Chapter 9
Trajectory Planning

In previous chapters we studied mathematical models of robot mechanisms. First of all we were interested in robot kinematics and dynamics. Before applying this knowledge to robot control, we must become familiar with the planning of robot motion. The aim of trajectory planning is to generate the reference inputs to the robot control system, which will ensure that the robot end-effector will follow the desired trajectory.

Robot motion is usually defined in the rectangular world coordinate frame placed in the robot workspace most convenient for the robot task. In the simplest task we only define the initial and the final point of the robot end-effector. The inverse kinematic model is then used to calculate the joint variables corresponding to the desired position of the robot end-effector.

9.1 Interpolation of the Trajectory Between Two Points

When moving between two points, the robot manipulator must be displaced from the initial to the final point in a given time interval t_f. Often we are not interested in the precise trajectory between the two points. Nevertheless, we must determine the time course of the motion for each joint variable and provide the calculated trajectory to the control input.

The joint variable is either the angle ϑ for the rotational or the displacement d for the translational joint. When considering the interpolation of the trajectory we shall not distinguish between the rotational and translational joints, so that the joint variable will be more generally denoted as q. With industrial manipulators moving between two points we most often select the so called trapezoidal velocity profile. The robot movement starts at $t = 0$ with constant acceleration, followed by the phase of constant velocity and finished by the constant deceleration phase (Fig. 9.1).

M. Mihelj et al., *Robotics*, https://doi.org/10.1007/978-3-319-72911-4_9

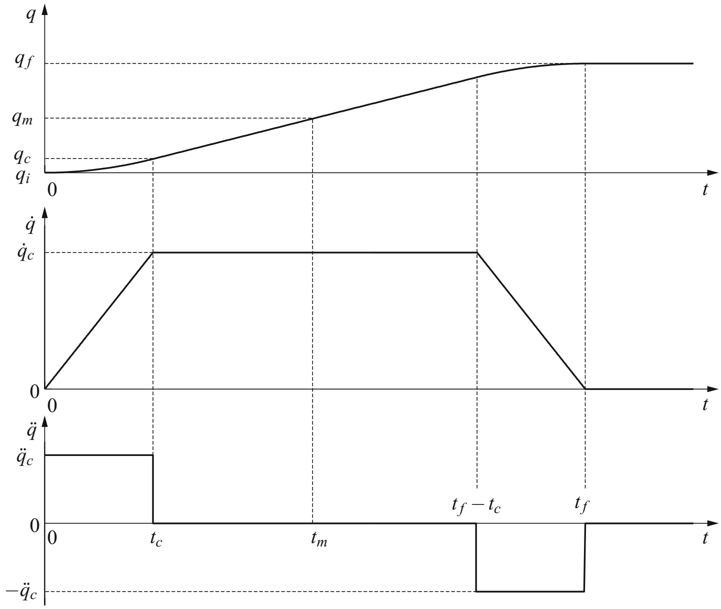

Fig. 9.1 The time dependence of the joint variables with trapezoidal velocity profile

The resulting trajectory of either the joint angle or displacement consists of the central linear interval, which is started and concluded with a parabolic segment. The initial and final velocities of the movement between the two points are zero. The duration of the constant acceleration phase is equal to the interval with the constant deceleration. In both phases the magnitude of the acceleration is $\ddot{q}_c$. In this way we deal with a symmetric trajectory, where

$$q_m = \frac{q_f + q_i}{2} \quad \text{at the moment} \quad t_m = \frac{t_f}{2}. \tag{9.1}$$

The trajectory $q(t)$ must satisfy several constraints in order that the robot joint will move from the initial point q_i to the final point q_f in the required time interval t_f. The velocity at the end of the initial parabolic phase must be equal to the constant velocity in the linear phase. The velocity in the first phase is obtained from the equation describing the constant acceleration motion

$$\dot{q} = \ddot{q}_c t. \tag{9.2}$$

At the end of the first phase we have

$$\dot{q}_c = \ddot{q}_c t_c. \tag{9.3}$$

The velocity in the second phase can be determined with the help of Fig. 9.1

$$\dot{q}_c = \frac{q_m - q_c}{t_m - t_c}, \tag{9.4}$$

where q_c represents the value of the joint variable q at the end of the initial parabolic phase (i.e., at the time t_c). Until that time the motion with constant acceleration $\ddot{q}_c$ takes place, so the velocity is determined by Eq. (9.2). The time dependence of the joint position is obtained by integrating Eq. (9.2)

$$q = \int \dot{q} dt = \ddot{q}_c \int t dt = \ddot{q}_c \frac{t^2}{2} + q_i, \tag{9.5}$$

where the initial joint position q_i is taken as the integration constant. At the end of the first phase we have

$$q_c = q_i + \frac{1}{2}\ddot{q}_c t_c^2. \tag{9.6}$$

The velocity at the end of the first phase (9.3) is equal to the constant velocity in the second phase (9.4)

$$\ddot{q}_c t_c = \frac{q_m - q_c}{t_m - t_c}. \tag{9.7}$$

By inserting Eq. (9.6) into Eq. (9.7) and considering the expression (9.1), we obtain, after rearrangement, the following quadratic equation

$$\ddot{q}_c t_c^2 - \ddot{q}_c t_f t_c + q_f - q_i = 0. \tag{9.8}$$

The acceleration $\ddot{q}_c$ is determined by the selected actuator and the dynamic properties of the robot mechanism. For chosen q_i, q_f, $\ddot{q}_c$, and t_f the time interval t_c is

$$t_c = \frac{t_f}{2} - \frac{1}{2}\sqrt{\frac{t_f^2 \ddot{q}_c - 4(q_f - q_i)}{\ddot{q}_c}}. \tag{9.9}$$

To generate the movement between the initial q_i and the final position q_f the following polynomial must be generated in the first phase

$$q(t) = q_i + \frac{1}{2}\ddot{q}_c t^2 \qquad 0 \le t \le t_c. \tag{9.10}$$

In the second phase a linear trajectory must be generated starting in the point (t_c, q_c), with the slope $\dot{q}_c$

$$(q - q_c) = \dot{q}_c(t - t_c). \tag{9.11}$$

After rearrangement we obtain

$$q(t) = q_i + \ddot{q}_c t_c (t - \frac{t_c}{2}) \qquad t_c < t \leq (t_f - t_c). \tag{9.12}$$

In the last phase the parabolic trajectory must be generated similarly to the first phase, only that now the extreme point is in (t_f, q_f) and the curve is turned upside down

$$q(t) = q_f - \frac{1}{2}\ddot{q}_c (t - t_f)^2 \qquad (t_f - t_c) < t \leq t_f. \tag{9.13}$$

In this way we obtained analytically the time dependence of the angle or displacement of the rotational or translational joint moving from point to point.

9.2 Interpolation by Use of via Points

In some robot tasks, movements of the end-effector more complex than point to point motions, are necessary. In welding, for example, the curved surfaces of the objects must be followed. Such trajectories can be obtained by defining, besides the initial and the final point, also the so called via points through which the robot end-effector must move.

In this chapter we shall analyze the problem, where we wish to interpolate the trajectory through n via points $\{q_1, \ldots, q_n\}$, which must be reached by the robot in time intervals $\{t_1, \ldots, t_n\}$. The interpolation will be performed with the help of trapezoidal velocity profiles. The trajectory will consist of a sequence of linear segments describing the movements between two via points and parabolic segments representing the transitions through the via points. In order to avoid the discontinuity of the first derivative at the moment t_k, the trajectory $q(t)$ must have a parabolic course in the vicinity of q_k. By doing so the second derivative in the point q_k (acceleration) remains discontinuous.

The interpolated trajectory, defined as a sequence of linear functions with parabolic transitions through the via points (the transition time Δt_k), is analytically described by the following constraints

$$q(t) = \begin{cases} a_{1,k}(t - t_k) + a_{0,k} & t_k + \frac{\Delta t_k}{2} \leq t < t_{k+1} - \frac{\Delta t_{k+1}}{2} \\ b_{2,k}(t - t_k)^2 + b_{1,k}(t - t_k) + b_{0,k} & t_k - \frac{\Delta t_k}{2} \leq t < t_k + \frac{\Delta t_k}{2} \, . \end{cases} \tag{9.14}$$

The coefficients $a_{0,k}$ and $a_{1,k}$ determine the linear parts of the trajectory, where k represents the index of the corresponding linear segment. The coefficients $b_{0,k}$, $b_{1,k}$ and $b_{2,k}$ belong to the parabolic transitions. The index k represents the consecutive number of a parabolic segment.

First, the velocities in the linear segments will be calculated from the given positions and the corresponding time intervals, as shown in Fig. 9.2. We assume that the initial and final velocities are equal to zero. In this case we have

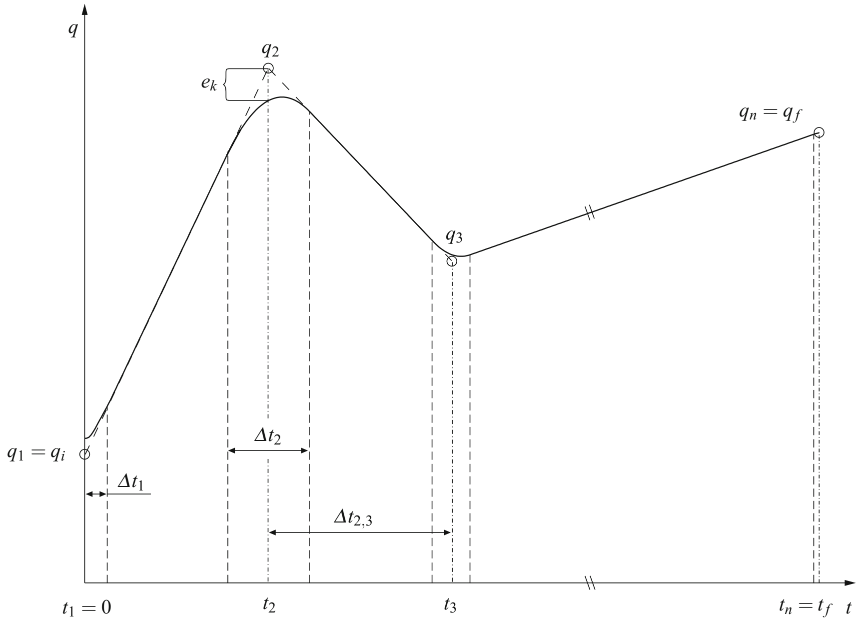

Fig. 9.2 Trajectory interpolation through n via points—linear segments with parabolic transitions are used

$$\dot{q}_{k-1,k} = \begin{cases} 0 & k = 1 \\ \frac{q_k - q_{k-1}}{t_k - t_{k-1}} & k = 2, \ldots, n \\ 0 & k = n+1. \end{cases} \tag{9.15}$$

Further, we must determine the coefficients of the linear segments $a_{0,k}$ and $a_{1,k}$. The coefficient $a_{0,k}$ can be found from the linear function (9.14), by taking into account the known position at the moment t_k, when the robot segment approaches the point q_k

$$q(t_k) = q_k = a_{1,k}(t_k - t_k) + a_{0,k} = a_{0,k}, \tag{9.16}$$

therefore

$$t = t_k \quad \Rightarrow \quad a_{0,k} = q_k \qquad k = 1, \ldots, n-1\,. \tag{9.17}$$

The coefficient $a_{1,k}$ can be determined from the time derivative of the linear function (9.14)

$$\dot{q}(t) = a_{1,k}. \tag{9.18}$$

By considering the given velocities in the time interval $t_{k,k+1}$, we obtain

$$a_{1,k} = \dot{q}_{k,k+1} \qquad k = 1, \ldots, n-1. \tag{9.19}$$

In this way the coefficients of the linear segments of the trajectory are determined and we can proceed with the coefficients of the parabolic functions. We shall assume that the transition time is predetermined as Δt_k. If the transition time is not prescribed, the absolute value of the acceleration $|\ddot{q}_k|$ in the via point must be first defined and then the transition time is calculated from the accelerations and velocities before and after the via point. In this case only the sign of the acceleration must be determined by considering the sign of the velocity difference in the via point

$$\ddot{q}_k = sign(\dot{q}_{k,k+1} - \dot{q}_{k-1,k})|\ddot{q}_k|, \tag{9.20}$$

where $sign(\cdot)$ means the sign of the expression in the brackets. Given the values of the accelerations in the via points and the velocities before and after the via point, the time of motion through the via point Δt_k is calculated (deceleration and acceleration)

$$\Delta t_k = \frac{\dot{q}_{k,k+1} - \dot{q}_{k-1,k}}{\ddot{q}_k}\,. \tag{9.21}$$

We shall proceed by calculating the coefficients of the quadratic functions. The required continuity of the velocity during the transition from the linear into the parabolic trajectory segment at the instant $(t_k - \frac{\Delta t_k}{2})$ and the required velocity continuity during the transition from the parabolic into the linear segment at $(t_k + \frac{\Delta t_k}{2})$ represents the starting point for the calculation of the coefficients $b_{1,k}$ and $b_{2,k}$. First, we calculate the time derivative of the quadratic function (9.14)

$$\dot{q}(t) = 2b_{2,k}(t - t_k) + b_{1,k}\,. \tag{9.22}$$

Assuming that the velocity at the instant $(t_k - \frac{\Delta t_k}{2})$, is $\dot{q}_{k-1,k}$, while at $(t_k + \frac{\Delta t_k}{2})$, it is $\dot{q}_{k,k+1}$, we can write

$$\begin{aligned} \dot{q}_{k-1,k} &= 2b_{2,k}\left(t_k - \frac{\Delta t_k}{2} - t_k\right) + b_{1,k} = -b_{2,k}\Delta t_k + b_{1,k} \qquad t = t_k - \frac{\Delta t_k}{2} \\ \dot{q}_{k,k+1} &= 2b_{2,k}\left(t_k + \frac{\Delta t_k}{2} - t_k\right) + b_{1,k} = b_{2,k}\Delta t_k + b_{1,k} \qquad t = t_k + \frac{\Delta t_k}{2}\,. \end{aligned} \tag{9.23}$$

By adding Eq. (9.23), the coefficient $b_{1,k}$ can be determined

$$b_{1,k} = \frac{\dot{q}_{k,k+1} + \dot{q}_{k-1,k}}{2} \qquad k = 1, \ldots, n \tag{9.24}$$

and by subtracting Eq. (9.23), the coefficient $b_{2,k}$ is calculated

$$b_{2,k} = \frac{\dot{q}_{k,k+1} - \dot{q}_{k-1,k}}{2\Delta t_k} = \frac{\ddot{q}_k}{2} \qquad k = 1, \ldots, n\,. \tag{9.25}$$

By taking into account the continuity of the position at the instant $(t_k + \frac{\Delta t_k}{2})$, the coefficient $b_{0,k}$ of the quadratic polynomial can be calculated. At $(t_k + \frac{\Delta t_k}{2})$, the position $q(t)$, calculated from the linear function

$$q\left(t_k + \frac{\Delta t_k}{2}\right) = a_{1,k}\left(t_k + \frac{\Delta t_k}{2} - t_k\right) + a_{0,k} = \dot{q}_{k,k+1}\frac{\Delta t_k}{2} + q_k \tag{9.26}$$

equals the position $q(t)$, calculated from the quadratic function

$$\begin{aligned} q\left(t_k + \frac{\Delta t_k}{2}\right) &= b_{2,k}\left(t_k + \frac{\Delta t_k}{2} - t_k\right)^2 + b_{1,k}\left(t_k + \frac{\Delta t_k}{2} - t_k\right) + b_{0,k} \\ &= \frac{\dot{q}_{k,k+1} - \dot{q}_{k-1,k}}{2\Delta t_k}\left(\frac{\Delta t_k}{2}\right)^2 + \frac{\dot{q}_{k,k+1} + \dot{q}_{k-1,k}}{2} \cdot \frac{\Delta t_k}{2} + b_{0,k}\,. \end{aligned} \tag{9.27}$$

By equating (9.26) and (9.27) the coefficient $b_{0,k}$ is determined

$$b_{0,k} = q_k + (\dot{q}_{k,k+1} - \dot{q}_{k-1,k})\frac{\Delta t_k}{8}\,. \tag{9.28}$$

It can be verified that the calculated coefficient $b_{0,k}$ ensures also continuity of position at the instant $(t_k - \frac{\Delta t_k}{2})$. Such a choice of the coefficient $b_{0,k}$ prevents the joint trajectory going through point q_k. The robot only more or less approaches this point. The distance of the calculated trajectory from the reference point depends mainly on the decelerating and accelerating time interval Δt_k, which is predetermined by the required acceleration $|\ddot{q}_k|$. The error e_k of the calculated trajectory can be estimated by comparing the desired position q_k with the actual position $q(t)$ at the instant t_k, which is obtained by inserting t_k into the quadratic function (9.14)

$$e_k = q_k - q(t_k) = q_k - b_{0,k} = -(\dot{q}_{k,k+1} - \dot{q}_{k-1,k})\frac{\Delta t_k}{8}\,. \tag{9.29}$$

It can be noticed that the error e_k equals zero only when the velocities of the linear segments before and after the via points are equal or when the time interval Δt_k is zero, meaning infinite acceleration (which in reality is not possible).

The described approach to the trajectory interpolation has a minor deficiency. From Eq. (9.29) it can be observed that, instead of reaching the via point, the robot goes around it. As the initial and final trajectory points are also considered as via points, an error is introduced into the trajectory planning. At the starting point of the trajectory, the actual and the desired position differ by the error e_1 (Fig. 9.3, the light curve shows the trajectory without correction), arising from Eq. (9.29). The error represents a step in the position signal, which is not desired in robotics. To avoid this abrupt change in position, the first and the last trajectory point must be handled separately from the via points.

The required velocities in the starting and the final points should be zero. The velocity at the end of the time interval Δt_1 must be equal to the velocity in the first

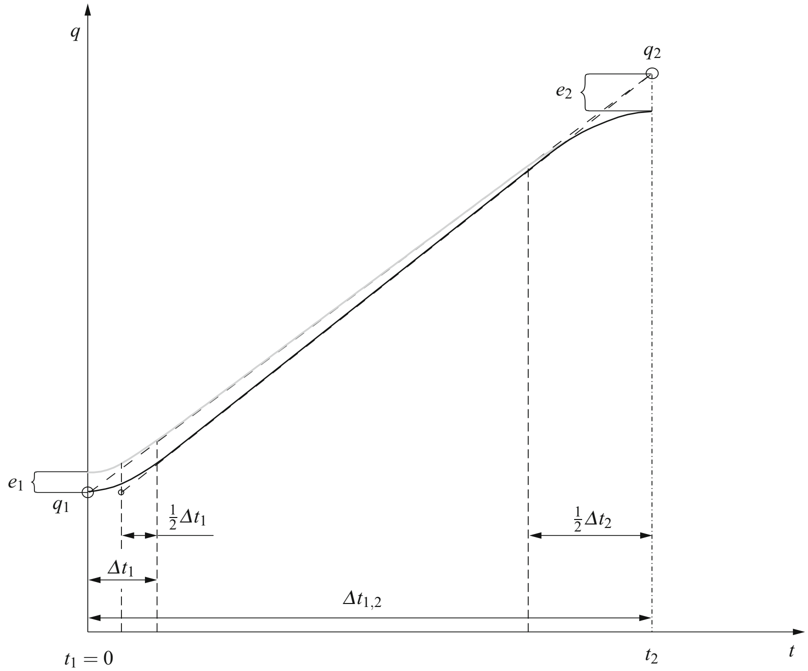

Fig. 9.3 Trajectory interpolation—enlarged presentation of the first segment of the trajectory shown in Fig. 9.2. The lighter curve represents the trajectory without correction, while the darker curve shows the corrected trajectory

linear segment. First, we calculate the velocity in the linear part

$$\dot{q}_{1,2} = \frac{q_2 - q_1}{t_2 - t_1 - \frac{1}{2}\Delta t_1}. \tag{9.30}$$

Equation (9.30) is similar to Eq. (9.15), only that now $\frac{1}{2}\Delta t_1$ is subtracted in the denominator, as in the short time interval (the beginning of the parabolic segment in Fig. 9.3) the position of the robot changes only to a very small extent. By doing so, a higher velocity in the linear segment of the trajectory is obtained. At the end of the acceleration interval Δt_1 we have

$$\frac{q_2 - q_1}{t_2 - t_1 - \frac{1}{2}\Delta t_1} = \ddot{q}_1 \Delta t_1 \tag{9.31}$$

We must determine also the acceleration $\ddot{q}_1$ at the starting point of the trajectory. Assuming that its absolute value $|\ddot{q}_1|$ was predetermined, only the sign must be adequately selected. The choice of the sign will be performed on the basis of the positional difference. In principle the velocity difference should be taken into account when determining the sign of acceleration, however the initial velocity is zero, and the sign can therefore depend on the difference in positions.

$$\ddot{q}_1 = sign(q_2 - q_1)|\ddot{q}_1|. \tag{9.32}$$

From Eq. (9.31), the time interval Δt_1 is calculated

$$(q_2 - q_1) = \ddot{q}_1 \Delta t_1 (t_2 - t_1 - \frac{1}{2}\Delta t_1). \tag{9.33}$$

After rearrangement we obtain

$$-\frac{1}{2}\ddot{q}_1 \Delta t_1^2 + \ddot{q}_1 (t_2 - t_1)\Delta t_1 - (q_2 - q_1) = 0, \tag{9.34}$$

so the time interval Δt_1 is

$$\Delta t_1 = \frac{-\ddot{q}_1(t_2 - t_1) \pm \sqrt{\ddot{q}_1^2 (t_2 - t_1)^2 - 2\ddot{q}_1 (q_2 - q_1)}}{-\ddot{q}_1}, \tag{9.35}$$

and after simplifying Eq. (9.35)

$$\Delta t_1 = (t_2 - t_1) - \sqrt{(t_2 - t_1)^2 - \frac{2(q_2 - q_1)}{\ddot{q}_1}}. \tag{9.36}$$

In Eq. (9.36), the minus sign was selected before the square root, because the time interval Δt_1 must be shorter than $(t_2 - t_1)$. From Eq. (9.30), the velocity in the linear part of the trajectory can be calculated. As is evident from Fig. 9.3 (the darker curve represents the corrected trajectory), the introduced correction eliminates the error in the initial position.

Similarly, as for the first segment, the correction must be calculated also for the last segment between points q_{n-1} and q_n. The velocity in the last linear segment is

$$\dot{q}_{n-1,n} = \frac{q_n - q_{n-1}}{t_n - t_{n-1} - \frac{1}{2}\Delta t_n}. \tag{9.37}$$

In the denominator of Eq. (9.37) the value $\frac{1}{2}\Delta t_n$ was subtracted, as immediately before the complete stop of the robot, its position changes only very little. At the transition from the last linear segment into the last parabolic segment the velocities are equal

$$\frac{q_n - q_{n-1}}{t_n - t_{n-1} - \frac{1}{2}\Delta t_n} = \ddot{q}_n \Delta t_n. \tag{9.38}$$

The acceleration (deceleration) of the last parabolic segment is determined on the basis of the positional difference

$$\ddot{q}_n = sign(q_{n-1} - q_n)|\ddot{q}_n|. \tag{9.39}$$

By inserting the above equation into Eq. (9.38), we calculate, in a similar way as for the first parabolic segment, also the duration of the last parabolic segment

$$\Delta t_n = (t_n - t_{n-1}) - \sqrt{(t_n - t_{n-1})^2 - \frac{2(q_n - q_{n-1})}{\ddot{q}_n}}. \tag{9.40}$$

From Eq. (9.37), the velocity of the last linear segment can be determined. By considering the corrections at the start and at the end of the trajectory, the time course through the via points is calculated. In this way the entire trajectory was interpolated at the n points.