Chapter 10

Games and Strategies

You have to learn the rules of the game. And then you have to play better than anyone else.

<div style="text-align: right">- Albert Einstein.</div>

Games-of-chance derive this title from the fact that *luck* plays a part in deciding the winner of a play of the game. Sometimes the game consists solely of luck, as with COIN-FLIPPING (*"heads wins"*) or CARD-CUTTING (*"highest card wins"*). Typically, though, this isn't the case, and a sensible *strategy* is needed to beat a good player who isn't burdened by a string of extraordinarily bad deals of the cards (in the case of, e.g., POKER or BRIDGE), or throws of the dice (in the case of, e.g., BACKGAMMON or MONOPOLY). However, casinos operate (very successfully!) on the premise that (most of) their clientele do not play with luck on their side.

What we might call **games-of-no-chance** are those games for which the winner is decided based solely on *ability*. Examples of such games are CHESS and GO. They involve no decisions taken on the results of random events such as the deal of cards or the throw of dice, and no information is hidden from the players (apart from what moves the other player will choose to make during the play of the game).

For example, in the children's paper-and-pencil game NOUGHTS AND CROSSES (also known as TIC-TAC-TOE), two players alternately place crosses (×) and noughts (○) in nine square spaces arranged in a 3 × 3 grid. The goal of the first player is to align three crosses in a line (row, column or diagonal), and the goal of the second player is to align three noughts in a line (row, column or diagonal). A player wins the game if they achieve their goal before the other player does so. A game that ends with a full grid without a line of crosses or noughts is a draw.

When children first learn to play this game, the outcomes will be variable; sometimes the first player wins, sometimes the second player wins, and sometimes the game ends in a draw. However, every child eventually becomes bored with this game, as they discover that they can only win if their opponent makes a silly error. This is regardless of whether they are

playing as first player or second player, though it seems that the first player should have a distinct advantage.

(10.1) Strategies for Games-of-No-Chance

In this chapter we shall be interested in such two player games-of-no-chance. We shall typically refer to the first player as **A** (for Alice) for whom we shall use female pronouns (she, her), and the second player as **B** (for Bob) for whom we shall use male pronouns (he, his). Furthermore, these will be games of *perfect information*, meaning that both players will be aware of all aspects of the game: at every point in the game, both players know what moves have been made up to that point in time, as well as what moves their opponent can make in response to any move that they themselves make. The game of PAPER-SCISSORS-ROCK, for example, is not a game of perfect information, as neither player has information regarding the move being made by the other player. While there is no element of chance in the players' decision making, as each player is free to choose whatever move they wish, the lack of information about the opponent's move makes luck a factor in this game.

Another typical feature of the games that we shall consider is finiteness. A *finite game* is one that is guaranteed to terminate within a finite number of steps. This isn't true of many games, for example CHESS (unless some rule is introduced which declares a game to be a draw if it continues indefinitely, the standard rule being that a draw is declared if 50 consecutive moves have been made without a piece being captured nor a pawn being moved). If a play of a particular game may continue indefinitely, we will rule infinite plays to be predetermined in some way; that is, either the first player wins every infinite play, or the second player wins every infinite play, or the game is declared to be a draw. For example, we may declare that every infinite play of the game of CHESS is ruled to be a draw.

We shall at times consider games in which the first player is in the role of an attacker; she makes attacking moves which the second player, in the role of a defender, must defend against with his responses. We may refer to such games as *attacker-defender games*. The first player's aim is to achieve some goal (which will end the game), while the second player's aim is to prevent her from doing this. The important aspect of these games is that a play which continues forever is a positive result for the second player; that is, every infinite play of an attacker-defender game is ruled to be a win for the second player.

A *strategy* for a player in a game is a rule which tells that player what move to make each time it is their turn to move. A strategy which guarantees that you will win the game regardless of what moves your opponent

makes is referred to as a ***winning strategy***. If a game may end in a draw, then a strategy which guarantees that your opponent will not win (without guaranteeing that you will win) is referred to as a ***drawing strategy***.

A position in a game is a ***winning position*** if the player whose turn it is has a winning strategy from this position; it is a ***losing position*** if the other player (whose turn it is not) has a winning strategy from this position; and finally, it is a drawing position if neither player has a winning strategy from this position. Clearly, from a winning position there must be a move to a losing position, while every move from a losing position must lead to a winning position. From a drawing position there must be some move to a drawing position, perhaps some moves to winning positions, but no moves to losing positions.

For a given game it is not possible for both players to have a winning strategy, though it is possible that neither player has one. For example, we noted above that neither player has a winning strategy in NOUGHTS AND CROSSES; the game can be played out through the maximum nine moves filling in all nine squares in the grid without either player winning, regardless of how cleverly they play. The *first* player does not have a winning strategy because:

1. no matter what the first player does

2.   there is something that the second player can do such that

3.   no matter what the first player does

4.   there is something that the second player can do such that

5.   no matter what the first player does

6.   there is something that the second player can do such that

7.   no matter what the first player does

8.   there is something that the second player can do such that

9.   no matter what the first player does
she will not have formed a line of crosses.

Similarly, the *second* player does not have a winning strategy because:

1. there is something that the first player can do such that

2.   no matter what the second player does

3.   there is something that the first player can do such that

4.   no matter what the second player does

5.   there is something that the first player can do such that

6.   no matter what the second player does

7.   there is something that the first player can do such that

8.   no matter what the second player does
he will not have formed a line of noughts.

The simplicity of this game makes it easy to analyse; a play consists of (at most) nine moves, and the game is very symmetric. Thus a drawing strategy for both players is easy to discover, which ultimately renders the game uninteresting to play.

All such games are boring in this sense. At most one of the two players has a winning strategy; and by following this strategy, they ensure that the other player cannot do anything to avoid losing. If draws are possible, then both players may have strategies which prevent the other from winning. This important fact is recorded by the following theorem.

Theorem 10.1

In any two-player game-of-no-chance of perfect information, either one of the two players has a winning strategy, or they both have drawing strategies.

Proof: Clearly, if one of the two players has a winning strategy, then the other player cannot have a winning strategy: Fixing the strategies of the two players, only one of the two players can win the game, so only one of these two strategies can be a winning strategy.

Assume, then, that neither player has a winning strategy. That the first player does not have a winning strategy means that the second player may respond to each move made by the first player in such a way that either:

- the game ends in a draw or as a win for the second player;   or
- The game continues forever, and infinite games are either ruled to be draws or ruled to be wins for the second player.

That is to say, the second player has a strategy for ensuring that the first player does not win. Equally, that the second player does not have a winning strategy means that the first player has a strategy for ensuring that the second player does not win. Each of these strategies, therefore, must be a drawing strategy for its associated player.   □

Corollary 10.1

If a game cannot end in a draw, then one of the two players has a winning strategy.

Example 10.1

Consider the following game: starting with a pile of 10 coins, two players take turns removing either 2 coins or 3 coins from the pile. The player who takes the last coin wins; if one coin remains, then the game is a draw.

We can systematically analyse this game from the end backwards as follows:

(1) If there is 1 coin left, then the game is a draw. This is thus a *drawing* position.

(2) If there are 2 coins left, then you can win the game by taking both coins. This is thus a *winning* position.

(3) If there are 3 coins left, then you can win the game by taking all three coins. This is thus a *winning* position.

(4) If there are 4 coins left, then you can either:

- take 2 coins and leave 2, thus leaving the other player in what we know from (2) above is a winning position; or
- take 3 coins and leave 1, thus leaving the other player in what we know from (1) above is a drawing position.

Clearly the latter option is the correct one to make, and this is thus a *drawing* position.

(5) If there are 5 coins left, then you can either:

- take 2 coins and leave 3, thus leaving the other player in what we know from (3) above is a winning position; or
- take 3 coins and leave 2, again leaving the other player in what we know from (2) above is a winning position.

Whatever you do will leave the other player in a winning position. This is thus a *losing* position.

(6) If there are 6 coins left, then you can either:

- take 2 coins and leave 4, thus leaving the other player in what we know from (4) above is a drawing position; or
- take 3 coins and leave 3, thus leaving the other player in what we know from (3) above is a winning position.

Clearly the first option is the correct one to make, and this is thus a *drawing* position.

(7) If there are 7 coins left, then you can take 2 coins and leave 5, which we know from (5) above is a losing position. This is thus a *winning* position.

(8) If there are 8 coins left, then you can take 3 coins and leave 5, which we know from (5) above is a losing position. This is thus a *winning* position.

(9) If there are 9 coins left, then you can either:

– take 2 coins and leave 7, thus leaving the other player in what we know from (7) above is a winning position; or

– take 3 coins and leave 6, thus leaving the other player in what we know from (6) above is a drawing position.

Clearly the latter option is the correct one to make, and this is thus a *drawing* position.

(10) If there are 10 coins left (that is, if you are at the start of the game), then you can either:

– take 2 coins and leave 8, thus leaving the other player in what we know from (8) above is a winning position; or

– take 3 coins and leave 7, again leaving the other player in what we know from (7) above is a winning position.

Whatever you do will leave the other player in a winning position. This is thus a *losing* position.

We can summarise this analysis concisely in the following table:

0	1	2	3	4	5	6	7	8	9	10
L	D	W	W	D	L	D	W	W	D	L
–	–	2	3	3	–	2	2	3	3	–

The top row indicates the running total; the middle row indicates whether the current player is in a winning position (W), or a losing position (L), or in a drawing position (D); and the bottom row indicates how many coins (2 or 3) the player should remove from the pile in that turn (there being no entry in the cases where the player is in a losing position).

Figure 10.1 depicts this game as a so-called **game tree** for this game. The nodes of this tree represent positions in the game (labelled by the number of coins remaining in the pile), and the arrows represent the possible moves which a player can make in the given position (labelled by the number of coins removed by that move). The winning positions are depicted by circled nodes, while the losing positions are depicted by boxed nodes; the nodes which are neither circled nor boxed depict drawing positions. The important observations to make are:

1. every winning position has at least one move leading to a losing position, (that is, every circled node has an arrow leading to a boxed node, emphasised in the figure by a double arrow);

2. every move from a losing position leads to a winning position (that is, every arrow from a boxed node leads to a circled node); and

3. every drawing position has a move to another drawing position; possibly a move to a winning position; but no move to a losing position

Figure 10.1: The game tree for a simple coin game.
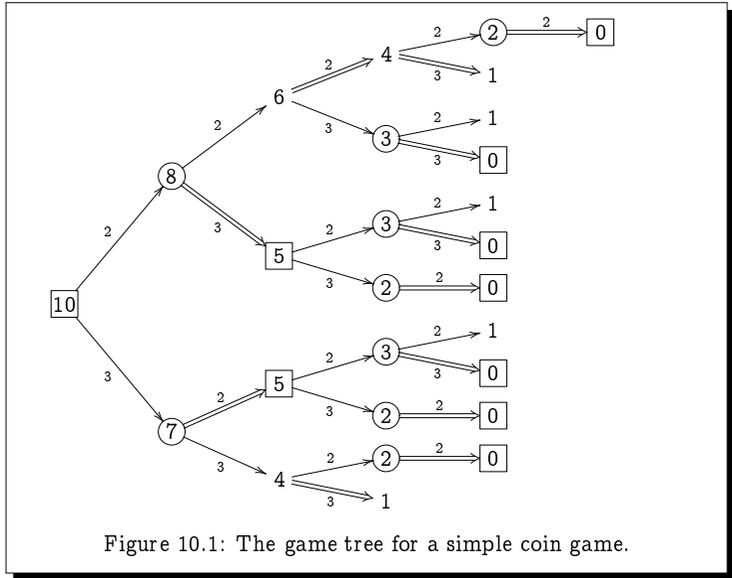
(that is, every undecorated node has an arrow to another undecorated node, emphasised in the figure by a double arrow; possibly an arrow to a circled node; but no arrow to a boxed node).

These three observations respectively define what it means for a position to be a winning position, a losing position, or a drawing position.

Exercise 10.1   (Solution on page 453)

In the game of TAKE-3, there is a single pile of coins, and two players alternately remove either 1, 2, or 3 coins from the pile. The player who takes the last coin wins.

1. For each number n from 1 to 10, explain who has the winning strategy in TAKE-3 starting from a pile of n coins. In the cases in which the first player has the winning strategy, state how many coins (1, 2 or 3) the first player should take.

2. Generalise the above by explaining who has the winning strategy in TAKE-3 starting from a pile of n coins for an arbitrary n.

3. Generalise the above further by explaining who has the winning strategy in TAKE-k starting from a pile of n coins for an arbitrary n, but where players may alternately remove between 1 and k coins (above, we had $k=3$).

4. MISÈRE TAKE-3 is played exactly like TAKE-3, but the object of the game is to ***not*** take the last coin; that is, you wish to force your opponent to take the last coin. How does this change the above analysis?

Exercise 10.2   (Solution on page 454)

In the game of MISÈRE NOUGHTS AND CROSSES, the aim is to *avoid* placing three of your symbols in a row, but rather to force your opponent to place three of their symbols in a row.

1. The first player does not have a winning strategy in this game. Explain how the second player can play to avoid losing.

   (Hint: It is a good idea to occupy two adjacent side squares first, and then a square which is aligned with only one of these two side squares. Why is this possible, and why does it work?)

2. The second player also does not have a winning strategy in this game. Explain how the first player can play to avoid losing.

   (Hint: Start by placing the first cross in the middle, and then "mirroring" every move of the second player by placing each subsequent cross directly opposite to where the second player places his noughts. Why is this a good idea?)

Exercise 10.3   (Solution on page 454)

The game of CLOCK-2-3 is played on a board which looks like the face of a 12-hour clock such as depicted as follows:
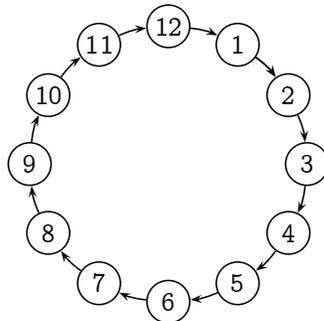

A token is placed on one of the hours (1 through 12) and the players take turns moving the token either 2 or 3 hours forward (i.e., in a clockwise fashion). The player who moves the token onto the 12 o'clock slot wins the game.

Explain who has the winning strategy in Clock-2-3 starting from each of the 12 hours. In the cases in which the first player has the winning strategy, state how many hours forward (2 or 3) the first player should move the token. (As a start, the first player clearly has a winning strategy starting from either 9 o'clock or 10 o'clock, by moving 3 and 2 hours, respectively, to land on 12 o'clock. Thus the second player has the winning strategy starting from 7 o'clock, as the first player will be forced to move the token to either 9 o-clock or 10 o'clock.)

Exercise 10.4   (Solution on page 456)

The following depicts a simple variant of the children's board game Snakes and Ladders.
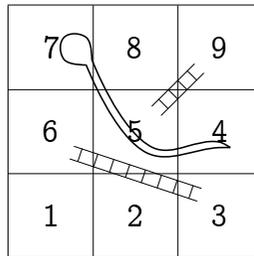
In this game, a single shared counter is started on square 1, and two players take turns moving the counter either *one* or *two* spaces forward (with the player moving deciding whether to move one or two spaces). If the counter lands at the foot of a ladder, it climbs to the top of the ladder; and if the counter lands on the head of a snake, it slides down to the tail of the snake. The object of this game is to be the one to move the counter to the final square number 9.

Identify which of the positions are winning positions; which are losing positions; and which are drawing positions. (Recall that a winning position is one from which there is a move to a losing position, whereas a losing position is one from which every move leads to a winning position; all other positions are drawing positions, as from these you cannot force a win nor be forced to lose.) As a start, 9 is a losing position in both games, while 8 is a winning position in both games, as you can win by moving one space forward. For the non-losing positions, indicate the optimal move(s).

The Chess-playing computer Deep Blue attributes a large part of its success in its ability to search for a winning strategy in a manner similar to the above analysis. The salvation for such games comes from the fact that there are astronomically-many configurations to consider, far too many for

a modern (and indeed any conceivable) computer to analyse. Today's Kas-parovs are safe in the fact that CHESS-playing computers such as Deep Blue must still invoke questionable decision-making procedures, but perhaps one day a Very Deep Blue will render CHESS-playing, like playing NOUGHTS AND CROSSES, a pointless activity.

In the rest of this chapter we consider several moderately-simple two-player games-of-no-chance, and try to understand the strategies which a player should use in order to win them.

10.2 Nim

NIM is a simple and ancient game played with coins, thought to be Chinese in origin. To play this game, an arbitrary number of piles of coins are formed, each with an arbitrary number of coins in them, and two players alternate in removing one or more coins from any one pile. Whoever takes the last coin is declared to be the winner.

This game is trivial when played with only one or two piles of coins, or with three very small piles. The analysis of the game in these cases is as follows.

1. In the one-pile game, the *first* person has a trivial winning strategy: take all the coins in the first move.

2. In the two-pile game,

    (a) if the piles contain an equal number of coins then the *second* player has a winning strategy: always take the same number of coins as the first player, repeatedly leaving the first player with equal-sized piles.

    (b) if the piles contain an unequal number of coins, then the *first* player has a winning strategy: start by taking coins from the larger pile to leave equal-sized piles, and then use the strategy described in 2(a) for the second player.

3. In the three-pile game,

    (a) if two of the piles are equal, then the *first* player has a winning strategy: take all of the coins in the third pile, leaving just the two equal-sized piles (and one empty pile), and then use the strategy described in 2(a) for the second player.

    (b) if the piles contain one, two, and three coins, respectively, then the *second* player has a winning strategy:

      i. if the first person takes the whole of one of the piles, then there will be just two unequal piles left (and one empty pile), and the second player can win using the strategy described in 2(b) for the first player.

      ii. if the first player takes just part of one of the piles, then there will be three non-empty piles remaining, two of which must be equal, and the second player can win using the strategy described in 3(a) for the first player.

The game is traditionally played with three piles, containing three, four, and five coins, respectively, and even here its complexity starts to become convincing; after playing many times, it remains difficult to glean any good long-term strategy. The only approach to the game which comes immediately to mind, reminiscent of games like CHESS, is to look ahead several moves, anticipating the moves of the other player, in order to avoid bad positions. With time, it is possible to recognise more and more bad positions, and become better at avoiding these. However, the character of the game changes with four, five, or more piles.

In fact there is a straightforward winning strategy for this game, either for the first player or the second player, depending on the number of piles and the number of coins in each pile. To see this, write out the numbers of coins in the piles in binary notation, one above the other, and add up the columns modulo 2; that is, compute the sum of a column to be 0 if it has even parity (i.e., there are an even number of 1's in the column), and 1 if it has odd parity (i.e., there are an odd number of 1's in the column). If all columns have an even parity, we shall say that the position is *balanced*; otherwise we say that the position is *unbalanced*. The following observations can be made:

1. If every column has even parity (i.e., we are in a balanced position), then *every* move will result in some column having odd parity (i.e., every move leads to an unbalanced position).

2. If one or more columns have odd parity (i.e., we are in an unbalanced position), then *some* move will result in every column having even parity (i.e., some move leads to a balanced position).

For example, in the 3-4-5-7 game, the first and third columns have odd parity (while the second column has even parity). By taking 3 coins from the second pile, we give the first and third columns even parity (while leaving the parity of the second column even).

$$
\begin{array}{ll}
3: 0\ 1\ 1 & 3: 0\ 1\ 1 \\
4: 1\ 0\ 0 \implies & 1: 0\ 0\ 1 \\
5: 1\ 0\ 1 & 5: 1\ 0\ 1 \\
7: \underline{1\ 1\ 1} & 7: \underline{1\ 1\ 1} \\
\quad 1\ 0\ 1 & \quad 0\ 0\ 0 \\
\ \uparrow\ \ \uparrow &
\end{array}
$$

From this new position, whatever coins are removed, there will result at least one column with odd parity.

With the above two observations, along with the insight that the ultimate goal of the game is to make all columns add up to zero, and hence an even

number, it is clear that:

1. the ***first*** player has a winning strategy if one or more of the columns has odd parity: the correct move is to remove coins so as to leave all columns with even parity;

2. the ***second*** player has a winning strategy if all of the columns have even parity: regardless of what move is made, the resulting parity of at least one column will be odd.

$\boxed{\textbf{Exercise 10.5}}$ (Solution on page 456)

If a player is in a winning position in NIM, then there will in general be more than one winning move. (Two moves are different if they involve different piles, or if they involve the same pile but removing different numbers of coins.) What is the maximum number of different winning moves possible from a NIM position with n piles? Justify your answer.

$\boxed{\textbf{Exercise 10.6}}$ (Solution on page 456)

Suppose we change the rules of NIM slightly so that the first player, instead of removing some coins from a pile, has the additional option of *creating a new pile* of any size (with at least one coin in it); the first player may do this at most once during a play of the game. Under which circumstances can the first player force a win with the help of this extra move? (Consider, in particular, the two situations in which the game starts from an unbalanced, respectively a balanced, position.)   Justify your answer.

★ $\boxed{\textbf{10.3}}$ Fibonacci Nim

The next game we consider is a variation on NIM called FIBONACCI NIM. In this game we have a single pile containing $n \geq 2$ coins. The first player removes one or more coins but not the whole pile. From then on, the players alternate moves, each person removing one or more coins, but not more than twice as many coins as the other player has taken in the preceding move. The player who removes the last coin wins.

The analysis of this game is complicated by the fact that a player's available moves depend on the opponent's last move. However, we can nonetheless easily analyse small instances of this game:

2 coins: the first player must take 1 coin, leaving the second player to take the last coin. Hence in this case, the ***second*** player has a (trivial) winning strategy.

3 coins: The first player must take 1 or 2 coins; in either case the second player can take all remaining coins. Hence in this case, the ***second*** player again has a (trivial) winning strategy.

4 coins: The first player can take 1 coin, leaving the second player to take either 1 or 2 of the remaining 3 coins; the second player can thus not avoid losing as described in the 3-coin game for the first player. Hence in this case, the ***first*** player has a winning strategy.

5 coins: If the first player takes more than 1 coin, then the second player will be able to take all remaining coins; thus, in order to win the first player must take only 1 coin, leaving 4 coins. But then the second player can win by using the strategy described in the 4-coin case for the first player. Hence in this case, the ***second*** player has a winning strategy.

6 coins: The first player can take 1 coin, leaving 5 coins to the second player. The second player can then not avoid losing as described in the 5-coin case for the first player. Hence in this case, the ***first*** player has a winning strategy.

7 coins: The first player can take 2 coins, leaving 5 coins to the second player. The second player can then not avoid losing as described in the 5-coin case for the first player. Hence in this case, the ***first*** player has a winning strategy.

8 coins: If the first player takes more than 2 coins, then the second player will be able to take all remaining coins; thus, in order to win the first player must take only 1 or 2 coins, leaving either 7 or 6 coins. The second player can then win by using the strategy described in the 7-coin or 6-coin case for the first player. Hence in this case, the ***second*** player has a winning strategy.

9 coins: The first player can take 1 coin, leaving 8 coins to the second player. The second player can then not avoid losing as described in the 8-coin case for the first player. Hence in this case, the ***first*** player has a winning strategy.

We can exhaustively work out winning strategies this way, but the reasoning is indeed exhausting. It would be a major effort, for example, to work out if we have a winning strategy as the first player starting with 100 coins, and if so how many coins we should take. There is, however, a straightforward way to work out who has the winning strategy, and what the winning move is if one exists. To determine this, we first recall the following.

Theorem 10.6   The Fibonacci Number System

Every integer $N \geq 0$ can be expressed uniquely as a sum of Fibonacci numbers

$$N \;=\; f_{k_1} \;+\; f_{k_2} \;+\; f_{k_3} \;+\; \cdots \;+\; f_{k_n}$$

where $0 \ll k_1 \ll k_2 \ll k_3 \ll \cdots \ll k_n$.   (Here, $i \ll j$ means that $i \leq j-2$.)

For example, $100 \;=\; 3+8+89 \;=\; f_4 + f_6 + f_{11}$.

Proof: This is Zeckendorf's Theorem which we proved in Example 9.15.

$\square$

Theorem 10.7

The first player has a winning strategy in FIBONACCI NIM starting with n coins if, and only if, n is **not** a Fibonacci number. In this case, the winning strategy, when n coins remain, is always to take f_{k_1} coins, where $n = f_{k_1} + f_{k_2} + \cdots + f_{k_r}$ (with $0 \ll k_1 \ll k_2 \ll \cdots \ll k_r$) is the representation of n in the Fibonacci number system.

For example, in the game starting with $100 = f_4 + f_6 + f_{11}$ coins, the winning opening move is to take $f_4 = 3$ coins.

Exercise 10.7   (Solution on page 457)

Prove Theorem 10.7.

10.4 Chomp

In the game of CHOMP, we have an $m \times n$ chocolate bar, in which the leftmost-topmost square $(1, 1)$ is poisonous. Two players take turns taking bites out of the chocolate bar, with each player having to choose a remaining square and eat it along with all remaining squares below and to the right. The goal is to force the other player to eat the poisonous square.

As before, we can easily analyse small instances of this game.

1. In the 1×1 case, the first player loses right away; hence the *second* player has a trivial winning strategy.

2. In the $1 \times n$ case with $n > 1$ (or, similarly, the $m \times 1$ case with $m > 1$), the *first* player has a trivial winning strategy: bite off all but the poisonous square, leaving just the poisonous square for the second player to take.

3. In the $2 \times n$ (or $m \times 2$) case, the *first* player has a simple winning strategy: bite off one square, leaving a $2 \times n$ rectangle with the bottom-right square missing.

   (a) if the remaining chocolate is a $2 \times k$ rectangle with the bottom-right square missing, then *every* move will result in a shape *different* from this.
   (b) if the remaining chocolate is *not* in the shape of a $2 \times k$ rectangle with only the bottom-right square missing, then *some* move will result in a shape of this form.

   With this observation, along with the insight that the ultimate goal of the game is to leave just the poisonous square which has the shape of a 2×1 rectangle with the bottom-right square missing, it is clear that the first person has a winning strategy.

4. In the $n \times n$ (square) case, the *first* player again has a simple winning strategy: bite off the $(n-1) \times (n-1)$ sub-square, leaving just the top row and left column. From here, just mimic every move of the second player, biting off as many squares from the row (respectively, column) as the second player bites off the column (respectively, row).

Apart from these special cases, very little is known about winning strategies in this game. The only way to find the winning strategy is to explore moves, and responses to moves, and responses to responses to moves, etc. For example consider the 3×4 game. In this case, it is not a good idea to take just one square (as was the strategy in case 3 above), nor to take all but the first row and first column (as was the strategy in case 4 above). However, the first player does have a winning strategy, which starts by biting off a 2×2 square. This leaves the second player with 7 moves to choose from; whatever move the second person takes, though, will be bad, as can be seen in Figure 10.2.
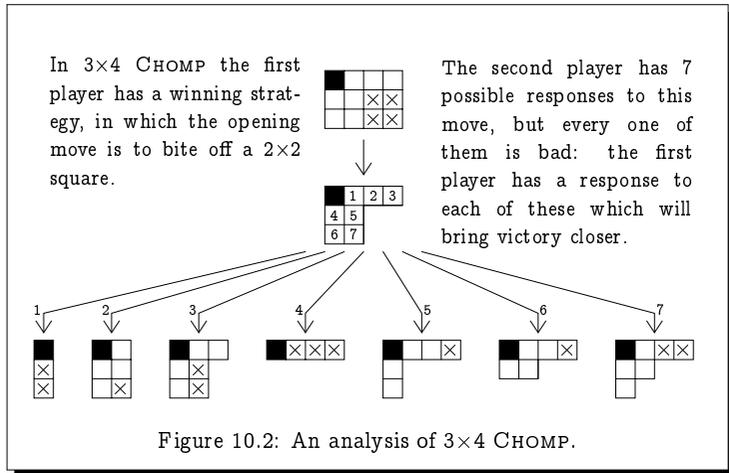
Despite the difficulty of this game, we can easily prove the following remarkable fact.

Theorem 10.8

Except for the degenerate 1×1 case, the first player always has a winning strategy.

Proof: Suppose, for the sake of argument, that the *second* player has a winning strategy. This means, in particular, that whatever move the first person opens the game with, the second person has a response which will leave the chocolate in a configuration from which the first person cannot win.

Consider the response that the second person makes using this winning strategy if the first person opens by biting off just a single square. Whatever

In 3×4 CHOMP the first player has a winning strategy, in which the opening move is to bite off a 2×2 square.

The second player has 7 possible responses to this move, but every one of them is bad: the first player has a response to each of these which will bring victory closer.

Figure 10.2: An analysis of 3×4 CHOMP.

this response, it is a move which the first player could equally have opened the game with, thus leaving the second player to play from the losing configuration.

This contradicts the assumption that the second player has the winning strategy. □

This is indeed an interesting state of affairs. In this game, we know that the first player has a winning strategy, but apart from exhaustively analysing all possible plays of the game, there is no way of knowing how to win as the first player.

10.5 Hex

The game of HEX is played on a board consisting of an $n \times n$ grid of hexagons, as shown in Figure 10.3. At the beginning of the game, the first player is considered to own the territories to the North-East and South-West of the board (the two sides labelled with crosses × in the figure), while the second player is considered to own the territories to the North-West and South-East of the board (the two sides labelled with noughts ○ in the figure). The object of the game for each player is to create a path through the board joining their disconnected territories. The players alternate moves; the first player places a cross × in a vacant hexagon, and the second player follows on by placing a nought ○ in a vacant hexagon. The winner of the game is the first player to connect their two sides of the board with a contiguous chain of hexagons labelled with their symbol.
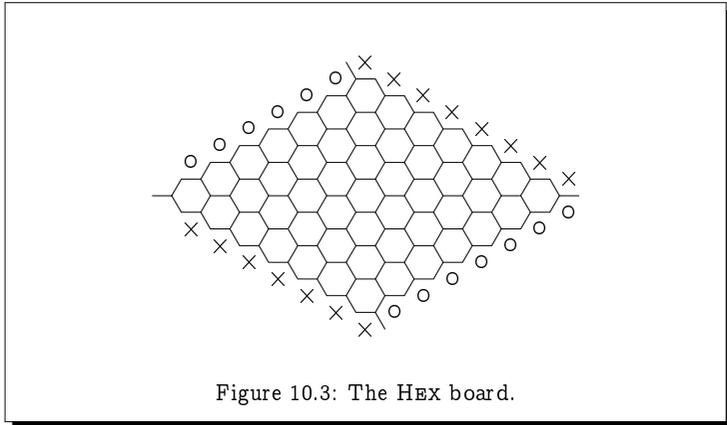
Figure 10.3: The HEX board.

Theorem 10.9

The game of HEX *can never end in a draw.*

Proof: An informal argument runs as follows. Think of the crosses as land
and the circles as water; when all the hexagons are labelled, either there is an
isthmus connecting the two continents. or else water flows between the two
oceans, In the first case, the first player has a winning chain of ×-labelled
hexagons, while in the latter case the second player has a winning chain of
○-labelled hexagons.

A formal proof would require a fair amount of explanation; here we pro-
vide only an outline. Assuming that every hexagon is labelled with a cross
× or a nought ○, we show that one of the opposite pairs of sides is con-
nected in a winning fashion. To see this, we imagine tracing a path along
the boundaries of the hexagons, entering the grid at the left-most corner. At
each junction we look at the territory we are facing; if it is labelled × then
we turn left, and if it is labelled ○ then we turn right. If we do this, then
we shall trace a path which always has × on its right and ○ on its left, and
the path will exit the grid either at the top or at the bottom. In the first
case, the ×-hexagons to the right of the path include a winning path for the
first player, and in the second case, the ○-hexagons to the left of the path
include a winning path for the second player. Figure 10.4 gives an example
in which the second player has a winning chain.   □

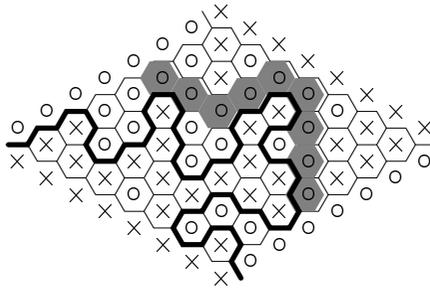Knowing that this game can never ends in a draw, we can then prove
the following.

Figure 10.4: HEX never ends in a draw.

Theorem 10.10

The first player always has a winning strategy for HEX.

Proof: Suppose for the purpose of argument that the second player has a winning strategy. Then the first player may play as follows.

1. She may label *any* hexagon chosen at random, and then forget that she has done this.

2. She may then pretend from this point on that she is playing the game as the *second* player, using the (supposed) winning strategy for the second player.

3. If at any time this strategy dictates that she should label the pre-labelled hexagon, then she should simply label any other unlabelled hexagon at random, pretend that *it* isn't labelled, and pretend that her response was to label the pre-labelled hexagon.

In this fashion, the first player is *stealing* the winning strategy from the second player, and using it to win the game. This proves that if the second player has a winning strategy, then the first player has a winning strategy, which of course is a contradiction.   □

Again, as with CHOMP, we are able to prove that the first player has the winning strategy in HEX, but our proof gives no indication as to what that strategy might be!

Figure 10.5: The BRIDG-IT board.
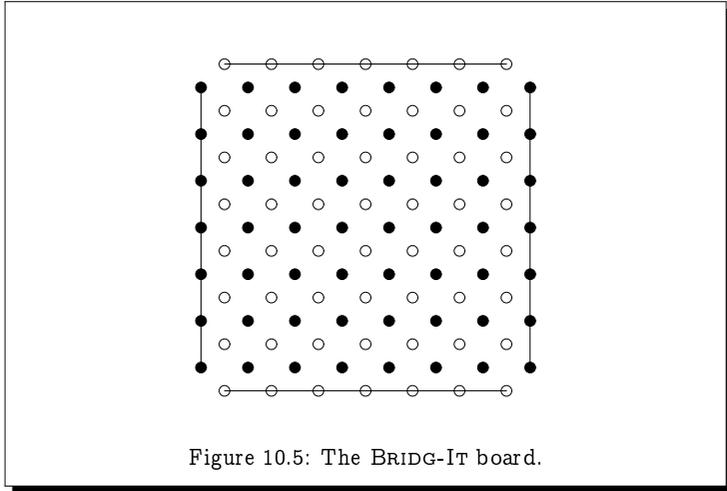
10.6 Bridg-It

The game of BRIDG-IT is similar to HEX, but is played on a staggered $n \times n$ board as depicted in Figure 10.5. The goal of the first player is to link the left- and right-hand borders, while the goal of the second player is to link the top and bottom borders. The two players alternate moves; the first player joins two neighbouring spots • either horizontally or vertically, and the second player joins two neighbouring circles ○ either horizontally or vertically. Neither player can cross a link previously made by the other player.

Theorem 10.11

The game of BRIDG-IT *can never end in a draw.*

Proof: Assuming that no further moves can be made, the board will depict a simple maze pattern, such as that given in Figure 10.6. Entering the maze from the bottom left, there is a unique path through the maze, which always has the first player's •-links on its left and the second player's ○-links on its right. This path must exit the maze at either the bottom right or the top left. (The path cannot exit the maze at the top right, as then it would end with ○-links on its left and •-links on its right.) In the first case, the •-links to the left of the path contain a winning path for the first player, and in the second case, the ○-links to the right of the path contain a winning path for the second player. □

Figure 10.6: BRIDG-IT never ends in a Draw.
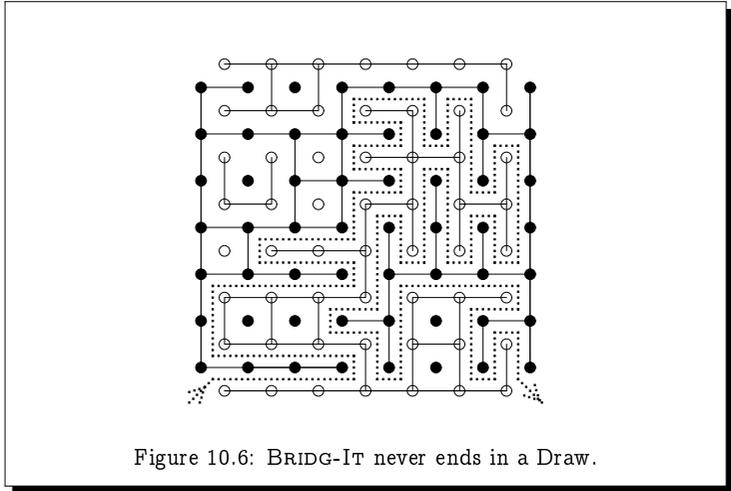
Knowing that this game can never ends in a draw, we can then prove the following.

Theorem 10.12

The first player always *has a winning strategy for* BRIDG-IT.

Proof: The reasoning is identical to that used in the proof of the analogous result for HEX.   □

Yet again this proves that the first player has a winning strategy without giving any indication as to what that strategy might be. However, in this case, we can explicitly describe a winning strategy for the first player. Referring to Figure 10.7, the first player should open with the link indicated. From that point onwards, each link that the second player makes will touch the end of one of the dotted lines depicted in Figure 10.7; in response, the first player should add the link which touches the other end of this dotted line. In this way, the first player will successfully block any attempt by the second player to create a path linking the top and bottom borders, and hence she will herself eventually win.

Exercise 10.12   (Solution on page 459)

Argue that the above does indeed describe a winning strategy for the first player.

Figure 10.7: The winning strategy for BRIDG-IT.
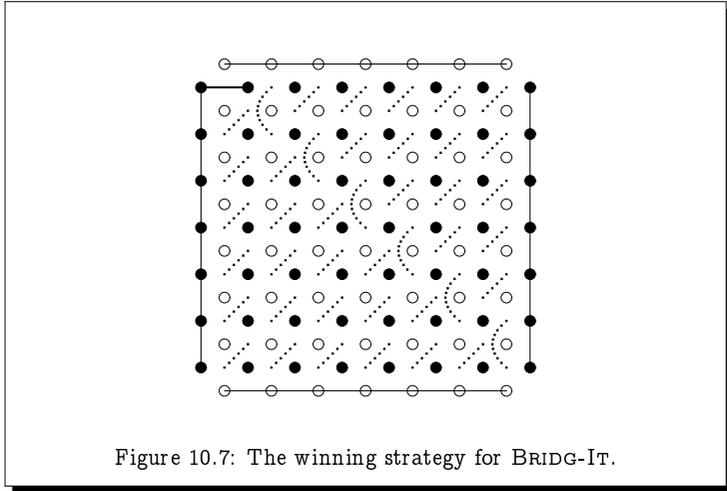
10.7 Additional Exercises

1. Consider the following game:

   > Starting with 5 coins, two players take turns taking 1 or 2 coins; and whoever ends up with an *odd* number of coins wins.

   (a) Draw the complete game tree for this game, and determine who has the winning strategy.

   (b) Who has the winning strategy in this game when started with n coins, where n is an arbitrary odd number? (Hint: for each $n = 1, 2, 3, \ldots$, determine whether or not you have a winning move if it is your turn and there are n coins left. and you are currently holding an even number of coins; in parallel to this, determine whether or not there is a winning move if it is your turn and there are n coins left. and you are currently holding an odd number of coins. Look for a pattern.)

2. Consider the following game:

   > Starting with 5 coins, each player takes turns taking 1, 2 or 3 coins; and whoever ends up with an *odd* number of coins wins.

   (a) Draw the complete game tree for this game, and determine who has the winning strategy.

(b) Who has the winning strategy in this game when started with n coins, where n is an arbitrary odd number? (The same hint as for question 1(b) applies.)

3. Consider the following game:

   Starting with a single pile of coins, two players alternate taking either 1 coin or half of the remaining coins, including the leftover coin if there is an odd number of coins remaining. Thus, for example, if there are 25 coins in the pile then a move consists of taking either 1 coin or 13 coins; if 13 coins are taken leaving 12 in the pile, then the next move will consist of taking either 1 coin or 6 coins. The player who takes the last coin wins.

   (a) For each number n from 1 to 10, explain who has the winning strategy in this game starting from a pile of n coins. In the cases in which the first player has the winning strategy, state how many coins the first player should take.
   (b) Argue that the first player has a winning strategy in the game starting with n coins if, and only if, the binary representation of n ends in an even number of 0's. Specifically,
       - if the binary representation of n ends in an even number of 0's, then either $n=1$ and you can win by taking the single coin, or there is a move which leaves a number of coins whose binary representation ends in an odd number of 0's; and
       - if the binary representation of n ends in an odd number of 0's, then every move leaves a number of coins whose binary representation ends in an even number of 0's.

4. Consider the following game:

   Starting with a pile of n coins, two players alternately remove a number of coins which is a power of 2. That is, a player may take 1 coin, or 2 coins, or 4 coins, or 8 coins, or 16 coins, or 2^k coins for any k. The player who takes the last coin wins.

   Argue that the second player has a winning strategy if, and only if, n is a multiple of 3.

5. Consider the following game:

   Starting with a pile of n coins, two players alternately remove either 1 or 3 or 8 coins. The player who takes the last coin wins.

   Argue that the second player has a winning strategy if, and only if, n is of the form $11k$ or $11k+2$ or $11k+4$ or $11k+6$.

6. (a) The game of CLOCK-1-3 is identical to the game of CLOCK-2-3 from Exercise 10.3 (page 258) except in this game the token moves either 1 or 3 hours forward.

Work out who has the winning strategy in the game of CLOCK-1-3 starting from each of the 12 hours. In the cases in which the first player has the winning strategy, state how many hours forward (1 or 3) the first player should move the token.

   (b) The game of CLOCK-1-4 is identical to the game of CLOCK-2-3 from Exercise 10.3 (page 258) except in this game the token moves either 1 or 4 hours forward.

Work out who has the winning strategy in the game of CLOCK-1-4 starting from each of the 12 hours. In the cases in which the first player has the winning strategy, state how many hours forward (1 or 4) the first player should move the token.

7. (a) The game of DAYS-OF-THE-YEAR is played by two players who take turns naming a date of the year starting from January 1st. On any move a player may increase the month or the day but not both. Thus, for example, the first player can start the game by naming any day in January (apart from the 1st), or the 1st of any month of the year (apart from January). The player who names December 31st wins.

Work out who has the winning strategy for this game. For a start, you can note that there is a winning move from any date in December (apart from the 31st), as well as from the 31st of any month (apart from December).

   (b) The game of DAYS-OF-THE-CENTURY is played by two players who take turns naming a date in the 20th century starting from 1st January 1900. On any move a player may increase the month or the day or the year, but only one of these three. The player who names 31st December 1999 wins.

Work out who has the winning strategy for this game.

8. In MISÈRE NOUGHTS AND CROSSES, it would seem sensible to avoid placing the first cross in the centre, as the centre is involved in the most winning lines. However, as the Hint in Exercise 10.2 suggests, a sensible opening move in MISÈRE NOUGHTS AND CROSSES is to place a cross in the centre.

This is in fact the only sensible opening move. Suppose that the first player starts by placing a cross somewhere other than the centre, that is, in a corner or side square. Show that the second player has a winning strategy from this position.

9. The following depicts a simple variant of the children's board game
   SNAKES AND LADDERS.
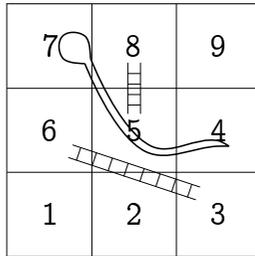
The rules of the game are as described in Exercise 10.4 (page 259).

Identify which of the positions are winning positions; which are losing
positions; and which are drawing positions. (The game is played as
described in Exercise 10.4 on page 259.) For the non-losing positions,
indicate the optimal move(s).

10. Who has the winning strategy in NIM when you start with n piles each
    containing an equal number of coins? Justify your answer without
    referring to the general theory of NIM, that is, without referring to
    balanced versus unbalanced positions.

11. In MISÈRE NIM, the objective is to *not* take the last coin. What is
    the winning strategy for this variation?

12. The game of NIM-k is played just like NIM except that in a single
    move a player can remove (a different number of) coins from up to k
    different piles. Thus NIM-1 is the usual game of NIM.

    Prove that there is a winning strategy in NIM-k from a given collec-
    tion of piles if, and only if, when writing out the numbers of coins
    in the piles in binary notation, one above the other, and adding up
    the columns, the sum of at least one of the columns is not divisible
    by $k+1$.

13. Does the first player have any other safe opening moves in 3×4 CHOMP
    apart from the one outlined in Figure 10.2?

14. What are the possible safe opening moves in 3×5 CHOMP?

15. In this exercise we use a simple game to prove the result from Exam-
    ple 6.16 that the set $[0,1] = \{\, x \,:\, 0 \le x \le 1 \,\}$ is uncountable.

    In this game, a subset $S \subseteq [0,1]$ of real numbers between 0 and 1 is
    fixed, and the two players A and B take turns choosing real numbers
    $a_0, b_0, a_1, b_1, a_2, b_2, \ldots$ – with A choosing the a_is and B choosing the b_is

– starting from $a_0 = 0$ and $b_0 = 1$. When choosing a_i, A must choose a value satisfying $a_{i-1} < a_i < b_{i-1}$; and when choosing b_i, B must choose a value satisfying $a_i < b_i < b_{i-1}$. That is, A starts at 0 and B starts at 1, and they take turns moving towards – but never reaching – the other.

The increasing sequence $a_0, a_1, a_2, \ldots$ which A is choosing must converge towards a limit value a; that is, a is the smallest real value which is bigger than every a_i. If $a \in S$ then A wins the game; otherwise B wins the game.

(a) Prove that if S is countable then B has a winning strategy in this game. (Hint: Given an enumeration $s_1, s_2, s_3, \ldots$ of S, B's winning strategy is to choose $b_i = s_i$ whenever possible.)

(b) Deduce from the above that $[0, 1]$ is uncountable. (Hint: A clearly has the winning strategy when $S = [0, 1]$.)

16. This exercise exposes a paradox devised by the mathematician William Zwicker.

Professor Bertrand likes *every* game which can *never* be played forever; and he hates any game which may potentially go on forever. For example, he likes NIM, but he hates lawn tennis, as it could potentially get into an infinite "advantage-deuce" cycle.

Consider the game of **Russell** whose rules are as follows:

- The first player chooses *any* game that must terminate.
- The two players play the chosen game, with the second player making the first move in the chosen game.

Does Professor Bertrand like the game of RUSSELL? Explain.