29 Forecast Methods

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In Chap. 29 we will show how demand planning can be done when seasonality and trend are given. For a comprehensive introduction to forecasting in general the reader is referred to Hanke and Wichern (2009) or Makridakis et al. (1998).

29.1 Forecasting for Seasonality and Trend

This section introduces *Winters' method* which is appropriate for multiplicative seasonal models (see Chap. 7). In Sect. 29.2 the parameters of Winters' method are initialized. This incorporates the introduction of *linear regression*, too. A working example illustrates the explanations.

29.1.1 Working Example

Figure 29.1 shows the sales volume of a supplementary product of a large German shoe retailer. The data are aggregated over the whole sales region and comprise a time horizon of 4 weeks. In our working example we use the first 3 weeks (days $-20, \ldots, 0$) as input and — starting with day 1 — try to estimate day by day the sales of the fourth week.

Two observations are striking when analyzing the data:

- There seems to be a common sales pattern with weekly repetition. Saturdays usually show the highest, Sundays the lowest sales volume of a week. So weekly seasonality can be assumed with a cycle length of $T = 7$ days.

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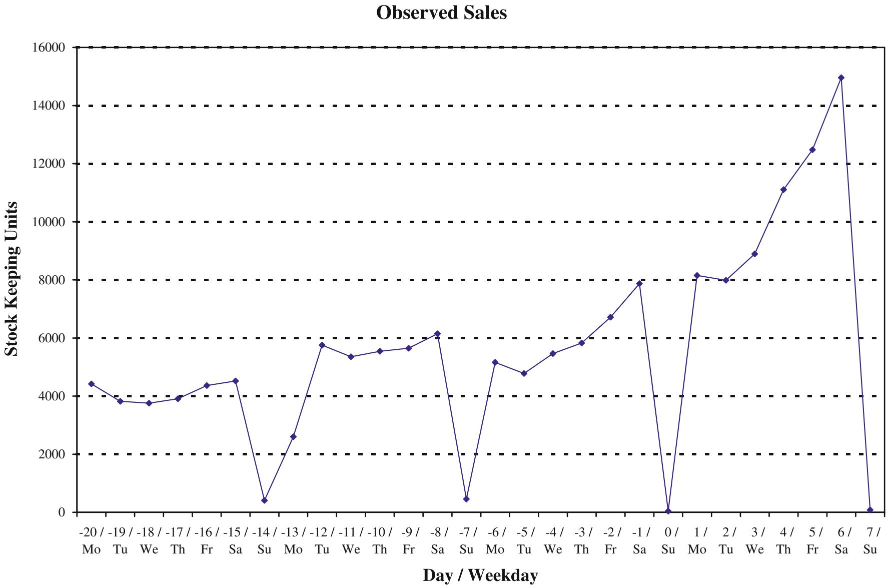

Fig. 29.1 Sales volume of a supplementary product of a German shoe retailer

- Sales per week appear to be continuously increasing. This is obvious when all 4 weeks are considered. But even within the first 3 weeks a (weaker) trend of growing sales is visible.

Since the amplitude of seasonality is increasing, too, a seasonal multiplicative forecast model seems justified. All subsequent explanations will be demonstrated by use of this working example.

29.1.2 Modeling Seasonality and Trend

As already shown in Sect. 7.4 a multiplicative seasonal model is characterized by the parameters a and b describing the trend and the seasonal coefficients c_t modeling the seasonality of period t. Figure 29.2 makes clear that the trend is expressed by the linear function $a + b \cdot t$ with t denoting periods of time (e.g., days in our working example).

A sales volume x_t observed in period t is modeled by

$$x_t = (a + b \cdot t) \cdot c_t + u_t \tag{29.1}$$

with the seasonal coefficients c_t in- or decreasing the trend. Please note, if all seasonal coefficients are equal to 1, seasonality disappears and the model reduces to a simple trend model (see, e.g., Silver et al. 1998, p. 93). The erratic noise u_t makes things difficult. Because of the randomness that is represented by u_t the other

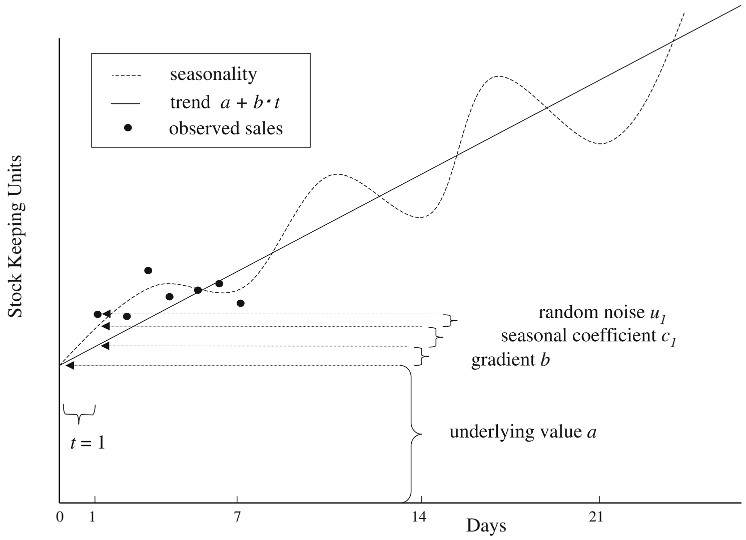

Fig. 29.2 Modeling seasonality and trend

parameters cannot be measured exactly, but have to be predicted. In the following the superscript ˆ is used to distinguish between an observation (no superscript) that has been measured and its forecast being estimated without this knowledge.

Let $\hat{a}_t, \hat{b}_t$ and $\hat{c}_{t-T+1}, \ldots, \hat{c}_t$ denote the forecasts of $a.$, $b.$ and the seasonal coefficients $c.$ that are valid in period t. Then (29.1) can be engaged to estimate the sales volume $\hat{x}^t_{t+s}$ of all subsequent periods $t + s$ $(s = 1, 2, \ldots)$. For example, the sales volume of the next seasonal cycle is predicted in period t by

$$\hat{x}^t_{t+s} = (\hat{a}_t + \hat{b}_t \cdot s) \cdot \hat{c}_{t+s-T} \quad (s = 1, \ldots, T). \tag{29.2}$$

The method of Winters described in the next subsection iteratively computes the sales estimation of only the subsequent period $t + 1$. For this reason we can use the simpler notation $\hat{x}_{t+1}$ instead of $\hat{x}^t_{t+1}$.

29.1.3 Winters' Method

The method of Winters (1960) basically builds on (29.2) and the principle of exponential smoothing which has been introduced in Chap. 7. Since sales are predicted indirectly via $\hat{a}.$, $\hat{b}.$ and $\hat{c}.$ in (29.2), these three types of parameters have to be estimated by means of exponential smoothing instead of the sales volume itself (as it has been done by (7.5) for models without trend and seasonality). Remember the generic principle of exponential smoothing:

$$\text{new forecast} = sc \cdot \text{latest observation} + (1 - sc) \cdot \text{last forecast}. \tag{29.3}$$

Table 29.1 Exponential smoothing applied in Winters' method

New forecast	Smoothing constant sc	Latest observation	Last forecast
$\hat{a}_{t+1}$	α	$x_{t+1}/\hat{c}_{t+1-T}$	$\hat{a}_t + \hat{b}_t \cdot 1$
$\hat{b}_{t+1}$	β	$\hat{a}_{t+1} - \hat{a}_t$	$\hat{b}_t$
$\hat{c}_{t+1}$	γ	$x_{t+1}/\hat{a}_{t+1}$	$\hat{c}_{t+1-T}$

Table 29.2 Forecasting the fourth week using Winters' method

t	Weekday	x_t	$\hat{x}_t$	$\hat{a}_t$	$\hat{b}_t$	$\hat{c}_t$
−6	Monday					1.245693
−5	Tuesday					1.115265
−4	Wednesday					1.088853
−3	Thursday		Initialization			1.135378
−2	Friday					1.178552
−1	Saturday					1.229739
0	Sunday			5, 849.0	123.3	0.006520
1	Monday	8,152	**7,440**	**6,429.8**	**489.3**	**1.2523...**
2	Tuesday	7,986	**7,717**	7, 112.3	643.9	1.1175...
3	Wednesday	8,891	8,445	8, 083.6	905.8	1.0922...
4	Thursday	11,107	10,206	9, 624.0	1, 413.5	1.1410...
5	Friday	12,478	13,008	10, 677.6	1, 125.5	1.1756...
6	Saturday	14,960	14,515	12, 092.8	1, 357.3	1.2319...
7	Sunday	81	88	12, 628.5	700.0	0.0065...

The *new forecast* of the current period estimating the subsequent period(s) can be calculated by smoothing the *latest observation*, i.e., the observation in the current period and the *last forecast* that has been made to predict the current period's observation. The smoothing constant $sc \in (0; 1)$ determines the weight the new observation has. The higher the smoothing constant the more importance is given to the latest observation. Table 29.1 summarizes how Winters applies exponential smoothing in period $t+1$ to estimate the parameters $\hat{a}_{t+1}$, $\hat{b}_{t+1}$ and $\hat{c}_{t+1}$ determining the sales forecast $\hat{x}_{t+2}$ of the subsequent period (29.2).

These three types of equations become clear when looking at our working example. We start our computation at the end of day $t = 0$. Table 29.2 further illustrates this proceeding:

1. **Initialization**:

   In order to get things work initial values $\hat{a}_0$, $\hat{b}_0$ and $\hat{c}_{-6}, \ldots, \hat{c}_0$ (seasonal coefficients for each weekday) have to be given. As examples Sects. 29.2.2 (for $\hat{c}_.$) and 29.2.3 (for $\hat{a}_0$, $\hat{b}_0$) show how these values can be computed from the sales observations of the first 3 weeks (day −20, ..., 0). For the moment we will accept in blank the values $\hat{a}_0 = 5,849.0$, $\hat{b}_0 = 123.3$ and $\hat{c}_{-6} = 1.245693$ that are used in Table 29.2.[1]

2. **Estimating the sales volume of period** $t + 1$:
   Applying (29.2) we can estimate the sales volume $\hat{x}_1$ of period 1:

$$\hat{x}_1 = (\hat{a}_0 + \hat{b}_0 \cdot 1) \cdot \hat{c}_{-6} = (5,849.0 + 123.3) \cdot 1.245693 = 7,440$$

   The linear trend $(\hat{a}_0 + \hat{b}_0 \cdot 1)$ does not consider any seasonal influences and will therefore be called "*deseasonalized*". Since sales on Mondays are (estimated to be) about 25 % higher than average weekly sales ($c_{-6} = 1.245693$), the trend has to be increased accordingly. Please note that at the end of day 0 sales of day 2 (Tuesday) could roughly be estimated to amount to $(\hat{a}_0 + \hat{b}_0 \cdot 2) \cdot \hat{c}_{-5} = 6,798$. However, a more accurate forecast of $\hat{x}_2$ can be given at the end of day 1 because the sales observation x_1 of day 1 offers further information.
3. **Observation in period** $t + 1$:
   In day 1 sales x_1 of 8,152 stock keeping units (SKU) are observed.
4. **Using the latest observation to update trend and seasonal coefficients:**
   The latest observation x_1 improves the forecast of the trend and Monday's seasonal coefficient. So the smoothing constants $\alpha = 0.8$, $\beta = 0.8$ and $\gamma = 0.3$ are applied to the three exponential smoothing equations defined in Table 29.1:
   (a) The underlying value $\hat{a}.$ of the trend is updated as follows:

$$\hat{a}_1 = \alpha \frac{x_1}{\hat{c}_{-6}} + (1 - \alpha)(\hat{a}_0 + \hat{b}_0) = 0.8 \cdot \frac{8,152}{1.245693} + 0.2 \cdot 5,972.3 = 6,429.8$$

   Thereby, the "deseasonalized" sales volume $\frac{x_1}{\hat{c}_{-6}}$ of day 1 serves as a new observation for the underlying value, while $(\hat{a}_0 + \hat{b}_0 \cdot 1)$ was the forecast of the deseasonalized sales of day 1 which has been obtained in period 0 (29.1).
   (b) Using $\hat{a}_1$, the new gradient $\hat{b}_1$ can be calculated:

$$\hat{b}_1 = \beta(\hat{a}_1 - \hat{a}_0) + (1 - \beta)\hat{b}_0 = 0.8(6,429.8 - 5,849) + 0.2 \cdot 123.3 = 489.3$$

   Between day 0 and day 1 the underlying value $a.$ has been increased from $\hat{a}_0$ to $\hat{a}_1$. Since $\hat{a}_1$ is based on the latest sales observation x_1, this is interpreted as the "new observation" of the gradient $b.$ which again has to be exponentially smoothed.
   (c) The same procedure is applied to the seasonal coefficient $\hat{c}_1$:

[1] Note that the initial seasonal coefficients $\hat{c}_{-6}, \ldots, \hat{c}_0$ are printed with two additional digits in order to indicate that high precision floating point arithmetic — commonly used in APS, programming languages, and spreadsheets — is applied throughout the working example.

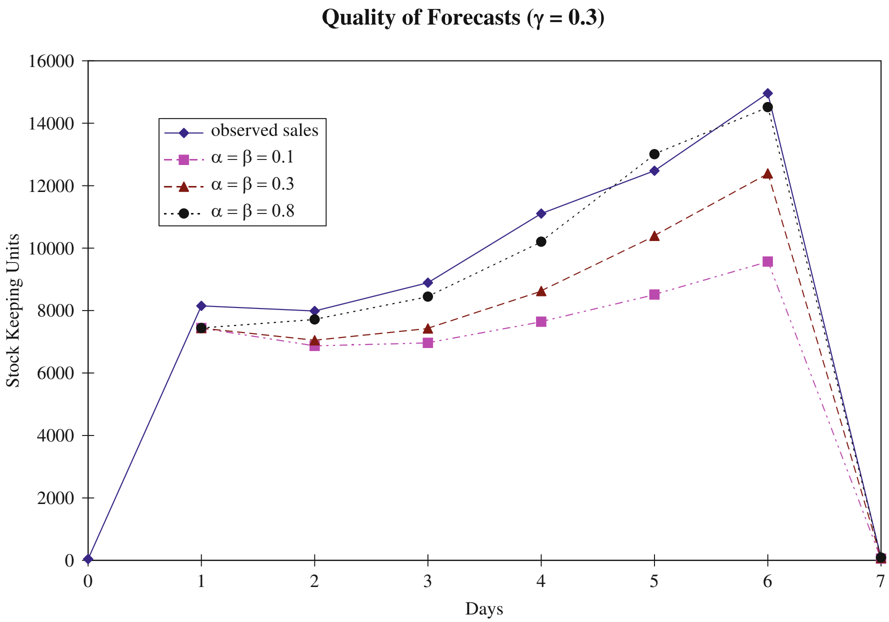

Fig. 29.3 Variation of the smoothing constants α and β

$$\hat{c}_1 = \gamma \frac{x_1}{\hat{a}_1} + (1-\gamma)\hat{c}_{-6} = 0.3 \frac{8,152}{6,429.8} + 0.7 \cdot 1.245693 = 1.2523$$

$\hat{c}_{-6}$ was the last forecast of the Monday's seasonal coefficient. The new observation of the seasonal influence of a Monday, however, is achieved by dividing the observed sales volume x_1 (including seasonal influences) by $\hat{a}_1$ (deseasonalized).

5. **Stepping forward in time:**
   Now we can go one day ahead (increasing t by 1) and repeat the steps (2) to (5). At the end of day 1 the sales volume $\hat{x}_2$ of day 2 is estimated by

$$\hat{x}_2 = (\hat{a}_1 + \hat{b}_1 \cdot 1) \cdot \hat{c}_{-5} = (6,429.8 + 489.3 \cdot 1) \cdot 1.115265 = 7,717$$

   and so on…

Table 29.2 shows the results of Winters' method when applied to the days 2–7.

Figure 29.3 illustrates the consequences of a variation of the smoothing constants α and β of the trend. Generally, smoothing constants out of the intervals $\alpha \in [0.02; 0.51]$, $\beta \in [0.005; 0.176]$ and $\gamma \in [0.05; 0.5]$ are recommended (see Silver et al. 1998, p. 108). In our working example, however, $\alpha = \beta = 0.8$ perform best, i.e., the few latest observations get a very high weight and smoothing is only weak. Thus, the forecast is able to react quickly to the progressively rising sales of the fourth week.

29.2 Initialization of Trend and Seasonal Coefficients

Until now we have not shown how the trend and seasonal coefficients can be initialized using the information that is given by the sales volume of the first 3 weeks. The next subsection demonstrates how the data basis can be improved if additional information is considered. Sections 29.2.2 and 29.2.3 finally present the initialization of the seasonal coefficients $\hat{c}_.$ and the trend parameters $\hat{a}_.$ and $\hat{b}$.

29.2.1 Consideration of Further Information

When looking at the data of the first 3 weeks (see Fig. 29.1) two phenomena seem to be contradictory to the assumption of a linear trend with seasonality:

1. Sales on Monday −13 are unexpectedly low. In weeks 1 and 3 sales on Mondays are clearly higher than sales on Tuesdays.
2. While the trend of weekly increasing sales is obvious, sales on Sunday 0 are much lower than sales on the respective Sundays of the first 2 weeks (days −14 and −7).

We want to know whether these inconsistencies are purely random or due to an identifiable actuator and get the following information:

1. In some parts of Germany Monday −13 was a holiday. Therefore, 58 % of the stores of the shoe retailer were closed this day.
2. Usually, shoe stores have to be closed on Sundays in Germany. Some few cities, however, granted a special authorization for sale. Starting with the third week $93\frac{1}{3}$ % of these cities do not grant such an authorization any more.

We can now improve our data basis by exploiting this information about special influences in our further investigations. Therefore, the sales volume of day −13 is increased by 138.1 % ($x_{-13} = 2,600 \cdot \frac{100}{100-58} = 6,190.4761$) and sales on Sundays −14 (410 SKU) and −7 (457 SKU) are decreased by $93\frac{1}{3}$ % so that $x_{-14} = 27.\bar{3}$ and $x_{-7} = 30.4\bar{6}$. In the next two subsections original sales are replaced by these corrected sales.

29.2.2 Determination of Seasonal Coefficients by the Ratio-to-Moving Averages Decomposition

The ratio-to-moving averages decomposition (see, e.g., Makridakis et al. 1998, p. 109) is used as an example to determine the initial seasonal coefficients of Winters' method. In Sect. 29.1.3 we already applied the equation:

$$\text{observed sales in } t = (\text{deseasonalized sales in } t) \cdot (\text{seasonal coefficient of } t).$$

In other words, if we want to isolate seasonal coefficients, we have to compute

Table 29.3 Ratio-to-moving averages decomposition

Week	Day	Weekday	(Corr.) x_t	Moving aver. (ma_t)	$o^{week}_{weekday}(t) = \frac{x_t}{ma_t}$
1	−20	Monday	4,419		
1	−19	Tuesday	3,821		
1	−18	Wednesday	3,754		
1	−17	Thursday	3,910	**3,544.6**	1.1031
1	−16	Friday	4,363	**3,797.7**	1.1489
1	−15	Saturday	4,518	4,074.0	1.1090
1	−14	Sunday	($27.33\bar{3}$)	4,302.3	0.0064
2	−13	Monday	(6,190.4761)	4,535.1	1.3650
2	−12	Tuesday	5,755	4,719.0	1.2195
2	−11	Wednesday	5,352	4,951.1	1.0810
2	−10	Thursday	5,540	**4,951.6**	1.1188
2	−9	Friday	5,650	4,804.1	1.1761
2	−8	Saturday	6,143	4,664.6	1.3169
2	−7	Sunday	($30.466\bar{6}$)	4,680.6	0.0065
3	−6	Monday	5,158	4,721.8	1.0924
3	−5	Tuesday	4,779	4,873.8	0.9806
3	−4	Wednesday	5,464	5,120.8	1.0670
3	−3	Thursday	5,828	**5,122.4**	1.1377
3	−2	Friday	6,714		
3	−1	Saturday	7,872		
3	0	Sunday	42		

$$\text{seasonal coefficient of period } t = \frac{\text{observed sales in } t}{\text{deseasonalized sales in } t} \tag{29.4}$$

where the *deseasonalized sale in period t* is a sales volume that does not contain any seasonal influences. But how to determine such a value?

Considering our working example, the sales volume of a full week is apparently not influenced by daily sales peaks. So the most intuitive way to obtain sales data without seasonal influences is to compute daily sales averaged over a full week. This leads to average daily sales $\frac{4{,}419+\ldots+27.\bar{3}}{7} = 3,544.6, 4,951.6$ and $5,122.4$ SKU for the weeks 1–3 (see Table 29.3). Thereby, the Thursday is settled in the middle of each week.

But we can employ the same procedure for each other time period of 7 days, e.g., day $-19, \ldots, -13$, and assign the average daily sales 3,797.7 to the medium Friday −16. By doing so we compute moving averages over a full seasonal cycle of 7 days for each day $-17, \ldots, -3$ which represent deseasonalized daily sales volumes. Table 29.3 illustrates the whole procedure.

In a next step we apply (29.4), thus setting the observed sales x_t in *ratio to* the deseasonalized *moving averages* (remember the name of the algorithm). The result are multiple observations of seasonal coefficients $o^{week}_{weekday}(t)$ for each day of the week (three for a Thursday and two for each other weekday) which still contain the random noise u_t.

Table 29.4 Reducing randomness of seasonal coefficients

Week	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Suday	$\sum$
1				**1.1031**	1.1489	1.1090	0.0064	
2	1.3650	1.2195	1.0810	**1.1188**	1.1761	1.3169	0.0065	
3	1.0924	0.9806	1.0670	**1.1377**				o^{total}:
$o^{aver}_{weekday}$	1.2287	1.1000	1.0740	**1.1199**	1.1625	1.2130	0.0064	**6.9045**
$\hat{c}.$	1.2457	1.1153	1.0889	1.1354	1.1786	1.2297	0.0065	7.00

In order to reduce this randomness, now we compute the average seasonal coefficients $o^{aver}_{weekday}$ of each weekday (Table 29.4). For example, for the Thursday we get

$$o^{aver}_{Thursday} = \frac{o^{week\,1}_{Thursday}(-17) + o^{week\,2}_{Thursday}(-10) + o^{week\,3}_{Thursday}(-3)}{number\ of\ weeks} =$$

$$= \frac{1.1031 + 1.1188 + 1.1377}{3} = 1.1199$$

If a pure trend without any seasonal influence is given, one would expect all seasonal coefficients to equal 1 (see Sect. 29.1.2), thus summing up to 7 for a weekly seasonal cycle. As we can see in Table 29.4, the sum of our average seasonal coefficients $o^{total} = \sum_{day=Monday}^{Sunday} o^{aver}_{day} = 6.9045$ falls short of 7. To reflect the trend correctly, we have to normalize our $o^{aver}_{.}$ by multiplying them with the constant $7/o^{total}$. The resulting final seasonal coefficients for *Monday* ... *Sunday* are already known as $\hat{c}_{-6}, \ldots, \hat{c}_0$ from Table 29.2.

29.2.3 Determining the Trend by Linear Regression

Finally it will be shown how the trend parameters a and b can be determined. When "deseasonalizing" the observed sales by dividing through c_t one can see from (29.5) that the trend $a + b \cdot t$, distorted by some random noise $\frac{u_t}{c_t}$, results:

$$d_t = \frac{x_t}{c_t} = \frac{(a + b \cdot t) \cdot c_t + u_t}{c_t} = a + b \cdot t + \frac{u_t}{c_t}. \qquad (29.5)$$

The parameters a and b can be estimated by means of *linear regression* (see Wood and Fildes 1976, p. 76). As Fig. 29.4 shows, appropriate estimators $\hat{a}$ and $\hat{b}$ are computed by minimizing the (squared) vertical distances between the deseasonalized sales $d_t = \frac{x_t}{\hat{c}_t}$ and the trend line $\hat{a} + \hat{b} \cdot t$. This useful way of eliminating the random noise is also applied in causal forecasts and has already been introduced in Sect. 7.4.2.

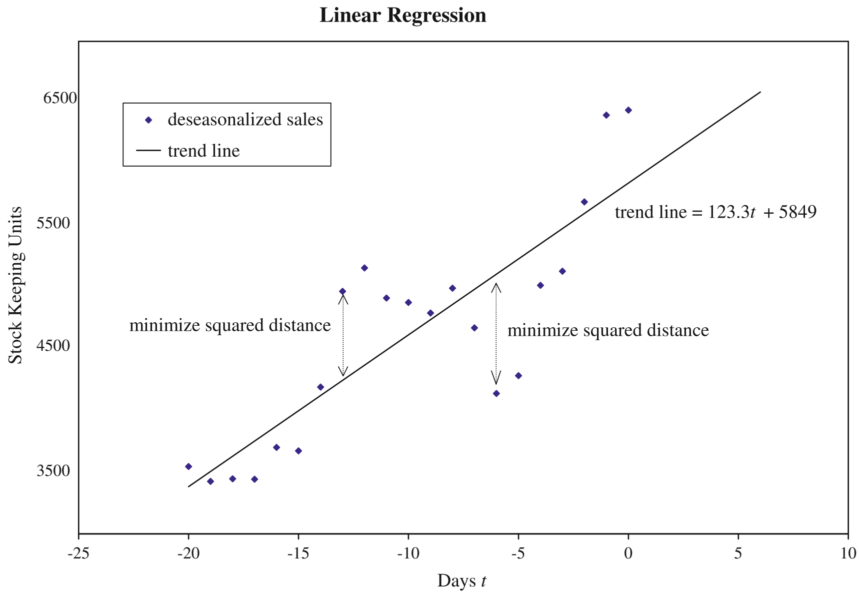

Fig. 29.4 Visualization of linear regression

Table 29.5 and Eqs. (29.6) and (29.7) illustrate how the trend parameters $\hat{a}_0$ and $\hat{b}_0$ have been calculated by linear regression to initialize Winters' method in Sect. 29.1.3:

$$\hat{b}_0 = \frac{\sum_t (t-\bar{t})(d_t-\bar{d})}{\sum_t (t-\bar{t})^2} = \frac{94,943}{770} = 123.3 \tag{29.6}$$

$$\hat{a}_0 = \bar{d} - \hat{b}_0 \cdot \bar{t} = 4,616 - 123.3 \cdot (-10) = 5,849 \tag{29.7}$$

Here $\bar{t} = \frac{1}{21} \cdot \sum_t t = \frac{-210}{21} = -10$ and $\bar{d} = \frac{1}{21} \cdot \sum_t d_t = \frac{96,936}{21} = 4,616$ represent the average values of t and d_t over the first weeks of our working example.

Please note that similar deseasonalized sales have been obtained by the moving averages computation in the last subsection. These could also be used to estimate $\hat{a}$ and $\hat{b}$ by linear regression. In this case, however, only 15 instead of 21 observations of deseasonalized sales would have been available, thus preparing a noticeably smaller sample to overcome randomness.

Table 29.5 Calculation of linear regression

Week	Day	(Corr.) x_t	$\hat{c}_t$	$d_t = \frac{x_t}{\hat{c}_t}$	$(t-\bar{t})^2$	$(t-\bar{t})(d_t-\bar{d}_t)$
1	−20	4,419	1.2457	3,547	100	10,686
1	−19	3,821	1.1153	3,426	81	10,709
1	−18	3,754	1.0889	3,448	64	9,347
1	−17	3,910	1.1354	3,444	49	8,206
1	−16	4,363	1.1786	3,702	36	5,484
1	−15	4,518	1.2297	3,674	25	4,710
1	−14	$(27.333\bar{3})$	0.0065	4,192	16	1,695
2	−13	(6,190.4761)	1.2457	4,970	9	−1060
2	−12	5,755	1.1153	5,160	4	−1088
2	−11	5,352	1.0889	4,915	1	−299
2	−10	5,540	1.1354	4,879	0	0
2	−9	5,650	1.1786	4,794	1	178
2	−8	6,143	1.2297	4,995	4	759
2	−7	$(30.466\bar{6})$	0.0065	4,673	9	170
3	−6	5,158	1.2457	4,141	16	−1,901
3	−5	4,779	1.1153	4,285	25	−1,655
3	−4	5,464	1.0889	5,018	36	2,413
3	−3	5,828	1.1354	5,133	49	3,620
3	−2	6,714	1.1786	5,697	64	8,646
3	−1	7,872	1.2297	6,401	81	16,068
3	0	42	0.0065	6,442	100	18,256
$\sum$	−210			96,936	770	94,943

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