

Chapter 1

Electrostatic Field

1.1 Electric Charge in Vacuum

When we touch a metal doorknob after walking on a carpet on a dry day, we sometimes feel a shock on the fingertips as a small crackle. If it is dark, we can see a spark when inserting a key into a keyhole. This is the same phenomenon as thunder. This phenomenon is brought about by **electric charge** in substances. The usual frictional electricity we experience also comes from electric charges.

Electric charge build-up in a substance that causes various kinds of electric phenomena, including the above examples. Matter is a substance that obeys universal gravity laws, and its magnitude is quantitatively described in terms of mass. In the case of electric phenomena, an amount of electric charge quantitatively describes the phenomena and the same term, “electric charge”, is also used to mean the amount of electric charge.

Unlike mass, there are two kinds of electric charge, positive and negative. The components of the electric charge are the proton with positive charge and electron with negative charge. The electric charge of a proton is called the **elementary electric charge** and its magnitude is

$$e = 1.602\,189\,2 \times 10^{-19} \text{ C},$$

where the unit [C] is **coulomb**. The electric charge of an electron is $-e$. The elementary electric charge is the minimum amount of electric charge, and any electric charge is its integral multiple. Since e is sufficiently small, electric charge can be regarded as a continuous quantity in many cases. This is similar to the fact that an amount of water can be regarded as a continuous quantity in usual cases.

On an atomic scale the nucleus of an atom is composed of protons and neutrons, which are electrically neutral, and electrons stay in orbits around the nucleus. There are innumerable positive and negative electric charges in substances. Since the size of each atom is very small, electrons and protons can be regarded as being

in the center of each atom on a macroscopic scale. As a result, the positive and negative charges cancel each other to yield an electrically neutral state. Ionic crystals composed of equal amounts of positive and negative ions can also be regarded as electrically neutral on the macroscopic scale, since the distance between these ions is sufficiently small. Sometimes the electric charge is not balanced. In such a case the electric charge that remains after cancellation causes various electric phenomena.

There are two kinds of electric charge that cause electric phenomena: one is **true electric charge**, which can be transferred outside a substance and the other is **polarization charge**, which is locally bound around a nucleus and cannot be transferred outside. The former charge appears on the surface of a conductor and will be covered in Chap. 2, and the latter appears on the surface of a dielectric and will be covered in Chap. 4. These charges that contribute to electric phenomena are called **free electric charge**.

Electric charge is generally distributed with some density in the interior or on the surface of matter. Electric charge small enough to be regarded as a point is called **point charge**. This is similar to a material particle in mechanics. Electric charge distributed along a thin line with negligible cross-sectional area is **line charge**, and electric charge distributed on a surface with negligible thickness is **surface charge**.

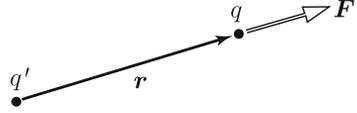
The **principle of conservation of charge** is a fundamental principle for electric charge, which is similar to the law of conservation of mass in mechanics. It states that the amount of electric charge is constant in a closed system. Even when positive and negative electric charges cancel each other, resulting in an electrically neutral state, the algebraic sum of electric charge is unchanged.

1.2 Coulomb's Law

Electric force works between electric charges, and this force is called the **Coulomb force**. This force is analogous to universal gravitation between two particles with masses. The Coulomb force on two point charges in vacuum is expressed as follows:

- The force between two electric charges of the same kind (i.e. both positive or both negative) is repulsive and the force between electric charges of different kinds (i.e. one positive and one negative) is attractive.
- The magnitude of the force is proportional to the product of the two electric charges.
- The magnitude of the force is inversely proportional to the square of the distance between the two electric charges.
- The direction of the force lies on the straight line connecting the two electric charges.

Fig. 1.1 The Coulomb force exerted on point charge q by q'



The first property is different from the property of universal gravitation whereby the force between two masses is always attractive. The Coulomb force between two point charges, q and q' , separated by distance d is mathematically expressed as

$$F = \frac{qq'}{4\pi\epsilon_0 r^2}, \quad (1.1)$$

where ϵ_0 is a constant called the **permittivity of vacuum**,

$$\epsilon_0 = \frac{10^7}{4\pi c_0^2} = 8.8542 \times 10^{-12} \text{ C}^2/\text{Nm}^2 \quad (1.2)$$

with $c_0 = 2.998 \times 10^8$ m/s denoting the speed of light in vacuum. The force in Eq. (1.1) is repulsive when $F > 0$ and attractive when $F < 0$. This equation is called **Coulomb's law**.

Since force is a vector, the Coulomb force can be expressed as a vector. We denote the direction vector of point charge q measured from the position of q' as \mathbf{r} , as shown in Fig. 1.1. Then, its magnitude is $r = |\mathbf{r}|$ and the unit vector pointing from q' to q is $\mathbf{i}_r = \mathbf{r}/r$. Hence, the force that works on q is

$$\mathbf{F} = \frac{qq'\mathbf{i}_r}{4\pi\epsilon_0 r^2} = \frac{qq'\mathbf{r}}{4\pi\epsilon_0 r^3}. \quad (1.3)$$

The force on q' is given by $-\mathbf{F}$, and the law of action and reaction is satisfied.

When there are more than two material particles, the gravitational force on one particle is the linear sum of the gravitational forces exerted on it by all other particles, and the principle of superposition holds. The same principle holds also for the Coulomb force. Assume that n point charges, q_1, q_2, \dots, q_n , are distributed in vacuum, as shown in Fig. 1.2. The total Coulomb force on another point charge, q , is given by the sum of each individual Coulomb force exerted by each point charge. If the direction vectors of point charges q_i and q are denoted by \mathbf{r}_i ($i = 1, 2, \dots, n$) and \mathbf{r} , respectively, the Coulomb force exerted by q_i on q is

$$\mathbf{F}_i = \frac{qq_i(\mathbf{r} - \mathbf{r}_i)}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_i|^3}. \quad (1.4)$$

Hence, the total Coulomb force on q is

$$\mathbf{F} = \sum_{i=1}^n \mathbf{F}_i = \frac{q}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}. \quad (1.5)$$

Fig. 1.2 The Coulomb force exerted on point charge q by more than one point charge

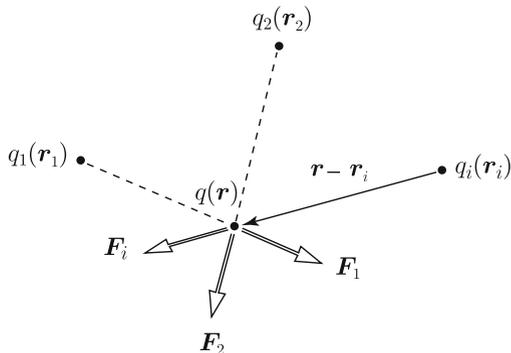
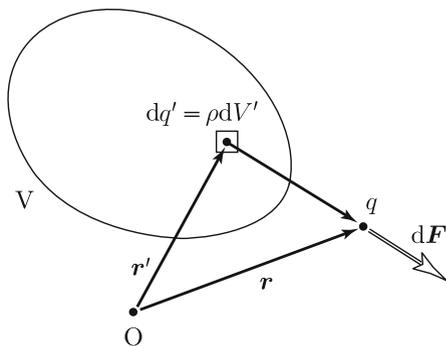


Fig. 1.3 The Coulomb force exerted on point charge q by electric charge inside small volume



This result can be extended to the case where electric charge is continuously distributed. Suppose that electric charge is distributed with the density ρ within a region, V , in vacuum, as shown in Fig. 1.3. We treat the electric charge, $dq' = \rho dV'$, in an infinitesimal volume dV' as a point charge at the position \mathbf{r}' . The Coulomb force this charge exerts on the point charge, q , at \mathbf{r} is given by

$$d\mathbf{F} = \frac{q(\rho dV')(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^3}. \quad (1.6)$$

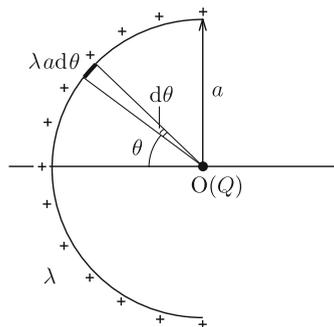
Hence, the Coulomb force from all electric charges is

$$\mathbf{F} = \frac{q}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'. \quad (1.7)$$

In the above, $\int dV'$ denotes a volume integral with respect to \mathbf{r}' .

Example 1.1. Electric charge is uniformly distributed with a linear density, λ , along a semicircle of radius a . Determine the Coulomb force on a point charge, Q , placed at the center of curvature of the semicircle.

Fig. 1.4 Electric charge distributed uniformly with linear density λ on a semicircle and point charge Q placed at center O



Solution 1.1. An angle is defined as shown in Fig. 1.4. We treat an electric charge $\lambda a d\theta$ between θ and $\theta + d\theta$ as a point charge. The Coulomb force it exerts on charge Q at the center is

$$dF = \frac{Q\lambda a d\theta}{4\pi\epsilon_0 a^2} = \frac{Q\lambda d\theta}{4\pi\epsilon_0 a}.$$

From symmetry, the vertical components of the Coulomb forces exerted by infinitesimal arc elements on Q cancel out, and only the horizontal component remains. This component is $dF' = dF \cos \theta$. Hence, the total Coulomb force is

$$F = \frac{Q\lambda}{4\pi\epsilon_0 a} \int_{-\pi/2}^{\pi/2} \cos \theta d\theta = \frac{Q\lambda}{4\pi\epsilon_0 a} [\sin \theta]_{-\pi/2}^{\pi/2} = \frac{Q\lambda}{2\pi\epsilon_0 a}.$$

This force is directed to the right in the figure. ◇

1.3 Electric Field

When a point charge, Q , is placed at the origin, the Coulomb force on another point charge, q , at the position \mathbf{r} is given by Eq. (1.3) as

$$\mathbf{F} = \frac{qQ\mathbf{r}}{4\pi\epsilon_0 r^3}. \quad (1.8)$$

This expression for force holds for arbitrary q and for arbitrary position \mathbf{r} . Hence, the space can be regarded as exerting the following force on a point charge, q :

$$\mathbf{F} = q\mathbf{E}. \quad (1.9)$$

We can consider the operation \mathbf{E} to be caused by an electrical distortion of the space due to the charge, Q . That is, the Coulomb force on q can be understood as a force that this charge feels under the electrical distortion of the space. This operation on q , \mathbf{E} , is called the **electric field**, and its magnitude is called **electric field strength**. In the above case, where Q is at the origin, the electric field strength is

$$\mathbf{E} = \frac{Q\mathbf{r}}{4\pi\epsilon_0 r^3}. \quad (1.10)$$

The unit of electric field strength is [N/C]. This is also expressed as [V/m] using the unit [V] (**volt**) of electrostatic potential, which will be defined later. The **electrostatic field**, which is the title of this chapter, is the electric field that comes from electric charges, and does not include the electric field caused by electromagnetic induction (see Chap. 10).

The electric field strength is the Coulomb force on a unit electric charge. Hence, calculating the electric field strength is equivalent to calculating the Coulomb force.

Here, we calculate the electric field strength for electric charges distributed in space. When an electric charge, q_i , is placed at position \mathbf{r}_i ($i = 1, 2, \dots, n$), the electric field strength at \mathbf{r} is given by Eq. (1.5) as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}. \quad (1.11)$$

When an electric charge is continuously distributed in a region, V , with density $\rho(\mathbf{r}')$, as shown in Fig. 1.3, the electric field strength at \mathbf{r} from Eq. (1.7) is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'. \quad (1.12)$$

Equations (1.11) and (1.12) for the electric field strength are also called Coulomb's law.

The electric field is generally complicated depending on the distribution of electric charges. However, we can visualize the field using **electric field lines (lines of electric force)**, which help us to understand the field easily. When a point charge is put in an electric field of strength \mathbf{E} , the charge experiences the Coulomb force directed parallel to \mathbf{E} . If this charge is sufficiently small, its movement driven by the force will not appreciably change the electric field. The direction of the electric field line is defined as the direction of the force on a positive charge. Therefore, we can refer to a line of electric force as an electric field line, and take the tangent to an

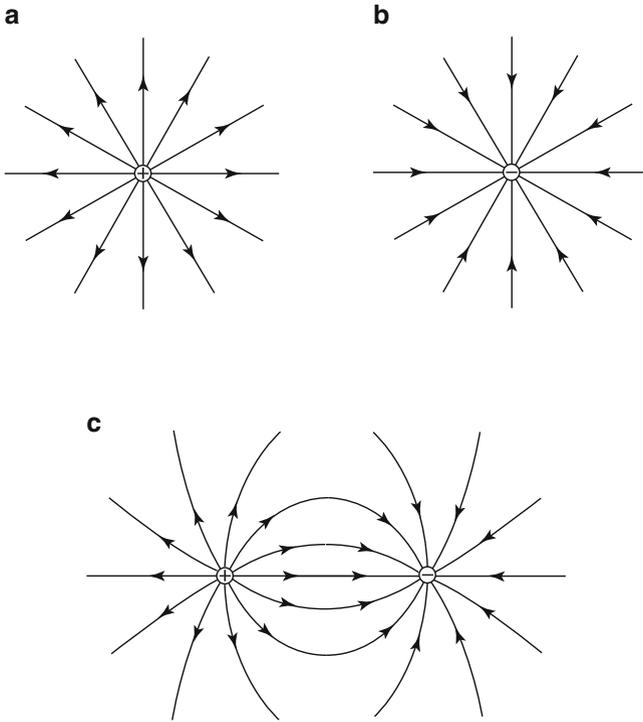


Fig. 1.5 Electric field lines of (a) single positive charge, (b) single negative charge and (c) pair of positive and negative charges

electric field line at an arbitrary point as being parallel to the direction of the electric field at this point.

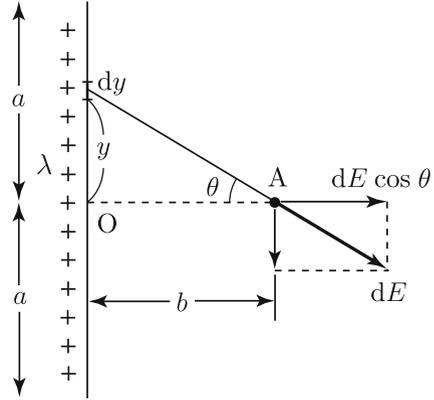
It can be shown that electric field lines never cross each other. That is, if two field lines cross at a certain point, two forces must work on a point charge put on this point. However, the electric field at any point has a single strength.

Figure 1.5 shows examples of electric field lines. A field line always starts from a positive charge and ends at a negative charge. One can see that a field line never begins or ends at a point with no charge. We will cover mathematical expressions of these facts in Sect. 1.4.

In principle, we can draw an arbitrary number of field lines. Hence, we define $E = |\mathbf{E}|$ as the number of field lines through a unit area perpendicular to the electric field. Thus, the electric field strength can be expressed through the density of field lines. For example, if the distance from the point charge is doubled in Fig. 1.5a, the spacing between two field lines is also doubled. This means that the field strength becomes one quarter as great, which is directly derived from Eq. (1.10).

Example 1.2. Electric charge is uniformly distributed with a linear density, λ , on a straight segment of length $2a$ parallel to the y -axis, as shown in Fig. 1.6. Determine the electric field strength at a point, A, at distance b from the center of the segment to the direction of the x -axis.

Fig. 1.6 Electric charge distributed uniformly with linear density λ on a straight segment of length $2a$



Solution 1.2. We define the y -axis along the length of the segment with the origin at its center. The electric field strength at A produced by the charge, λdy , in the region from y to $y + dy$ is

$$dE = \frac{\lambda dy}{4\pi\epsilon_0(y^2 + b^2)}.$$

The angle θ is defined as shown in the figure. From symmetry, the y -component of the electric field is cancelled and only the x -component, $dE \cos \theta$, remains. The relationship $y = b \tan \theta$ gives $dy = b d\theta / \cos^2 \theta$ and $y^2 + b^2 = b^2 / \cos^2 \theta$. The electric field strength is given by

$$E = \frac{\lambda}{4\pi\epsilon_0 b} \int_{-\theta_a}^{\theta_a} \cos \theta d\theta$$

with $\theta_a = \tan^{-1}(a/b)$. After a simple calculation we have

$$E = \frac{\lambda}{4\pi\epsilon_0 b} [\sin \theta]_{-\theta_a}^{\theta_a} = \frac{\lambda a}{2\pi\epsilon_0 b(a^2 + b^2)^{1/2}}.$$

For an infinitely long line ($a \rightarrow \infty$) this result gives

$$E = \frac{\lambda}{2\pi\epsilon_0 b}.$$

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1.4 Gauss' Law

Suppose a closed surface, S , which includes a point charge, q , inside, as shown in Fig. 1.7. The number of electric field lines produced by this charge that penetrate S is given by

$$N = \int_S \mathbf{E} \cdot d\mathbf{S}. \quad (1.13)$$

Since these lines do not terminate halfway, N is also the number of lines that penetrate the sphere, S_0 , of radius r_0 with its center on q . Thus, we have

$$N = \int_{S_0} \mathbf{E} \cdot d\mathbf{S}. \quad (1.14)$$

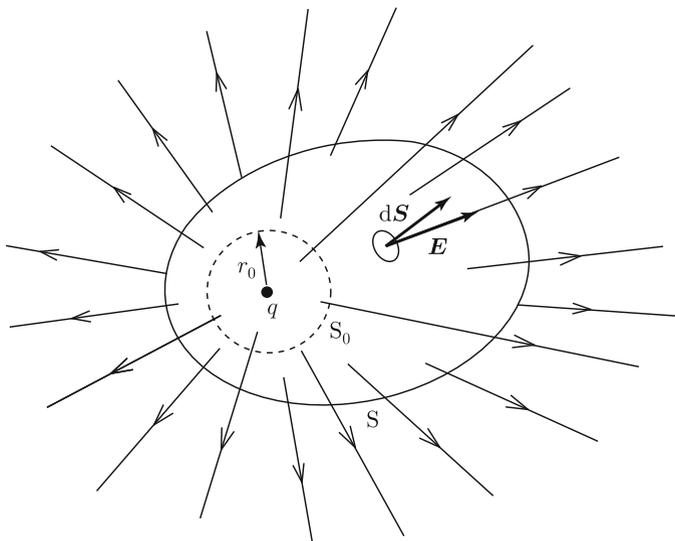


Fig. 1.7 Point charge q and closed surfaces S and S_0 containing q

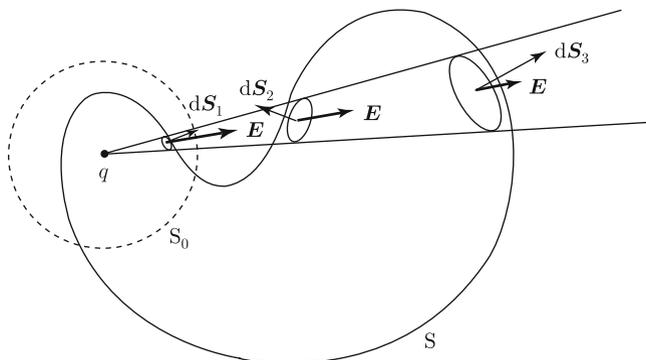


Fig. 1.8 Case where closed surface S containing point charge q has a complicated shape

Since \mathbf{E} is parallel to $d\mathbf{S}$ and $|\mathbf{E}| = q/(4\pi\epsilon_0 r_0^2)$ is constant on S_0 , a simple calculation gives

$$N = \frac{q}{4\pi\epsilon_0 r_0^2} \int_{S_0} dS = \frac{q}{\epsilon_0}. \quad (1.15)$$

In the above $\int_{S_0} dS = 4\pi r_0^2$ is the surface area of S_0 . This surface area divided by r_0^2 , i.e., the surface area of a unit sphere, is equal to 4π , the full solid angle. The reason why the full solid angle appears in Coulomb's law, Eq. (1.1), is to make Gauss' law simple as in Eq. (1.15). Thus, the relationship

$$N = \int_S \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\epsilon_0} \quad (1.16)$$

holds for S in Fig. 1.7. Thus, when $q > 0$, $N > 0$ and field lines go outside S , and when $q < 0$, $N < 0$ and field lines go inside S .

Now consider the case where the shape of sphere S is complicated in such a way that points on S and points on S_0 do not correspond one to one, as shown in Fig. 1.8. Suppose a thin cone with a top on the point charge, q . We denote small areas on S cut by the cone as dS_1 , dS_2 and dS_3 . If the number of field lines that go out of S through the area, dS_1 , is dN , the number of field lines that go out through the areas dS_2 and dS_3 are $-dN$ and dN , respectively. Hence, the number of field lines inside the cone is dN and is equal to the number of field lines through the area of S_0 cut by the cone. Thus, we can see that Eq. (1.16) holds when the point charge is included in S , even if the shape of S is complicated.

Next, consider the case where a point charge, q , is placed outside a closed surface, S , as shown in Fig. 1.9. From the above discussion, the number of field lines that enter S through dS_1 is equal to the number of field lines that go out of S through dS_2 . Hence, we have

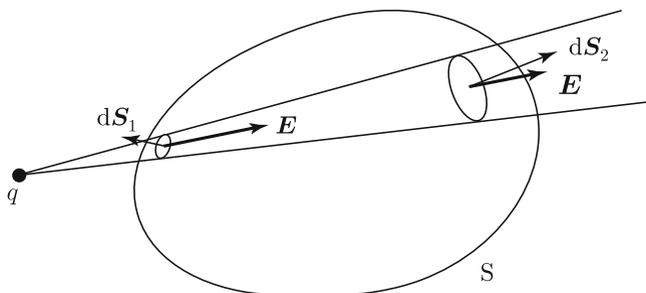


Fig. 1.9 Case where point charge q is outside closed surface S

$$N = \int_S \mathbf{E} \cdot d\mathbf{S} = 0. \quad (1.17)$$

Based on the above discussion and the principle of superposition, we can easily obtain the number of field lines when point charges are distributed. When point charges q_1, q_2, \dots, q_n are distributed inside S and $q_{n+1}, q_{n+2}, \dots, q_{n+m}$ are distributed outside S , the number of field lines that go out of S is

$$N = \int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \sum_{i=1}^n q_i. \quad (1.18)$$

When the electric charge is continuously distributed with the density, $\rho(\mathbf{r})$, the number of field lines is

$$N = \int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{r}) dV, \quad (1.19)$$

where V is the region surrounded by S . Equations (1.18) and (1.19) show that the number of field lines that go out of S is equal to the total sum of electric charge inside S divided by ϵ_0 . These equations are called **Gauss' law**. Gauss' law describes the global relationship between the distributed electric charge and the electric field, while Coulomb's law describes the local electric field caused by individual electric charges. These laws are equivalent to each other.

Using Gauss' theorem, we rewrite Eq. (1.19) as

$$\int_V \nabla \cdot \mathbf{E} dV = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{r}) dV \quad (1.20)$$

for continuously distributed electric charge. Since this equation holds for arbitrary V , we obtain the relation

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\epsilon_0}. \quad (1.21)$$

This is **Gauss' divergence law**.

The left side of Eq. (1.21) represents a source of electric field lines. That is, field lines come out of positive electric charges and go into negative electric charges. An electric field line never starts or ends at a point where there is no electric charge ($\rho = 0$), since $\nabla \cdot \mathbf{E} = 0$.

Example 1.3. Electric charge is uniformly distributed with a density, ρ , inside an infinitely long cylinder of radius a . Determine the electric field strength inside and outside the cylinder.

Solution 1.3. It is possible to calculate the electric field strength using Coulomb's law, Eq. (1.12). However, the calculation is not easy even if we use the result in Example 1.2. Gauss' law can be used to calculate the electric field strength when the geometry is highly symmetric as in this problem.

We apply Gauss' law, Eq. (1.19), to an imaginary infinite cylindrical closed surface, S , of radius R and length l with a common axis with the infinite cylinder (see Fig. 1.10). The electric field, \mathbf{E} , is directed radially from the central axis and perpendicular to it. Hence, \mathbf{E} is perpendicular to the elementary surface vector, $d\mathbf{S}$, on the top and bottom surfaces, and there is no contribution to the surface integral of the electric field strength from these surfaces. On the other hand, \mathbf{E} is parallel to $d\mathbf{S}$ on the side surface and the strength, E , is constant and depends only on the distance from the axis. Hence, the surface integral in Eq. (1.19) gives

$$\int_S \mathbf{E} \cdot d\mathbf{S} = 2\pi R l E.$$

The total charge inside S is $\pi R^2 l \rho$ for $R < a$ and $\pi a^2 l \rho$ for $R > a$. Hence, the electric field strength is

$$E = \frac{\rho}{2\epsilon_0} R; \quad R < a,$$

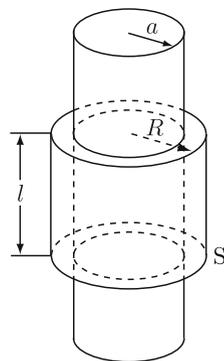


Fig. 1.10 Cylinder with distributed charge and cylindrical closed surface S (case for $R > a$)

$$= \frac{\rho a^2}{2\epsilon_0 R}; \quad R > a.$$

The result for $R > a$ is identical with the result of Example 1.2 for an infinitely long line charge with $\lambda = \pi a^2 \rho$, i.e., the case where all electric charge is concentrated on the central axis. When the length of the cylinder is finite, the electric field strength is not uniform along the length, and Gauss' law cannot be used to calculate the strength.

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Example 1.4. Electric charge is uniformly distributed with a density, σ , on a wide flat plane. Determine the electric field strength at a point, A, at distance h from the plane.

Solution 1.4. This problem can also be easily solved using Gauss' law. From symmetry we can assume that the electric field, \mathbf{E} , is directed normally to the plane with its strength dependent only on the distance from the plane. Assume a closed cylindrical surface, S, of radius a and length $2h$, as shown in Fig. 1.11: Its side surface is normal to and the top and bottom surfaces parallel to the plane, and A is on the top surface. We apply Gauss' law to this cylindrical surface. The field, \mathbf{E} , is parallel to the side surface, and there are no field lines passing through this surface. The numbers of field lines that go out of the top and bottom surfaces are the same from symmetry. Thus, we have

$$\int_S \mathbf{E} \cdot d\mathbf{S} = 2ES$$

with $S = \pi a^2$ denoting the area of the top or bottom surface. Since the total electric charge included inside S is σS , the electric field strength is

$$E = \frac{\sigma}{2\epsilon_0}. \quad (1.22)$$

This result shows that the electric field strength does not change with the distance from the plane. This can be understood from the fact that the distance between field lines does not change with the distance from the plane, as shown in Fig. 1.12. ◇

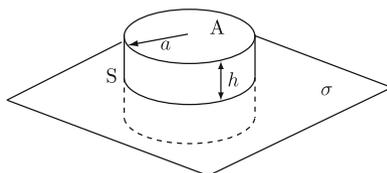
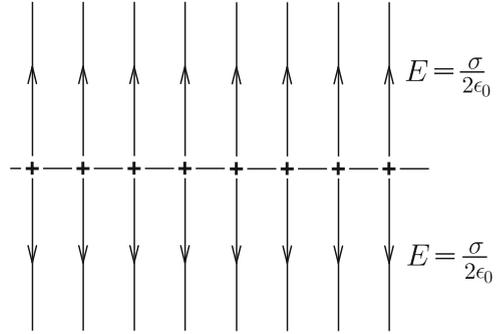


Fig. 1.11 Cylindrical closed surface S with point A on top surface

Fig. 1.12 Electric field lines from uniformly distributed electric charge on a plane



1.5 Electric Potential

The Coulomb force works between any electric charges, and any electric field originates from electric charges. Hence, the general nature of the electric field can be deduced from one point charge. This nature can be extended to any case using the principle of superposition. The electric field strength caused by an isolated point charge, Q , placed at the origin is given by Eq. (1.10). If we note the expression,

$$\nabla \left(\frac{1}{r} \right) = -\frac{\mathbf{r}}{r^3}, \quad (1.23)$$

using polar coordinates, we write this electric field strength as

$$\mathbf{E} = -\nabla\phi \quad (1.24)$$

with

$$\phi = \frac{Q}{4\pi\epsilon_0 r}. \quad (1.25)$$

This scalar function, ϕ , is called the **electric potential** or **electrostatic potential**. The unit of electric potential is [Nm/C] and is defined as [V] (volt).

The electric potential produced by a positive electric charge is illustrated in Fig. 1.13a. This shows the electric potential on a plane that contains the electric charge. Figure 1.13b illustrates the electric potential given by a pair of positive and negative electric charges of equal amounts on a plane that contains both charges.

When there are two or more electric charges, the principle of superposition holds for the electric field strength. A similar principle of superposition holds for the electric potential. While the superposition is with vector quantities for the electric field, it is with scalar quantities for the electric potential. For discontinuously distributed electric charges that provide the electric field strength of Eq. (1.11), the electric potential is given by

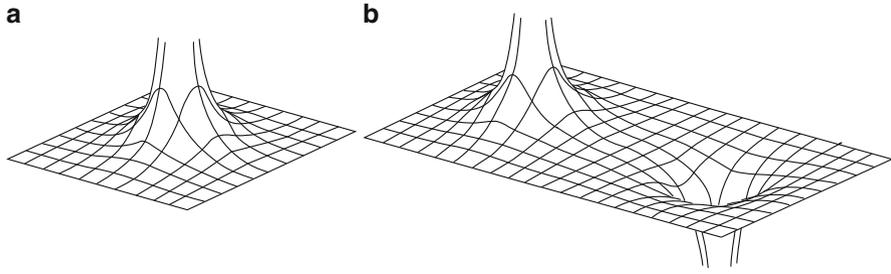


Fig. 1.13 Electric potential on a plane with (a) positive electric charge and (b) pair of positive and negative electric charges of equal amounts

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|}. \quad (1.26)$$

For continuously distributed electric charge that provides the electric field strength of Eq. (1.12), the electric potential is given by

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'. \quad (1.27)$$

Since the electric field obeys Eq. (1.24), we obtain

$$\nabla \times \mathbf{E} = 0 \quad (1.28)$$

using Eq. (A1.45) in the Appendix. This shows that the electrostatic field is an irrotational field, that is, a field without a vortex. Equations (1.21) and (1.28) describe the fundamental properties of the electrostatic field.

Since the inverse operation of a gradient is a curvilinear integral, the electric potential is generally given by

$$\phi(\mathbf{r}) = - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{s}, \quad (1.29)$$

where \mathbf{r}_0 is a reference point satisfying $\phi(\mathbf{r}_0) = 0$ and is usually taken at infinity. It should be noted that Eqs. (1.25)–(1.27) satisfy this requirement. Suppose a closed loop, C , with the surface in it denoted by S (see Fig. 1.14). The curvilinear integral of the electric field strength on C is

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = 0. \quad (1.30)$$

Fig. 1.14 Closed loop C with surface S in it

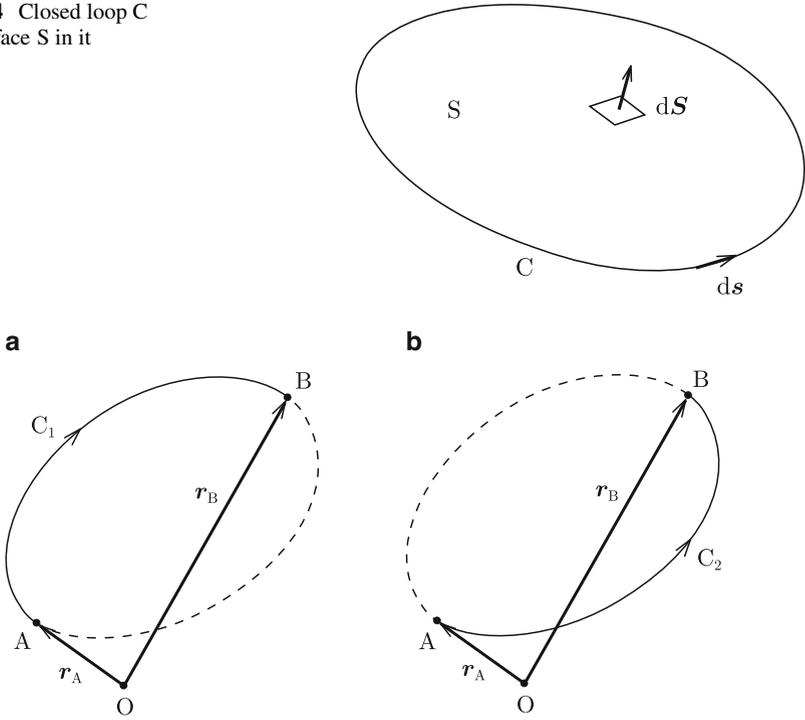


Fig. 1.15 Curvilinear integral of electric field strength from point A to point B over closed loop C through (a) C_1 and (b) C_2

The above uses Stokes' theorem given by Eq.(A1.73) in the Appendix and Eq. (1.28). This holds for any closed loops and can also be derived from Eqs. (1.24) and (A1.55).

Next, assume two points, A and B , on a closed loop, C , as shown in Fig. 1.15, and the positions of these points are denoted by \mathbf{r}_A and \mathbf{r}_B , respectively. We denote the line from A to B on one side as C_1 and that on the other side as C_2 . Thus, Eq. (1.30) gives

$$\int_{C_1(A \rightarrow B)} \mathbf{E} \cdot d\mathbf{s} = \int_{C_2(A \rightarrow B)} \mathbf{E} \cdot d\mathbf{s}. \quad (1.31)$$

Since this equation holds for an arbitrary closed loop C , the curvilinear integral of the electric field strength is determined only by the starting point, A , and terminal point, B , and is independent of the path. In fact, Eq. (1.31) is given by

$$\int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{E} \cdot d\mathbf{s} = \phi(\mathbf{r}_A) - \phi(\mathbf{r}_B), \quad (1.32)$$

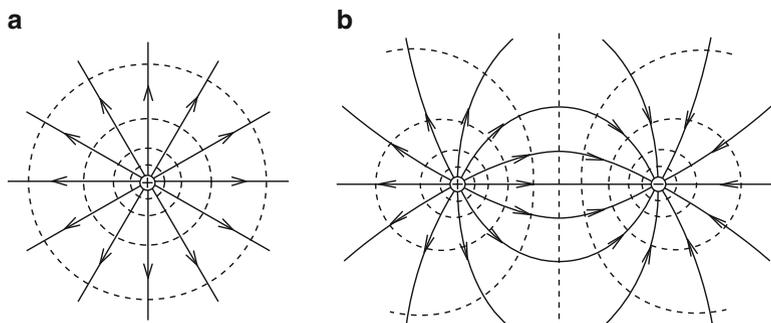


Fig. 1.16 Electric field lines (*solid lines*) and equipotential surfaces for (a) positive electric charge and (b) pair of positive and negative electric charges of equal amounts. These illustrate the conditions on a plane containing the electric charge(s), and the *dotted lines* are the crossing lines between the plane and equipotential surfaces

using Eq. (1.29). Such a vector is called a conservative vector. The electric field belongs to this kind of vector, and the electric field is also called conservative field.

Here, assume that an electric charge is forced to move in the electric field. The Coulomb force, QE , is exerted on a charge, Q , in the electric field of strength E . Hence, it is necessary to apply an opposite force, $-QE$, to the electric charge to prevent it from moving, and then to add an infinitesimal force to move it in a desired direction. The work necessary to carry slowly the electric charge from point A to point B is

$$W = -Q \int_{r_A}^{r_B} \mathbf{E} \cdot d\mathbf{s} = Q[\phi(r_B) - \phi(r_A)]. \quad (1.33)$$

Thus, the work necessary to carry the electric charge from point A to point B is proportional to the difference in electric potential between the two points but is independent of the particular path. Specifically, the work needed to carry the electric charge from a reference point such as infinity to position \mathbf{r} is

$$W = Q\phi(\mathbf{r}). \quad (1.34)$$

This work can be regarded as the potential energy of the electric charge, Q .

A virtual surface composed of points with the same electric potential is called an **equipotential surface**. The work necessary to carry an electric charge, q , a small distance, $\delta\mathbf{r}$, on an equipotential surface is zero from Eq. (1.33). Since this work is given by $-q\mathbf{E} \cdot \mathbf{r}$, we have $\mathbf{E} \cdot \mathbf{r} = 0$. That is, \mathbf{E} vector is normal to the equipotential surface. This can also be expressed by saying that the electric field lines are normal to the equipotential surface. Figure 1.16 shows examples of field lines and equipotential surfaces.

Human beings live in the earth's gravitational field. The gravity we experience is the universal gravitation between matter and the earth, and is similar to the electric force discussed here. Hence, we can define a surface on which gravitational potential is constant, and this surface is spherical and concentric with the earth. The direction of gravity, i.e., the direction of free motion of matter, is toward the center of the earth and is perpendicular to the sphere. Contour lines in maps are interfacial lines between spheres with different potential energies and the surface of the earth.

Elimination of the electric field strength, \mathbf{E} , using Eqs. (1.21) and (1.24) gives

$$\nabla \cdot (\nabla \phi) = -\frac{\rho}{\epsilon_0}. \quad (1.35)$$

The left side is written as $\nabla^2 \phi$. This is expressed as $\Delta \phi$, and the operator Δ is called **Laplacian**. This operator is

$$\Delta = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (1.36)$$

in Cartesian coordinates. Sections A1.17 and A1.18 in the Appendix give its expressions in other coordinates. Using this operator, we rewrite Eq. (1.35) as

$$\Delta \phi = -\frac{\rho}{\epsilon_0}. \quad (1.37)$$

This equation is called **Poisson's equation**, which the electrostatic potential satisfies. When there is no electric charge ($\rho = 0$), Eq. (1.37) gives

$$\Delta \phi = 0, \quad (1.38)$$

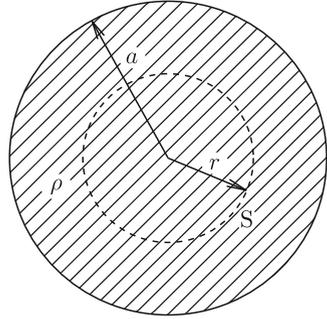
which is called **Laplace's equation**. We can directly prove that the electric potential given by Eq. (1.27) satisfies Eq. (1.37) (see Sect. A2.1). Chapter 2 describes the method of solving Laplace's equation.

Example 1.5. Electric charge is distributed uniformly with a density, ρ , inside a sphere of radius a . Determine the electric potential inside and outside the sphere.

Solution 1.5. We apply Gauss' law to a virtual sphere, S , of radius, r , concentric with the charged sphere (see Fig. 1.17). The electric field, \mathbf{E} , is parallel to the elementary surface vector, $d\mathbf{S}$, with a constant strength, E , on the surface of S . Thus, we have

$$\int_S \mathbf{E} \cdot d\mathbf{S} = 4\pi r^2 E.$$

Fig. 1.17 Virtual sphere S with radius r smaller than a



Since the total electric charge inside S is $(4\pi/3)r^3\rho$ and $(4\pi/3)a^3\rho$ for $r < a$ and $r > a$, respectively, we obtain the electric field strength as

$$E = \frac{\rho}{3\epsilon_0}r; \quad r < a,$$

$$= \frac{\rho a^3}{3\epsilon_0 r^2}; \quad r > a.$$

Substituting these results into Eq. (1.29) gives

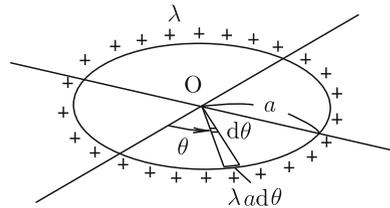
$$\phi = -\int_{\infty}^r \frac{\rho a^3}{3\epsilon_0 r^2} dr = \frac{\rho a^3}{3\epsilon_0 r}; \quad r > a,$$

$$= -\int_a^r \frac{\rho}{3\epsilon_0} r dr + \frac{\rho a^2}{3\epsilon_0} = \frac{\rho}{2\epsilon_0} \left(a^2 - \frac{r^2}{3} \right); \quad r < a.$$

◇

Example 1.6. Electric charge is uniformly distributed with a linear density, λ , around a circle of radius a , as shown in Fig. 1.18. Determine the electric potential at the center, O.

Fig. 1.18 Electric charge distributed uniformly around a circle of radius r



Solution 1.6. Here, we directly calculate the electric potential with Eq. (1.27). The electric charge in an infinitesimal region between θ and $\theta + d\theta$, $\lambda a d\theta$, is regarded as a point charge, $\rho dV'$, with θ denoting the azimuthal angle. Thus, we have

$$\phi = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \lambda d\theta = \frac{\lambda}{2\epsilon_0}.$$

◇

Example 1.7. Determine the electric potential inside and outside the cylinder in Example 1.3 when the position $R = R_0 (> a)$ outside the cylinder is the reference point with zero electric potential.

Solution 1.7. Under the given condition the electric potential outside the cylinder is

$$\phi(R) = - \int_{R_0}^R \frac{\rho a^2}{2\epsilon_0 R} dR = \frac{\rho a^2}{2\epsilon_0} \log \frac{R_0}{R}; \quad R > a. \quad (1.39)$$

Then, the electric potential inside the cylinder is given by

$$\phi(R) = \phi(a) - \int_a^R \frac{\rho R}{2\epsilon_0} dR = \frac{\rho a^2}{2\epsilon_0} \log \frac{R_0}{a} + \frac{\rho}{4\epsilon_0} (a^2 - R^2); \quad R < a. \quad (1.40)$$

The reason why the reference point is not infinity is that the electric potential diverges because of the infinite total electric charge. In practice, we easily find that the electric potential directly estimated from Eq. (1.27) diverges. This divergence comes from assuming an infinitely long cylinder; the theory of electromagnetism itself does not contain any defect. ◇

We rewrite the electric field and electric potential in Example 1.5 as

$$E(r = a) = \frac{Q}{4\pi\epsilon_0 a^2}, \quad (1.41)$$

$$\phi(r = a) = \frac{Q}{4\pi\epsilon_0 a} \quad (1.42)$$

with the total electric charge $Q = (4\pi/3)a^3\rho$. Both of them diverge in the limit of a point charge, i.e., $a \rightarrow 0$, indicating an abnormal situation. However, this comes from the problematic assumption that finite electric charge exists in an infinitesimal volume, and hence, it is not an essential problem in the theory itself. In practice the space in which electric charge exists is finite. The concept of “point charge” is an approximation of the fact that the size of the electric charge is much smaller than the treated system.

1.6 Electric Dipole

Most materials are electrically neutral with equal amounts of positive and negative electric charges, as mentioned in the beginning of this chapter. When an electric field is applied to such materials, positive and negative charges are displaced in and against the direction of the electric field, respectively, resulting in local deviations from neutrality. As a result, some electric phenomena are observed, as will be discussed in Chaps. 2 and 4. The fundamental element for such electric phenomena is a pair of positive and negative point charges that are separated by a small distance. This is called an **electric dipole**.

Suppose that electric charges q and $-q$ are displaced by $d/2$ in the positive and negative directions from the origin along the z -axis, as shown in Fig. 1.19. We determine the electric potential due to the pair of charges at a point, P, sufficiently far from the origin. In polar coordinates the distance of P from the origin is denoted by r ($\gg d$) and the angle of P from the z -axis is denoted by θ . The distance from the positive point charge to P is $r' = [r^2 + (d/2)^2 - rd \cos \theta]^{1/2} \simeq r - (d/2) \cos \theta$, and the electric potential at P due to this charge is

$$\phi_+(r) \simeq \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{r - (d/2) \cos \theta} \simeq \frac{q}{4\pi\epsilon_0 r^2} \left(r + \frac{d}{2} \cos \theta \right). \quad (1.43)$$

Similarly, the electric potential due to the negative charge is

$$\phi_-(r) \simeq -\frac{q}{4\pi\epsilon_0} \cdot \frac{1}{r + (d/2) \cos \theta} \simeq -\frac{q}{4\pi\epsilon_0 r^2} \left(r - \frac{d}{2} \cos \theta \right). \quad (1.44)$$

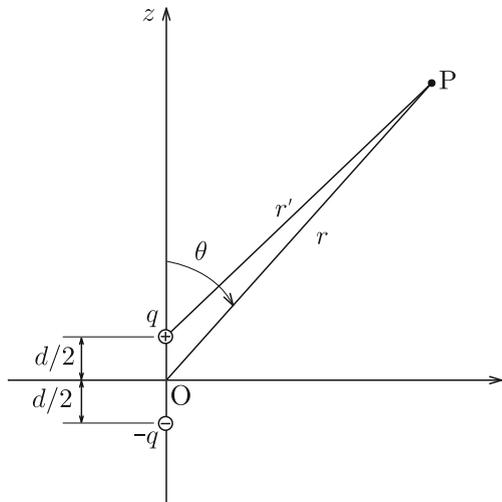


Fig. 1.19 Pair of positive and negative electric charges near the origin and point P sufficiently far from the charges

Thus, the electric potential caused by the electric dipole is given by

$$\phi(r) = \phi_+(r) + \phi_-(r) \simeq \frac{qd \cos \theta}{4\pi\epsilon_0 r^2}. \quad (1.45)$$

Here, the **electric dipole moment** is defined as the product of the direction vector from the negative electric charge to the positive electric charge, $d\mathbf{i}_z$, and the amount of charge, q :

$$\mathbf{p} = qd\mathbf{i}_z. \quad (1.46)$$

In terms of the electric dipole moment, we rewrite the electric potential caused by the dipole as

$$\phi(r) = \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\epsilon_0 r^3} = \frac{p \cos \theta}{4\pi\epsilon_0 r^2}, \quad (1.47)$$

where $p = qd$ is the magnitude of the electric dipole moment. We calculate the resultant electric field to be

$$E_r = -\frac{\partial\phi}{\partial r} = \frac{p \cos \theta}{2\pi\epsilon_0 r^3}, \quad (1.48a)$$

$$E_\theta = -\frac{1}{r} \cdot \frac{\partial\phi}{\partial\theta} = \frac{p \sin \theta}{4\pi\epsilon_0 r^3}, \quad (1.48b)$$

$$E_\varphi = -\frac{1}{r \sin \theta} \cdot \frac{\partial\phi}{\partial\varphi} = 0. \quad (1.48c)$$

Figure 1.20 shows the electric potential and electric field produced by the dipole.

Suppose that electric charge is distributed in a region, V , around the origin, O . We determine the electric potential caused by this charge at a point, P , with a position vector \mathbf{r} with $r = |\mathbf{r}|$ sufficiently large compared with V (see Fig. 1.21). The electric potential is formally given by Eq. (1.27). We denote by θ the angle between the vector \mathbf{r}' from the origin to a point, A , in V and \mathbf{r} . Since $r' = |\mathbf{r}'| \ll r$, we have

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} \simeq \frac{1}{r} \left(1 + \frac{r'}{r} \cos \theta \right) + \dots = \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \dots$$

Hence, we rewrite the electric potential at P as

$$\phi = \frac{Q}{4\pi\epsilon_0 r} + \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\epsilon_0 r^3} + \dots \quad (1.49)$$

Fig. 1.20 Equipotential surfaces (*dotted lines*) and electric field lines (*solid lines*) produced by an electric dipole on a plane including positive and negative electric charges

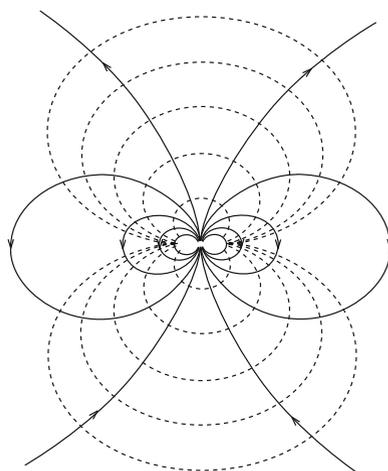
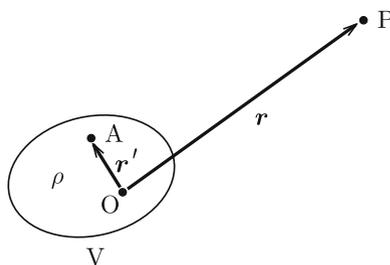


Fig. 1.21 Region V in which electric charge is distributed and point P sufficiently far from it



In the above,

$$Q = \int_V \rho(\mathbf{r}') dV' \tag{1.50}$$

is the total amount of electric charge and

$$\mathbf{p} = \int_V \mathbf{r}' \rho(\mathbf{r}') dV' \tag{1.51}$$

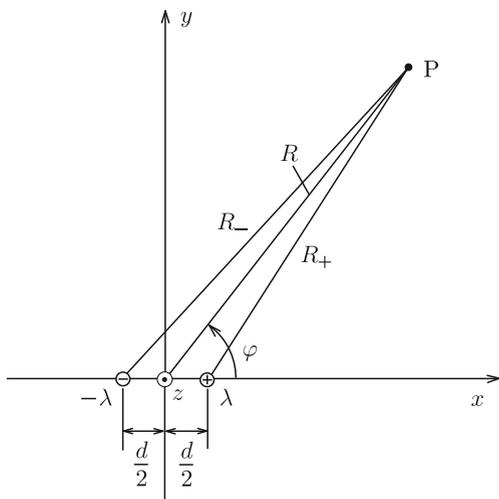
is the total electric dipole moment caused by non-uniform distribution of electric charge. An expansion such as that in Eq. (1.49) is called a **multipole expansion**.

For a single electric dipole, shown in Fig. 1.19, the position vectors of charges q and $-q$ are $(d/2)\mathbf{i}_z$ and $-(d/2)\mathbf{i}_z$, respectively. Thus, we can easily show that the electric dipole moment of Eq. (1.46) is derived from Eq. (1.51).

Example 1.8. Electric charge is uniformly distributed with linear densities λ and $-\lambda$ on the lines at $x = d/2$ and $x = -d/2$ parallel to the z -axis, respectively, as

shown in Fig. 1.22. Determine the electric potential at a point sufficiently far from these lines. Such a pair of electric charges is called an **electric dipole line**.

Fig. 1.22 Electric dipole line



Solution 1.8. We define the cylindrical coordinates as in Fig. 1.22, where the azimuthal angle φ is measured from the x -axis. The distance between the point, P, and the line $x = d/2$ is denoted by R_+ . The electric field at P produced by this line charge is

$$E = \frac{\lambda}{2\pi\epsilon_0 R_+}.$$

Electric field lines start radially from the line charge. The resultant electric potential is

$$\phi_+ = \frac{\lambda}{2\pi\epsilon_0} \log \frac{R_0}{R_+},$$

where R_0 is the distance to the reference point with zero electric potential. In fact the above result satisfies the condition $\phi_+ = 0$ at $R_+ = R_0$.

Similarly, the electric potential due to the line charge at $x = -d/2$ is

$$\phi_- = -\frac{\lambda}{2\pi\epsilon_0} \log \frac{R_0}{R_-}$$

with R_- denoting the distance from the line charge to P. Thus, the total electric potential is given by

$$\phi = \phi_+ + \phi_- = \frac{\lambda}{2\pi\epsilon_0} \log \frac{R_-}{R_+}.$$

When $R \gg d$, we approximate R_+ as

$$R_+ = \left[R^2 + \left(\frac{d}{2} \right)^2 - Rd \cos \varphi \right]^{1/2} \simeq R \left(1 - \frac{d}{2R} \cos \varphi \right)$$

and have $R_- \simeq R \{ 1 + [d/(2R)] \cos \varphi \}$. Thus, the electric potential is reduced to

$$\phi(R, \varphi) \simeq \frac{\lambda}{2\pi\epsilon_0} \log \frac{1 + [d/(2R)] \cos \varphi}{1 - [d/(2R)] \cos \varphi} \simeq \frac{\lambda}{2\pi\epsilon_0} \cdot \frac{d}{R} \cos \varphi.$$

Using the **moment of an electric dipole line** given by

$$\hat{p} = \lambda d, \quad (1.52)$$

we write the electric potential as

$$\phi(R, \varphi) \simeq \frac{\hat{p} \cos \varphi}{2\pi\epsilon_0 R}. \quad (1.53)$$

The equality holds at the far distance ($R \gg d$). In this case the total electric charge is 0 and the electric potential is 0 at infinity. Hence, there is no problem of divergence.

We determine the electric field strength to be

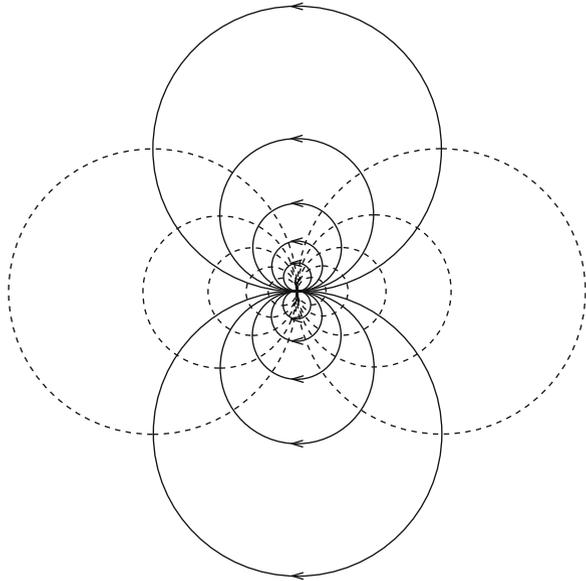
$$E_R = \frac{\hat{p} \cos \varphi}{2\pi\epsilon_0 R^2}, \quad (1.54a)$$

$$E_\varphi = \frac{\hat{p} \sin \varphi}{2\pi\epsilon_0 R^2}, \quad (1.54b)$$

$$E_z = 0. \quad (1.54c)$$

Figure 1.23 shows interfacial lines between the equipotential surfaces and the sheet and electric field lines. Both are circles. \diamond

Fig. 1.23 Equipotential surfaces (*dotted lines*) and electric field lines (*solid lines*) in a plane normal to the line charges



Column: Electric Charge Density in Electric Phenomena

Here we estimate the amount of electric charges involved in a particular electric phenomenon. Equation (1.22) gives the relationship between surface electric charge and resultant electric field strength in Example 1.4. For a conductor surface the electric field strength is doubled, as Sect. 2.1 will show.

Here we examine the dielectric breakdown of air as an extreme case. The dielectric breakdown of air occurs when the applied electric field strength is as high as $3 \times 10^6 \text{ V/m}$. Realizing such a field strength requires a surface charge density of $\sigma = \epsilon_0 E \simeq 2.7 \times 10^{-5} \text{ C/m}^2$. Since the electric charge of one electron is $1.6 \times 10^{-19} \text{ C}$, we estimate the number density of electrons or holes on the surface to be $1.7 \times 10^{14} \text{ m}^{-2}$.

The surface number density of metal atoms such as copper is about $0.6 \times 10^{20} \text{ m}^{-2}$. If one of every 3.6×10^5 atoms on the surface gets or loses one electron, such a high electric field is produced. The number density of electric charge involved in usual electric phenomena is much lower.

Exercises

- 1.1. Electric charge, Q , is uniformly distributed on a bar, AB, of length L , as shown in Fig. E1.1. Determine the force on a point charge, q , put at point P.
- 1.2. Electric charge is uniformly distributed with a linear density, λ , on a bar of length a parallel to the y -axis. Determine the electric field strength at point A at distance b along the x -axis from the bottom of the bar (see Fig. E1.2).
- 1.3. Electric charge is uniformly distributed with a linear density, λ , around a square of width a . Determine the electric field strength at point P at distance z above the center of the square (see Fig. E1.3).
- 1.4. Electric charge is uniformly distributed with a surface density, σ , on a long thin slab of width $2a$. Determine the electric field strength at points A and B (see Fig. E1.4). A and B are located at distance b above the center and at distance $d(> a)$ from the center in the same plane as the slab, respectively.
- 1.5. A solid sphere of diameter a contains a spherical void of radius b . Electric charges uniformly distributed in the solid portion with a density, ρ (see Fig. E1.5). In the figure, $a > b + d$. Determine the electric field strength at the center, A, of the spherical void and at point B outside the sphere. Suppose that O, A and B are on the same line.
- 1.6. Determine the electric potential at point A in Example 1.2.
- 1.7. Electric charge is uniformly distributed with a density, σ , on the surface of a sphere of radius a . Determine the electric potential at the center of the sphere.

Fig. E1.1 Uniformly distributed charge Q on a bar and point charge q put at a position extrapolated from the bar

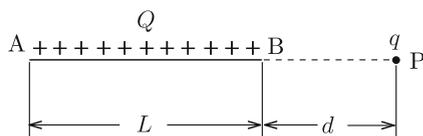


Fig. E1.2 Bar with uniformly distributed electric charge and point A

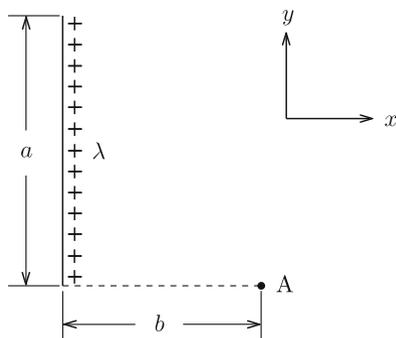


Fig. E1.3 Square with uniformly distributed electric charge and point P above the center

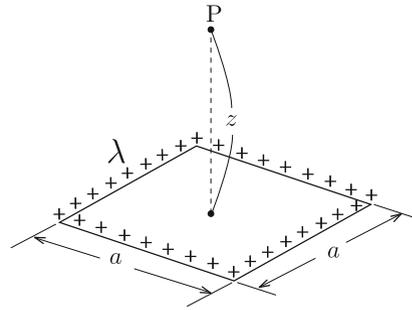


Fig. E1.4 Long thin slab with uniformly distributed electric charge and points A and B

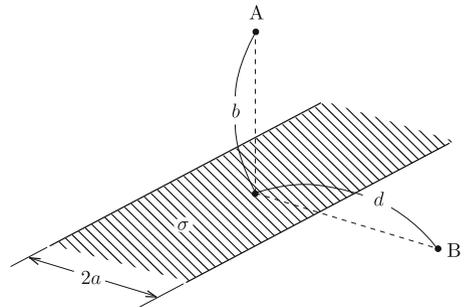


Fig. E1.5 Solid sphere containing a spherical void, uniformly distributed electric charge in the solid portion

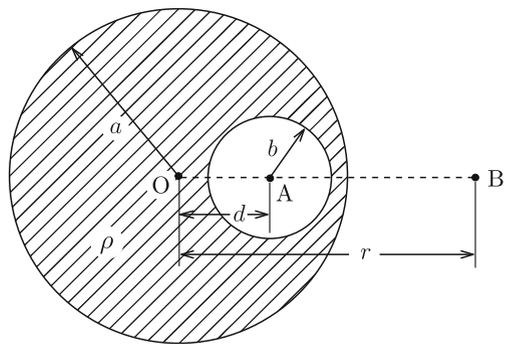
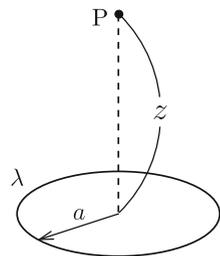


Fig. E1.6 Circle with uniformly distributed electric charge and point P above the center



1.8. Suppose that electric charge is uniformly distributed on a circle of radius a with a linear density, λ , as illustrated in Fig. E1.6. Determine the electric field strength and electric potential at a point P at distance z from the center of the circle.

1.9. Electric charge, Q , is uniformly distributed in a sphere of radius a . Determine the work necessary to carry a point charge, q , from point A to point B. These points are outside the sphere, and the distances of A and B from the sphere's center are r_A and r_B , respectively. Suppose that the electric charge distribution in the sphere is not influenced by the movement of the point charge.