. This holds for all planar electromagnetic waves including the elliptically polarized wave. Hence, the Poynting vector expresses the energy that flows through a unit area in unit time as defined in Sect. 11.5.

12.4 Wave Guide

Hollow metal tubes called **wave guides** are used to transmit electromagnetic waves such as microwaves. The cross-section of a wave guide is usually rectangular or circular. Here we treat a rectangular wave guide for simplicity. Assume that the wave guide is uniformly extended along the z-axis and the internal vacuum region is $0 \leq x \leq a$ and $0 \leq y \leq b$, as shown in Fig. 12.5.

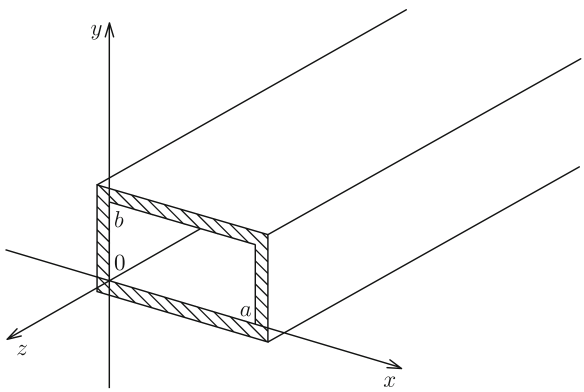

Fig. 12.5 Rectangular wave guide

We assume that the factors for time variation and spatial variation are given by $\mathrm{e}^{\mathrm{i}\omega t}$ and $\mathrm{e}^{-\mathrm{i}\gamma z}$, respectively. Equations (11.7) and (11.8) then reduce to

$$\frac{\partial E_z}{\partial y} + \mathrm{i}\gamma E_y = -\mathrm{i}\omega B_x, \tag{12.49a}$$

$$-\mathrm{i}\gamma E_x - \frac{\partial E_z}{\partial x} = -\mathrm{i}\omega B_y, \tag{12.49b}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mathrm{i}\omega B_z, \tag{12.49c}$$

and

$$\frac{\partial B_z}{\partial y} + \mathrm{i}\gamma B_y = \mathrm{i}\frac{\omega}{c_0^2} E_x, \tag{12.50a}$$

$$-\mathrm{i}\gamma B_x - \frac{\partial B_z}{\partial x} = \mathrm{i}\frac{\omega}{c_0^2} E_y, \tag{12.50b}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mathrm{i}\frac{\omega}{c_0^2} E_z. \tag{12.50c}$$

Using these equations, the equations for the z-components, E_z and B_z, are obtained as

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k^2 E_z = 0, \tag{12.51a}$$

$$\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + k^2 B_z = 0. \tag{12.51b}$$

If these equations can be solved, we obtain other components from

$$E_x = -\frac{\mathrm{i}}{k^2}\left(\gamma \frac{\partial E_z}{\partial x} + \omega \frac{\partial B_z}{\partial y}\right), \tag{12.52a}$$

$$E_y = -\frac{\mathrm{i}}{k^2}\left(\gamma \frac{\partial E_z}{\partial y} - \omega \frac{\partial B_z}{\partial x}\right), \tag{12.52b}$$

$$B_x = \frac{\mathrm{i}}{k^2}\left(\frac{\omega}{c_0^2} \cdot \frac{\partial E_z}{\partial y} - \gamma \frac{\partial B_z}{\partial x}\right), \tag{12.52c}$$

$$B_y = -\frac{\mathrm{i}}{k^2}\left(\frac{\omega}{c_0^2} \cdot \frac{\partial E_z}{\partial x} + \gamma \frac{\partial B_z}{\partial y}\right), \tag{12.52d}$$

where

$$k^2 = \left(\frac{\omega}{c_0}\right)^2 - \gamma^2, \tag{12.53}$$

and $\nabla \cdot \boldsymbol{E} = 0$ is used. Equations (12.52a)–(12.52d) hold for $k \neq 0$.

In the case of $k = 0$, we have $\gamma = \pm\omega/c_0$, and we may consider that there is an electromagnetic wave propagating along the z-axis at light speed. For example, we assume an electromagnetic wave without z-components ($E_z = B_z = 0$) similar to a planar electromagnetic wave. This is called the **transverse electromagnetic (TEM) wave**. If we choose $\gamma = \omega/c_0$, Eqs. (12.49a) and (12.49b) lead to

$$E_x = c_0 B_y, \qquad E_y = -c_0 B_x, \tag{12.54}$$

showing that the electric field $\boldsymbol{E}$ and magnetic flux density $\boldsymbol{B}$ are perpendicular to each other. However, Eq. (12.49c) gives

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0. \tag{12.55}$$

This shows that the electric field is a dynamic two-dimensional field with no rotation. The spatial structure of the irrotational field is the same as that of the electrostatic field, and hence, we conclude that such an electric field cannot exist in a space surrounded by a conductor like a rectangular wave guide. That is, TEM wave cannot exist in simple rectangular or circular wave guides. Such a field can exist only when the guide is composed of two or more conductors like those in Fig. 12.6, and a potential difference can appear between conductors with electric field lines extending from one conductor to another.

From the above discussion, we know that either the electric field $\boldsymbol{E}$ or magnetic flux density $\boldsymbol{B}$ has at least one component in the propagation direction. The electromagnetic wave with a zero longitudinal component of the electric field is called the **transverse electric (TE) wave** and that with a zero longitudinal component of the magnetic flux density is called the **transverse magnetic (TM) wave**. The general electromagnetic wave is given by a linear combination of these waves.

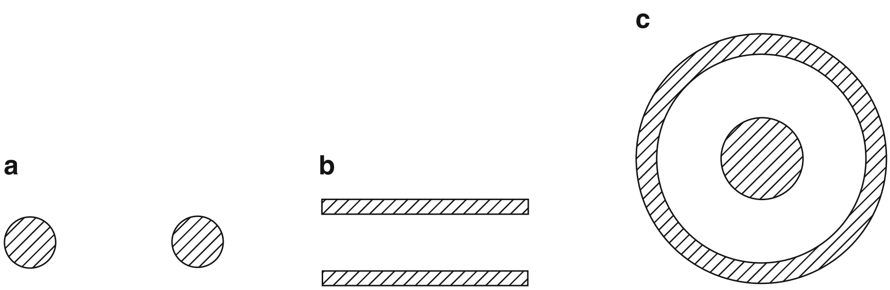

Fig. 12.6 Examples of the cross-section of a wave guide in which a TEM wave exists: (**a**) parallel cylindrical conductor, (**b**) parallel-plate conductor and (**c**) coaxial conductor

Here we consider a TM wave. In this case $B_z = 0$. Since any electromagnetic wave of high frequency does not penetrate the conductor, the electric field is perpendicular to and the magnetic flux density is parallel to the conductor surface. That is,

$$\begin{aligned} E_y = E_z = B_x = 0; \qquad & x = 0,\ a, \\ E_x = E_z = B_y = 0; \qquad & y = 0,\ b. \end{aligned} \tag{12.56}$$

The general solution of Eq. (12.51a) is given by

$$E_z(x, y, z, t) = K \exp[\pm \mathrm{i}(k_x x + k_y y)] \exp[\mathrm{i}(\omega t - \gamma z)] \tag{12.57}$$

with

$$k_x^2 + k_y^2 = k^2. \tag{12.58}$$

The dependence on x can be written as

$$E_z = K_1 \mathrm{e}^{\mathrm{i}k_x x} + K_2 \mathrm{e}^{-\mathrm{i}k_x x}. \tag{12.59}$$

From Eq. (12.56), the following conditions should be satisfied:

$$K_1 + K_2 = 0, \qquad \mathrm{e}^{\mathrm{i}2k_x a} = 1. \tag{12.60}$$

The latter condition gives

$$k_x = \frac{m\pi}{a}; \qquad m = 1, 2, \cdots. \tag{12.61}$$

The case of $m = 0$ also satisfies this condition. However, we have $E_z = 0$, which is meaningless. Thus, Eq. (12.59) reduces to

$$E_z = 2K_1' \sin\left(\frac{m\pi x}{a}\right) \tag{12.62}$$

with $K_1' = \mathrm{i}K_1$. We similarly obtain the y-dependence with $k_y = n\pi/b$ ($n = 1, 2, \cdots$), and Eq. (12.57) is rewritten as

$$E_z(x, y, z, t) = A \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \exp[\mathrm{i}(\omega t - \gamma z)], \tag{12.63}$$

where A is a constant, and there is a relationship between m and n written as

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \left(\frac{\omega}{c_0}\right)^2 - \gamma^2. \tag{12.64}$$

The mode of the electromagnetic wave is different depending on the set of integers, (m, n), and each mode of the TM wave is expressed as TM_{mn}. The mode of the TE wave is represented similarly as TE_{mn}.

So that the TM wave propagates through the wave guide along the z-axis without damping, γ must be a real number and we have

$$\left(\frac{\omega}{c_0}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 \geq 0. \tag{12.65}$$

That is, the angular frequency ω should be larger than the **cut-off frequency** given by

$$\omega_0 = c_0 \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}. \tag{12.66}$$

Example 12.3. Determine other components of electromagnetic fields of the above TM wave.

Solution 12.3. Substituting E_z in Eq. (12.63) and $B_z = 0$ into Eqs. (12.52a)–(12.52d), we have

$$E_x = -\mathrm{i}A\frac{m\pi\gamma}{k^2 a}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),$$
$$E_y = -\mathrm{i}A\frac{n\pi\gamma}{k^2 b}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$
$$B_x = \mathrm{i}A\frac{n\pi\omega}{k^2 c_0^2 b}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$
$$B_y = -\mathrm{i}A\frac{m\pi\omega}{k^2 c_0^2 a}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),$$

where the factor $\exp[\mathrm{i}(\omega t - \gamma z)]$ is neglected. The real parts of these expressions give practical physical quantities. That is, the above factor is replaced by $\cos(\omega t - \gamma z)$ for E_z, and $\mathrm{i}\exp[\mathrm{i}(\omega t - \gamma z)]$ is replaced by $\sin(\omega t - \gamma z)$ for the above quantities. We easily find that $\boldsymbol{E} \cdot \boldsymbol{B} = 0$ is satisfied, indicating that the electric field and magnetic flux density are perpendicular to each other.

◇

12.5 Spherical Wave

Here we consider the case in which an electromagnetic wave propagates radially. This wave is a **spherical wave**. We assume that the factor of time variation is given by $\mathrm{e}^{\mathrm{i}\omega t}$. Then, from Eq. (11.23), we write the equation for the electric field as

$$\Delta\boldsymbol{E} + \frac{\omega^2}{c^2}\boldsymbol{E} = 0. \tag{12.67}$$

We assume spherical symmetry of the electromagnetic quantities except in the vicinity of the source of the electromagnetic wave. We also assume that the electric field has only the zenithal component, E_θ, dependent only on the radius r:

$$E_\theta(r,t) = E_\theta(r)\mathrm{e}^{\mathrm{i}\omega t}. \tag{12.68}$$

Then, the above equation leads to

$$\frac{1}{r}\cdot\frac{\partial^2}{\partial r^2}(rE_\theta) + \left(\frac{\omega}{c}\right)^2 E_\theta = 0, \tag{12.69}$$

and we obtain a general solution as

$$E_\theta(r,t) = \frac{K_1}{r}\exp\left[\mathrm{i}\omega\left(t+\frac{r}{c}\right)\right] + \frac{K_2}{r}\exp\left[\mathrm{i}\omega\left(t-\frac{r}{c}\right)\right]. \tag{12.70}$$

The first term represents an electromagnetic wave propagating to the origin at speed c, but such a wave concentrating to one point is unrealistic. On the other hand, the second term represents an electromagnetic wave radiating at speed c from the origin, and this is the solution to be obtained. Taking the real part, we have

$$E_\theta(r,t) = \frac{K_2}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]. \tag{12.71}$$

Equation (11.7) shows that the magnetic flux density has only the azimuthal component, B_φ, and reduces to

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}(rE_\theta) = -\frac{\partial B_\varphi}{\partial t}. \tag{12.72}$$

This is easily solved, and we have

$$B_\varphi(r,t) = \frac{K_2}{cr}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]. \tag{12.73}$$

From the above results, we obtain the Poynting vector as

$$\boldsymbol{S}_\mathrm{P} = \frac{1}{\mu}E_\theta B_\varphi \boldsymbol{i}_r = \left(\frac{\epsilon}{\mu}\right)^{1/2}\frac{K_2^2}{r^2}\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\boldsymbol{i}_r, \tag{12.74}$$

showing that it is directed radially. The energy density is

$$u = \frac{1}{2}\epsilon E_\theta^2 + \frac{1}{2\mu}B_\varphi^2 = \epsilon\,\frac{K_2^2}{r^2}\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right] = \frac{1}{c}\boldsymbol{S}_\mathrm{P}\cdot\boldsymbol{i}_r. \tag{12.75}$$

This relationship is similar to that for a planar electromagnetic wave. The energy density and Poynting vector are proportional to r^{-2} because the wave front expands and yield constants when integrated on the spherical surface.

12.6 Retarded Potential

When electric charge (of density ρ) or current (of density $\boldsymbol{i}$) changes with time, an electromagnetic wave is emitted into the surrounding space. At a distance far from the source, the electromagnetic wave propagates as a spherical wave as treated in Sect. 12.5. The electromagnetic potential describing such time-dependent electromagnetic fields is given by Eqs. (11.31) and (11.32). In the static limit, these reduce to Eqs. (4.17) and (9.21), and the solutions are given by Eqs. (1.27) and (6.33) when ϵ_0 and μ_0 are replaced with ϵ and μ.

Using polar coordinates, Eq. (11.31) for the electric potential reduces to

$$\frac{1}{r} \cdot \frac{\partial^2}{\partial r^2}(r\phi) - \frac{1}{c^2} \cdot \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon}. \tag{12.76}$$

We denote by $\phi(r) = f(t)/r$ the solution for the corresponding quasi-static equation, i.e., the equation with the second term on the left side omitted. Referring to the solution of Eq. (12.70) for Eq. (11.23) of the same form, we can prove that the general solution of Eq. (12.76) is given by

$$\phi(r, t) = \frac{1}{r} f\left(t - \frac{r}{c}\right). \tag{12.77}$$

Hence, we expect that the solution of Eq. (11.31) is given by

$$\phi(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon} \int_{\mathrm{V}} \frac{\rho(\boldsymbol{r}', t - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}V' \tag{12.78}$$

(see Exercise 12.8). We also obtain the vector potential as

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu}{4\pi} \int_{\mathrm{V}} \frac{\boldsymbol{i}(\boldsymbol{r}', t - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}V'. \tag{12.79}$$

The above results show that, since the speed of propagation of a variation in the fields due to the change in the source is finite (light speed c), the change is transmitted with a delay in time of R/c at distance $R = |\boldsymbol{r} - \boldsymbol{r}'|$. For this reason, the above potentials are called **retarded potentials**. When the variation with time is slow, we can neglect this delay, resulting in the static potentials given by Eqs. (1.27) and (6.33).

Column: To the Theory of Relativity

Newton's equation of motion is unchanged between coordinate system K with spatial coordinates (x, y, z) and time t and another coordinate system K* with the dimensions

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t.$$

The coordinate systems K and K* move with constant velocity relatively to each other and this transformation is called the Galilean transformation. That is, Newton's equation is unchanged under the Galilean transformation.

We assume that the equation for the electromagnetic potential [e.g., Eq. (11.31)] holds in coordinate system K:

$$\Delta\phi - \epsilon\mu\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon}.$$

Under the Galilean transformation, however, this equation does not hold in the same form in coordinate system K*:

$$\Delta'\phi' - \epsilon\mu\frac{\partial^2\phi'}{\partial t'^2} = -\frac{\rho}{\epsilon},$$

where Δ' represents the Laplacian with respect to x', y' and z'.

Since there is no rotation of space, the scalar potential ϕ' is equal to ϕ. From each relationship such as

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial x'} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial x'} + \frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial x'} + \frac{\partial}{\partial t}\cdot\frac{\partial t}{\partial x'} = \frac{\partial}{\partial x},$$

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial t'} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial t'} + \frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial t'} + \frac{\partial}{\partial t}\cdot\frac{\partial t}{\partial t'} = v\frac{\partial}{\partial x} + \frac{\partial}{\partial t},$$

we have

$$\Delta' = \Delta$$

and

$$\frac{\partial^2}{\partial t'^2} = v^2\frac{\partial^2}{\partial x^2} + 2v\frac{\partial^2}{\partial x\partial t} + \frac{\partial^2}{\partial t^2} \neq \frac{\partial^2}{\partial t^2}.$$

The above results indicate that Maxwell's equations hold correctly only in one coordinate system. However, it was experimentally demonstrated that Maxwell's equations hold in all coordinate systems that move relatively to each other with constant velocities. This means that the Galilean transformation is not a correct transformation connecting two coordinate systems. The correct transformation is the Lorentz transformation and, in this case, the time is no longer independent of the velocity of the coordinate system.

In 1905, Einstein proposed the special theory of relativity based on the principle of relativity that all coordinate systems are equivalent, and the principle of the constancy of light speed that the light speed observed in any coordinate system is the same.

Exercises

12.1. Discuss the reflection and refraction of a planar electromagnetic wave at the boundary when medium 1 is a vacuum and medium 2 is a conductor. Assume that the electric field of the incident wave is normal to the plane of incidence.

12.2. Discuss the same problem as in Exercise 12.1 when the electric field of the incident wave is parallel to the plane of incidence.

12.3. Discuss the energy flow using the Poynting vector for the reflection and refraction of a planar electromagnetic wave treated in Sect. 12.2. Assume that the electric field in the incident wave is normal to the place of incidence.

12.4. Discuss the energy flow for the TM_{mn} wave in the wave guide treated in Sect. 12.4 and Example 12.3.

12.5. Determine the electromagnetic fields of the TE wave in the rectangular wave guide in Fig. 12.5.

12.6. Determine the electric charge density and current density on the inner surface $x = 0$ of the rectangular wave guide for the TM wave discussed in Example 12.3. Discuss the relation between them.

12.7. Determine the electromagnetic fields of a TEM wave propagating along the length (z-axis) for the case of two parallel cylindrical conductors of radius a and mean distance d in Fig. E12.1. Disregard the factor $\mathrm{e}^{\mathrm{i}(\omega t - \gamma z)}$ with $\gamma/\omega = 1/c_0$. (Hint: Since the arrangement of conductors is the same as that in Exercise 5.8, the electric field has the same form as in that case.)

12.8. Prove that Eq. (12.78) satisfies Eq. (11.31). See Sect. A2.1 in the Appendix.

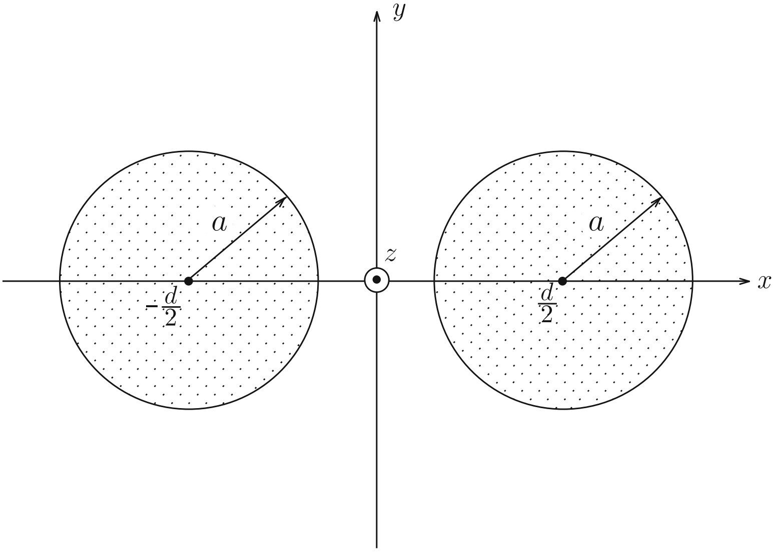

Fig. E12.1 Parallel cylindrical conductors