Chapter 7
Superconductors

7.1 Magnetic Properties of Superconductors

A **superconductor** is a material that loses its electric resistance when cooled below a characteristic temperature called the critical temperature. Many elements, alloys and compounds are superconductors. This state of zero resistivity is called the **superconducting state**. A superconductor has not only this property but also **perfect diamagnetism**. That is, when a magnetic flux density is applied to a superconductor, the interior magnetic flux density is zero:

$$\boldsymbol{B} = 0. \tag{7.1}$$

When the temperature is above the critical temperature, the superconductor is in a **normal state** with nonzero resistance. In this case the magnetic flux density penetrates the superconductor. If the superconductor is cooled below the critical temperature, the magnetic flux is expelled from the superconductor and perfect diamagnetism occurs. These diamagnetic phenomena are called the **Meissner–Ochsenfeld effect**.

The perfect diamagnetic state is realized by a current that flows on the surface of the superconductor, as will be mentioned. This is similar to the electric property of a conductor that the inside is completely shielded by an electric charge induced on the surface when an external electric field is applied. In this chapter we study the magnetic phenomena around a superconductor in the perfect diamagnetic state. The perfect diamagnetic state of $\boldsymbol{B} = 0$ occurs in a type 1 superconductor or type 2 superconductor in a magnetic flux density below the lower critical flux density (see Sect. A3.2 in the Appendix). Special knowledge is needed to understand the physical mechanism that causes the zero resistivity and perfect diamagnetism. This is not discussed in this textbook. Section A3.1 in the Appendix gives a brief explanation of phenomenological theory. For a more detailed understanding, it is recommended to read technical books.

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Materials are roughly classified into conductors and insulators (dielectric materials) with respect to their electric properties. For the magnetic properties, materials are classified into superconductors and **magnetic materials**. The latter will be covered in Chap. 9.

Since Eq. (7.1) holds inside the superconductor, from Eq. (6.27), we have

$$\boldsymbol{i} = 0. \tag{7.2}$$

Hence, current flows only on the surface of the superconductor. Equation (6.29) generally gives

$$\boldsymbol{A} = \nabla\alpha \tag{7.3}$$

with α denoting a scalar function. However, it is no problem to assume as

$$\boldsymbol{A} = \text{const.} \tag{7.4}$$

in most cases. This is a special case of Eq. (7.3). If the superconducting region is not simply connected but a magnetic flux penetrates a space surrounded by the superconductor, the vector potential in the superconductor is not a constant. Equations (7.1), (7.2) and (7.4) correspond to Eqs. (2.1), (2.2) and (2.3) for conductors, respectively.

Here, we discuss the magnetic flux density in the vicinity of the superconductor surface. Suppose a small closed surface of a pellet that includes the interface between the superconductor and vacuum, as shown in Fig. 7.1. We denote the height of the pellet and the area of surface inside the pellet by Δh and ΔS, respectively. Assume that the upper and lower surfaces of the pellet are parallel to the superconductor surface. We apply Gauss' law, Eq. (6.20), to the surface of the pellet, ΔS, and we have

$$\int_{\Delta \mathrm{S}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0. \tag{7.5}$$

When the height Δh is sufficiently small, we can neglect the magnetic flux going out of the side surface. Since $\boldsymbol{B} = 0$ on the lower surface in the superconductor, there

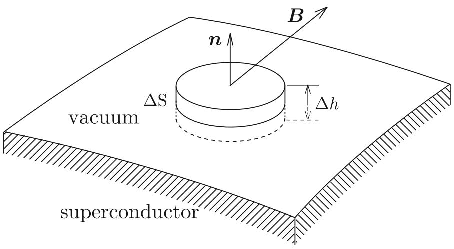

Fig. 7.1 Small closed surface that includes a part of superconductor surface

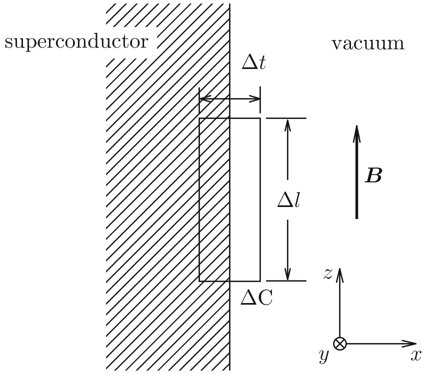

Fig. 7.2 Magnetic flux density parallel to wide superconductor surface

is no magnetic flux going out from this surface. Hence, we can conclude that there is no magnetic flux going out from the upper surface. This means that the magnetic flux density is parallel to the superconductor surface. That is,

$$\boldsymbol{B} \cdot \boldsymbol{n} = 0, \tag{7.6}$$

where $\boldsymbol{n}$ is the unit vector normal to the superconductor surface and directed outward. This is in contrast with the electric field, which is normal to the conductor surface.

Next, we discuss the current that flows on the superconductor surface. Suppose that the magnetic flux density B is parallel to a wide surface of the superconductor, as shown in Fig. 7.2. We define the coordinates as in the figure. We apply Ampere's law, Eq. (6.25), to a small rectangular closed loop, ΔC, with two sides of length Δl parallel to the z-axis and two sufficiently short sides of length Δt parallel to the x-axis. Since the magnetic flux density inside the superconductor is zero and there is no contribution from the two sides of length Δt, the closed curvilinear integral gives

$$\oint_{\Delta \mathrm{C}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = B\Delta l. \tag{7.7}$$

This is equal to the current flowing along the negative y-axis inside ΔC multiplied by μ_0, that is, $\mu_0 \tau \Delta l$ with τ denoting the surface current density. Thus, we have

$$B = \mu_0 \tau. \tag{7.8}$$

It should be noted that the directions of the current and magnetic flux density follow the right hand rule. This relationship is also in contrast with Eq. (2.5) for conductors.

As has been discussed above, magnetic phenomena with perfect diamagnetism ($\boldsymbol{B} = 0$) in superconductors are similar to electric phenomena with perfect electrostatic shielding ($\boldsymbol{E} = 0$) in conductors. This similarity is prominent in the

$\boldsymbol{E}$–$\boldsymbol{B}$ analogy in electromagnetism. In addition, the correspondence of Eqs. (7.2), (7.4) and (7.8) to Eqs. (2.2), (2.3) and (2.5) shows that the similarity between electricity and magnetism is considerably deep. Especially, it should be noted that shielding current must continue to flow on the surface of a superconductor and hence, the electric resistance of the superconductor must be zero to achieve perfect diamagnetism in the superconductor (see Exercise 7.11). With the $\boldsymbol{E}$–$\boldsymbol{B}$ analogy, it was possible for someone even in the 19th Century to predict the existence of a material with perfect diamagnetism, i.e., a superconductor, just after completion of the Maxwell theory. In reality the superconductivity with zero resistivity was discovered independently of the above consideration in 1911, and perfect diamagnetism was discovered 22 years later in 1933.

Here, we show an example of magnetic phenomena associated with a superconductor. Suppose that current I is applied to a long cylindrical superconductor of radius a. We determine the magnetic flux density and vector potential inside and outside the superconductor. We define cylindrical coordinates with the z-axis on the central axis of the superconductor. The current flows uniformly only on the superconductor surface and the magnetic flux density does not appear inside the superconductor. Thus, the surface current density is $\tau = I/(2\pi a)$.

We define a circle, C, of radius R from the central axis on a plane normal to the axis (see Fig. 7.3). We apply Ampere's law to C. Since the current distribution is cylindrically symmetric, we can assume the magnetic flux density also has cylindrical symmetry. Hence, the magnetic flux density is parallel to C and its magnitude B is constant on C. Thus, the left side of Eq. (6.25) is $2\pi R B(R)$. All the current I flows through C and the right side of Eq. (6.25) is $\mu_0 I$ for $R > a$. We obtain the magnetic flux density as

$$B(R) = \frac{\mu_0 I}{2\pi R}; \qquad R > a. \tag{7.9}$$

The magnetic flux density outside the superconductor is the same as that when all the current is concentrated along the central axis. For $R < a$ the right side of Eq. (6.25) is zero. This gives

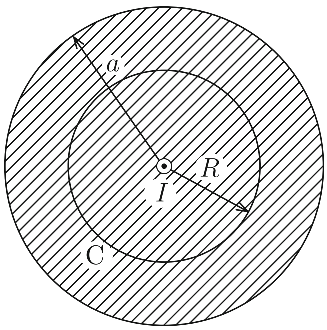

Fig. 7.3 Cross-section of cylindrical superconductor and closed circle C (for $R < a$)

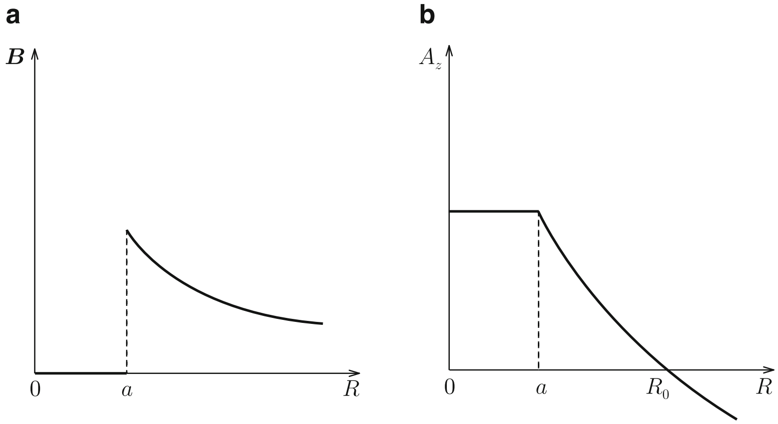

Fig. 7.4 (**a**) Magnetic flux density and (**b**) vector potential inside and outside the current-carrying cylindrical superconductor

$$B(R) = 0; \qquad 0 \le R < a. \tag{7.10}$$

Thus, Eq. (7.1) is fulfilled inside the superconductor. We can also show that Eq. (7.9) satisfies Eq. (7.8) on the superconductor surface ($R = a$) with the surface current density determined above.

Now we determine the vector potential using the above results. From Eq. (6.33) we find that the vector potential has only the z-component, A_z, which is given by

$$A_z(R) = -\int B(R)\mathrm{d}R. \tag{7.11}$$

We choose the reference point for zero vector potential at distance $R_0(> a)$ from the central axis, as done in Example 6.6. Then, the vector potential is determined to be

$$A_z(R) = \frac{\mu_0 I}{2\pi} \log \frac{R_0}{R}; \qquad R > a, \tag{7.12a}$$

$$= \frac{\mu_0 I}{2\pi} \log \frac{R_0}{a}; \qquad 0 \le R < a. \tag{7.12b}$$

Figure 7.4a, b shows the obtained magnetic flux density and vector potential. The vector potential is constant inside the superconductor and Eq. (7.4) is satisfied.

Example 7.1. Suppose a long superconducting coaxial transmission line, as shown in Fig. 7.5. Determine the magnetic flux density and vector potential when we apply current I only to the inner superconductor.

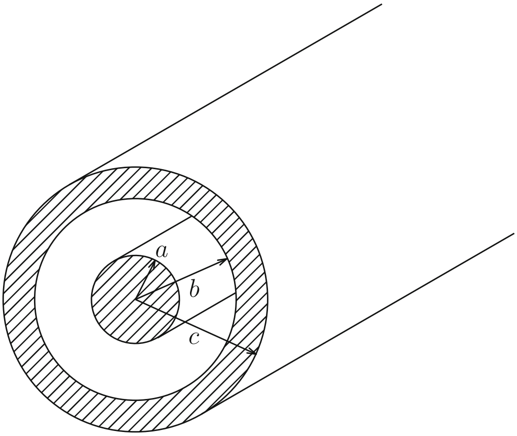

Fig. 7.5 Long superconducting coaxial transmission line

Solution 7.1. We use cylindrical coordinates. From cylindrical symmetry the current I flows uniformly on the surface of the inner superconductor ($R = a$) and the internal magnetic flux density ($R < a$) is zero. The current is induced on the inner surface ($R = b$) of the outer superconductor so that the magnetic flux does not penetrate the outer superconductor. This current is denoted by I_b. We apply Ampere's law to a circle C of radius $R(a < R < b)$ from the central axis on a plane perpendicular to the axis. Then, we have

$$\oint_{\mathrm{C}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = \mu_0(I + I_b).$$

Since $\boldsymbol{B} = 0$ on C, this gives $I_b = -I$. Since the total current is zero in the outer superconductor, an opposite current, i.e., I flows on the outer surface ($R = c$) of the outer superconductor.

If we denote the total current passing through C of radius R as I_R, Ampere's law gives

$$B(R) = \frac{\mu_0 I_R}{2\pi R}.$$

Since I_R is equal to I, 0 and I for $a < R < b$, $b < R < c$ and $R > c$, respectively, we determine the magnetic flux density to be

$$\begin{aligned} B(R) &= 0; && 0 \le R < a, \\ &= \frac{\mu_0 I}{2\pi R}; && a < R < b, \\ &= 0; && b < R < c, \\ &= \frac{\mu_0 I}{2\pi R}; && R > c. \end{aligned}$$

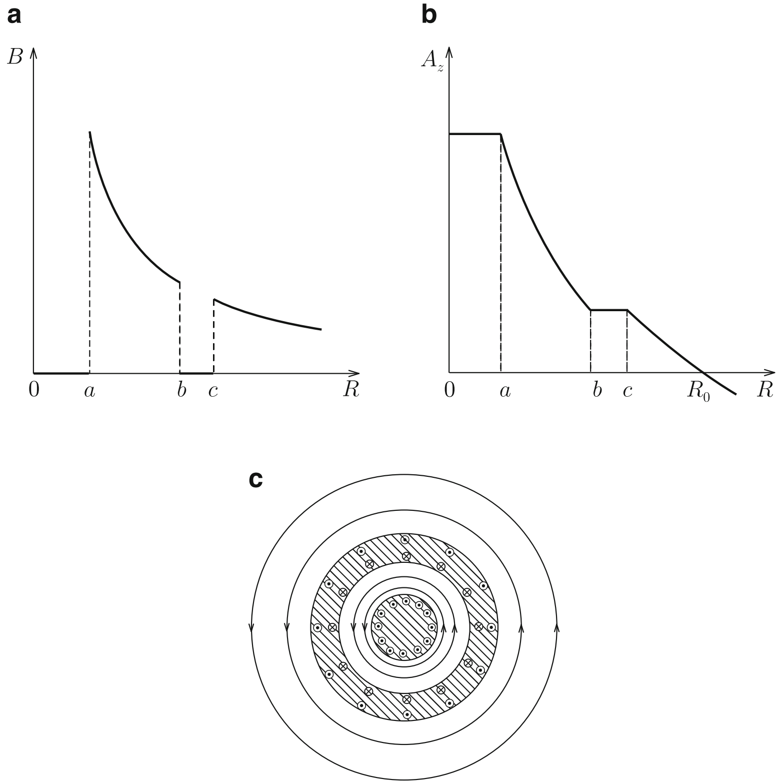

Fig. 7.6 (**a**) Magnetic flux density, (**b**) z-component of vector potential and (**c**) magnetic flux lines when current is applied to the inner superconductor of a superconducting coaxial transmission line

The vector potential has only the z-component and is determined to be

$$\begin{aligned} A_z(R) &= \frac{\mu_0 I}{2\pi} \log \frac{R_0}{R}; && R > c, \\ &= \frac{\mu_0 I}{2\pi} \log \frac{R_0}{c}; && b < R < c, \\ &= \frac{\mu_0 I}{2\pi} \log \frac{bR_0}{cR}; && a < R < b, \\ &= \frac{\mu_0 I}{2\pi} \log \frac{bR_0}{ac}; && 0 \leq R < a. \end{aligned}$$

In the above, $R_0(> c)$ is the distance to the reference point.

Figure 7.6a–c shows the obtained magnetic flux density, vector potential and magnetic flux lines, respectively. The above results for the magnetic flux density and vector potential are formally the same as for the electric field and electric potential of the coaxial cylindrical conductor when electric charge is given to the inner conductor (see Example 2.2).

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There is a treatment called grounding for a conductor. What is the corresponding treatment for a superconductor? It is to connect the superconductor with a current path with infinitely large capacity. In the above case a connection of the outer superconductor to infinity moves the back current on the surface at $R = c$ to infinity. We also call this grounding. Then, the current is I at $R = a$ and $-I$ at $R = b$. The magnetic flux density and vector potential are

$$
\begin{aligned}
B(R) &= 0; && 0 \le R < a, \\
&= \frac{\mu_0 I}{2\pi R}; && a < R < b, \\
&= 0; && R > b.
\end{aligned}
$$

and

$$
\begin{aligned}
A_z(R) &= \frac{\mu_0 I}{2\pi} \log \frac{b}{a}; && 0 \le R < a, \\
&= \frac{\mu_0 I}{2\pi} \log \frac{b}{R}; && a < R < b, \\
&= 0; && R > b.
\end{aligned}
$$

Figure 7.7a, b shows the obtained results.

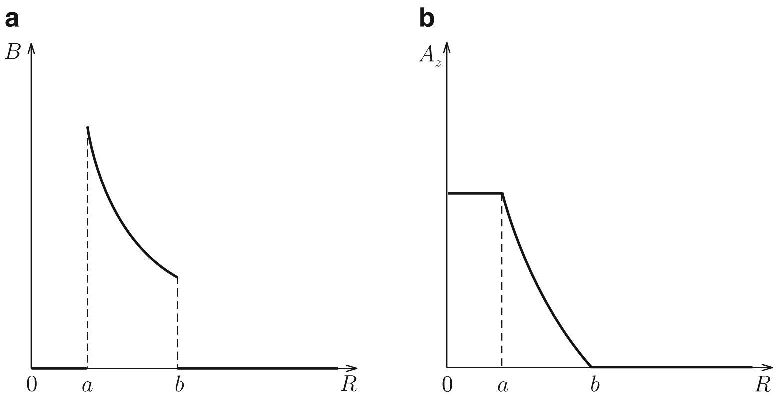

Fig. 7.7 (**a**) Magnetic flux density and (**b**) z-component of vector potential when we apply a current to the inner superconductor of superconducting coaxial transmission line and ground the outer superconductor

7.2 Special Solution Method for Magnetic Flux Density

We determine the current distribution on the superconductor surface or magnetic flux density around a superconductor when the superconductor is placed in an applied magnetic flux density. The vector potential in the superconductor is constant in space, as shown by Eq. (7.4). Outside the superconductor, there is no current and the vector potential $\boldsymbol{A}$ satisfies Laplace's equation (6.38).

When we are given the boundary condition on the surface of a treated area, Laplace's equation can be solved uniquely, as mentioned in Sect. 2.2. Hence, there is only one solution for $\boldsymbol{A}$ in the space outside the superconductor, which becomes a constant value on the surface of the superconductor. Hence, if some function satisfies the boundary condition, it is a solution. This is the same as for the electric potential around a conductor. Some solution methods are introduced here. It may be helpful for readers to compare these methods with those mentioned in Sect. 2.2 for conductors.

First, suppose we apply current I through a thin straight line placed at distance a from a wide flat surface of a superconductor, as shown in Fig. 7.8a. A current of opposite direction appears on the surface of the superconductor to shield it, and exerts a repulsive force on I. We determine the distribution of the induced current similarly to the electric charge distribution on the conductor surface in Fig. 2.9.

We define the x-y plane on the superconductor surface. Suppose that current I flows on the line at $x = 0$ and $z = a$ along the positive y-axis. The magnetic flux density on the surface ($z = 0$) must be parallel to the superconductor, as shown in Sect. 7.1. We can realize this situation by virtually removing the superconductor and then applying a straight current with the same magnitude and opposite direction at the symmetric position ($x = 0$, $z = -a$) with respect to the superconductor surface. This will be confirmed below. The vector potential in the vacuum region ($z > 0$) produced by the two straight currents has only the y-component parallel to the currents, and we calculate it as

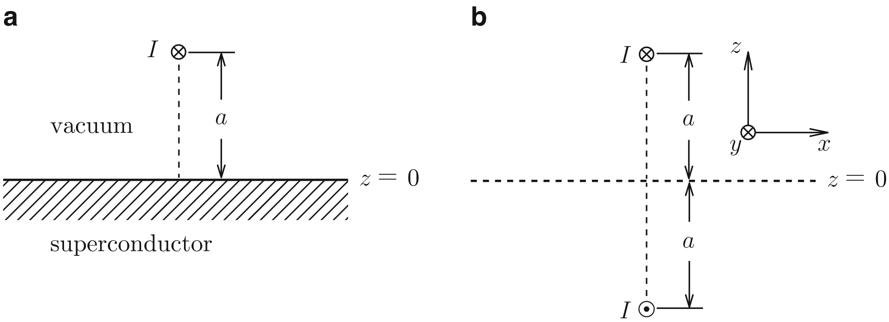

Fig. 7.8 (**a**) Thin straight current I placed at distance a from wide flat surface of superconductor and (**b**) imaginary current placed at a mirror position with respect to the superconductor surface

$$A_y(x, y, z) = \frac{\mu_0 I}{2\pi}\left\{\log\frac{R_0}{[x^2 + (z-a)^2]^{1/2}} - \log\frac{R_0}{[x^2 + (z+a)^2]^{1/2}}\right\}$$
$$= \frac{\mu_0 I}{4\pi}\log\frac{x^2 + (z+a)^2}{x^2 + (z-a)^2} \tag{7.13}$$

using Eq. (7.12a). In the above, R_0 is the distance from the current to the reference point. We can easily show that $A_y = 0$ on the superconductor surface ($z = 0$). Thus, the condition, Eq. (7.4), is satisfied. Laplace's equation is satisfied except at the position of the current in the vacuum region ($z > 0$). Hence, we conclude that Eq. (7.13) is the solution for the vector potential. The vector potential inside the superconductor ($z < 0$) is $A_y = 0$. Thus, the method of images is useful also for superconductors. The imaginary current placed at the mirror position is called an **image current**.

Using Eq. (7.13), we obtain the magnetic flux density in the vacuum region as

$$B_x = -\frac{\mu_0 I}{2\pi}\left[\frac{z+a}{x^2 + (z+a)^2} - \frac{z-a}{x^2 + (z-a)^2}\right],$$
$$B_y = 0, \tag{7.14}$$
$$B_z = \frac{\mu_0 I x}{2\pi}\left[\frac{1}{x^2 + (z+a)^2} - \frac{1}{x^2 + (z-a)^2}\right].$$

On the superconductor surface it reduces to

$$B_x(x, y, 0) = -\frac{\mu_0 I a}{\pi(x^2 + a^2)}, \qquad B_y(x, y, 0) = B_z(x, y, 0) = 0, \tag{7.15}$$

showing that the magnetic flux density is parallel to the surface. Figure. 7.9 shows the magnetic flux lines. Then, from Eq. (7.15) we determine the density of current

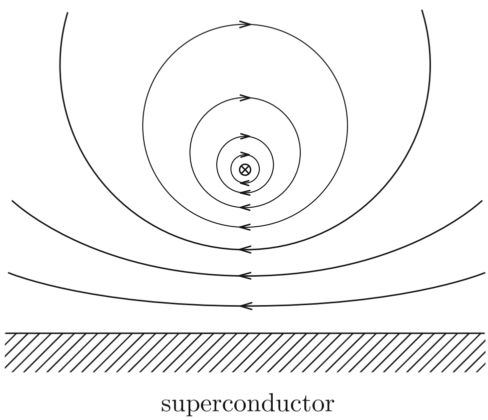

Fig. 7.9 Magnetic flux lines produced by given current and current induced on the superconductor surface

induced on the superconductor surface to be

$$\tau = -\frac{Ia}{\pi(x^2 + a^2)}. \tag{7.16}$$

The total current is

$$\int \mathrm{d}x\, \tau = -\frac{Ia}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 + a^2} = -I. \tag{7.17}$$

That is, the total current is equal to the image current. The force on the given current I caused by the current induced on the superconductor surface is equal to the force by the image current, and its magnitude in a unit length is

$$F' = \frac{\mu_0 I^2}{4\pi a}. \tag{7.18}$$

This force is repulsive ($F' > 0$). This is called the image force.

One can show that the current induced on the superconductor surface completely cancels the magnetic flux density produced by the given current in the superconductor ($z < 0$).

Suppose we apply current I through a line, A, separated by distance d from the central axis O of a grounded parallel long superconducting cylinder of radius a, as shown in Fig. 7.10a. Now we determine the vector potential outside the superconductor. We virtually remove the superconductor and apply an image current I' through a line, B, separated by h from the central axis of the superconducting cylinder, as shown in Fig. 7.10b, similarly to the example in Sect. 2.2. The vector potential at point P on the superconductor surface is given by

$$A_z = \frac{\mu_0 I}{2\pi} \log \frac{R_0}{(a^2 + d^2 - 2ad\cos\varphi)^{1/2}} + \frac{\mu_0 I'}{2\pi} \log \frac{R_0'}{(a^2 + h^2 - 2ah\cos\varphi)^{1/2}}, \tag{7.19}$$

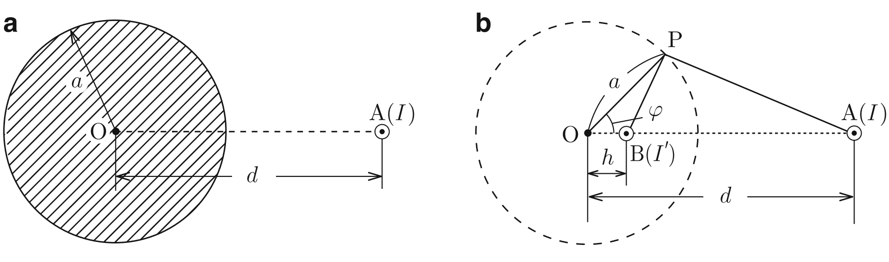

Fig. 7.10 (**a**) Long superconducting cylinder and straight current parallel to it and (**b**) image current placed on line B after removing the superconductor

where φ is the angle POA. In the above, R_0 and R'_0 are distances to a suitable reference point and are not important quantities. For the vector potential to be constant and independent of φ, the following conditions should be satisfied:

$$I' = -I, \tag{7.20}$$

$$\frac{2ad}{a^2 + d^2} = \frac{2ah}{a^2 + h^2}. \tag{7.21}$$

Equation (7.21) reduces to

$$h = \frac{a^2}{d}. \tag{7.22}$$

In this case the current-carrying wire and superconductor are infinitely long and hence, the current induced in the superconductor has the same magnitude. This is different from the case of the spherical conductor treated in Sect. 2.2.

The above results give the vector potential at point (R, φ),

$$A_z(R, \varphi) = \frac{\mu_0 I}{2\pi} \log \frac{d[R^2 + (a^2/d)^2 - 2(a^2 R/d)\cos\varphi]^{1/2}}{a(R^2 + d^2 - 2Rd\cos\varphi)^{1/2}} \tag{7.23}$$

when it is zero on the superconductor surface ($R = a$). That is, $R'_0 = (a/d)R_0$. We can calculate the magnetic flux density outside the superconductor using Eq. (7.23) (see Exercise 7.6). Figure 7.11 shows the magnetic flux lines. The current density on the superconductor surface is obtained as

$$\tau(\varphi) = \frac{1}{\mu_0} B_\varphi(R = a) = -\frac{1}{\mu_0}\left(\frac{\partial A_z}{\partial R}\right)_{R=a} = -\frac{I(d^2 - a^2)}{2\pi a(a^2 + d^2 - 2ad\cos\varphi)}. \tag{7.24}$$

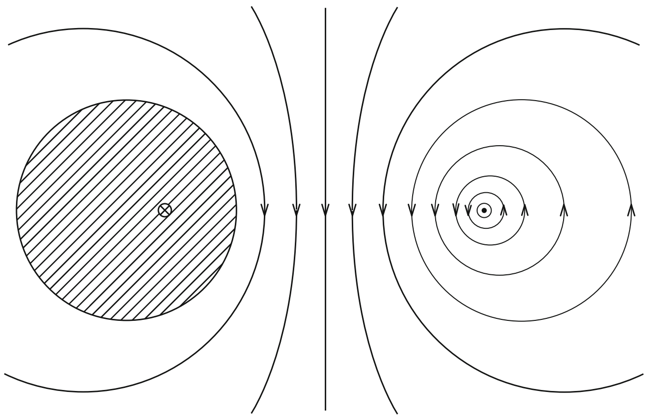

Fig. 7.11 Magnetic flux lines produced by long superconducting cylinder and straight current parallel to it

The total current is

$$\int_0^{2\pi} \tau(\varphi)\, a\mathrm{d}\varphi = -\frac{I(d^2-a^2)}{\pi}\int_0^{\pi}\frac{\mathrm{d}\varphi}{a^2+d^2-2ad\cos\varphi} = -I \tag{7.25}$$

and agrees with the image current, Eq. (7.20). In the above the following formula was used.

$$\int_0^{\pi}\frac{\mathrm{d}\varphi}{1-k\cos\varphi} = \frac{\pi}{(1-k^2)^{1/2}}. \tag{7.26}$$

This problem corresponds to estimating electric potential in Exercise 2.8.

Example 7.2. In the above we discussed the vector potential when straight current I is placed outside the grounded superconducting cylinder, as shown in Fig. 7.10. Determine the vector potential when the superconductor is isolated.

Solution 7.2. In this situation the total current flowing in the superconductor must be zero. Hence, we use a superposition. That is, we obtain the current distribution by superposing currents $-I$ and I in a way that makes the vector potential of the superconductor constant. Current $-I$ is distributed according to Eq. (7.24) and I has uniform distribution on the superconductor surface. The former current and the external current give the vector potential, Eq. (7.23), which we denote as $A_{1z}(R,\varphi)$. The latter current gives the vector potential, Eq. (7.12a). Both of them satisfy the requirement to be constant on the superconductor surface. Hence, the sum of these components gives the unique solution. If we determine the vector potential so that it is zero at the superconductor surface $R=a$, we have

$$A_z(R,\varphi) = A_{1z}(R,\varphi) + \frac{\mu_0 I}{2\pi}\log\frac{d}{R}.$$

◇

7.3 Magnetization

Suppose that a superconducting sphere of radius a is put in a uniform magnetic flux density, B_0 (see Fig. 7.12). The superconducting current flows on the surface to cancel the applied magnetic flux density inside the superconductor. Here we determine the density of this current and the magnetic flux density outside the superconductor. We use polar coordinates: the origin is at the center of the superconductor and the z-axis is parallel to the direction of the applied magnetic flux density.

When a conducting sphere is in a uniform electric field, the boundary condition is fulfilled by assuming an electric dipole of suitable moment placed at the center of the conductor, as discussed in Sect. 2.3. It seems useful to use a similar method

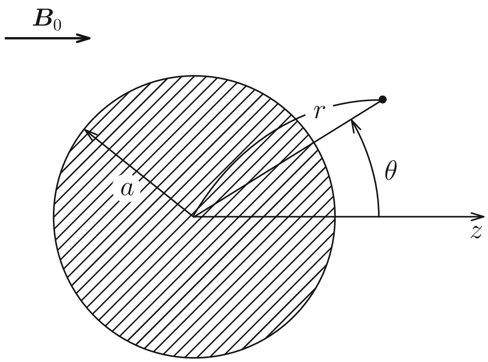

Fig. 7.12 Superconducting sphere in uniform magnetic flux density

assuming a magnetic moment placed at the center of the superconductor. In fact, the dipole moment of a pair of magnetic charges, which correspond to a pair of electric charges, is equivalent to the magnetic moment of a closed current, as shown in Sect. 6.8. This virtual closed current at the center flows along the azimuthal direction similarly to the real current on the surface.

The vector potential outside the superconducting sphere has only the azimuthal component, A_φ, corresponding to the current. This component consists of $A_{f\varphi}$ caused by the uniform magnetic flux density B_0 and $A_{d\varphi}$ caused by the magnetic moment m. The former component is given by

$$A_{f\varphi} = \frac{B_0 r}{2} \sin\theta. \tag{7.27}$$

Confirm for yourself that the following conditions are satisfied:

$$\frac{1}{r\sin\theta} \cdot \frac{\partial}{\partial\theta}(\sin\theta A_{f\varphi}) = B_0\cos\theta, \qquad -\frac{1}{r} \cdot \frac{\partial}{\partial r}(rA_{f\varphi}) = -B_0\sin\theta.$$

The latter component is given by Eq. (6.43),

$$A_{d\varphi} = \frac{\mu_0 m \sin\theta}{4\pi r^2}. \tag{7.28}$$

Thus, the vector potential is

$$A_\varphi = A_{f\varphi} + A_{d\varphi} = \left(\frac{B_0 r}{2} + \frac{\mu_0 m}{4\pi r^2}\right)\sin\theta. \tag{7.29}$$

The requirement that it is zero on the superconductor surface ($r = a$) gives

$$m = -\frac{2\pi a^3 B_0}{\mu_0}. \tag{7.30}$$

Hence, we determine the vector potential to be

$$A_\varphi = \frac{B_0}{2}\left(r - \frac{a^3}{r^2}\right)\sin\theta. \tag{7.31}$$

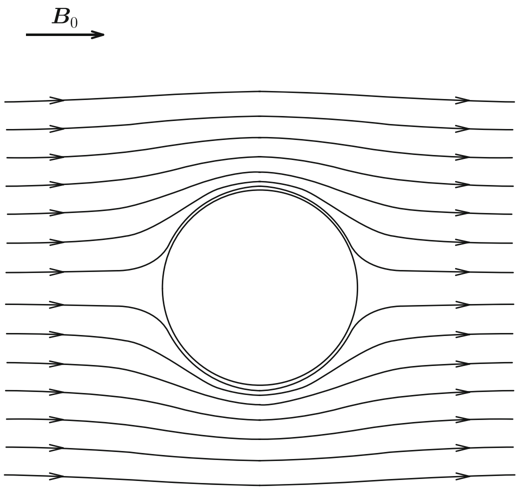

Fig. 7.13 Magnetic flux lines around superconducting sphere in uniform magnetic flux density

This satisfies the boundary condition on the superconductor surface and Laplace's equation. Hence, this is the solution. Inside the superconductor the solution is $A_\varphi = 0$.

We obtain the magnetic flux density outside the superconductor as

$$B_r = \frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\theta}(\sin\theta A_\varphi) = B_0\left(1-\frac{a^3}{r^3}\right)\cos\theta, \tag{7.32a}$$

$$B_\theta = -\frac{1}{r}\cdot\frac{\partial}{\partial r}(rA_\varphi) = -B_0\left(1+\frac{a^3}{2r^3}\right)\sin\theta, \tag{7.32b}$$

$$B_\varphi = 0. \tag{7.32c}$$

Figure 7.13 shows magnetic flux lines on the plane including the z-axis. Equation (7.32a) shows that the magnetic flux lines are parallel to the superconductor surface, $B_r(r = a) = 0$. We find from Eq. (7.32b) that the magnitude of the magnetic flux density takes the maximum value, $3B_0/2$, on the equator ($\theta = \pi/2$). The current density in the azimuthal direction on the superconductor surface is

$$\tau = \frac{B_\theta(r = a)}{\mu_0} = -\frac{3B_0}{2\mu_0}\sin\theta. \tag{7.33}$$

This azimuthal current produces a magnetic flux density opposite to the applied one. Using Eq. (7.30) we have the magnetic moment in a unit volume of the superconductor,

$$M = -\frac{3B_0}{2\mu_0}. \tag{7.34}$$

This is called **magnetization**. This corresponds formally to the magnetization of magnetic materials.

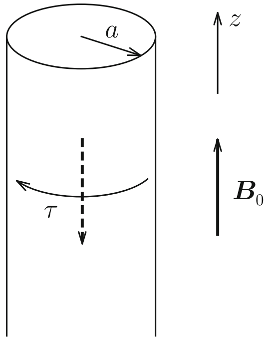

Fig. 7.14 Surface current and resultant magnetic moment (*broken line*) when magnetic flux density is applied parallel to long superconducting cylinder

The magnetization in a superconductor is given by the magnetic moment in a unit volume of the superconductor as defined above. In the above example, the magnetic flux density that the superconductor experiences on its surface is different from the applied value because of the geometry of the superconductor. Here we consider a simple case where there is no such geometrical effect. Suppose we apply magnetic flux density B_0 parallel to a long superconducting cylinder along the z-axis of radius a (see Fig. 7.14). The current flows along the negative azimuthal direction and produces a magnetic moment along the negative z-axis. The surface current density is

$$\tau = \frac{B_0}{\mu_0}. \tag{7.35}$$

The magnetic moment in a unit length due to this current is

$$m' = -\pi a^2 \tau = -\frac{\pi a^2 B_0}{\mu_0}. \tag{7.36}$$

Hence, the magnetization is

$$M = -\frac{B_0}{\mu_0}. \tag{7.37}$$

We assume that the superconductor is a type 2 superconductor and is in the mixed state with the internal magnetic flux density B (see Sect. A3.2 in the Appendix). Then, a similar discussion derives $\tau = (B_0 - B)/\mu_0$ and we have

$$M = -\frac{B_0 - B}{\mu_0}. \tag{7.38}$$

This is the common definition of magnetization of a superconductor when it is not influenced by the geometrical effect.

Example 7.3. An infinitely long superconducting cylinder of radius a is in a uniform perpendicular magnetic flux density of B_0. Determine the vector potential and magnetic flux density outside the superconductor and current density on the superconductor surface.

Solution 7.3. We use cylindrical coordinates with the z-axis at the central axis of the superconductor and the azimuthal angle measured from the direction of the applied magnetic flux density. From a similarity with Example 2.4 we can expect the following treatment to be useful for determining the vector potential: We virtually remove the superconductor and place a magnetic dipole line produced by a pair of anti-parallel straight currents at the axis. We denote the magnitude of the magnetic moment in a unit length along the z-axis by $\hat{m}$. From Eq. (6.52) the vector potential is given by

$$A_z(R,\varphi) = \left(B_0 R + \frac{\mu_0 \hat{m}}{2\pi R} \right) \sin\varphi,$$

where the first and second terms are components of the applied magnetic flux density and magnetic moment, respectively. The requirement $A_z(R = a) = 0$ gives

$$\hat{m} = -\frac{2\pi a^2 B_0}{\mu_0}.$$

Thus, the vector potential outside the superconductor is

$$A_z(R,\varphi) = B_0 \left(R - \frac{a^2}{R} \right) \sin\varphi.$$

We obtain the magnetic flux density as

$$\begin{aligned} B_R &= \frac{1}{R} \cdot \frac{\partial A_z}{\partial \varphi} = B_0 \left(1 - \frac{a^2}{R^2} \right) \cos\varphi, \\ B_\varphi &= -\frac{\partial A_z}{\partial R} = -B_0 \left(1 + \frac{a^2}{R^2} \right) \sin\varphi, \\ B_z &= 0. \end{aligned}$$

We can show that $B_R(R = a) = 0$. The surface current density is

$$\tau = \frac{B_\varphi(R = a)}{\mu_0} = -\frac{2B_0}{\mu_0} \sin\varphi.$$

The magnetization of the superconducting cylinder is

$$M = -\frac{2B_0}{\mu_0}.$$

◇

From the above examples we can understand that the interior of a superconductor is completely shielded with zero magnetic flux density. This situation is unchanged even for the case of a hollow superconductor. Hence, if we completely surround a space with a superconductor, the effect of external magnetic flux density can be completely shielded in the space. This is called **magnetic shielding** and corresponds to the electrostatic shielding attained by a conductor.

Column: (1) Penetration of Magnetic Flux into Superconducting Hollow Cylinder with Tilted Slit

Suppose we apply a magnetic flux density parallel to a long superconducting hollow cylinder with a tilted slit as shown in Fig. 7.15. Does the magnetic flux penetrate into the interior of the superconducting hollow cylinder without passing through the superconducting region?

If axial magnetic flux lines move along the radial direction, they surely have to pass through the superconducting region, indicating that the magnetic flux cannot penetrate into the interior. Is this true? In practice the magnetic flux penetrates into the interior and its density is the same as that of the external one.

Remember here that a superposition holds for electric and magnetic quantities. That is, the resultant magnetic flux density is the sum of the applied one and one produced by the shielding current. The shielding current flows on the surface of the superconductor, as schematically shown in Fig. 7.16a. This is composed of the current flowing on the surface of a virtual hollow cylinder with no slit in Fig. 7.16b and that flowing in the opposite direction only in the region of the slit in Fig. 7.16c. The current in Fig. 7.16b produces a magnetic flux density of the same magnitude and opposite direction to the external one in the superconducting region, resulting in complete shielding there. The important thing is that the current in Fig. 7.16c produces a tilted magnetic flux at the slit. This shows the magnetic flux structure during the penetration. That is, the magnetic flux is tilted when it penetrates into the interior.

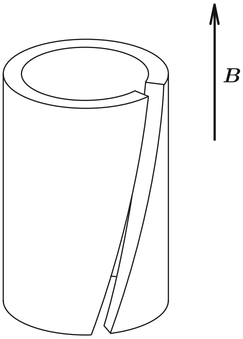

Fig. 7.15 Long superconducting hollow cylinder with a tilted slit in parallel magnetic flux density

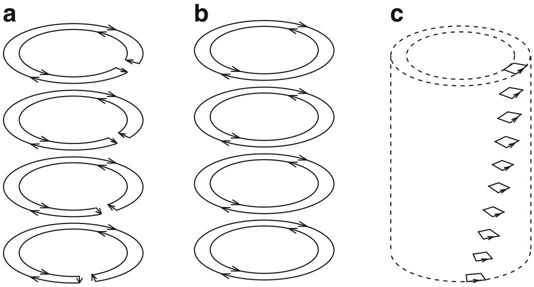

Fig. 7.16 (**a**) Flow of current on the superconductor surface, (**b**) that on the surface of virtual superconductor with no slit and (**c**) that in the opposite direction in the slit region. The current in (**a**) is obtained by superposing those in (**b**) and (**c**)

(2) Intermediate State

When we apply an external magnetic flux density to a superconducting sphere, the superconductor expels the magnetic flux from its interior. In this case some part of the superconductor experiences a magnetic flux density higher than the applied value because of the geometrical effect, as shown in Fig. 7.13. Hence, even when the applied magnetic flux density is below the critical value, B_c, the local magnetic flux density can exceed B_c, resulting in a breakdown of the superconductivity. If we assume that the superconductivity is completely broken, the magnetic flux will completely penetrate the superconductor, resulting in a magnetic flux density of the same value as the external one. This will bring about a recovery of the superconductivity. However, this is contradictory. In reality the superconductor goes into a state in which the superconductivity is partially broken. This is called the **intermediate state**. In this state the superconductor is in the layered structure composed of the superconducting region with perfect extrusion of magnetic flux and normal region with penetration of magnetic flux, as shown in Fig. 7.17. Since the size of these layers is of the order of several 10 μm, the magnetic structure can be regarded as a uniform partially diamagnetic structure on a macroscopic scale. However, we cannot use the

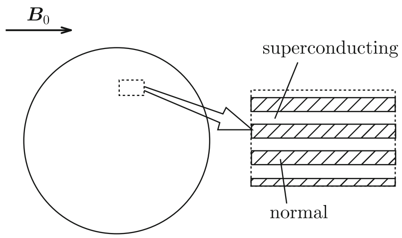

Fig. 7.17 Multilayered structure composed of superconducting and normal layers in the intermediate state

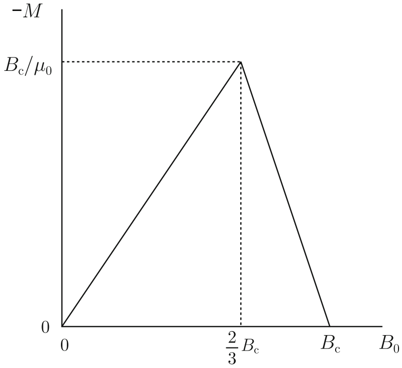

Fig. 7.18 Magnetization curve of spherical type 1 superconductor.

condition (7.4), so we need another method to determine the internal magnetic flux density and surface current density. This is the method using the boundary conditions on the magnetic flux density and magnetic field, which will be described in Chap. 9 (see Exercise 9.9). Figure 7.18 shows the magnetization curve of a spherical type 1 superconductor. The ascending line starting from the origin represents the perfect diamagnetic characteristic given by Eq. (7.34) and the descending line shows the characteristic in the intermediate state.

Exercises

7.1. Determine the magnetic flux density and vector potential when we apply currents I_1 and I_2 to the inner and outer superconductors, respectively, for the coaxial superconductor in Fig. 7.5.

7.2. Two wide slab superconductors are parallel to each other, as shown in Fig. E7.1, and current I is applied along the y-axis to the left superconductor. The length along the z-axis of each slab is l. Determine the current that appears on each superconductor surface, the magnetic flux density and vector potential inside and outside the superconductors.

7.3. When a current flows uniformly with a surface density τ on a thin sheet conductor, the magnetic flux density near the sheet is given by Eq. (6.28). However, Eq. (7.8) yields double this magnetic flux density near the superconductor surface with the same current density. Discuss the reason for the difference.

7.4. When we put straight current I at a distance a from a wide superconductor surface, the current given by Eq. (7.16) is induced on the superconductor surface.

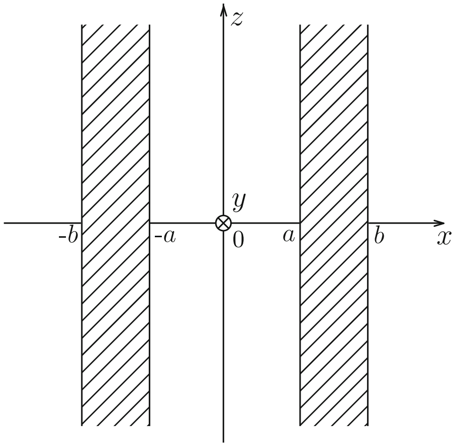

Fig. E7.1 Two parallel slab superconductors

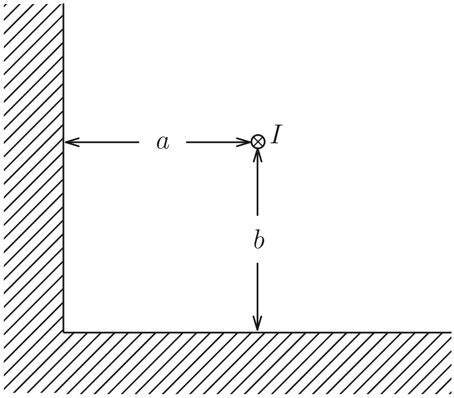

Fig. E7.2 Two perpendicular flat superconductor surfaces and straight current I

Prove that the Lorentz force exerted on I by the induced current is given by Eq. (7.18).

7.5. Straight current I is placed at distances a and b from two flat superconductor surfaces that are perpendicular to each other, as shown in Fig. E7.2. Determine the vector potential and magnetic flux density in the vacuum.

7.6. Determine the magnetic flux density in the space around a superconducting cylinder using the vector potential given by Eq. (7.23).

7.7. The vector potential is given by Eq. (7.13) for the case of a straight current and a wide superconductor surface. Determine the equivector-potential surface.

7.8. A long superconducting cylinder of radius a is placed at distance $l(>a)$ from an infinite flat superconductor surface, as shown in Fig. E7.3, and a current I is applied to the superconducting cylinder. Determine the current density on the surfaces of the two superconductors.

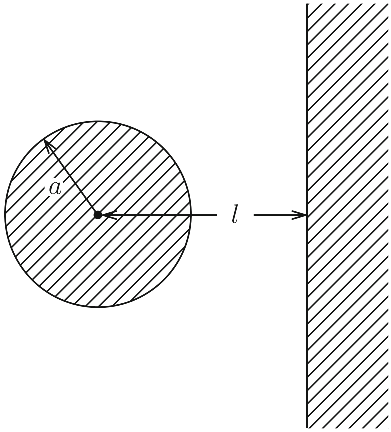

Fig. E7.3 Superconducting cylinder parallel to infinite flat superconductor surface

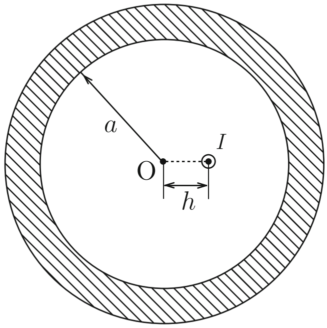

Fig. E7.4 Hollow cylindrical superconductor and straight current inside the superconductor

7.9. Straight current I is placed at distance h from the central axis, O, of a hollow cylindrical superconductor, as shown in Fig. E7.4. Determine the vector potential in the vacuum and current density on the inner surface of the superconductor.

7.10. Derive Eqs. (7.30) and (7.33) for a superconducting sphere in a uniform magnetic flux density using Eq. (7.1) with the boundary conditions, Eqs. (7.6) and (7.8).

7.11. It is stated in Sect. 7.1 that someone might have predicted the existence of superconducting material in the 19th Century. Prove this prediction that a material with perfect magnetism ($\boldsymbol{B} = 0$) has no resistivity. Note that the proof using the magnetization of a hollow cylindrical material, as done in Sect. 9.1, cannot be used, since no one can experimentally determine whether the magnetic flux density in the hollow is zero for an undiscovered material.