Chapter 3
Geometric Description of the Robot Mechanism

The geometric description of the robot mechanism is based on the usage of translational and rotational homogenous transformation matrices. A coordinate frame is attached to the robot base and to each segment of the mechanism, as shown in Fig. 3.1. Then, the corresponding transformation matrices between the consecutive frames are determined. A vector expressed in one of the frames can be transformed into another frame by successive multiplication of intermediate transformation matrices.

Vector **a** in Fig. 3.1 is expressed relative to the coordinate frame x_3–y_3–z_3, while vector **b** is given in the frame x_0–y_0–z_0 belonging to the robot base. A mathematical relationship between the two vectors is obtained by the following homogenous transformation

$$\begin{bmatrix} \mathbf{b} \\ 1 \end{bmatrix} = {}^0\mathbf{H}_1\,{}^1\mathbf{H}_2\,{}^2\mathbf{H}_3 \begin{bmatrix} \mathbf{a} \\ 1 \end{bmatrix}. \tag{3.1}$$

3.1 Vector Parameters of a Kinematic Pair

Vector parameters will be used for the geometric description of a robot mechanism. For simplicity we shall limit our consideration to the mechanisms with either parallel or perpendicular consecutive joint axes. Such mechanisms are by far the most frequent in industrial robotics.

In Fig. 3.2, a kinematic pair is shown consisting of two consecutive segments of a robot mechanism, segment $i-1$ and segment i. The two segments are connected by the joint i including both translation and rotation. The relative pose of the joint is determined by the segment vector $\mathbf{b}_{i-1}$ and unit joint vector $\mathbf{e}_i$, as shown in Fig. 3.2. The segment i can be translated with respect to the segment $i-1$ along the vector $\mathbf{e}_i$ for the distance d_i and can be rotated around $\mathbf{e}_i$ for the angle ϑ_i. The coordinate

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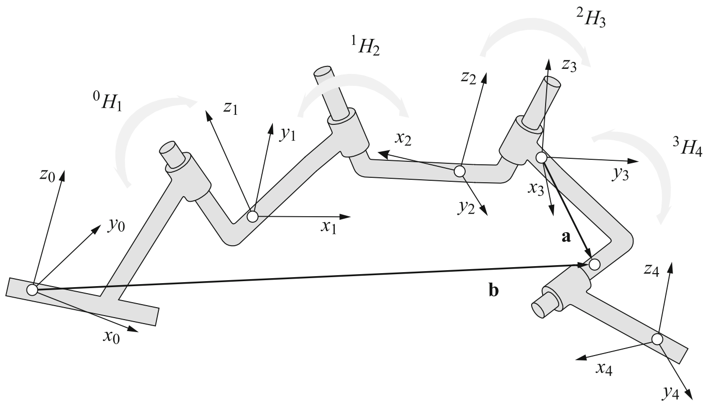

Fig. 3.1 Robot mechanism with coordinate frames attached to its segments

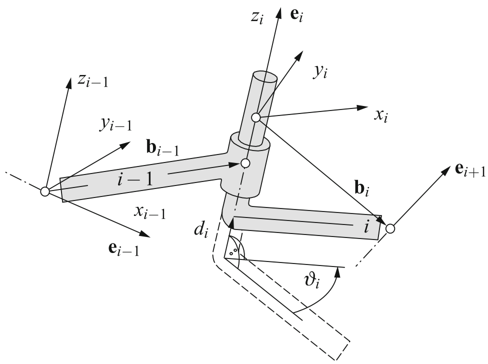

Fig. 3.2 Vector parameters of a kinematic pair

frame x_i–y_i–z_i is attached to the segment i, while the frame x_{i-1}–y_{i-1}–z_{i-1} belongs to the segment $i-1$.

The coordinate frame x_i–y_i–z_i is placed into the axis of the joint i in such a way that it is parallel to the previous frame x_{i-1}–y_{i-1}–z_{i-1} when the kinematic pair is in its initial pose (both joint variables are zero $\vartheta_i = 0$ and $d_i = 0$).

The geometric relations and the relative displacement of two neighboring segments of a robot mechanism are determined by the following parameters:

$\mathbf{e}_i$ —unit vector describing either the axis of rotation or direction of translation in the joint i and is expressed as one of the axes of the frame x_i–y_i–z_i. Its components are the following

$$\mathbf{e}_i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$

$\mathbf{b}_{i-1}$ —segment vector describing the segment $i-1$ expressed in the frame x_{i-1}–y_{i-1}–z_{i-1}. Its components are the following

$$\mathbf{b}_{i-1} = \begin{bmatrix} b_{i-1,x} \\ b_{i-1,y} \\ b_{i-1,z} \end{bmatrix};$$

ϑ_i —rotational variable representing the angle measured around the $\mathbf{e}_i$ axis in the plane which is perpendicular to $\mathbf{e}_i$ (the angle is zero when the kinematic pair is in the initial position);

d_i —translational variable representing the distance measured along the direction of $\mathbf{e}_i$ (the distance equals zero when the kinematic pair is in the initial position).

If the joint is only rotational (Fig. 3.3 above), the joint variable is represented by the angle ϑ_i, while $d_i = 0$. When the robot mechanism is in its initial pose, the joint angle equals zero $\vartheta_i = 0$ and the coordinate frames x_i–y_i–z_i and x_{i-1}–y_{i-1}–z_{i-1} are parallel. If the joint is only translational (Fig. 3.3 below), the joint variable is d_i,

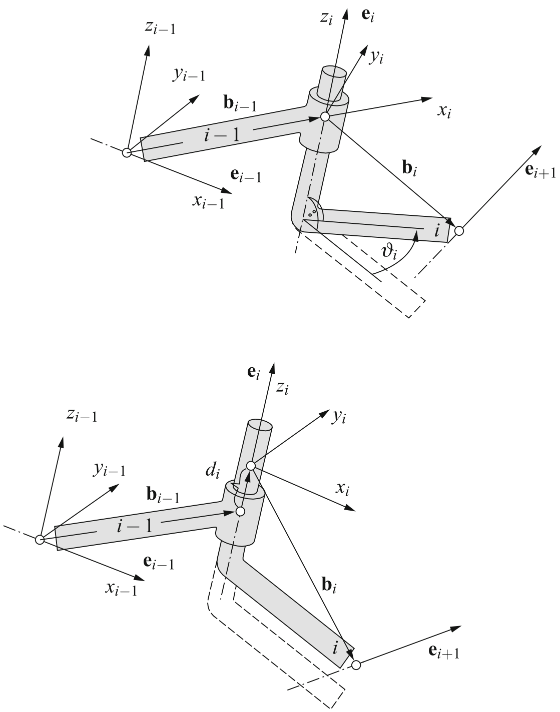

Fig. 3.3 Vector parameters of a kinematic pair

while $\vartheta_i = 0$. When the joint is in its initial position, then $d_i = 0$. In this case the coordinate frames x_i–y_i–z_i and x_{i-1}–y_{i-1}–z_{i-1} are parallel irrespective of the value of the translational variable d_i.

By changing the value of the rotational joint variable ϑ_i, the coordinate frame x_i–y_i–z_i is rotated together with the segment i with respect to the preceding segment $i-1$ and the corresponding frame x_{i-1}–y_{i-1}–z_{i-1}. By changing the translational variable d_i, the displacement is translational, where only the distance between the two neighboring frames is changing.

The transformation between the coordinate frames x_{i-1}–y_{i-1}–z_{i-1} and x_i–y_i–z_i is determined by the homogenous transformation matrix taking one of the three possible forms regarding the direction of the joint vector $\mathbf{e}_i$. When the unit vector $\mathbf{e}_i$ is parallel to the x_i axis, there is

$$ {}^{i-1}\mathbf{H}_i = \begin{bmatrix} 1 & 0 & 0 & d_i + b_{i-1,x} \\ 0 & \cos\vartheta_i & -\sin\vartheta_i & b_{i-1,y} \\ 0 & \sin\vartheta_i & \cos\vartheta_i & b_{i-1,z} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.2} $$

when $\mathbf{e}_i$ is parallel to the y_i axis, we have the following transformation matrix

$$ {}^{i-1}\mathbf{H}_i = \begin{bmatrix} \cos\vartheta_i & 0 & \sin\vartheta_i & b_{i-1,x} \\ 0 & 1 & 0 & d_i + b_{i-1,y} \\ -\sin\vartheta_i & 0 & \cos\vartheta_i & b_{i-1,z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.3} $$

When $\mathbf{e}_i$ is parallel to the z_i axis, the matrix has the following form

$$ {}^{i-1}\mathbf{H}_i = \begin{bmatrix} \cos\vartheta_i & -\sin\vartheta_i & 0 & b_{i-1,x} \\ \sin\vartheta_i & \cos\vartheta_i & 0 & b_{i-1,y} \\ 0 & 0 & 1 & d_i + b_{i-1,z} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.4} $$

In the initial pose the coordinate frames x_{i-1}–y_{i-1}–z_{i-1} and x_i–y_i–z_i are parallel ($\vartheta_i = 0$ and $d_i = 0$) and displaced only for the vector $\mathbf{b}_{i-1}$

$$ {}^{i-1}\mathbf{H}_i = \begin{bmatrix} 1 & 0 & 0 & b_{i-1,x} \\ 0 & 1 & 0 & b_{i-1,y} \\ 0 & 0 & 1 & b_{i-1,z} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.5} $$

3.2 Vector Parameters of the Mechanism

The vector parameters of a robot mechanism are determined in the following four steps:

step 1 —the robot mechanism is placed into the desired initial (reference) pose. The joint axes must be parallel to one of the axes of the reference coordinate frame x_0–y_0–z_0 attached to the robot base. In the reference pose all values of joint variables equal zero, $\vartheta_i = 0$ and $d_i = 0$, $i = 1, 2, \ldots, n$;

step 2 —the centers of the joints $i = 1, 2, \ldots, n$ are selected. The center of joint i can be anywhere along the corresponding joint axis. A local coordinate frame x_i–y_i–z_i is placed into the joint center in such a way that its axes are parallel to the axes of the reference frame x_0–y_0–z_0. The local coordinate frame x_i–y_i–z_i is displaced together with the segment i;

step 3 —the unit joint vector $\mathbf{e}_i$ is allocated to each joint axis $i = 1, 2, \ldots, n$. It is directed along one of the axes of the coordinate frame x_i–y_i–z_i. In the direction of this vector the translational variable d_i is measured, while the rotational variable ϑ_i is assessed around the joint vector $\mathbf{e}_i$;

step 4 —the segment vectors $\mathbf{b}_{i-1}$ are drawn between the origins of the frames x_i–y_i–z_i, $i = 1, 2, \ldots, n$. The segment vector $\mathbf{b}_n$ connects the origin of the frame x_n–y_n–z_n with the robot end-point.

Sometimes an additional coordinate frame is positioned in the reference point of a gripper and denoted as x_{n+1}–y_{n+1}–z_{n+1}. There exists no degree of freedom between the frames x_n–y_n–z_n and x_{n+1}–y_{n+1}–z_{n+1}, as both frames are attached to the same segment. The transformation between them is therefore constant.

The approach to geometric modeling of robot mechanisms will be illustrated by an example of a robot mechanism with four degrees of freedom shown in Fig. 3.4. The selected initial pose of the mechanism together with the marked positions of the joint centers is presented in Fig. 3.5. The corresponding vector parameters and joint variables are gathered in Table 3.1.

The rotational variables ϑ_1, ϑ_2 and ϑ_4 are measured in the planes perpendicular to the joint axes $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_4$, while the translational variable d_i is measured along the axis $\mathbf{e}_3$. Their values are zero when the robot mechanism is in its initial pose. In Fig. 3.6 the robot manipulator is shown in a pose where all four variables are positive and nonzero. The variable ϑ_1 represents the angle between the initial and momentary y_1 axis, the variable ϑ_2 the angle between the initial and momentary z_2 axis, variable d_3 is the distance between the initial and actual position of the x_3 axis, while ϑ_4 represents the angle between the initial and momentary x_4 axis.

The selected vector parameters of the robot mechanism are inserted into the homogenous transformation matrices (3.2)–(3.4)

$$
{}^0\mathbf{H}_1 = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & h_0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

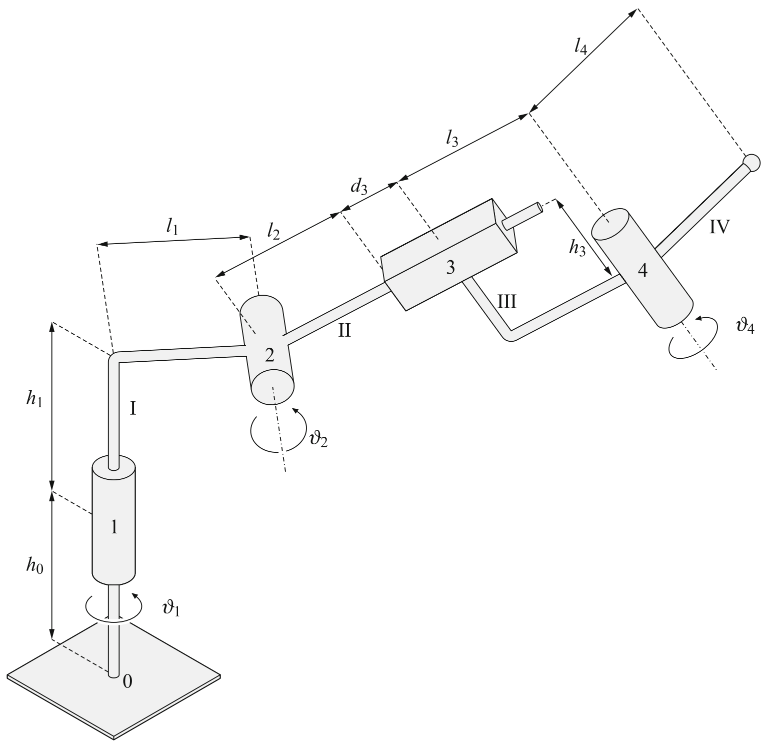

Fig. 3.4 Robot mechanism with four degrees of freedom

$$
{}^{1}\mathbf{H}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c2 & -s2 & l_1 \\ 0 & s2 & c2 & h_1 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^{2}\mathbf{H}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_3 + l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^{3}\mathbf{H}_4 = \begin{bmatrix} c4 & -s4 & 0 & 0 \\ s4 & c4 & 0 & l_3 \\ 0 & 0 & 1 & -h_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

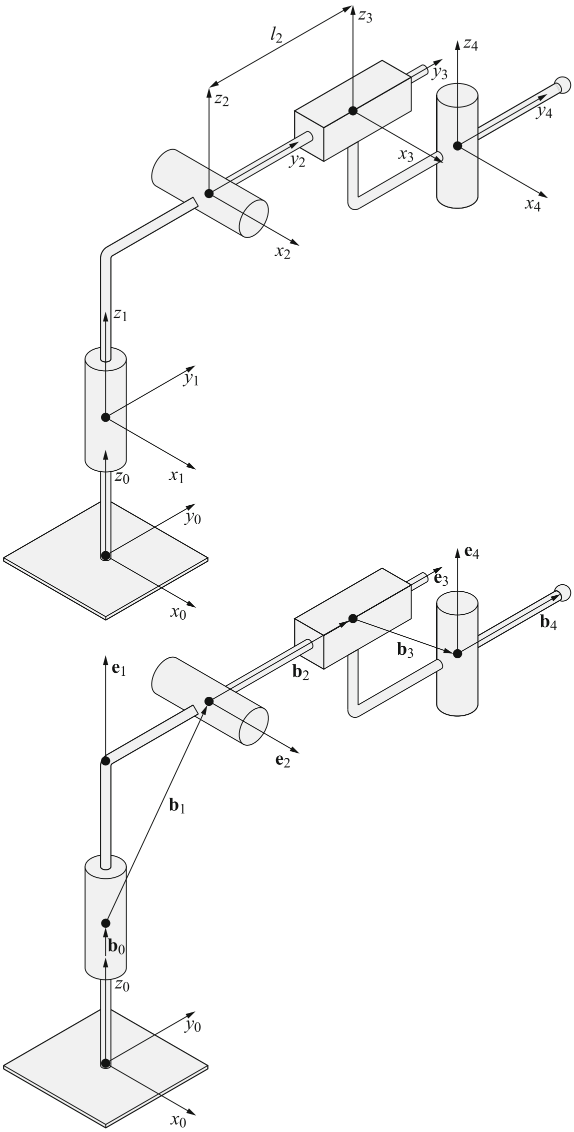

Fig. 3.5 Positioning of the coordinate frames for the robot mechanism with four degrees of freedom

An additional homogenous matrix describes the position of the gripper reference point where the coordinate frame x_5–y_5–z_5 can be allocated

$$ {}^4\mathbf{H}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

Table 3.1 Vector parameters and joint variables for the robot mechanism in Fig. 3.5

i	1	2	3	4
ϑ_i	ϑ_1	ϑ_2	0	ϑ_4
d_i	0	0	d_3	0

i	1	2	3	4
$\mathbf{e}_i$	0	1	0	0
	0	0	1	0
	1	0	0	1

i	1	2	3	4	5
$\mathbf{b}_{i-1}$	0	0	0	0	0
	0	l_1	l_2	l_3	l_4
	h_0	h_1	0	$-h_3$	0

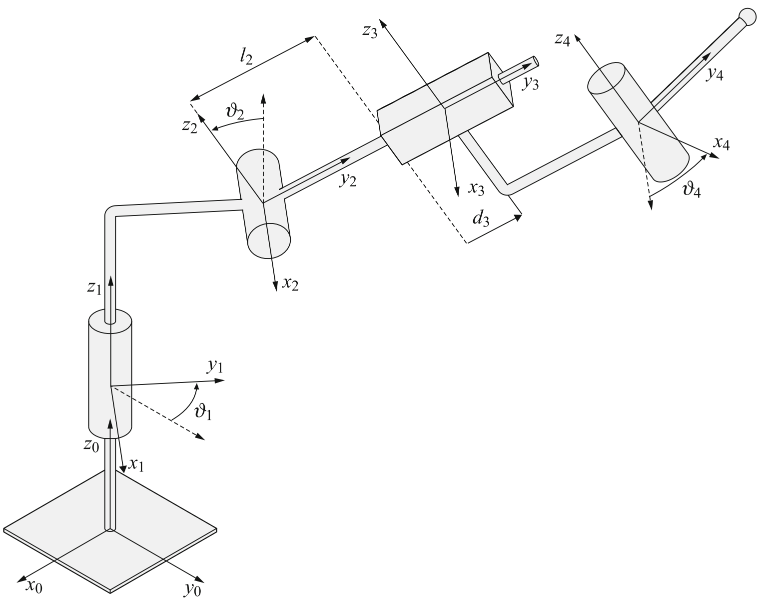

Fig. 3.6 Determining the rotational and translational variables for the robot mechanism with four degrees of freedom

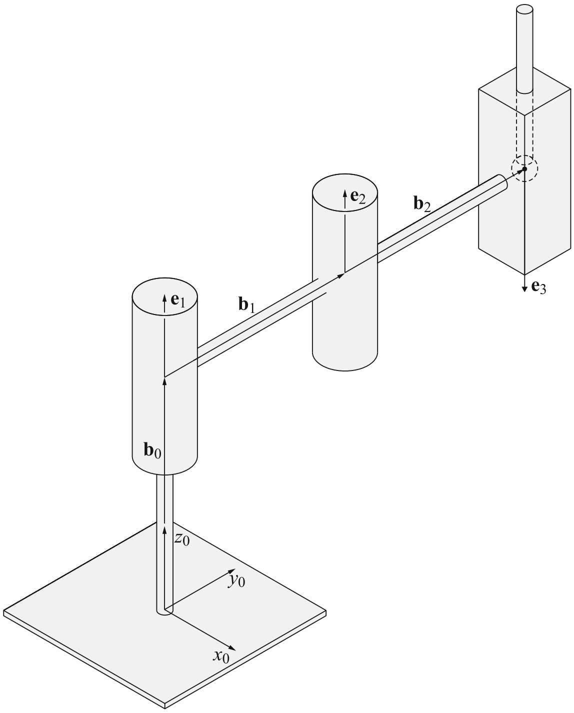

Fig. 3.7 The SCARA robot manipulator in the initial pose

This last matrix is constant as the frames x_4–y_4–z_4 and x_5–y_5–z_5 are parallel and displaced for the distance l_4. Usually this additional frame is not even attached to the robot mechanism, as the position and orientation of the gripper can be described in the frame x_4–y_4–z_4.

When determining the initial (home) pose of the robot mechanism we must take care that the joint axes are parallel to one of the axes of the reference coordinate frame. The initial pose should be selected in such a way that it is simple and easy to examine, that it corresponds well to the anticipated robot tasks and that it minimizes the number of required mathematical operations included in the transformation matrices.

As another example we shall consider the SCARA robot manipulator whose geometric model was developed already in the previous chapter and is shown in Fig. 2.10. The robot mechanism should be first positioned into the initial pose in such a way that the joint axes are parallel to one of the axes of the reference frame x_0–y_0–z_0. In this way the two neighboring segments are either parallel or perpendicular. The translational joint must be in its initial position ($d3 = 0$). The SCARA robot in the selected initial pose is shown in Fig. 3.7.

The joint coordinate frames x_i–y_i–z_i are all parallel to the reference frame. Therefore, we shall draw only the reference frame and have the dots indicate the joint centers. In the centers of both rotational joints, unit vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ are placed

Table 3.2 Vector parameters and joint variables for the SCARA robot manipulator

i	1	2	3	4
ϑ_i	ϑ_1	ϑ_2	0	ϑ_4
d_i	0	0	d_3	0

i	1	2	3	4
	0	1	0	0
$\mathbf{e}_i$	0	0	1	0
	1	0	0	1

i	1	2	3	4	5
	0	0	0	0	0
$\mathbf{b}_{i-1}$	0	l_1	l_2	l_3	l_4
	h_0	h_1	0	$-h_3$	0

along the joint axes. The rotation around the $\mathbf{e_1}$ vector is described by the variable ϑ_1, while ϑ_2 represents the angle about the $\mathbf{e_2}$ vector. Vector $\mathbf{e_3}$ is placed along the translational axis of the third joint. Its translation variable is described by d_3. The first joint is connected to the robot base by the vector $\mathbf{b_0}$. Vector $\mathbf{b_1}$ connects the first and the second joint and vector $\mathbf{b_2}$ the second and the third joint. The variables and vectors are gathered in the three tables (Table 3.2).

In our case all $\mathbf{e_i}$ vectors are parallel to the z_0 axis, the homogenous transformation matrices are therefore written according Eq. (3.4). Similar matrices are obtained for both rotational joints.

$$
{}^{0}\mathbf{H}_1 = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

$$
{}^{1}\mathbf{H}_2 = \begin{bmatrix} c2 & -s2 & 0 & 0 \\ s2 & c2 & 0 & l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

For the translational joint, $\vartheta_3 = 0$ must be inserted into Eq. (3.4), giving

$$^{2}\mathbf{H}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & 1 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

With postmultiplication of all three matrices the geometric model of the SCARA robot is obtained

$$^{0}\mathbf{H}_3 = {}^{0}\mathbf{H}_1 {}^{1}\mathbf{H}_2 {}^{2}\mathbf{H}_3 = \begin{bmatrix} c12 & -s12 & 0 & -l_3 s12 - l_2 s1 \\ s12 & c12 & 0 & l_3 c12 + l_2 c1 \\ 0 & 0 & 1 & l_1 - d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We obtained the same result as in previous chapter, however in a much simpler and more clearer way.