

# Chapter 5

## Steady Current

### 5.1 Current

A conductor contains freely moving electric charge and the Coulomb force can move the electric charge when an electric field is directly applied to the conductor. This movement of electric charge is **current**. In this chapter we discuss electric phenomena when a **steady current** that does not change with time flows. As distinct from magnetization current and displacement current discussed in Chaps. 9 and 11, the current of true electric charge is sometimes called **true current**.

The current is a vector with a magnitude and direction. When electric charge  $dQ$  passes through a cross-section within time  $dt$ , the current is given by

$$I = \frac{dQ}{dt}. \tag{5.1}$$

Its unit is [C/s] and is denoted [A] (**ampere**).

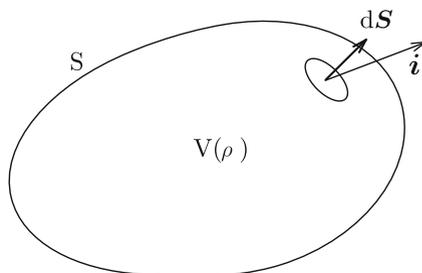
Although the current is an amount of electric charge that passes through a certain cross-section in a unit time, it is not a quantity representing strength. We define **current density** as a quantity representing the strength of current. The current density  $\mathbf{i}$  is also a vector. Its direction is the same as that of the current, and its magnitude is given by

$$i = \frac{dI}{dS}, \tag{5.2}$$

when current  $dI$  flows through a small normal cross-section of area  $dS$ . Its unit is [A/m<sup>2</sup>]. When the direction of current is tilted from elementary surface vector  $d\mathbf{S}$ , the current that flows through the elementary surface is

$$dI = i dS \cos \theta = \mathbf{i} \cdot d\mathbf{S}. \tag{5.3}$$

**Fig. 5.1** Electric charge in region  $V$  and current through surface  $S$



Since the current is a flow of electric charge, we can describe it using the density and velocity of electric charge. Suppose that particles of electric charge  $q$  and density  $n$  move with velocity  $\mathbf{v}$ . The current density is then given by

$$\mathbf{i} = qn\mathbf{v}. \quad (5.4)$$

Since the electric charge density is given by  $\rho = qn$ , the current density is expressed as

$$\mathbf{i} = \rho\mathbf{v}. \quad (5.5)$$

The amount of electric charge is conserved similarly to the mass of materials. That is, the algebraic sum of positive and negative charges is conserved. We suppose a region  $V$  surrounded by a closed surface  $S$  (see Fig. 5.1) and denote the electric charge density inside it by  $\rho$ . The total electric charge in  $V$  is

$$Q = \int_V \rho \, dV. \quad (5.6)$$

When current of density  $\mathbf{i}$  flows across the surface, the electric charge that passes out of  $V$  through a small area  $dS$  in unit time is  $\mathbf{i} \cdot d\mathbf{S}$ . Hence, the electric charge that goes out of  $S$  in unit time is given by

$$\frac{dQ'}{dt} = \int_S \mathbf{i} \cdot d\mathbf{S}. \quad (5.7)$$

The principle of conservation of electric charge requires that this should be equal to the decrease in electric charge in region  $V$  in unit time,  $-dQ/dt$ . This gives

$$\frac{d}{dt} \int_V \rho \, dV + \int_S \mathbf{i} \cdot d\mathbf{S} = 0. \quad (5.8)$$

Since region  $V$  does not change with time, we can change the order of the time derivative and spatial integral in the first term. Using Gauss' theorem for the second term, Eq. (5.8) is written as

$$\int_V \left( \nabla \cdot \mathbf{i} + \frac{\partial \rho}{\partial t} \right) dV = 0. \quad (5.9)$$

Since this relationship holds for arbitrary  $V$ , we have

$$\nabla \cdot \mathbf{i} + \frac{\partial \rho}{\partial t} = 0. \quad (5.10)$$

This is called the **continuity equation of current**.

For a steady current that does not change with time Eq. (5.10) reduces to

$$\nabla \cdot \mathbf{i} = 0. \quad (5.11)$$

That is, the current density does not diverge and follows the same continuity equation as an incompressible fluid.

## 5.2 Ohm's Law

It is necessary to apply an electric potential difference to a material such as a metal to get a current. In many cases it is empirically known that there is a proportional relationship between the electric potential difference  $V$  and the current  $I$ :

$$V = R_r I. \quad (5.12)$$

The proportional constant  $R_r$  is called **electric resistance** or simply **resistance**. This constant is determined by the shape and property of the material that carries current. The unit of electric resistance is [V/A] and is denoted [ $\Omega$ ] (**ohm**). Equation (5.12) is called **Ohm's law**. For a material of length  $l$  and uniform cross-sectional area  $S$ , the electric resistance is given by

$$R_r = \rho_r \frac{l}{S}, \quad (5.13)$$

where  $\rho_r$  is a constant inherent to material and is called **resistivity** or **specific resistance**. Its unit is [ $\Omega\text{m}$ ]. Table 5.1 lists values of the resistivity for various materials. The electric property of a material differs dramatically depending on the resistivity: materials with resistivity less than  $10^{-6} \Omega\text{m}$  that can easily transport a current are called conductors, and materials with resistivity above  $10^8 \Omega\text{m}$  that can hardly transport a current are classified as insulators. Materials with intermediate resistivity are called semiconductors.

The relationship between the current and electric potential difference is also written as

**Table 5.1** Resistivity of various materials at 20°C

<i>Metal</i>	( $\times 10^{-8} \Omega\text{m}$ )	<i>Semiconductor</i>	( $\Omega\text{m}$ )
Silver (Ag)	1.62	Germanium (Ge) <sup>a</sup>	$4.8 \times 10^{-1}$
Copper (Cu)	1.72	Silicon (Si) <sup>a</sup>	$3.2 \times 10^3$
Gold (Au)	2.4	<i>Insulator</i>	( $\Omega\text{m}$ )
Aluminum (Al)	2.75	Epoxy resin	$10^{11}$ – $10^{14}$
Brass (Cu–Zn)	5–7	Aluminum oxide	$10^{12}$ – $10^{13}$
Iron (Fe)	9.8	Mica	$10^{12}$ – $10^{15}$
Platinum (Pt)	10.6	Natural rubber	$10^{13}$ – $10^{15}$
Constantan	50	Polyethylene	$> 10^{14}$
Mercury (Hg)	95.8	Paraffin	$10^{14}$ – $10^{17}$
Nichrome	109	Quartz glass	$> 10^{15}$

<sup>a</sup>Values at 27 °C

$$I = GV. \quad (5.14)$$

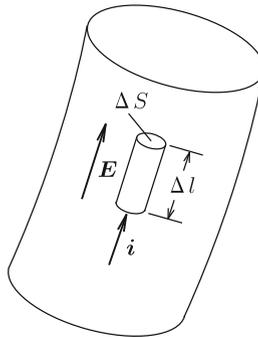
In the above the proportional constant  $G = 1/R_r$  is called **conductance**. Its unit is [S] (**Siemens**). Using Eq. (5.13), the conductance is written as

$$G = \sigma_c \frac{S}{l}. \quad (5.15)$$

The constant  $\sigma_c = 1/\rho_r$  is called **electric conductivity**. Its unit is [S/m].

Suppose a small region in which the current flows under electric potential difference (see Fig. 5.2). The length of this region along the current is  $\Delta l$  and the cross-sectional area normal to the current is  $\Delta S$ . When the electric field strength is  $E$  and the current density is  $i$ , the potential difference and the current in this small region are  $\Delta V = E \Delta l$  and  $\Delta I = i \Delta S$ , respectively. Hence, the electric resistance is written as

$$R_r = \frac{\Delta V}{\Delta I} = \frac{\Delta l}{\Delta S} \cdot \frac{E}{i}. \quad (5.16)$$

**Fig. 5.2** Small region in which current flows under an electric potential difference

On the other hand, the electric resistance is defined as

$$R_r = \frac{\Delta I}{\sigma_c \Delta S}. \quad (5.17)$$

Hence we have  $i = \sigma_c E$ . Since the current and electric field have the same direction, we can write this relationship in the vector form as

$$\mathbf{i} = \sigma_c \mathbf{E} \quad (5.18)$$

or

$$\mathbf{E} = \rho_r \mathbf{i}. \quad (5.19)$$

These are Ohm's law for electromagnetism.

The current is realized by a movement of electric charge, most of which is electrons. Therefore, current is a dynamic phenomenon and is not a static phenomenon, even if it is in a steady state. The reason why some electric field can remain in a conductor is that the phenomenon is in a dynamic state. Even in this case, we can express the electric field using the electric potential as in Eq. (1.24).

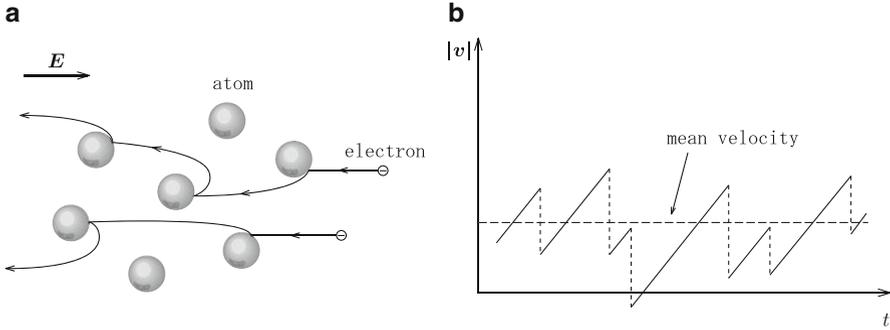
### 5.3 Microscopic Investigation of Electric Resistance

As mentioned in Sect. 5.2, Ohm's law is an empirical law for various kinds of materials, and it should be noted that this law cannot be explained by any physical principle. This is because the resistance is associated with energy dissipation, which cannot be derived theoretically. Hence, Ohm's law is a kind of phenomenological model. On the other hand, in a superconductor, which is introduced as one kind of magnetic material in Chap. 7, Ohm's law does not hold and the current obeys a physical principle. Hence, we can say that a superconductor is a pure material from the viewpoint of physics.

Here we investigate microscopically the occurrence of electric resistance, although it is not rigorously based on a physical principle. When a current flows steadily inside a metal, electrons that compose the current are driven by the electric field. If the mass of an electron is  $m$ , the equation of motion of the electron in electric field  $\mathbf{E}$  is expected to be

$$m \frac{d\mathbf{v}}{dt} = -e\mathbf{E}, \quad (5.20)$$

where  $\mathbf{v}$  is the velocity of the electron. However, this equation requires the electron to be accelerated by the electric field, which would bring about increasing current with time, resulting in a contradiction with the assumption of a steady state.



**Fig. 5.3** (a) Motion of electrons in a metal and (b) variation in velocity with time

In a practical condition, every time an electron is accelerated by the electric field, it will collide with atoms in the metal and lose the energy given by the electric field, as illustrated in Fig. 5.3a. Thus, the velocity of the electron will have some mean value without increasing appreciably (see Fig. 5.3b). The effect of collision can be introduced by assuming a **viscous force** acting on electrons. Although this force cannot be derived from any physical principle, this phenomenological assumption is known to be useful in many examples.

Here we average the motion of an electron within a suitable time scale as in Fig. 5.3b in such a way that we can describe a gradual variation in the average velocity  $v$  on a much longer time scale. Then, the viscous force is defined. This force is directed opposite to the motion of the electron and its magnitude is proportional to the velocity. Hence, when the electron moves with a higher velocity, it is subjected to a stronger force that reduces the velocity. This force is expressed as  $-\eta v$  and constant  $\eta$  is called the **coefficient of viscosity**. Hence, the equation of motion of the electron is now given by

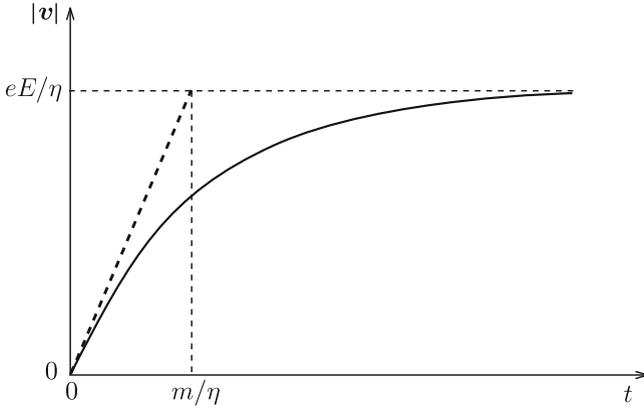
$$m \frac{dv}{dt} = -eE - \eta v. \quad (5.21)$$

We can easily solve this equation; the solution under the initial condition of  $v = 0$  at  $t = 0$  is

$$v = -\frac{eE}{\eta} \left[ 1 - \exp\left(-\frac{\eta t}{m}\right) \right]. \quad (5.22)$$

Figure 5.4 shows the variation in the velocity with time. After a sufficiently long time we can neglect the exponential term and the velocity reaches a constant value,

$$v = -\frac{eE}{\eta}. \quad (5.23)$$



**Fig. 5.4** Time variation in velocity of an electron derived from the equation of motion

Hence, using Eq. (5.4), the current density in the steady state is given by

$$\mathbf{i} = -en_e\mathbf{v} = \frac{n_e e^2}{\eta} \mathbf{E}, \quad (5.24)$$

where  $n_e$  is the density of electrons. Thus, we obtain Ohm's law. In the above model the electric conductivity is given by

$$\sigma_c = \frac{n_e e^2}{\eta}. \quad (5.25)$$

From Eq. (5.22) the time needed for the current to reach the steady-state value is approximately given by

$$t = t_0 = \frac{m}{\eta} = \frac{\sigma_c m}{n_e e^2}. \quad (5.26)$$

Substituting  $m \simeq 0.9 \times 10^{-30}$  kg,  $e \simeq 1.6 \times 10^{-19}$  C,  $n_e \simeq 1 \times 10^{29}$  m<sup>-3</sup> for usual metals and  $\sigma_c \simeq 0.6 \times 10^8$  S/m for copper, we have  $t_0 \simeq 2 \times 10^{-14}$  s. Since the time required for observation using measurement instruments is of the order of  $10^{-10}$  s, the above variation in current cannot be observed. This means that Ohm's law is always observed.

When we apply a current to a material with electric resistance, energy dissipation takes place. Suppose that a current  $I$  flows in a material under an electric potential difference  $V$  given by a power source. Here we estimate the work done by the power source during a period  $\Delta t$ . The amount of electric charge that is transferred during this period is  $\Delta Q = I \Delta t$ . The work done on this electric charge is

$$\Delta W = V \Delta Q = VI \Delta t. \quad (5.27)$$

Hence, the work done in unit time is

$$P = \frac{\Delta W}{\Delta t} = VI. \quad (5.28)$$

This is called **electric power**. Its unit is [VA] and denoted [W] (**watt**). Using Ohm's law this is rewritten as

$$P = R_r I^2 = \frac{V^2}{R_r}. \quad (5.29)$$

In this case the steady state of the current is maintained and the stored electric energy does not change. Hence, the work done by the power source is dissipated to heat.

Suppose a small region of length  $\Delta l$  and cross-sectional area  $\Delta S$ . We assume that a current,  $\Delta I = i \Delta S$ , flows through the cross-section under an electric potential difference,  $\Delta V = E \Delta l$ , along the length. The electric power dissipated in this region is

$$\Delta P = \Delta V \Delta I = E i \Delta l \Delta S. \quad (5.30)$$

Hence, the dissipated electric power in a unit volume (i.e., the electric power density) is

$$\frac{\Delta P}{\Delta l \Delta S} = p = E i = \sigma_c E^2 = \rho_r i^2. \quad (5.31)$$

## 5.4 Fundamental Equations for Steady Electric Current

We have discussed the equation describing the phenomena associated with a steady current. The fundamental physical quantity for the steady current is the current density  $\mathbf{i}$ . The quantity that causes it is the electric field,  $\mathbf{E}$ . Here we summarize the fundamental equations for these quantities.

The continuity equation for a steady current is given by Eq. (5.11). The electric field is derived from the electric potential and hence, Eq. (1.28) holds. Ohm's law that connects these quantities is Eq. (5.18).

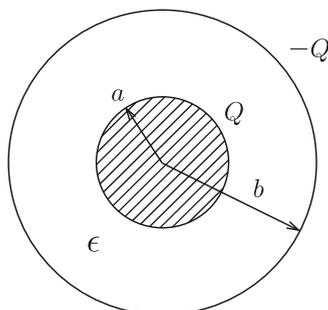
These equations have the same forms as fundamental equations describing the electrostatic field in a space in which there is no electric charge, as compared in Table 5.2. That is, the electric field  $\mathbf{E}$  is common to the two cases, and the current density  $\mathbf{i}$  corresponds to the electric flux density  $\mathbf{D}$  and the electric conductivity  $\sigma_c$  corresponds to the dielectric constant  $\epsilon$ . However, it should be noted that this correspondence is mathematical, and similarity is sometimes broken in real cases as will be shown later.

Here we suppose that electric charges  $\pm Q$  are given to the outer and inner electrodes of a concentric spherical capacitor with a dielectric material of dielectric

**Table 5.2** Comparison of fundamental equations for steady current and electrostatic field in the absence of electric charge

Steady current		Electrostatic field in the absence of electric charge
$\nabla \cdot \mathbf{i} = 0$		$\nabla \cdot \mathbf{D} = 0$
$\mathbf{i} = \sigma_c \mathbf{E}$		$\mathbf{D} = \epsilon \mathbf{E}$
$\nabla \times \mathbf{E} = 0$		$\nabla \times \mathbf{E} = 0$
$\mathbf{i}$	$\longleftrightarrow$	$\mathbf{D}$
$\sigma_c$	$\longleftrightarrow$	$\epsilon$

**Fig. 5.5** Concentric spherical capacitor



constant  $\epsilon$  in Fig. 5.5. We have learned the method with which to determine the electric potential difference between the electrodes. The electric flux density is directed outward from the inner electrode, and using Gauss' law we can determine its magnitude as

$$D = \frac{Q}{4\pi r^2} \quad (5.32)$$

at position  $r$  ( $a < r < b$ ) from the center. Since the electric field is  $E = D/\epsilon$ , the electric potential difference is given by

$$V = \int_a^b \frac{Q}{4\pi\epsilon r^2} dr = \frac{Q}{4\pi\epsilon} \left( \frac{1}{a} - \frac{1}{b} \right). \quad (5.33)$$

Thus, the capacitance is

$$C = \frac{Q}{V} = \frac{4\pi\epsilon ab}{b-a}. \quad (5.34)$$

The dielectric material in the concentric spherical capacitor is replaced by a substance with electric conductivity  $\sigma_c$ . Now we determine the electric resistance between the two electrodes under the electric potential difference  $V$ . We denote the current by  $I$ . Since the current density does not have a divergence similarly to the electric flux density, Gauss' law gives the current density as

$$i = \frac{I}{4\pi r^2}, \quad (5.35)$$

corresponding to Eq. (5.32). Since the electric field is  $E = i/\sigma_c$ , the electric potential difference is given by

$$V = \int_a^b \frac{I}{4\pi\sigma_c r^2} dr = \frac{I}{4\pi\sigma_c} \left( \frac{1}{a} - \frac{1}{b} \right). \quad (5.36)$$

The electric resistance is

$$R_r = \frac{V}{I} = \frac{b-a}{4\pi\sigma_c ab}. \quad (5.37)$$

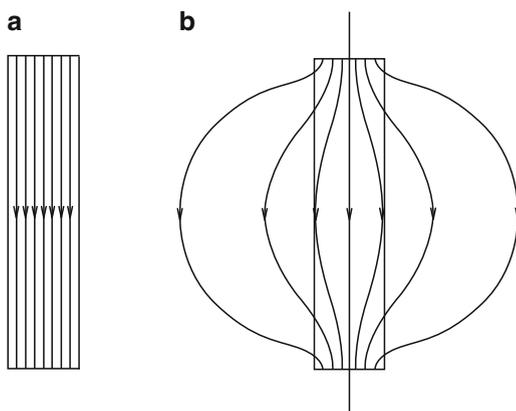
Thus, the above two problems are formally identical. Eliminating  $V$  common to each example from Eqs. (5.34) and (5.37), we have

$$CR_r = \frac{\epsilon}{\sigma_c}. \quad (5.38)$$

This quantity—the product of capacitance and electric resistance—does not depend on the shape of capacitor or resistor and is given only by the dielectric constant and electric conductivity. This relationship of Eq. (5.38) generally holds for a capacitor and resistor having electrodes of the same shape. However, this is limited to the case in which we can obtain a rigorous solution for the field.

Similarity is rarely found between phenomena of steady current and an electrostatic field. This is explained by a quite large difference in the electric conductivity between a conductor and vacuum, while the difference in the dielectric constant between a dielectric material and vacuum is a factor of several tens at most.

For example, Fig. 5.6a shows the current when an electric potential difference is applied to the two edges of a long thin resistor. In this case the current flows



**Fig. 5.6** (a) Lines of current in a long thin resistor and (b) electric flux lines in a capacitor with long thin dielectric material

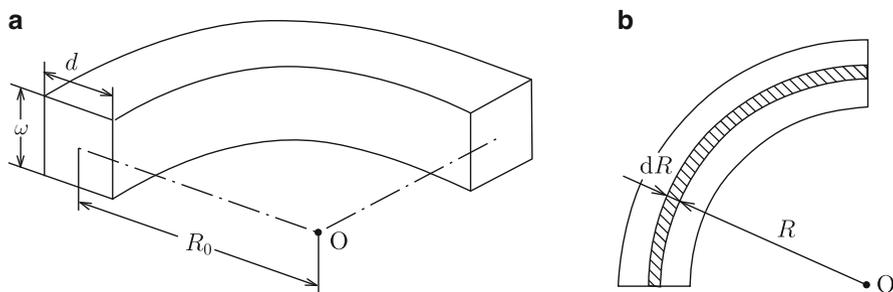
uniformly. On the other hand, Fig. 5.6b shows the electric flux when an electric potential difference is applied to the two electrodes of a capacitor composed of a long thin dielectric material. The electric flux lines spread out of the dielectric material, although they pass more easily through the dielectric material than through the vacuum. This difference arises from the large difference in material constants mentioned above. The reason why Eq. (5.38) holds in the case of Fig. 5.5 is that the electric field has completely the same form under the rigorous symmetry in both cases.

Here we discuss the boundary condition to be satisfied for the steady current at an interface between substances with different electric resistivities. Since the equation for the current density  $\mathbf{i}$  is formally the same as that for the electric flux density  $\mathbf{D}$  in the absence of electric charge, the boundary condition is also the same. That is, from Eq. (4.19) we have

$$\mathbf{n} \cdot (\mathbf{i}_1 - \mathbf{i}_2) = 0, \quad (5.39)$$

where  $\mathbf{n}$  is a unit vector normal to the interface. This shows that the normal component of the current density is continuous at the interface.

*Example 5.1.* Determine the electric resistance when current flows along a quarter of a circle of radius  $R_0$  with a rectangular cross-section, as shown in Fig. 5.7a. The electric resistivity is  $\rho_r$ .



**Fig. 5.7** (a) Shape of a quarter circular prism and (b) part of a thin region of radius  $R$  to  $R + dR$

**Solution 5.1.** We apply electric potential difference  $V$  between the two edges. The electric field at an arc of radius  $R$  from the center in Fig. 5.7b is

$$E(R) = \frac{2V}{\pi R}$$

and the current density is

$$i(R) = \frac{2V}{\pi\rho_r R}.$$

The current that flows in the region of  $R$  to  $R + dR$  is  $i(R)wdR$  and the total current is

$$I = \int_{R_0-d/2}^{R_0+d/2} \frac{2wV}{\pi\rho_r R} dR = \frac{2wV}{\pi\rho_r} \log \frac{R_0 + d/2}{R_0 - d/2}.$$

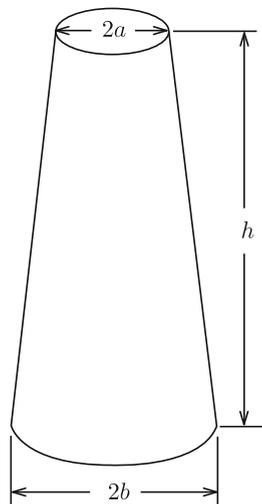
Thus, the electric resistance is given by

$$R_r = \frac{\pi\rho_r}{2w \log[(R_0 + d/2)/(R_0 - d/2)]}.$$

◇

*Example 5.2.* Determine the electric resistance along the length of the truncated cone in Fig. 5.8. The electric resistivity is  $\rho_r$ .

**Fig. 5.8** Long truncated cone



**Solution 5.2.** The cross-sectional area at position  $x$  from the bottom is

$$S(x) = \pi \left( b - \frac{b-a}{h}x \right)^2.$$

If the applied current is  $I$ , the current density at this position is

$$i(x) = \frac{I}{S(x)}.$$

Since the electric field is  $E(x) = \rho_r i(x)$ , the electric potential difference between the two edges is

$$V = \int_0^h \rho_r \frac{I}{S(x)} dx = \frac{I \rho_r h}{\pi ab}.$$

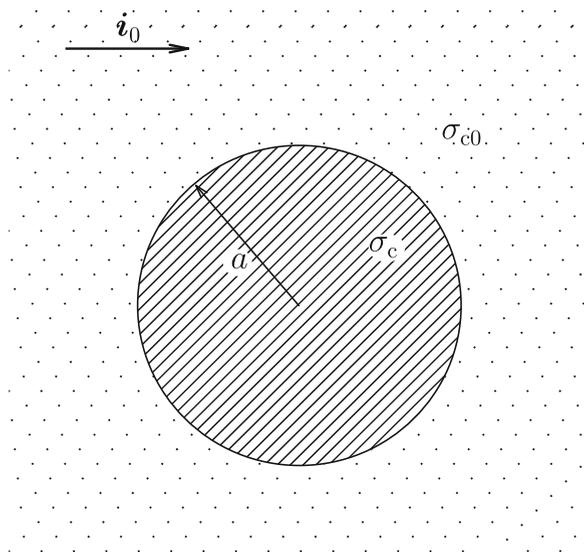
The electric resistance is

$$R_r = \frac{\rho_r h}{\pi ab}.$$

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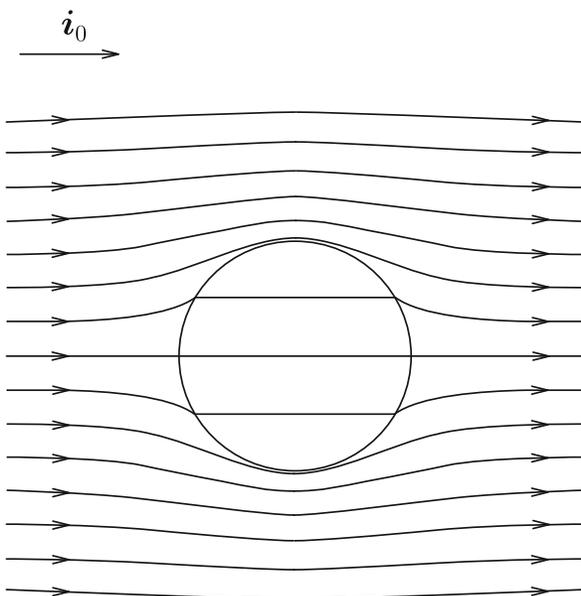
*Example 5.3.* Current of density  $i_0$  flows uniformly in a substance of electric conductivity  $\sigma_{c0}$ . When a part of a sphere of radius  $a$  in this substance is replaced by a different substance of electric conductivity  $\sigma_c$ , as shown in Fig. 5.9, determine the current density inside and outside the sphere.

**Fig. 5.9** Sphere of radius  $a$  with different electrical conductivity from the surrounding uniform substance



**Solution 5.3.** The equations describing the phenomena have the same form as those describing the electric flux density in the absence of an electric charge. Hence, the

**Fig. 5.10** Current around a sphere with different electric conductivity for  $\sigma_c = \sigma_{c0}/3$



solution for the current density and that for the electric flux density are formally the same. That is, we can use the solution for  $\mathbf{D}$  in Example 4.4 by replacing  $\epsilon_0$  and  $\epsilon$  with  $\sigma_{c0}$  and  $\sigma_c$ . In this case the uniform electric field  $E_0$  corresponds to  $i_0/\sigma_{c0}$ . We define polar coordinates with the origin at the center of the sphere. We denote by  $\theta$  the zenithal angle measured from the direction of the applied current. The current density outside the sphere ( $r > a$ ) is

$$i_r = \left(1 + \frac{\sigma_c - \sigma_{c0}}{\sigma_c + 2\sigma_{c0}} \cdot \frac{2a^3}{r^3}\right) i_0 \cos \theta, \quad i_\theta = -\left(1 - \frac{\sigma_c - \sigma_{c0}}{\sigma_c + 2\sigma_{c0}} \cdot \frac{a^3}{r^3}\right) i_0 \sin \theta$$

and that inside the sphere ( $0 \leq r < a$ ) is uniform:

$$i_r = \frac{3\sigma_c}{\sigma_c + 2\sigma_{c0}} i_0 \cos \theta, \quad i_\theta = -\frac{3\sigma_c}{\sigma_c + 2\sigma_{c0}} i_0 \sin \theta.$$

Figure 5.10 shows the current around the sphere. ◇

## 5.5 Electromotive Force

Suppose that a steady current,  $\mathbf{I}$ , flows in a closed electric circuit,  $C$ . When integrating the current along the circuit, we have

$$\oint_C \mathbf{I} \cdot d\mathbf{s} = lI, \quad (5.40)$$

where  $l$  is the perimeter of the circuit. On the other hand, with the aid of Ohm’s law the left side of this equation can be written as

$$S \oint_C \mathbf{i} \cdot d\mathbf{s} = S\sigma_c \oint_C \mathbf{E} \cdot d\mathbf{s} = 0, \tag{5.41}$$

where  $S$  and  $\sigma_c$  are the cross-sectional area and electric conductivity of the circuit, respectively. Thus, we have  $I = 0$ , which contradicts the assumption of a steady current. This gives  $\mathbf{E} = 0$ . On the other hand, under the initial condition of  $\mathbf{v} = \mathbf{v}_0$  ( $I = I_0 = -en_e|\mathbf{v}_0|S$ ) we obtain the solution to Eq. (5.21) as

$$I = I_0 \exp\left(-\frac{\eta t}{m}\right). \tag{5.42}$$

This also shows that  $I$  reduces to zero after a very short period. A steady current can continue to flow only in superconductors with  $\rho_r = 0$ .

Hence, to get a steady current in the general case, it is necessary to have an **electric power source** that applies an electric potential difference to force the current to flow in a circuit. The electric potential difference that the electric power source generates is called **electromotive force**. Table 5.3 lists practical electric power sources and the kinds of electromotive force. The unit of the electromotive force is [V]. Except in the case of generator, the electric energy of the electromotive force provided by the sources is transformed from chemical, mechanical, thermal or optical energy.

Suppose a closed circuit with an electric power source of electromotive force  $V$ . We denote the part of electric power source and the remaining part of the electric circuit as  $\Delta C$  and  $C'$ (= $C-\Delta C$ ), respectively, as shown in Fig. 5.11. The electric field due to the electromotive force is denoted  $\mathbf{E}_{em}$ . We then have

$$V = - \int_{\Delta C} \mathbf{E}_{em} \cdot d\mathbf{s}. \tag{5.43}$$

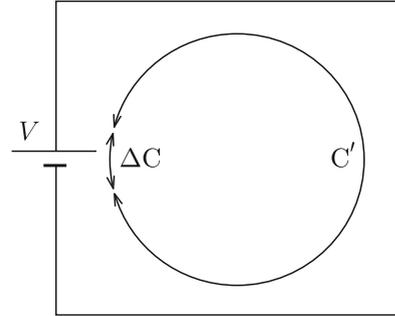
In the above the integral is directed along the current. We define the electric potential  $\phi$  that also includes the electromotive force. This satisfies

$$-\nabla\phi = \rho_r \mathbf{i} \tag{5.44}$$

**Table 5.3** Kinds of electric power source and electromotive force

Electric power source	Kind of electromotive force
Battery	Chemical electromotive force
Generator	Electromagnetic induction
Thermocouple	Thermoelectric power
Photo-electric cell	Photovoltaic effect

**Fig. 5.11** Closed electric circuit with electric power source



in  $C'$  and

$$-\nabla\phi = \mathbf{E}_{em} \quad (5.45)$$

in  $\Delta C$ . The condition

$$\oint_C \nabla\phi \cdot d\mathbf{s} = 0, \quad (5.46)$$

which is required for the electric potential, gives

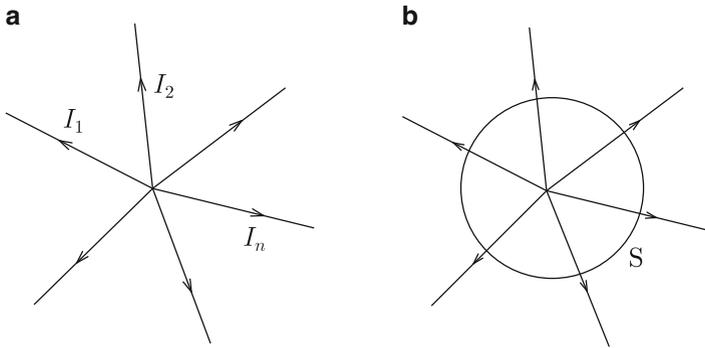
$$\frac{\rho_r}{S} \int_{C'} \mathbf{I} \cdot d\mathbf{s} = V. \quad (5.47)$$

Thus, the steady current is realized by the electric power source. The electric potential difference on the left side is due to the current that flows in an electric resistor and is called a **voltage drop**. The direction of the electric field is the same as that of the current but is opposite that of the electric field due to the electromotive force.

## 5.6 Kirchhoff's Law

In an electrical network composed only of resistors and DC (direct current) electric power sources, the current flows in a steady state. An important law that describes the steady current is **Kirchhoff's law**. We derive this law from the principles of electromagnetism.

Kirchhoff's first law states that the algebraic sum of currents passing out of an arbitrary node is zero. Here the currents that pass out and in are considered to be positive and negative, respectively. Applying this law to a node in Fig. 5.12a, we have



**Fig. 5.12** (a) Currents that flow out of and into a node and (b) closed surface that includes the node

$$\sum_n I_n = 0, \quad (5.48)$$

where  $I_n$  is the current that passes out of the node through the  $n$ -th branch. Suppose a closed surface  $S$  that includes the node (see Fig. 5.12b). The surface integral of the current density  $\mathbf{i}$  on  $S$  gives

$$\int_S \mathbf{i} \cdot d\mathbf{S} = \sum_n I_n. \quad (5.49)$$

Using Gauss' theorem and Eq. (5.11), the left side of Eq. (5.49) is rewritten as

$$\int_V \nabla \cdot \mathbf{i} \, dV = 0, \quad (5.50)$$

where  $V$  is the interior of  $S$ . Thus, we obtain Kirchhoff's first law, Eq. (5.48).

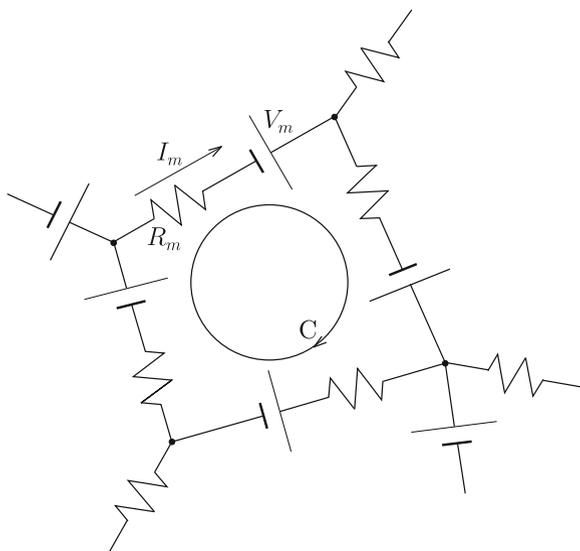
Kirchhoff's second law states that the sum of electromotive forces is equal to the sum of voltage drops in resistors in an arbitrary closed circuit composed of branches in an electrical network. This is expressed as

$$\sum_m V_m = \sum_m R_m I_m \quad (5.51)$$

for closed circuit  $C$  in Fig. 5.13. In the above  $V_m$ ,  $R_m$  and  $I_m$  are the electromotive force, electric resistance and current in the  $m$ -th branch, respectively. We denote the electric potential including the electromotive force by  $\phi$ . The potential difference between the two edges of the  $m$ -th branch is then given by

$$\Delta\phi_m = \int_m \nabla\phi \cdot d\mathbf{s} = V_m - R_m I_m. \quad (5.52)$$

**Fig. 5.13** Closed circuit in an electrical network



In the above the relationship  $\int_m (\rho_r/S) ds = R_m$  is used. From uniqueness of the electric potential we have

$$\oint_C \nabla\phi \cdot ds = \sum_m \Delta\phi_m = 0. \quad (5.53)$$

Substituting Eq. (5.52) into this equation gives Kirchoff's second law, Eq. (5.51).

### Column: Difference in Electric Resistivity

There are no material constants that differ so much depending on material as electric resistivity. Electric resistivity differs by a factor of  $10^{23}$  between silver and quartz glass. In superconducting materials covered in Chap. 7, the electric resistivity is theoretically predicted to be zero. However, it is impossible to experimentally prove that the resistivity is absolutely zero.

For example, if the electric potential difference is measured to be less than the noise level of 10 nV when current of 1 A is applied to a superconducting wire of cross-sectional area  $1 \text{ mm}^2$  and length 1 m, we can prove that the electric resistivity is less than  $10^{-14} \text{ } \Omega\text{m}$ . This is close to the limit of sensitivity of measurement instruments. It should be noted that the electric resistivity is of the order of  $10^{-11} \text{ } \Omega\text{m}$  even for high-purity copper at extremely low temperatures such as 4.2 K. Observation of the decay of current due to electric resistivity is much more sensitive. If the inductance

and electric resistance of a closed circuit are  $L$  and  $R_r$ , the current decays as

$$I(t) = I(0) \exp\left(-\frac{t}{\tau}\right); \quad \tau = \frac{L}{R_r},$$

as known for an electrical circuit. Hence, we can estimate the electric resistance  $R_r$  from the time constant  $\tau$ . For a closed circuit in which two parallel superconducting wires of length  $l = 0.5$  m and cross-sectional area  $S = 1 \text{ mm}^2$  separated by  $d = 10$  mm are connected at both edges with zero resistance, we calculate the inductance as

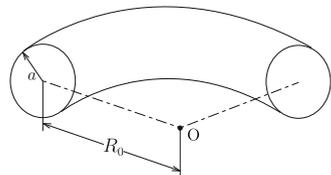
$$L \simeq \frac{\mu_0 l}{\pi} \log \frac{d}{a} \simeq 5.8 \times 10^{-7} \text{ H}$$

(see Example 8.1). In the above  $\mu_0$  is the magnetic permeability of vacuum. If no decay is observed with measurement uncertainty less than 0.1% over 3 years (approximately  $0.95 \times 10^8$  s),  $\tau$  is larger than  $0.95 \times 10^{11}$  s. Thus,  $R_r$  is less than  $6 \times 10^{-18} \Omega$  and we can say that the electric resistivity is less than  $6 \times 10^{-24} \Omega\text{m}$ .

Hence, the practical difference in electric resistivity reaches the level of  $10^{38}$ . The difference in size between a hydrogen atom and the universe (about 15 billion light years) is of the order of  $10^{36}$ , which gives an idea of the huge range of electric resistivity. Completely different electric properties of different materials are due to such a dramatic difference in the electric resistivity.

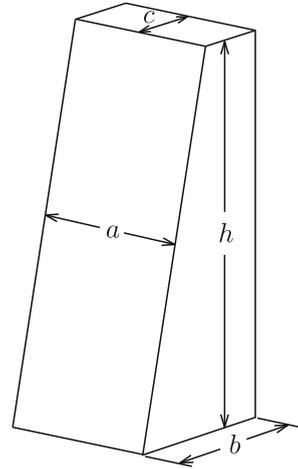
## Exercises

- 5.1.** Determine the electric resistance along the length of a quarter ring of radius  $R_0$  with a circular cross-section of radius  $a$ , as shown in Fig. E5.1. The electric resistivity is  $\rho_r$ .
- 5.2.** Determine the electric resistance along the length of a substance with electric resistivity  $\rho_r$ , as shown in Fig. E5.2.

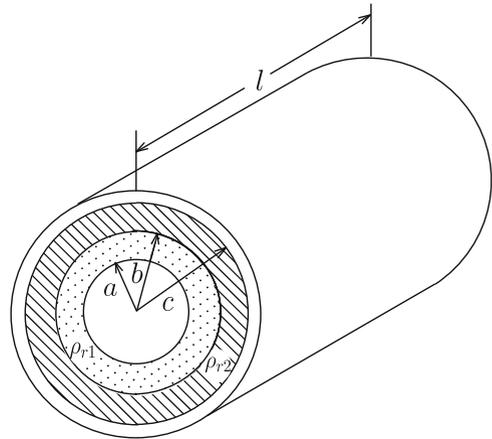


**Fig. E5.1** Quarter ring with circular cross-section

**Fig. E5.2** Long substance with rectangular cross-section



**Fig. E5.3** Long coaxial resistor composed of two types of substance with different electric resistivities



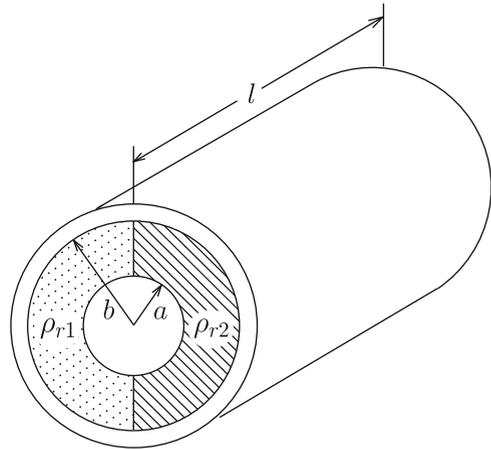
**5.3.** The space between two long coaxial electrodes is occupied by two types of substance with electric resistivities  $\rho_{r1}$  and  $\rho_{r2}$ , as shown in Fig. E5.3. Determine the electric resistance between the two electrodes.

**5.4.** The space between two long coaxial electrodes is occupied by two types of substance with electric resistivities  $\rho_{r1}$  and  $\rho_{r2}$ , as shown in Fig. E5.4. Determine the electric resistance between the two electrodes.

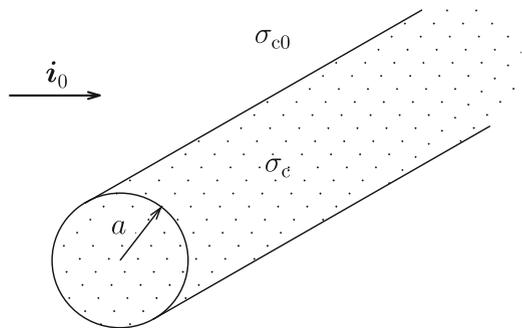
**5.5.** Suppose that, when we apply electric potential difference  $V$  between the two edges of the quarter circular prism shown in Fig. 5.7a, current  $I$  flows. Prove that the total electric power dissipated in this resistor is  $VI$ . (Hint: Integrate the loss power density given by Eq. (5.31).)

**5.6.** Prove that Eq. (5.38) holds also for two parallel plate electrodes with area  $S$  and distance  $d$ .

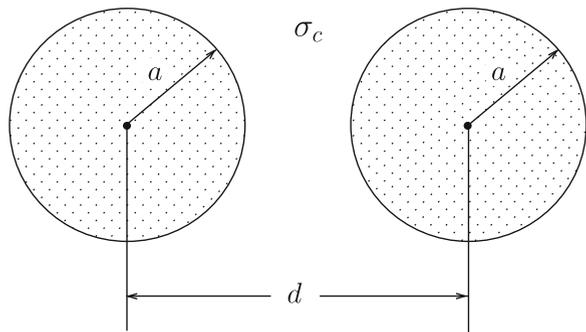
**Fig. E5.4** Long coaxial resistor composed of two types of substance with different electric resistivities



**Fig. E5.5** Current applied normal to a long cylindrical rod with a different electric conductivity embedded in a uniform substance



**Fig. E5.6** Cross-section of two parallel cylindrical conductors embedded in a uniform substance.



**5.7.** A long cylindrical rod of radius  $a$  and electric conductivity  $\sigma_c$  is embedded in a substance of electric conductivity  $\sigma_{c0}$ . Determine the current density inside and outside the rod, when we apply a uniform current of density  $i_0$  to an infinitely wide region, as shown in Fig. E5.5.

**5.8.** Two parallel long cylindrical conductors of radius  $a$  are embedded in a uniform substance of electric conductivity  $\sigma_c$ , as shown in Fig. E5.6. The distance between

the central axes of the two conductors is  $d(> 2a)$  and the electric resistivity of the two conductors is negligibly small. Determine the electric resistance in a unit length between the two conductors. (Hint: Convert this problem to the problem of determining the capacitance between two conductors with replacement of a substance having electric conductivity  $\sigma_c$  with a dielectric material having dielectric constant  $\epsilon$ . Place virtual positive and negative line charges having densities  $\pm\lambda$  at suitable positions so that the electric potential has constant values on the surfaces of the two conductors.)