Chapter 4
Orientation

We often describe our environment as a three-dimensional world. The world of the roboticist is, however, six-dimensional. He must not only consider the position of an object, but also its orientation. When a robot gripper or end-effector approaches an object to be grasped, the space angles between the gripper and the object are of the utmost importance.

Six parameters are required to completely describe the position and orientation of an object in a space. Three parameters refer to the position and the other three to the orientation of the object. There are three possible ways how to mathematically describe the orientation of the object. The first possibility is a rotation/orientation matrix consisting of nine elements. The matrix represents a redundant description of the orientation. A non-redundant description is given by RPY or Euler angles. In both cases we have three angles. The RPY angles are defined about the axes of a fixed coordinate frame, while the Euler angles describe the orientation about a relative coordinate frame. The third possible description of the orientation is enabled by four parameters of quaternion.

In the second chapter we already became acquainted with rotation matrices around x, y, and z axis of a rectangular frame. We found them useful when developing the geometrical model of a robot mechanism. It is not difficult to understand that there exists also a matrix describing the rotation around an arbitrary axis. This can be expressed in the following form

$$ {}^{0}\mathbf{R}_{1} = \begin{bmatrix} {}^{1}\mathbf{i}\,{}^{0}\mathbf{i} & {}^{1}\mathbf{j}\,{}^{0}\mathbf{i} & {}^{1}\mathbf{k}\,{}^{0}\mathbf{i} \\ {}^{1}\mathbf{i}\,{}^{0}\mathbf{j} & {}^{1}\mathbf{j}\,{}^{0}\mathbf{j} & {}^{1}\mathbf{k}\,{}^{0}\mathbf{j} \\ {}^{1}\mathbf{i}\,{}^{0}\mathbf{k} & {}^{1}\mathbf{j}\,{}^{0}\mathbf{k} & {}^{1}\mathbf{k}\,{}^{0}\mathbf{k} \end{bmatrix}. \tag{4.1} $$

The matrix of the dimension 3×3 does not only represent the rotation, but also the orientation of the frame x_1–y_1–z_1 with respect to the frame x_0–y_0–z_0, as it can be seen from Fig. 4.1. The reference frame x_0–y_0–z_0 is described by the unit vectors

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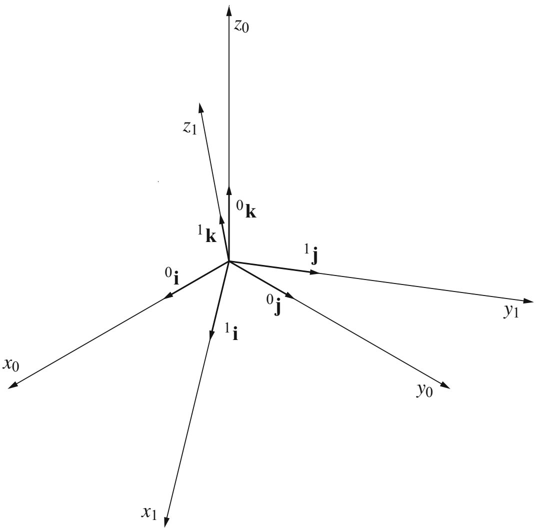

Fig. 4.1 Orientation of the coordinate frame x_1–y_1–z_1 with respect to the reference coordinate frame x_0–y_0–z_0

$^0\mathbf{i}$, $^0\mathbf{j}$, and $^0\mathbf{k}$ and the rotated frame x_1–y_1–z_1 with the unit vectors $^1\mathbf{i}$, $^1\mathbf{j}$, and $^1\mathbf{k}$. Both coordinate frames coincide in the same origin. As we are dealing with the unit vectors, the elements of the rotation/orientation matrix are simply the cosines of the angles appertaining to each pair of the axes.

Let us consider the example from Fig. 4.2 and calculate the matrix representing the orientation of the frame x_1–y_1–z_1, which is rotated for the angle $+\vartheta$ with respect to the frame x_0–y_0–z_0.

We are dealing with the following non-zero products of the unit vectors

$$\begin{aligned} {}^0\mathbf{i}\,{}^1\mathbf{i} &= 1, \\ {}^0\mathbf{j}\,{}^1\mathbf{j} &= \cos\vartheta, \\ {}^0\mathbf{k}\,{}^1\mathbf{k} &= \cos\vartheta, \\ {}^0\mathbf{j}\,{}^1\mathbf{k} &= -\sin\vartheta, \\ {}^0\mathbf{k}\,{}^1\mathbf{j} &= \sin\vartheta. \end{aligned} \tag{4.2}$$

The matrix describing the orientation of the frame x_1–y_1–z_1 with respect to x_0–y_0–z_0 is therefore

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\vartheta & -s\vartheta \\ 0 & s\vartheta & c\vartheta \end{bmatrix} \tag{4.3}$$

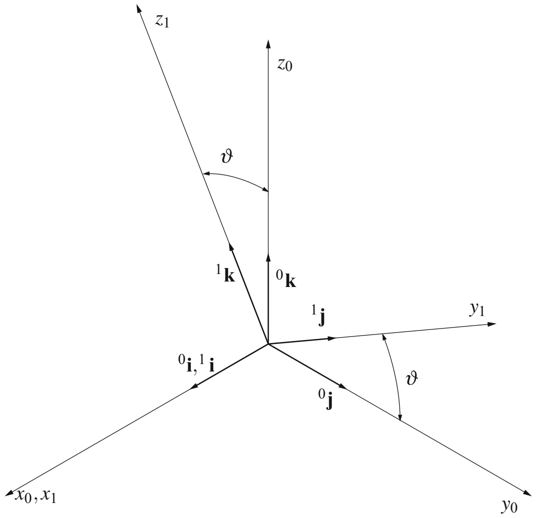

Fig. 4.2 Two coordinate frames rotated about the x_0 axis

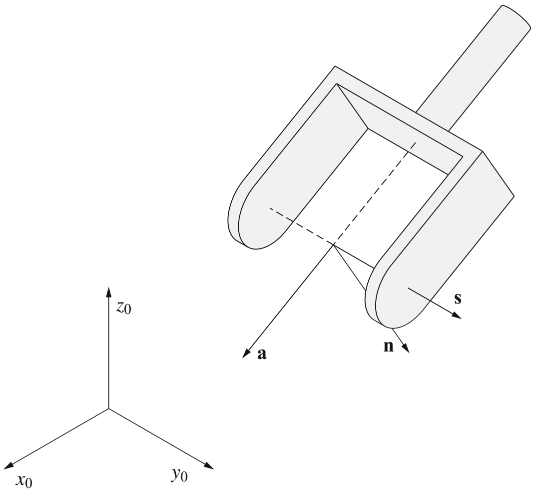

Fig. 4.3 Orientation of robot gripper

The matrix (4.3) can be interpreted also as the rotation matrix around the x axis that we already know as part of the homogeneous matrix (2.6) from the second chapter.

The notion of orientation is in robotics mostly related to the orientation of the robot gripper. A coordinate frame with three unit vectors $\mathbf{n}$, $\mathbf{s}$, and $\mathbf{a}$, describing the orientation of the gripper, is placed between two fingers of a simple robot gripper (Fig. 4.3).

The z axis vector lays in the direction of the approach of the gripper to the object. It is therefore denoted by vector **a** (approach). Vector, which is aligned with y axis, describes the direction of sliding of the fingers and is denoted as **s** (slide). The third vector completes the right-handed coordinate frame and is called normal. This can be shown as $\mathbf{n} = \mathbf{s} \times \mathbf{a}$. The matrix describing the orientation of the gripper with respect to the reference frame x_0–y_0–z_0 has the following form

$$\mathbf{R} = \begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix}. \tag{4.4}$$

The element n_x of the matrix (4.3) denotes the projection of the unit vector **n** on the x_0 axis of the reference frame. It equals the cosine of the angle between the axes x and x_0 and has the same meaning as the element ${}^1\mathbf{i}{}^0\mathbf{i}$ of the rotation/orientation matrix (4.1). The same is valid for the eight other elements of the orientation matrix **R** (4.3).

To describe the orientation of an object we do not need nine elements of the matrix. The left column vector is the cross product of vectors **s** and **a**. The vectors **s** and **a** are unit vectors which are perpendicular with respect to each other, so that we have

$$\begin{aligned} \mathbf{s} \cdot \mathbf{s} &= 1, \\ \mathbf{a} \cdot \mathbf{a} &= 1, \\ \mathbf{s} \cdot \mathbf{a} &= 0. \end{aligned} \tag{4.5}$$

Three elements are therefore sufficient to describe the orientation. This orientation is often described by the following sequence of rotations

R - roll - about z axis,
P - pitch - about y axis,
Y - yaw - about x axis.

This description is mostly used when describing the orientation of a ship or airplane. Let us imagine that the airplane flies along z axis and that the coordinate frame is positioned into the center of the airplane. Then, R represents the rotation φ about z axis, P refers to the rotation ϑ about y axis and Y to the rotation ψ about x axis, as shown in Fig. 4.4.

The use of the RPY angles for a robot gripper is shown in Fig. 4.5. As it can be realized from Figs. 4.4 and 4.5, the RPY orientation is defined with respect to a fixed coordinate frame. When developing the geometrical model of the SCARA robot manipulator in the second chapter, we were postmultiplying the homogenous transformation matrices describing the rotation (or translation) of each particular joint. The position and orientation of each joint frame was defined with respect to the preceding frame, appertaining to the joint axis which is not fixed. In this case, as

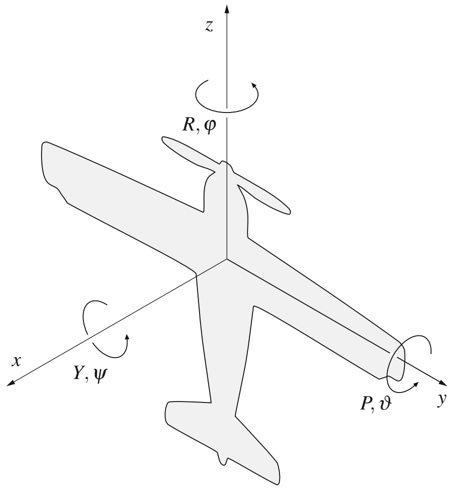

Fig. 4.4 RPY angles for the case of an airplane

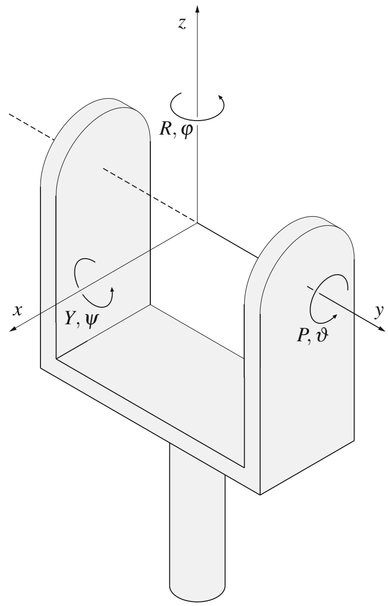

Fig. 4.5 RPY angles for the case of robot gripper

we have seen, we are multiplying the matrices from left to right. When we are dealing with consecutive rotations about the axes of the same coordinate frame, we make use of the premultiplication of the rotation matrices. In other words, the multiplications are performed in the reverse order from right to left.

We start with the rotation φ about z axis, continue with rotation ϑ around y axis and finish with the rotation ψ about x axis. The reverse order of rotations is also evident from the naming of RPY angles. The orientation matrix, which belongs to RPY angles, is obtained by the following multiplication of the rotation matrices

$$\begin{aligned}\mathbf{R}(\varphi, \vartheta, \psi) &= Rot(z, \varphi) Rot(y, \vartheta) Rot(x, \psi) = \\ &= \begin{bmatrix} c\varphi & -s\varphi & 0 \\ s\varphi & c\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\vartheta & 0 & s\vartheta \\ 0 & 1 & 0 \\ -s\vartheta & 0 & c\vartheta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\psi & -s\psi \\ 0 & s\psi & c\psi \end{bmatrix} = \\ &= \begin{bmatrix} c\varphi c\vartheta & c\varphi s\vartheta s\psi - s\varphi c\psi & c\varphi s\vartheta c\psi + s\varphi s\psi \\ s\varphi s\vartheta & s\varphi s\vartheta s\psi + c\varphi c\psi & s\varphi s\vartheta c\psi - c\varphi c\psi \\ -s\vartheta & c\vartheta s\psi & c\vartheta c\psi \end{bmatrix}.\end{aligned} \tag{4.6}$$

Equation (4.6) calculates the rotation matrix from the corresponding RPY angles.

We learned that rotation and orientation can be described either by rotation matrices or by RPY angles. In the first case we need 9 parameters, while only 3 parameters are required in the latter case. While matrices are convenient for computations, they do not however, provide a fast and clear image of, for example, the orientation of a robot gripper within a space. RPY and Euler angles do nicely present the orientation of a gripper, but they are not appropriate for calculations. In this chapter we shall learn that quaternions are appropriate for either calculation or description of orientation.

The quaternions represent extension of the complex numbers

$$z = a + \mathbf{i}b, \tag{4.7}$$

where $\mathbf{i}$ means the square root of -1, therefore $\mathbf{i}^2 = -1$. The complex numbers can be geometrically presented in a plane by introducing a rectangular frame with $\Re e$ (real) and $\Im m$ (imaginary) axis. When going from plane into space, two unit vectors $\mathbf{j}$ and $\mathbf{k}$ must be added to already existing $\mathbf{i}$. The following equality $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ is also valid. The quaternion has the following form

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}. \tag{4.8}$$

In the Eq. (4.8) q_i are real numbers, while $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ correspond to the unit vectors along the axes of the rectangular coordinate frame.

When describing the orientation by the RPY angles, the multiplications of the rotation matrices were needed. In a similar way we need to multiply the quaternions

$$pq = (p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k})(q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}). \tag{4.9}$$

Table 4.1 Rules for quaternion multiplications

*	1	**i**	**j**	**k**
1	1	**i**	**j**	**k**
i	**i**	−1	**k**	−**j**
j	**j**	−**k**	−1	**i**
k	**k**	**j**	−**i**	−1

The multiplication of quaternions is not commutative. When multiplying two quaternions we shall make use of the Table 4.1. Let us multiply two quaternions

$$\begin{aligned}
&(2+3\mathbf{i}-\mathbf{j}+5\mathbf{k})(3-4\mathbf{i}+2\mathbf{j}+\mathbf{k}) = \\
&= 6+9\mathbf{i}-3\mathbf{j}+15\mathbf{k}- \\
&-8\mathbf{i}-12\mathbf{i}^2+4\mathbf{ji}-20\mathbf{ki}+ \\
&+4\mathbf{j}+6\mathbf{ij}-2\mathbf{j}^2+10\mathbf{kj}+ \\
&+2\mathbf{k}+3\mathbf{ik}-\mathbf{jk}+5\mathbf{k}^2 = \\
&= 6+9\mathbf{i}-3\mathbf{j}+15\mathbf{k}- \\
&-8\mathbf{i}+12-4\mathbf{k}-20\mathbf{j}+ \\
&+4\mathbf{j}+6\mathbf{k}+2-10\mathbf{i}+ \\
&+2\mathbf{k}-3\mathbf{j}-\mathbf{i}-5 = \\
&= 15-10\mathbf{i}-22\mathbf{j}+19\mathbf{k}.
\end{aligned} \tag{4.10}$$

The following expression of a quaternion is specially appropriate to describe the orientation in the space

$$q = \cos\frac{\vartheta}{2} + \sin\frac{\vartheta}{2}\mathbf{s}. \tag{4.11}$$

In the Eq. (4.11) **s** is a unit vector aligned with the rotation axis, while ϑ is the angle of rotation. The orientation quaternion can be obtained from the RPY angles. Rotation R is described by the quaternion

$$q_{z\varphi} = \cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\mathbf{k}. \tag{4.12}$$

The following quaternion belongs to the rotation P

$$q_{y\vartheta} = \cos\frac{\vartheta}{2} + \sin\frac{\vartheta}{2}\mathbf{j}, \tag{4.13}$$

while rotation Y can be written as follows

$$q_{x\psi} = \cos\frac{\psi}{2} + \sin\frac{\psi}{2}\mathbf{i}. \tag{4.14}$$

After multiplying the above three quaternions (4.12–4.14), the resulting orientation quaternion is obtained

$$q(\varphi, \vartheta, \psi) = q_{z\varphi} q_{y\vartheta} q_{x\psi}. \tag{4.15}$$

Let us illustrate the three descriptions of the orientation, i.e. RPY angles, rotation matrix, and quaternions, by an example of description of gripper orientation. To make the example clear and simple, the plane of the two-finger gripper will be placed into the x_0–y_0 plane of the reference frame (Fig. 4.6). The RPY angles can be read from the Fig. 4.6. The rotations around z and y axis equal zero. The rotation for $-60°$ around the x axis can be seen from the Fig. 4.6. The orientation of the gripper can be, therefore, described by the following set of RPY angles

$$\varphi = 0, \vartheta = 0, \psi = -60°. \tag{4.16}$$

From the Fig. 4.6 we can read also the angles between the axes of the reference and gripper coordinate frame. Their cosines represent the orientation/rotation matrix $\mathbf{R}$

$$\begin{aligned} n_x &= \cos 0°, s_x = \cos 90°, a_x = \cos 90°, \\ n_y &= \cos 90°, s_y = \cos 60°, a_y = \cos 30°, \\ n_z &= \cos 0°, s_z = \cos 150°, a_z = \cos 60°. \end{aligned} \tag{4.17}$$

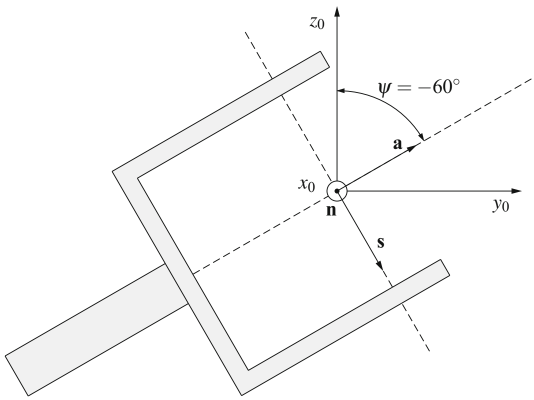

Fig. 4.6 Orientation of robot gripper

The matrix **R** can be calculated also by inserting the known RPY angles into the Eq. (4.6)

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.866 \\ 0 & -0.866 & 0.5 \end{bmatrix}. \tag{4.18}$$

In this way the correctness of our reading of the angles from the Fig. 4.6 was tested. We shall calculate the orientation quaternion by inserting the RPY angles into the Eqs. (4.12–4.14)

$$\begin{aligned} q_{z\varphi} &= 1 + 0\mathbf{k}, \\ q_{y\vartheta} &= 1 + 0\mathbf{j}, \\ q_{x\psi} &= 0.866 - 0.5\mathbf{i}. \end{aligned} \tag{4.19}$$

The orientation quaternion is obtained after multiplying the three above quaternions (4.15)

$$q_0 = 0.866, q_1 = -0.5, q_2 = 0, q_3 = 0. \tag{4.20}$$

The Eqs. (4.16), (4.18) and (4.20) demonstrate three different descriptions of the same gripper orientation.