

# Chapter 12

## Distinguishing Between Processes

*Satire is a lesson, parody is a game.*

- Vladimir Nabokov, *Strong Opinions*.

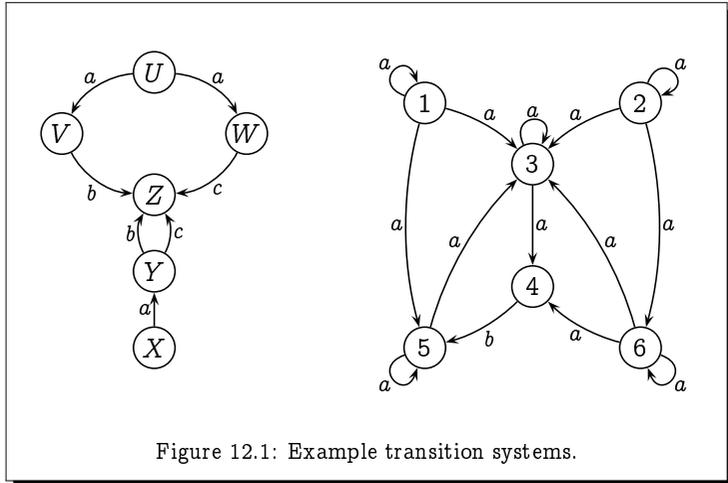
If we consider the properties which we used to distinguish between the vending machines from Section 11.4, we quickly notice an analogy with the way in which strategies for the two-player games of Chapter 10 were discussed; they both rely heavily on the use of modal verbs such as *may* and *must* to describe capabilities. In this chapter, we shall exploit this analogy by devising a two-player game for distinguishing between two given processes. In this game,

- the first player will aim to show that the two processes are *different*, by looking for an action that one process can do which the other cannot;
- the second player will aim to show that the two processes are *the same*, by showing that each process can copy every action made by the other.

In this game, one of the two players will always have a winning strategy (draws will not be possible); the two processes will be declared to be the same if the second player has a winning strategy, and different if the first has a winning strategy.

### 12.1 The Bisimulation Game

In this game we start by choosing two process states  $E$  and  $F$  (i.e., two designated states of some transition system). For example, we may consider the states  $X$  and  $U$  taken from the first of the two transition systems depicted in Figure 12.1. We may also define an *a priori* “time limit” of  $n \in \mathbb{N}$  moves, or declare that the game has no time limit (i.e., take  $n = \infty$ ). A game thus defined is represented either by  $G_n(E, F)$  or  $G_\infty(E, F)$ . The game is played between two players, who have the following goals.



1. The first player wishes to demonstrate that the two chosen states are in some way inherently different.
2. The second player wishes to demonstrate that the two chosen states are inherently the same.

To play the game, we start by placing tokens on the two states  $E$  and  $F$ , and then proceed as follows.

1. The first player chooses one of the two tokens, and moves it forward along an arrow to another state of her choosing; if this is impossible (that is, if there are no arrows leading out of either of the states on which the tokens sit), then the second player is declared to be the winner.
2. The second player must move the *other* token forward along an arrow which has *the same label* as the arrow used by the first player; if this is impossible, then the first player is declared to be the winner.

This exchange of moves is repeated for as long as neither player gets stuck, or for a total of  $n$  exchanges of moves in the case where a finite time limit  $n$  is defined. Note that the first player gets to choose which token to move *every time it is her turn*; she does not have to keep moving the same token. If the second player succeeds in matching every move of the first player, then he is declared to be the winner. If there is no time limit, then the second player is declared to be the winner of any play of the game that goes on forever. (It may seem rather strange to declare a player to be the winner of a play which lasts forever. However, there is nothing paradoxical

about this, and by doing so we ensure that there is always a winner; the game cannot end in a draw.)

### Example 12.1

Suppose we start with the tokens on states  $X$  and  $U$  of the first transition system of Figure 12.1, and we assume that the time limit is (at least) 2.

1. the first player can move the token on state  $U$  along the  $a$ -labelled arrow to state  $V$ ; in response the second player must move the token on state  $X$  along the  $a$ -labelled arrow to state  $Y$ .
2. The first player can then move the token on state  $Y$  along the  $c$ -labelled arrow to state  $Z$ ; the second player cannot respond to this move, as there are no  $b$ -labelled arrows leading out of state  $V$ , so the first player wins.

As the second player never has any options – and thus he can never have made a bad move – this defines a winning strategy for the first player.

### Example 12.2

Consider the following game played on the second transition system in Figure 12.1 with the tokens on states 1 and 2, where we assume that the time limit is (at least) 3.

1. The first player starts by moving the token on state 1 along the arrow labelled  $a$  to state 5. In response, the second player has to move the token on state 2 along an arrow labelled  $a$ ; there are three ways to do this: by moving the token to state 2, to state 3, or to state 6; after some thought, he chooses to move the token to state 6.
2. The first player then moves the token on state 6 along the arrow labelled  $a$  to state 4. In response, the second player has to move the token on state 5 along an arrow labelled  $a$ ; there are two ways to do this: by moving the token to state 3 or to state 5; he chooses to move the token to state 3.
3. The first player then moves the token on state 4 along the arrow labelled  $b$  to state 5. In response, the second player has to move the token on state 3 along an arrow labelled  $b$ ; however, this is impossible, so the first player is declared to be the winner.

In this case, the first player was lucky: the second player had several options open to him in response to the moves of the first player, and he simply chose poorly. Had the second player responded to the opening  $1 \xrightarrow{a} 5$  move of the first player by making the move  $2 \xrightarrow{a} 2$ , he could then have responded to all

subsequent moves of the first player. In fact, the second player has a winning strategy in this game. This fact will be made evident in Section 12.4

For such a simply-defined game, the fact that there is no possibility of a draw implies that one of the two players has a winning strategy. This fact is embodied in the following.

**Theorem 12.2**

*For any game  $G_n(E, F)$  or  $G_\infty(E, F)$ , either the first player has a winning strategy, or the second player has a winning strategy.*

**Exercise 12.2**

(Solution on page 470)

Prove Theorem 12.2 for finite games  $G_n(E, F)$ , by induction on  $n$ .

Induction cannot be used to prove the result for infinite games  $G_\infty(E, F)$ . Its proof is left as an exercise at the end of the chapter (Exercise 6, page 330).

**Definition 12.2**

*We say that two process states  $E$  and  $F$  are  $n$ -game equivalent, written  $E \sim_n F$ , if, and only if, the second player has a winning strategy in the game  $G_n(E, F)$ . Similarly, we say that  $E$  and  $F$  are  $\infty$ -game equivalent, written  $E \sim_\infty F$ , if, and only if, the second player has a winning strategy in the game  $G_\infty(E, F)$ .*

For example, if we again consider the three vending machines from Section 11.4, we can note that their starting states are pairwise 2-game equivalent but pairwise not 3-game equivalent.

1.  $V_i \sim_2 V_j$  for  $i, j \in \{1, 2, 3\}$ .

The second player has a winning strategy in the game which ends after the exchange of only two moves, as all three machines start with two consecutive 10p transitions.

2.  $V_1 \not\sim_3 V_2$  and  $V_1 \not\sim_3 V_3$ .

The first player has a winning strategy in the game which lasts for three exchanges of moves, namely to play arbitrarily for the first two exchanges of moves, and then to take the transition in the  $V_1$  process (coffee or tea) which is *not* available to the other process. The second player will be stuck at this point and lose the game.

3.  $V_2 \not\sim_3 V_3$ .

The first player has a winning strategy in the game which lasts for three exchanges of moves, namely to open with the transition in the

$V_3$  process towards the coffee transition, and then in the second move to take the transition in the  $V_2$  process towards the tea transition. The first player can then take the tea transition in the  $V_2$  process, which the second player cannot respond to.

**Exercise 12.3** (Solution on page 471)

Recall the following processes from Exercise 11.16.

$$\begin{array}{ll}
 A \stackrel{\text{def}}{=} b.c.0 + b.d.0 & C \stackrel{\text{def}}{=} a.B + a.A \\
 B \stackrel{\text{def}}{=} A + b.(c.0 + d.0) & D \stackrel{\text{def}}{=} a.B
 \end{array}$$

For which  $n$  do we have that  $C \sim_n D$ ? Justify your answer.

## 12.2 Properties of Game Equivalence

In this section, we explore various properties of game equivalences, beginning with the following characterisation which in particular provides an elegant inductive definition of finite game equivalences.

**Theorem 12.3**

1.  $E \sim_0 F$  for all processes  $E$  and  $F$ .
2.  $E \sim_{n+1} F$  if, and only if,
  - if  $E \xrightarrow{a} E'$  then  $F \xrightarrow{a} F'$  for some  $F'$  such that  $E' \sim_n F'$ ; and
  - if  $F \xrightarrow{a} F'$  then  $E \xrightarrow{a} E'$  for some  $E'$  such that  $E' \sim_n F'$ .
3.  $E \sim_\infty F$  if, and only if,
  - if  $E \xrightarrow{a} E'$  then  $F \xrightarrow{a} F'$  for some  $F'$  such that  $E' \sim_\infty F'$ ; and
  - if  $F \xrightarrow{a} F'$  then  $E \xrightarrow{a} E'$  for some  $E'$  such that  $E' \sim_\infty F'$ .

Pictorially, 2. and 3. can be represented as follows:



**Proof:** The first result about 0-game equivalence is trivially true, as the second player is immediately declared to be the winner of any game which lasts for only 0 exchanges of moves.

For the second result, we note that the second player has a winning strategy in the game  $G_{n+1}(E, F)$  if, and only if, regardless of what move the first player makes – either  $E \xrightarrow{a} E'$  or  $F \xrightarrow{a} F'$  – the second player can make a response – either  $F \xrightarrow{a} F'$  or  $E \xrightarrow{a} E'$  – in such a way that he still has a winning strategy in the game  $G_n(E', F')$ . But this is precisely what the statement in the theorem says.

Similarly for the third result, we note that the second player has a winning strategy in the game  $G_\infty(E, F)$  if, and only if, regardless of what move the first player makes – either  $E \xrightarrow{a} E'$  or  $F \xrightarrow{a} F'$  – the second player can make a response – either  $F \xrightarrow{a} F'$  or  $E \xrightarrow{a} E'$  – in such a way that he still has a winning strategy in the game  $G_\infty(E', F')$ . Again this is precisely what the statement in the theorem says.  $\square$

We can use Theorem 12.3 to prove that these game equivalence relations are indeed equivalence relations.

#### Theorem 12.4

*The relations  $\sim_n$  and  $\sim_\infty$  are all equivalence relations.*

**Proof:** To show that the relations  $\sim_n$  and  $\sim_\infty$  are reflexive, that is, that  $E \sim_n E$  and  $E \sim_\infty E$  for all  $E$ , we need to prove the following.

*The second player has a winning strategy in any game in which the two tokens start on the same state  $E$  of some transition system.*

This is obvious, as the second player need merely copy every move of the first player; wherever the first player moves one of the tokens, the second player moves the other token to the same place.

To show that the relations  $\sim_n$  and  $\sim_\infty$  are symmetric, that is, that  $F \sim_n E$  whenever  $E \sim_n F$  and that  $F \sim_\infty E$  whenever  $E \sim_\infty F$ , we need to prove the following.

*If the second player has a winning strategy in a game in which the tokens start on states  $E$  and  $F$  of some transition system, then he also has a winning strategy in the same game but with the tokens starting on states  $F$  and  $E$ .*

Again this is obvious, due to the symmetry of the game. The second player need merely use (essentially) the same winning strategy.

To show that the relations  $\sim_n$  and  $\sim_\infty$  are transitive, that is, that  $E \sim_n G$  whenever  $E \sim_n F$  and  $F \sim_n G$ , and that  $E \sim_\infty G$  whenever  $E \sim_\infty F$  and  $F \sim_\infty G$ , we need to prove the following.

*If the second player has a winning strategy in a game in which the tokens start on states  $E$  and  $F$  of some transition system, and*

he has a winning strategy in the same game but with the tokens starting on states  $F$  and  $G$ , then he also has a winning strategy in the same game but with the tokens starting on states  $E$  and  $G$ .

The details of this are left as an exercise.  $\square$

**Exercise 12.4** (Solution on page 471)

Prove, by induction on  $n$ , that the relation  $\sim_n$  is transitive for each  $n$ .

That the relation  $\sim_\infty$  is transitive cannot be proved by induction, but is proven in Exercise 12.7 (page 317).

**Theorem 12.5**

The relations  $\sim_n$  and  $\sim_\infty$  are strictly decreasing:  $\sim_0 \supset \sim_1 \supset \sim_2 \supset \dots \supset \sim_\infty$ . In particular, if  $E \sim_n F$  then  $E \sim_k F$  for all  $k \leq n$ .

**Proof:** If the first player has a winning strategy in a game of length  $n$ , then she can use that strategy to win any game with a longer time limit (and in particular, the game with no predetermined finite time limit). Alternatively, if the second player has a winning strategy in a game of length  $n$ , or one with no predetermined finite time limit, then he can use that strategy to win any game with a shorter time limit. This demonstrates the sequence of inclusions of the relations: if a pair of states is in  $\sim_i$  it will be in  $\sim_j$  for all  $j < i$ , and hence  $\sim_0 \supseteq \sim_1 \supseteq \sim_2 \supseteq \dots \supseteq \sim_\infty$ .

That these inclusions are strict can be noted by observing that for all  $n \in \mathbb{N}$ ,  $\text{Cl}_n \sim_n \text{Cl}$  but  $\text{Cl}_n \not\sim_{n+1} \text{Cl}$ ; and that for all  $n \in \mathbb{N}$ ,  $\text{Clock} \sim_n \text{Clock}_*$  but  $\text{Clock} \not\sim_\infty \text{Clock}_*$ , where these clock processes were defined in Example 11.11 (page 296).  $\square$

**Exercise 12.5** (Solution on page 472)

Prove the above claims, that for all  $n \in \mathbb{N}$ ,  $\text{Cl}_n \sim_n \text{Cl}$  but  $\text{Cl}_n \not\sim_{n+1} \text{Cl}$ ; and that for all  $n \in \mathbb{N}$ ,  $\text{Clock} \sim_n \text{Clock}_*$  but  $\text{Clock} \not\sim_\infty \text{Clock}_*$ .

## 12.3 Bisimulation Relations

We might expect  $\sim_\infty$  to be the “limit” of the  $\sim_n$  relations, that is, that the second player should have a winning strategy in the infinite game whenever he has a winning strategy for arbitrarily-long finite games. Alas, the above

example disproves this intuition, as the two clocks  $\text{Clock}$  and  $\text{Clock}_*$  are  $n$ -game equivalent for all  $n$  but they are not infinite-game equivalent.

Clearly these two clocks cannot be considered to be the same; the first one is guaranteed to stop after some indeterminate number of ticks, whereas the latter has the potential to tick forever. Infinite-game equivalence is thus the relation we wish to consider as defining equivalence between processes, and we shall henceforth generally refer to it as equivalence rather than infinite-game equivalence; that is, when we declare that two processes are equivalent, we shall mean that they are infinite-game equivalent.

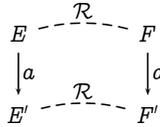
If our intuition had been right, then to demonstrate that two processes were equivalent we could exploit Theorem 12.3 and use induction to prove them to be  $n$ -game equivalent for all  $n \in \mathbb{N}$ . However, in general we need an alternative proof strategy to induction. Motivated by Theorem 12.3(3), we define the following notion to capture the essence of a winning strategy for the second player in an infinite game.

### Definition 12.5

A *bisimulation relation* is a binary relation  $\mathcal{R}$  over states which satisfies the following property: if  $E\mathcal{R}F$  then

- if  $E \xrightarrow{a} E'$  then  $F \xrightarrow{a} F'$  for some  $F'$  such that  $E'\mathcal{R}F'$ ; and
- if  $F \xrightarrow{a} F'$  then  $E \xrightarrow{a} E'$  for some  $E'$  such that  $E'\mathcal{R}F'$ .

We can represent this situation pictorially as follows:



As desired, a bisimulation relation  $\mathcal{R}$  represents a winning strategy for the second player in an infinite game: whenever the two tokens are on states which are related by  $\mathcal{R}$ , the second player can match any move of the first player in such a way as to ensure that the tokens once again end up on states related by  $\mathcal{R}$ . In this way, the second player can repeatedly match the moves of the first player *ad infinitum*.

### Theorem 12.6

The second player has a winning strategy in an infinite game with the tokens starting on states  $E$  and  $F$  if, and only if,  $E\mathcal{R}F$  for some bisimulation relation  $\mathcal{R}$ . Hence in particular,  $\mathcal{R} \subseteq \sim_\infty$  for any bisimulation relation  $\mathcal{R}$ .

**Proof:** If  $E\mathcal{R}F$  for some bisimulation  $\mathcal{R}$ , then the second player can merely use the winning strategy represented by  $\mathcal{R}$  as outlined above in order to win the infinite game with the tokens starting on states  $E$  and  $F$ .

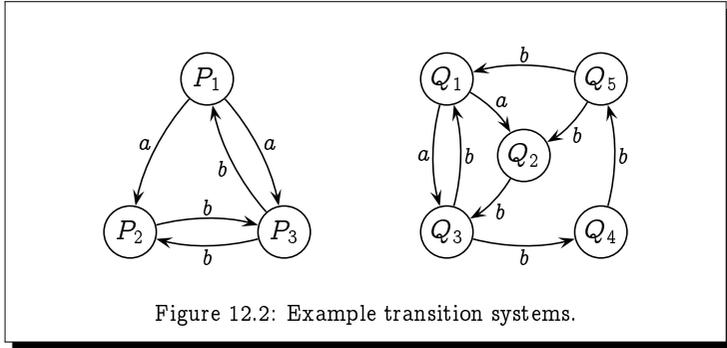


Figure 12.2: Example transition systems.

Conversely, by Theorem 12.3(3), the relation  $\sim_\infty$  itself is a bisimulation relation. Hence, if the second player has a winning strategy in an infinite game with the tokens starting on states  $E$  and  $F$ , then  $E\mathcal{R}F$  for the bisimulation relation  $\mathcal{R} = \sim_\infty$ .  $\square$

**Example 12.6**

Consider the two transition systems in Figure 12.2. It is straightforward to confirm, from Definition 12.5, that the following binary relation is a bisimulation relation:

$$\mathcal{R} = \{ (P_1, Q_1), (P_2, Q_2), (P_2, Q_4), (P_3, Q_3), (P_3, Q_5) \}.$$

As  $(P_1, Q_1) \in \mathcal{R}$ , by Theorem 12.6 we get that  $P_1 \sim_\infty Q_1$ .

**Exercise 12.6** (Solution on page 473)

Prove that the relation  $\mathcal{R}$  in Example 12.6 is a bisimulation relation.

**Exercise 12.7** (Solution on page 473)

Prove that if  $\mathcal{R}$  and  $\mathcal{S}$  are bisimulation relations over the states of a labelled transition system, then so is  $\mathcal{R} \circ \mathcal{S}$ . Infer from this that  $\sim_\infty$  is a transitive: that if  $E \sim_\infty F$  and  $F \sim_\infty G$  then  $E \sim_\infty G$ .

As a final observation regarding the relationship between the finite-game equivalences  $\sim_n$  and the infinite-game equivalence  $\sim_\infty$ , we note that the reason  $\sim_\infty \neq \bigcap_{n \in \mathbb{N}} \sim_n$  in the case of the two clocks – that is, that we can have  $\text{Clock} \sim_n \text{Clock}_x$  for all  $n \in \mathbb{N}$  but  $\text{Clock} \not\sim_\infty \text{Clock}_x$  – is solely due to

the fact that these clocks can perform their initial tick action in infinitely-many ways, leading to infinitely-many states. If this were not the case, then the relations would coincide. This is made precise as follows.

**Definition 12.7**

A process is *image-finite* if, and only if, for every state  $E$  of the process, and for every label  $a$ , the set  $\{F : E \xrightarrow{a} F\}$  is finite.

**Theorem 12.7**

For image-finite processes,  $\sim_\infty = \bigcap_{n \in \mathbb{N}} \sim_n$ .

**Proof:** Inclusion in one direction,  $\sim_\infty \subseteq \bigcap_{n \in \mathbb{N}} \sim_n$ , is guaranteed by Theorem 12.5:  $\sim_\infty \subseteq \sim_n$  for all  $n \in \mathbb{N}$ , so  $\sim_\infty \subseteq \bigcap_{n \in \mathbb{N}} \sim_n$ .

To show inclusion in the other direction,  $\bigcap_{n \in \mathbb{N}} \sim_n \subseteq \sim_\infty$ , it suffices to prove that the relation  $\mathcal{R} = \bigcap_{n \in \mathbb{N}} \sim_n$  is a bisimulation relation, for then by Theorem 12.6 we would have that  $\mathcal{R} \subseteq \sim_\infty$  as desired.

To this end, let  $E \mathcal{R} F$  be an arbitrary pair of states related by  $\mathcal{R}$ , that is,  $E \sim_n F$  for all  $n \in \mathbb{N}$ . Assume first that  $E \xrightarrow{a} E'$ . Since  $E \sim_{n+1} F$  for all  $n \in \mathbb{N}$ , by Theorem 12.3(2) we have that for each  $n \in \mathbb{N}$ ,  $F \xrightarrow{a} F_n$  for some  $F_n$  with  $E' \sim_n F_n$ . However, by image-finiteness there can be only finitely-many such  $F_n$ . Hence the same state  $F'$  must appear as  $F_n$  for infinitely-many values of  $n$ ; that is,  $F \xrightarrow{a} F'$  with  $E' \sim_n F'$  for infinitely-many  $n \in \mathbb{N}$ , and hence by Theorem 12.5 for *all*  $n \in \mathbb{N}$ . Hence  $E' \mathcal{R} F'$ .

By a symmetric argument, we can show that if  $F \xrightarrow{a} F'$  then  $E \xrightarrow{a} E'$  for some  $E'$  with  $E' \mathcal{R} F'$ . Hence  $\mathcal{R}$  is indeed a bisimulation.  $\square$

**Exercise 12.8** (Solution on page 474)

In the definition of the bisimulation game, the first player was free to move either token at each move. Suppose instead she must always move the same token with each move. For example, if for her first move she moves the token on state  $F$ , then she must always move that token in every move; at no time can she switch and move the token which started on state  $E$ . Let  $E \succsim_n F$  if, and only if, the second player has a winning strategy in this new game played for at most  $n$  rounds (where  $n$  may be  $\infty$ ).

1. Show that  $\succsim_n$  is an equivalence relation.
2. Show that  $E \sim_n F$  implies  $E \succsim_n F$ . That is, if the second player has a winning strategy in the bisimulation game, then he has a winning strategy in this new game.
3. Show that  $E \succsim_n F$  in general does not imply that  $E \sim_n F$ . (Hint: consider the processes  $a.b.0$  and  $a.b.0 + a.0$ .)

## 12.4 Bisimulation Colourings

Given that we cannot in general employ the inductive characterisation for finite-game equivalences to prove that two process states are (infinite-game) equivalent, we here devise an alternative approach to inferring if and when a winning strategy exists for the second player in an infinite game. The approach relies on colouring the states of the process in a particular fashion, thus partitioning the states into equivalence classes defined by colour.

### Definition 12.8

A *bisimulation colouring* of a transition system is a colouring of the states which satisfies the following property:

If *some* state with some colour  $C$  has a transition leading out of it into a state with some colour  $C'$ , then *every* state coloured  $C$  has an identically-labelled transition leading out of it into a state coloured  $C'$ .

For example, if some red state has an  $a$ -transition leading to a blue state, then *every* red state has an  $a$ -transition leading to a blue state.

That is to say, if  $E$  and  $F$  have the same colour, then

- if  $E \xrightarrow{a} E'$  then  $F \xrightarrow{a} F'$  for some  $F'$  such that  $E'$  and  $F'$  have the same colour; and
- if  $F \xrightarrow{a} F'$  then  $E \xrightarrow{a} E'$  for some  $E'$  such that  $E'$  and  $F'$  have the same colour.

Two states  $E$  and  $F$  are *bisimulation equivalent* or *bisimilar*, written  $E \sim F$ , if they have the same colour in some bisimulation colouring.

As a trivial example, if we assign each state its own unique colour, then this would clearly be a bisimulation colouring. However, finding a bisimulation colouring which assigns the same colour to two different states allows us to conclude that these two states are equivalent. This fact is recorded in the following.

### Theorem 12.8

$E \sim F$  if, and only if,  $E \sim_{\infty} F$ .

**Proof:** Given a bisimulation colouring of a transition system, the binary relation  $\mathcal{R}$  which relates like-coloured states is clearly a bisimulation relation (according to Definition 12.5), and hence, by Theorem 12.6, any two like-coloured states must be infinite-game equivalent. That is, if  $E \sim F$  (i.e.,  $E$  and  $F$  have the same colour in some bisimulation colouring) then  $E \sim_{\infty} F$ .

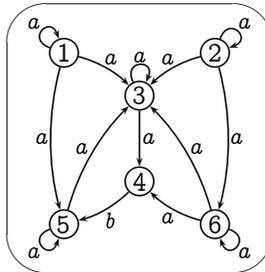
Conversely, consider colouring a transition system in such a way that any two states  $E$  and  $F$  have the same colour if, and only if,  $E \sim_\infty F$ . By Theorem 12.3(3), this colouring is clearly a bisimulation colouring (according to Definition 12.8). Thus, if  $E \sim_\infty F$  then  $E$  and  $F$  have the same colour in this bisimulation colouring, and hence  $E \sim F$ .  $\square$

This new characterisation of equivalence gives rise to the following approach to demonstrating that two states of a transition system are (or are not) equivalent. We start with all states being the same colour (white, say), and refine this colouring, always maintaining the following invariant:

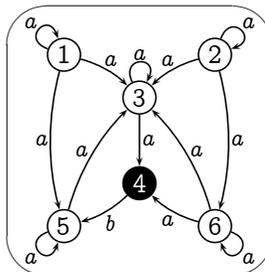
**Invariant:** *If  $E \sim F$  then  $E$  and  $F$  have the same colour.*

In this way, we start with a single equivalence class of states (ie, start with all states assigned the same colour), and refine this partition by subdividing the equivalence classes (by assigning some of the states in an equivalence class a new colour). This *partition refinement algorithm* can be effectively implemented to prove (or disprove) equivalences.

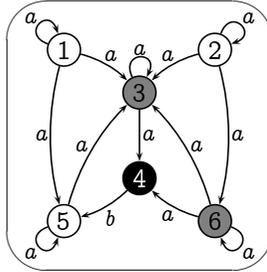
As an illustrative example, consider the second transition system of Figure 12.1.



The initial all-white colouring is *not* a bisimulation colouring, as the white state 4 has a  $b$ -transition to a white state 5, whereas the other white states 1, 2, 3, 5 and 6 do not have  $b$ -transitions to white states. Hence, by the invariant, state 4 cannot be equivalent to the other white states; in any bisimulation colouring, state 4 must have a different colour from states 1, 2, 3, 5 and 6. Hence we may safely refine our colouring by making state 4 a different colour (black, say).



This is still not a bisimulation colouring, as the white states 3 and 6 have  $a$ -transitions to black states, whereas the other white states 1, 2 and 5 do not. Hence, by the invariant, states 3 and 6 cannot be equivalent to the other white states; in any bisimulation colouring, states 3 and 6 must have a different colour from states 1, 2 and 5. Hence we may safely refine our colouring by making states 3 and 6 a different colour (grey, say).



This colouring *is* a bisimulation colouring, which by construction satisfies our invariant. To confirm this, we merely enumerate the possibilities.

1. every white state has an  $a$ -labelled arrow leading into a white state, and an  $a$ -labelled arrow leading into a grey state;
2. every grey state has an  $a$ -labelled arrow leading into a grey state, and an  $a$ -labelled arrow leading into a black state; and
3. every black state has a  $b$ -labelled arrow leading into a white state.

Hence, two states in this transition system are equivalent if, and only if, they have the same colour.

For the first transition system in Figure 12.1, a little reflection reveals that no bisimulation colouring of the states of this transition system exists in which the states  $X$  and  $U$  have the same colour.

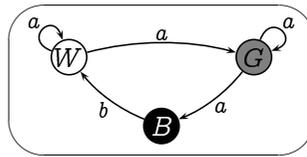
**Exercise 12.9** (Solution on page 475)

Prove the above claim that the states  $X$  and  $U$  of the first transition system in Figure 12.1 cannot have the same colour in any bisimulation colouring.

This completes the outline of our algorithm for determining whether two states of a transition system are equivalent. The algorithm works by partitioning the states into equivalence classes, by starting with the trivial partition consisting of a single class containing all states, and repeatedly refining the partition by splitting one of the classes into two separate sub-classes; it does this when it discovers that none of the states of one of the new sub-classes can be equivalent to any of the states of the other. If we carry this procedure out on a transition system with  $n$  states, then clearly it can perform no more than  $n$  refinements, as each refinement gives

rise to a new class and we cannot produce a partition with more than  $n$  classes. Furthermore, during each iteration we need only scan the edges of the transition system looking for a transition with which we can split a partition. Hence if there are  $k$  edges in the transition system, then this naive implementation of the algorithm would execute in time proportional to  $nk$ .

As a useful by-product, this algorithm produces a minimal-sized (in terms of the number of states) transition system which is equivalent to the original transition system. In the above example, the minimal-sized transition system has three states, which we might refer to as white, black and grey, and is depicted as follows.



**Exercise 12.10** (Solution on page 475)

Carry out the above bisimulation colouring algorithm on the first transition system of Figure 12.1, explaining each step in detail as above.

Note that the algorithm is nondeterministic; there may be several ways of splitting a set of like-coloured states. For example, starting with all states of the transition system in question white, there are three possible ways to proceed.

1. White states  $U$  and  $X$  both have an  $a$ -transition leading to a white state, while white states  $V$ ,  $W$ ,  $Y$  and  $Z$  do not.
2. White states  $V$  and  $Y$  both have a  $b$ -transition leading to a white state, while white states  $U$ ,  $W$ ,  $X$  and  $Z$  do not.
3. White states  $W$  and  $Y$  both have a  $c$ -transition leading to a white state, while white states  $U$ ,  $V$ ,  $X$  and  $Z$  do not.

It doesn't matter which choice you make; the end result will be the same.

## ★ 12.5 The Bisimulation Game Revisited: To Infinity and Beyond!

As we observed, the relations  $\sim_n$  representing  $n$ -game equivalence do not, in general, provide an adequate sequence of approximations to  $\sim_\infty$ , the  $\infty$ -game equivalence. This was demonstrated in Exercise 12.5 by the example

of the clocks, in which  $\text{Clock} \sim_n \text{Clock}_*$  for all  $n \in \mathbb{N}$  but  $\text{Clock} \not\sim_\infty \text{Clock}_*$ . All is not lost with the idea of approaching  $\sim_\infty$  by a sequence of approximations. The solution – which seems very odd on first encountering it – is to take the advice of Buzz Lightyear and go to infinity and beyond. The example of the clocks shows that it is not enough just to go to infinity through the natural numbers  $\sim_0, \sim_1, \sim_2, \sim_3, \dots$ . All we need to do is make sense of the idea of going beyond infinity.

Consider two young children playing the game of “*Who can name the largest number?*” in which they take turns naming larger and larger numbers. They quickly run up against the problem of what numbers come after *one-million, one-billion, one-trillion, one-quadrillion, . . .*, until one of them discovers the number *googol* ( $10^{100}$ , a one followed by 100 zeros); but the other quickly responds with an even bigger number: *googol-plus-one!* The first child’s argument that “*There’s no such thing, googol is the biggest number!*” is of course wrong. But then eventually, one of the children names the “number” *infinity*.

It is possible to accept the idea of naming *infinity* as a number, which by definition is bigger than any natural number, and even to give it its own symbol:  $\omega$ . But we will then be able to consider  $\omega+1$  as a bigger number, and  $\omega+2$  as an even bigger number, and even  $\omega+\omega$  as a far, far greater number; these are all infinitely-big numbers, but some are just bigger than others.

We already noted in Section 6.4, when comparing the sizes (cardinalities) of sets, that *infinity* comes in different varieties; in particular, the cardinality of the set of rational numbers is the same as the cardinality of the set of natural numbers (Exercise 6.16) but strictly smaller than the cardinality of the set of reals (Example 6.16). Infinite *counting* numbers (as opposed to *measuring* numbers) also exist as mathematical objects, and are collectively known as *ordinal numbers*. These are what will allow us to approximate  $\sim_\infty$ .

### 12.5.1 Ordinal Numbers

The ordinal numbers are an extension of the natural numbers as motivated above. The initial segment of ordinals is as follows:

$$0, 1, 2, \dots, \omega, \omega+1, \omega+2, \dots, \omega+\omega, \omega+\omega+1, \omega+\omega+2, \dots$$

Thus, after all finite ordinals have been listed (the natural numbers), the first infinite ordinal  $\omega$  is listed, and we can once again list ever-bigger ordinals by successively adding one; after adding each natural number to  $\omega$  we reach the ordinal  $\omega+\omega$ , or  $\omega \times 2$ , from which we continue the scheme, *ad infinitum*. The collection of ordinal numbers is denoted by  $\mathcal{O}$ . We shall not concern ourselves with the complete theory of ordinal numbers. All we will need to know about ordinals are the following four facts:

1. Every ordinal  $X$  has a *successor*  $X+1$ , whose *predecessor* is  $X$ .
2. An ordinal is either: zero (i.e., 0); or a successor ordinal (i.e.,  $X+1$  for some ordinal  $X$ ); or a *limit* ordinal which has a value which is greater than all previous ordinals, but has no predecessor.

The first limit ordinal – which is the first infinite ordinal – is  $\omega$ . It is the smallest ordinal greater than any finite ordinal (i.e., natural) number; the next limit ordinal is  $\omega+\omega$  (which is also written  $\omega\times 2$ ), then  $\omega+\omega+\omega$  (or  $\omega\times 3$ ), and so on.

3. Given any set  $S$  there is an ordinal  $X \in \mathcal{O}$  which represents the cardinality of  $S$ ; that is, there is a bijection between the set  $S$  and the set  $\{Y \in \mathcal{O} : Y < X\}$ .
4. In order to show that a property  $P(X)$  holds for all ordinals  $X$ , it suffices to show the following:

$$P(X) \text{ holds for } X \text{ whenever } P(Y) \text{ holds for all } Y < X; \text{ that is,}$$

$$(\forall Y < X : P(Y)) \Rightarrow P(X).$$

This principle is known as *transfinite induction* and is a restatement of the principle of strong induction from Section 9.4.

For those who find this brief initiation into the world of ordinal numbers confusing, you may find it helpful to concentrate on the natural numbers, and just think of  $\omega$  whenever limit ordinals are mentioned in what follows.

### Example 12.10

Consider the set  $\mathbb{N} \times \mathbb{N}$  of pairs of natural numbers ordered lexicographically:  $(i, j) < (p, q)$  if, and only if, either  $i < p$  or  $i = p$  and  $j < q$ . Thus, we can list these out in order as follows:

$$\begin{aligned} (0, 0) < (0, 1) < (0, 2) < \dots < (1, 0) < (1, 1) < (1, 2) < \dots \\ &< (2, 0) < (2, 1) < (2, 2) < \dots \\ &< (3, 0) < (3, 1) < (3, 2) < \dots \\ &< (4, 0) < (4, 1) < (4, 2) < \dots \\ &< \dots \end{aligned}$$

This gives us a way to view the start of the list of ordinal numbers, namely by associating the pair  $(i, j) \in \mathbb{N} \times \mathbb{N}$  with the ordinal number  $\omega \times i + j$ .

## 12.5.2 Ordinal Bisimulation Games

In Section 12.1 we defined the bisimulation game as either lasting for a predefined finite number  $n$  of exchanges of moves, denoting the game by

$G_n(E, F)$ ; or as continuing for as long as each player can make a move, denoting the game in this case by  $G_\infty(E, F)$ . We can refine this notion by defining the game  $G_X(E, F)$  for any ordinal number  $X$ . From now on, we shall use  $G_X(E, F)$  to denote both the game itself as well as the position that the game is in, with  $X$  denoting in a precise sense the length of the game.

1. From position  $G_0(E, F)$ , the second player is declared to be the winner.

This reflects the idea that the second player automatically wins any game of length 0, as he need not copy any moves of the first player.

2. From position  $G_{X+1}(E, F)$ , the two players exchange moves once as usual, and the play continues from  $G_X(E', F')$ , where  $E'$  and  $F'$  are the states to which the two tokens have been moved.

This reflects the usual idea that a game of length  $X+1$  consists of a single exchange of moves followed by a game of length  $X$ .

3. From position  $G_\lambda(E, F)$  where  $\lambda$  is a limit ordinal, the first player chooses a value  $X < \lambda$ , and the play continues from position  $G_X(E, F)$ .

This reflects the idea that  $G_\lambda(E, F)$  encompasses all games of length less than  $\lambda$ ; that is,  $G_X(E, F)$  for any  $X < \lambda$ . If the second player has a winning strategy in all such shorter games then he can force a win in any such game that the first player chooses, so the second player can force a win in this game  $G_\lambda(E, F)$ . However, if the first player has a winning strategy in some such shorter game, then she can choose that game and use her winning strategy to win the game  $G_\lambda(E, F)$ .

The following result corresponds to Theorem 12.2 (page 312), and is similarly proved but by transfinite induction rather than simple induction over the natural numbers.

### Theorem 12.10

*For any game  $G_X(E, F)$ , either the first player has a winning strategy, or the second player has a winning strategy.*

**Proof:** By transfinite induction. For the case  $X = 0$ , the second player clearly has a winning strategy for the game  $G_0(E, F)$ .

Suppose that  $X = Y+1$  is a successor ordinal, and that for any game  $G_Y(E', F')$  one of the two players has a winning strategy.

Suppose that  $X = Y+1$  is a successor ordinal, and that for any game  $G_Y(E', F')$  one of the two players has a winning strategy. The argument that one of the two players has a winning strategy in the game  $G_{Y+1}(E, F)$  is identical to the induction step in the proof of Theorem 12.2.

- Suppose that no matter what the first player does as her first move in the game  $G_{Y+1}(E, F)$ , the second player can respond in such a way

that he gets into a position in which he has a winning strategy in the game of length  $Y$ . This clearly provides a winning strategy for the second player in the game  $G_{Y+1}(E, F)$ .

- Hence, if the second player does *not* have a winning strategy in the game  $G_{Y+1}(E, F)$ , then the first player can make a move in such a way that any response the second player makes results in a position from which the second player does not have a winning strategy in the game of length  $Y$ ; but then by the inductive hypothesis, the first player has a winning strategy in the game of length  $Y$  from this resulting position, which means she has a winning strategy for the game  $G_{Y+1}(E, F)$ .

Suppose finally that  $X$  is a limit ordinal, and that for any game  $G_Y(E', F')$  with  $Y < X$  one of the players has a winning strategy.

- If there is some  $Y < X$  such that the first player has a winning strategy in the game  $G_Y(E, F)$ , then she can choose this value  $Y < X$  and use this winning strategy to win the game  $G_X(E, F)$ .
- If there is no  $Y < X$  such that the first player has a winning strategy in the game  $G_Y(E, F)$ , then by the induction hypothesis the second player has a winning strategy for the game  $G_Y(E, F)$  for each  $Y < X$ , and hence a winning strategy for the game  $G_X(E, F)$ .  $\square$

### Definition 12.10

We say that two process states  $E$  and  $F$  are  $X$ -game equivalent, written  $E \sim_X F$ , if, and only if, the second player has a winning strategy in the game  $G_X(E, F)$ .

### Example 12.11

From Exercise 12.5 we know that  $\text{Clock} \sim_\omega \text{Clock}_*$  since  $\text{Clock} \sim_n \text{Clock}_*$  for all  $n \in \mathbb{N}$  (i.e.,  $\text{Clock} \sim_X \text{Clock}_*$  for all  $X < \omega$ ).

However,  $\text{Clock} \not\sim_{\omega+1} \text{Clock}_*$ , since the move  $\text{Clock}_* \xrightarrow{\text{tick}} \text{C1}$  by the first player in the game  $G_{\omega+1}(\text{Clock}_*, \text{Clock})$  must be matched by a move  $\text{Clock} \xrightarrow{\text{tick}} \text{C1}_n$  for some  $n \in \mathbb{N}$ , but for *no*  $n \in \mathbb{N}$  do we have that  $\text{C1} \sim_\omega \text{C1}_n$ .

On the other hand, we do have that  $\text{tick}.\text{Clock}_* \sim_{\omega+1} \text{tick}.\text{Clock}$ .

### Exercise 12.11 (Solution on page 477)

Give process states  $E_n$  and  $F_n$  such that  $E_n \sim_{\omega+n} F_n$  but  $E_n \not\sim_{\omega+n+1} F_n$ .

We can now extend the results of Section 12.2 about game equivalence to ordinal game equivalence. We leave most of the proofs as exercises, as they

are straightforward adaptations of the proofs of the analogous results presented in Section 12.2. However, we prove the last result, which is the goal of this section: that the sequence of equivalences  $\sim_X$  does indeed properly approximate  $\sim_\infty$ .

**Theorem 12.11**

1.  $E \sim_0 F$  for all processes  $E$  and  $F$ .
2.  $E \sim_{X+1} F$  if, and only if,
  - if  $E \xrightarrow{a} E'$  then  $F \xrightarrow{a} F'$  for some  $F'$  such that  $E' \sim_X F'$ ; and
  - if  $F \xrightarrow{a} F'$  then  $E \xrightarrow{a} E'$  for some  $E'$  such that  $E' \sim_X F'$ .
3. For limit ordinals  $\lambda$ ,  $E \sim_\lambda F$  if, and only if,  $E \sim_X F$  for all  $X < \lambda$ .

**Theorem 12.12**

The relations  $\sim_X$  are all equivalence relations.

**Theorem 12.13**

The relations  $\sim_X$  are strictly decreasing. That is,  $\sim_X \subset \sim_Y$  whenever  $X > Y$ .

Specifically, if for each ordinal  $X \in \mathcal{O}$  we define the process

$$E_X \stackrel{\text{def}}{=} \sum_{Y < X} a.Y_Y,$$

then for all ordinals  $X$  and  $Y$  with  $X < Y$ ,  $E_X \sim_X E_Y$  but  $E_X \not\sim_{X+1} E_Y$ .

**Theorem 12.14**

$$\sim_\infty = \bigcap_{X \in \mathcal{O}} \sim_X.$$

**Proof:** Suppose that  $E \sim_\infty F$ ; we shall show by transfinite induction that  $E \sim_X F$  for all  $X \in \mathcal{O}$ .

- If  $X = 0$  then clearly  $E \sim_0 F$ .
- Suppose that  $X = Y + 1$  is a successor ordinal.
  - If  $E \xrightarrow{a} E'$  then  $F \xrightarrow{a} F'$  for some  $F'$  such that  $E' \sim_\infty F'$ , and hence by induction  $E' \sim_Y F'$ .
  - If  $F \xrightarrow{a} F'$  then  $E \xrightarrow{a} E'$  for some  $E'$  such that  $E' \sim_\infty F'$ , and hence by induction  $E' \sim_Y F'$ .

Thus we must have that  $E \sim_{Y+1} F$ .

- Suppose finally that  $X$  is a limit ordinal. Then by induction we have that  $E \sim_Y F$  for all  $Y < X$ , and hence  $E \sim_X F$ .

To show inclusion in the other direction,  $\bigcap_{X \in \mathcal{O}} \sim_X \subseteq \sim_\infty$ , it suffices to prove that the relation  $\mathcal{R} = \bigcap_{X \in \mathcal{O}} \sim_X$  is a bisimulation relation, for then by Theorem 12.6 we would have that  $\mathcal{R} \subseteq \sim_\infty$  as desired.

To this end, let  $E\mathcal{R}F$  be an arbitrary pair of states related by  $\mathcal{R}$ , that is,  $E \sim_X F$  for all  $X \in \mathcal{O}$ . Assume first that  $E \xrightarrow{a} E'$ . Since  $E \sim_{X+1} F$  for all  $X \in \mathcal{O}$ , by Theorem 12.11(2) we have that for each  $X \in \mathcal{O}$ ,  $F \xrightarrow{a} F_X$  for some  $F_X$  with  $E' \sim_X F_X$ . The set  $\{F_X : X \in \mathcal{O}\}$  can be no greater than the set of all states which are reachable from the state  $F$ , and there are ordinal numbers  $X$  which are arbitrarily larger than the cardinality of this set of states. Hence there must be some state  $F'$  which appears as  $F_X$  for arbitrarily-large values of  $X$ ; that is,  $F \xrightarrow{a} F'$  with  $E' \sim_X F'$  for arbitrarily-large  $X \in \mathcal{O}$ , and hence by Theorem 12.13 for all  $X \in \mathcal{O}$ . Hence  $E'\mathcal{R}F'$ .

By a symmetric argument, we can show that if  $F \xrightarrow{a} F'$  then  $E \xrightarrow{a} E'$  for some  $E'$  with  $E'\mathcal{R}F'$ . Hence  $\mathcal{R}$  is indeed a bisimulation.  $\square$

## 12.6 Additional Exercises

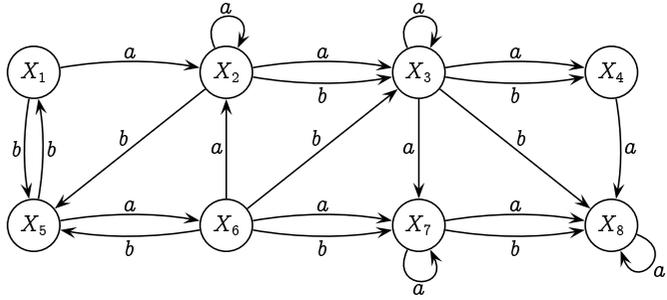
1. Carry out the bisimulation colouring algorithm step-by-step on the labelled transition system defined by the following process definition, and use this to provide an equivalent system with a minimal number of states.

$$\begin{aligned} W &\stackrel{\text{def}}{=} b.X + c.Z & X &\stackrel{\text{def}}{=} a.Y \\ Y &\stackrel{\text{def}}{=} c.X + b.Z & Z &\stackrel{\text{def}}{=} a.W + a.Y \end{aligned}$$

2. Carry out the bisimulation colouring algorithm step-by-step on the labelled transition system defined by the following process definition, and use this to provide an equivalent system with a minimal number of states.

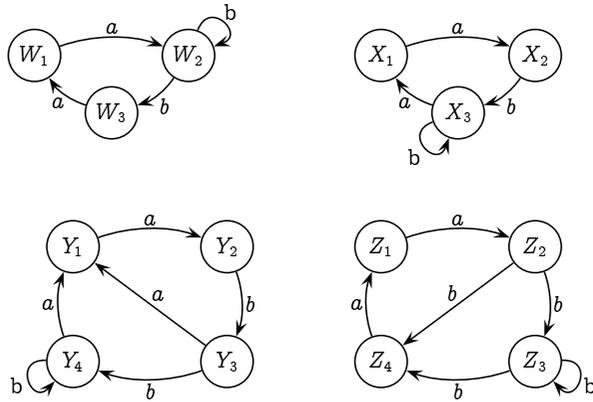
$$\begin{aligned} X_1 &\stackrel{\text{def}}{=} a.X_1 + b.X_3 & X_4 &\stackrel{\text{def}}{=} a.X_4 + b.X_3 \\ X_2 &\stackrel{\text{def}}{=} a.X_3 + a.X_6 + b.X_1 & X_5 &\stackrel{\text{def}}{=} a.X_3 + a.X_6 + b.X_1 \\ X_3 &\stackrel{\text{def}}{=} a.X_5 & X_6 &\stackrel{\text{def}}{=} a.X_3 + a.X_5 + b.X_4 \end{aligned}$$

3. Consider the following labelled transition system.



- (a) Which states are 2-game equivalent to state  $X_6$ ?
- (b) Which states are 2-game equivalent, but *not* 3-game equivalent, to state  $X_6$ ?
- (c) Which states are  $n$ -game equivalent to state  $X_5$  for all  $n$ ?

4. Consider the following labelled transition system.



- (a) For which  $n$  do we have  $W_1 \sim_n X_1$ ? Justify your answer.
- (b) For which  $n$  do we have  $W_1 \sim_n Y_1$ ? Justify your answer.
- (c) For which  $n$  do we have  $W_1 \sim_n Z_1$ ? Justify your answer.
- (d) For which  $n$  do we have  $X_1 \sim_n Y_1$ ? Justify your answer.
- (e) For which  $n$  do we have  $X_1 \sim_n Z_1$ ? Justify your answer.
- (f) For which  $n$  do we have  $Y_1 \sim_n Z_1$ ? Justify your answer.

5. Show that the algebraic laws from Section 11.5 are true of bisimulation equivalence:

$$(S_1) E + \mathbf{0} \sim E.$$

- $(S_2) E + E \sim E.$   
 $(S_3) E + F \sim F + E.$   
 $(S_4) (E + F) + G \sim E + (F + G).$   
 $(S_5) \text{ If } X \stackrel{\text{def}}{=} E \text{ then } X \sim E.$   
 $(C_1) \text{ If } E \sim F \text{ then } E + G \sim F + G.$   
 $(C_2) \text{ If } E \sim F \text{ then } a.E \sim a.F.$

6. Prove that the following binary relation on the states of a labelled transition system is a bisimulation relation:

$$R = \{ (E, F) : \text{the first player does not have a winning strategy in the game } G_\infty(E, F) \}.$$

Conclude from this the result from Theorem 12.2 that for any game  $G_\infty(E, F)$ , either the first player has a winning strategy, or the second player has a winning strategy.

7. In Theorem 12.7, only one of the two processes need be image-finite in order for the conclusion to be true. Prove this by showing that the relation

$$\mathcal{R} = \{ (E, F) : F \text{ is image-finite and } E \sim_k F \text{ for all } k \in \mathbb{N} \}$$

is a bisimulation relation.

8. The *trace set* of a state  $E$  is defined as

$$\mathcal{T}(E) = \{ s \in A^* : E \xrightarrow{s} F \text{ for some } F \}.$$

Two states  $E$  and  $F$  are *trace equivalent*, written  $E =_t F$ , if, and only if,  $\mathcal{T}(E) = \mathcal{T}(F)$ . Finally, a state  $E$  is *deterministic* if, and only if, for all  $s \in A^*$  there is at most one state  $F$  such that  $E \xrightarrow{s} F$ . That is, no state that is reachable from  $E$  has two transitions with the same label leading out of it.

- (a) Prove, by induction on the length of  $s$ , that if  $E \sim_n F$  and  $E \xrightarrow{s} E'$  with  $k = \text{length}(s) \leq n$ , then  $F \xrightarrow{s} F'$  for some  $F'$  with  $E' \sim_{n-k} F'$ . Deduce from this that if  $E \sim F$  then  $E =_t F$ .
- (b) Prove that  $\mathcal{R} = \{ (E, F) : E =_t F \text{ and } E, F \text{ are deterministic} \}$  is a bisimulation relation. Deduce from this that if  $E =_t F$  and  $E$  and  $F$  are deterministic, then  $E \sim F$ .

9. A *trace bisimulation relation* is a binary relation  $\mathcal{R}$  over states which satisfies the following property (where the extended transition relation  $\rightarrow \subseteq S \times A^* \times S$  is defined in the previous exercise):

If  $E \mathcal{R} F$  then

- if  $E \xrightarrow{s} E'$  then  $F \xrightarrow{s} F'$  for some  $F'$  such that  $E' \mathcal{R} F'$ ; and

- if  $F \xrightarrow{s} F'$  then  $E \xrightarrow{s} E'$  for some  $E'$  such that  $E' \mathcal{R} F'$ .

In terms of the bisimulation game, this reflects a change in the rules which allows the first player to make a sequence of transitions, rather than a single transition, which the second player must copy.

Prove that  $\mathcal{R}$  is a trace bisimulation relation if, and only if,  $\mathcal{R}$  is a bisimulation.

10. A set  $R \subseteq \Sigma$  is a *refusal set* of  $E$  if, and only if,  $E \not\xrightarrow{a}$  for any  $a \in R$ . A pair  $(w, R) \in \Sigma^* \times 2^\Sigma$  is a *failure* of  $E$  if, and only if,  $E \xrightarrow{w} F$  for some  $F$  such that  $R$  is a refusal set of  $F$ .  $E$  and  $F$  are *failures equivalent*, written  $E =_f F$ , if, and only if, they possess the same failures.
  - (a) Prove that  $E \sim F$  implies  $E =_f F$ , and that  $E =_f F$  implies  $E =_t F$ .
  - (b) Recalling the vending machines  $V_1, V_2$  and  $V_3$ , prove that  $V_1 \neq_f V_2$  but that  $V_2 =_f V_3$ , thus showing that the reverse implications do not hold in general.
  - (c) What is the relationship between  $=_f$  and  $\asymp$ , the simulation equivalence from Exercise 12.8?

11. Ordinal numbers, viewed as sets, can be defined as follows:

- if  $S$  is a set of ordinals, then so is  $\bigcup S$ ;
- if  $X$  is an ordinal, then so is  $X^+ = X \cup \{X\}$ ;
- nothing is an ordinal number unless it is constructed from the above two rules.

Thus we can construct the first few ordinals as follows:

- $0 = \bigcup \emptyset = \emptyset$
- $1 = 0^+ = 0 \cup \{0\} = \emptyset \cup \{0\} = \{0\}$
- $2 = 1^+ = 1 \cup \{1\} = \{0, 1\}$
- $3 = 2^+ = 2 \cup \{2\} = \{0, 1, 2\}$
- $\vdots$
- $n = \{0, 1, 2, \dots, n-1\}$
- $\vdots$
- $\omega = \bigcup \{0, 1, 2, \dots\} = \{0, 1, 2, \dots\}$
- $\omega+1 = \omega^+ = \omega \cup \{\omega\} = \{0, 1, 2, \dots, \omega\}$
- $\vdots$

Intuitively, an ordinal is the set of ordinals less than it; and the less-than relation corresponds to membership:  $X < Y$  if, and only if,  $X \in Y$ .

Prove the following facts about ordinal numbers  $X$ ,  $Y$  and  $Z$  as defined above.

- (a) Every element of an ordinal  $X$  is itself an ordinal.  
(Proof: By induction on  $X$ .)
- (b) If  $X \in Y$  and  $Y \in Z$  then  $X \in Z$ ; that is,  $\in$  is transitive.  
(Proof: By induction on  $Z$ .)
- (c) If  $X \in Y$  then  $X \subseteq Y$ .  
(Proof: Follows directly from previous result.)
- (d)  $X \notin X$ .  
(Proof: By induction on  $X$ .)
- (e)  $X \cap Y$  is an ordinal.  
(Proof: By induction on  $X$ .)

- 12. Prove Theorem 12.11. (page 327).
- 13. Prove Theorem 12.12 (page 327).
- 14. Prove Theorem 12.13 (page 327).