Chapter 4

Predicate Logic

Death is more universal than life; everyone dies but not everyone lives.

- Andrew Sachs.

Propositional logic allows us to express and reason about simple propositions. However, we quickly run into its limitations. For example, Augustus De Morgan put made the following deduction:

All horses are animals.

Therefore, all horse-heads are animal-heads.

This deduction is certainly valid. However, this cannot be demonstrated using propositional logic, as there is no way to discuss the properties of individual horses or animals, let alone their heads.

In this section we extend propositional logic to include ***predicates*** – properties which may be true or false of particular elements in a given universe – and ***quantifiers*** – the means by which we refer to elements which satisfy such properties.

4.1 Predicates and Free Variables

Recall how we defined the set of prime numbers:

$$\{\, x \;:\; x \textit{ is a prime number} \,\}.$$

We used this example to introduce the general scheme for defining sets as the collection of all objects which satisfy some property:

$$\{\, x \;:\; x \text{ has property } P \,\}$$

denotes the set of all objects x which satisfy the property P. Such a property is referred to as a ***predicate***, and we write $P(x)$ to say that "the object x has property P." A predicate is an indeterminate proposition which is true or false of any particular element x of a given universe. Thus, for example,

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$$Prime(x) = \text{“}x \text{ is a prime number”}$$

denotes the predicate which stipulates that the element x is a prime number; the universe of discourse in this instance, that is, the set of values which x may range over, would most naturally be the set of natural numbers $\mathbb{N}$ (though it could be anything; in the case that x were not a natural number, the predicate $Prime(x)$ would be false).

Predicates differ from propositions in that they do not have a fixed truth value, since we do not know the value of the object to which it refers: $Prime(x)$ may be true or false, depending on what value x refers to. The variable x is referred to as a ***free variable***. If we instantiate the free variable in such a predicate, we would get a proposition. For example, $Prime(7)$ is a true proposition (7 *is* a prime number), while $Prime(9)$ is a false proposition ($9 = 3{\cdot}3$ is *not* a prime number). The set of objects which satisfy a predicate, that is, which make the predicate true, is called the ***truth set*** of the predicate. Thus, for example, the truth set of the predicate $Prime(x)$ is the set of prime numbers. When we define a set by $\{\, x : P(x) \,\}$, we are defining it to be the truth set of the predicate $P(x)$.

Example 4.1

Let the universe of discourse be the Duck family:

$$\text{Ducks} = \Big\{ \text{Quackmore, Hortense, Scrooge, Donald, Della, Huey, Louis, Dewey} \Big\},$$

and define the following predicate:

$$Female(x) = \text{“}x \text{ is a female.”}$$

Then

- $Female$(Hortense) and $Female$(Della) are both true;
- $Female$(Quackmore), $Female$(Scrooge), $Female$(Huey), $Female$(Louis) and $Female$(Dewey) are all false;
- the truth set of the predicate $Female(x)$ is $\{$ Hortense, Della $\}$.

Predicates may range over more than one element. As familiar examples, *equality* and *set inclusion* are predicates that range over two elements. In these cases, *infix notation* “$x = y$” and “$x \subseteq y$” is more natural to use than prefix notation “$=(x, y)$” and “$\subseteq(x, y)$.” The statement $5 = 5$, for example, is true, whereas the statement $\{\emptyset\} = \emptyset$ is false.

The truth set of a predicate which ranges over more than one element consists of tuples of values; the number of coordinates of the tuples is equal to the number of free variables in the predicate. The tuples in the truth set represent those values that we can instantiate the free variables with in order to turn the predicate into a true proposition.

Example 4.2

We may use $Divides(x, y)$ to denote the two-place predicate over integers which stipulates that x divides evenly into y. In this case,

$$Divides(3, 15)$$

is true, since 3 divides evenly into 15 (5 times), while

$$Divides(4, 15)$$

is false, since 4 does not divide evenly into 15. The truth set of the predicate *Divides* is the set of pairs (x, y) such that x divides evenly into y:

$$\{ (x, y) : x \textit{ divides evenly into } y \}.$$

The standard mathematical symbol for this predicate is $\mid$ and is written in infix notation, as in $3 \mid 15$ and $4 \nmid 15$.

Exercise 4.2 (Solution on page 424)

What are the truth sets of the following predicates?

1. $Even(x)$ = "x is an even integer."
2. $EvenPrime(x)$ = "x is an even prime number."
3. $DeadlySin(x)$ = "x is a deadly sin."
4. $Sum(x, y, z)$ = "x, y and z are integers, and $x + y = z$."
5. $Sum(u, 5, v)$, where $Sum(x, y, z)$ is the predicate defined above.

4.2 Quantifiers and Bound Variables

Before Joel, Felix, Oskar and Amanda go to school in the morning, they have to remember to brush their teeth; that is, the predicate

$$Teeth(x),$$

which denotes that child x has brushed their teeth, must be true of *each* of them. To this end, each child is asked in turn if they have brushed their teeth, in order to ensure that the compound proposition

$$Teeth(\text{Joel}) \wedge Teeth(\text{Felix}) \wedge Teeth(\text{Oskar}) \wedge Teeth(\text{Amanda})$$

is true. The universe of discourse is the set consisting of the four children:

$$\text{Children} = \{\text{Joel, Felix, Oskar, Amanda}\}.$$

After this final check, they get into the car and head off to school. One of the children has to sit in the front passenger's seat, as there is only room for three passengers in the back seat. Thus, the predicate

$$\mathit{Front}(x),$$

which denotes that child x sits in the front seat, must be true of some *one* of them. They regularly argue over who this will be – either for, if they want to get away from their siblings, or against, to continue a joint activity – but the compound proposition

$$\mathit{Front}(\text{Joel}) \lor \mathit{Front}(\text{Felix}) \lor \mathit{Front}(\text{Oskar}) \lor \mathit{Front}(\text{Amanda})$$

must somehow be true.

In fact, as there is only room for one child in the front seat, the predicate $\mathit{Front}(x)$ must be true of *exactly* one child; that is, it must be true of one and false of all of the others. This means that the following proposition must be true:

$$\begin{array}{l}
(\ \ \mathit{Front}(\text{Joel}) \land \neg\mathit{Front}(\text{Felix}) \land \neg\mathit{Front}(\text{Oskar}) \land \neg\mathit{Front}(\text{Amanda})) \\
\lor \\
(\neg\mathit{Front}(\text{Joel}) \land \ \ \mathit{Front}(\text{Felix}) \land \neg\mathit{Front}(\text{Oskar}) \land \neg\mathit{Front}(\text{Amanda})) \\
\lor \\
(\neg\mathit{Front}(\text{Joel}) \land \neg\mathit{Front}(\text{Felix}) \land \ \ \mathit{Front}(\text{Oskar}) \land \neg\mathit{Front}(\text{Amanda})) \\
\lor \\
(\neg\mathit{Front}(\text{Joel}) \land \neg\mathit{Front}(\text{Felix}) \land \neg\mathit{Front}(\text{Oskar}) \land \ \ \mathit{Front}(\text{Amanda}))
\end{array}$$

That is: either Joel sits in the front seat and none of the others do; or Felix sits in the front seat and none of the others do; or Oskar sits in the front seat and none of the others do; or Amanda sits in the front seat and none of the others do.

These propositions are lengthy already when there are only four elements in the universe of discourse. Furthermore, we would not be able to write out formulæ to check if some or all elements satisfy a property if the universe of discourse is infinite. For example, to express the fact that every prime number greater than 2 is odd, using the predicate $\mathit{Odd}(x)$ to mean that x is an odd number, would require an infinitely-long conjunction:

$$\mathit{Odd}(3) \land \mathit{Odd}(5) \land \mathit{Odd}(7) \land \mathit{Odd}(11) \land \mathit{Odd}(13) \land \cdots$$

Similarly, to express the statement that some primes are square, using the predicate $\mathit{Square}(x)$ to mean that x is a prefect square, would require an infinitely-long disjunction:

$$Square(2) \lor Square(3) \lor Square(5) \lor Square(7) \lor \cdots$$

However, we cannot express infinite conjunctions and disjunctions in propositional logic.

Predicate logic provides two forms of ***quantification*** which allow you to express when properties are true of all elements in the universe of discourse, or true of at least some elements in the universe. These are outlined as follows.

4.2.1 Universal Quantification

When we want to express that a predicate $P(x)$ is true of *all* elements x of the universe of discourse, we can write:

$$\forall x\, P(x)$$

which is pronounced as

"for all x, $P(x)$";

it is true if, and only if, the predicate $P(x)$ is true of *all* possible values of x. This is called ***universal quantification***.

For example, instead of writing

$$Teeth(\text{Joel}) \land Teeth(\text{Felix}) \land Teeth(\text{Oskar}) \land Teeth(\text{Amanda})$$

to express that $Teeth(x)$ is true of all four children, we can simply write

$$\forall x\; Teeth(x)$$

which says the same thing, that *everyone* has brushed their teeth (assuming the universe of discourse is the set of the four children).

Notice that $\forall x\; Teeth(x)$ is a proposition: it has a definite truth value. The variable x is not a free variable in this case; it is a ***bound variable***; it is bound by the quantifier "$\forall x$".

Example 4.3

The statement

"Nobody did the homework"

is expressed as:

$$\forall x\, \neg H(x)$$

where $H(x)$ = *"x did the homework"*.

The universe of discourse is (assumed to be) the set of students who were assigned the homework to do.

Notice that saying something is true of *nobody* is a universal quantification: it is the same as saying that this something is *not* true of *everybody*. In this case, we are saying that everybody did *not* do their homework.

Example 4.4

The statement

> *"Every dog that has stayed in the kennel will have to go into quarantine"*

is expressed as:

$$\forall x\ (\ K(x) \Rightarrow Q(x)\ )$$

$$\text{where}\quad K(x) = \text{“}x \textit{ has stayed in the kennel”}$$

$$Q(x) = \text{“}x \textit{ will have to go into quarantine”}.$$

The universe of discourse is (assumed to be) the set of dogs, only some of which have stayed in the kennel in question.

This example demonstrates how to quantify universally over a subset of the universe of discourse: we simply stipulate that a property holds of something whenever it is a member of the subset of interest (that is, if it satisfies the predicate defining this subset). In this case, by using the implication

$$K(x) \Rightarrow Q(x)$$

we are not stating that every dog will have to go into quarantine, but only those dogs that have stayed in the kennel. If a particular dog x has not stayed in the kennel – that is, if $K(x)$ is not true – then that dog x need not go into quarantine – that is, $Q(x)$ need not be true. (Of course, this dog x might have to go into quarantine for some other reason; it is not necessarily the case that $Q(x)$ is false.)

Note that universal quantification is assumed to bind more strongly than all of the propositional connectives; that is, it is given higher precedence. For example, in the above example we wrote

$$\forall x\ (\ K(x) \Rightarrow Q(x)\ )$$

and not

$$\forall x\ K(x) \Rightarrow Q(x)$$

as the latter would be interpreted as

$$(\forall x\, K(x)) \Rightarrow Q(x)$$

which says *"if every dog has stayed in the kennel then x will have to go into quarantine."* This is certainly not what is intended; in particular, it is a predicate with a free variable x – appearing in $Q(x)$ – and is therefore not a proposition.

Example 4.5

The statement

"Nobody likes a sore loser"

is expressed as:

$$\forall x\, (\, S(x) \;\Rightarrow\; \forall y\, \neg L(y,x)\,)$$

$$\text{where} \quad S(x) = \text{“}x \textit{ is a sore loser”}$$

$$L(y,x) = \text{“}y \textit{ likes } x\text{”}.$$

The universe of discourse is (assumed to be) the collection of all people.

This proposition is saying the following is true of every person x: if x is a sore loser, then every person y does not like x.

Exercise 4.5 (Solution on page 424)

Using the predicates

$$B(x) = \text{“}x \textit{ is a bee”}$$

$$F(x) = \text{“}x \textit{ is a flower”}$$

$$L(x,y) = \text{“}x \textit{ likes } y\text{”}$$

write each of the following statements in predicate logic.

1. All bees like all flowers.
2. Bees only like flowers.
3. Only bees like flowers.

4.2.2 Existential Quantification

When we want to express that a predicate $P(x)$ is true of at least *some* element x of the universe of discourse, we can write:

$$\exists x\, P(x)$$

which is pronounced as

"there exists x such that $P(x)$";

it is true if, and only if, the predicate $P(x)$ is true of *some* value of x. For example, instead of writing

$$\mathit{Front}(\text{Joel}) \lor \mathit{Front}(\text{Felix}) \lor \mathit{Front}(\text{Oskar}) \lor \mathit{Front}(\text{Amanda})$$

to express that $\mathit{Front}(x)$ is true of at least one of the four children, we can simply write

$$\exists x\, \mathit{Front}(x)$$

which says the same thing, that *someone* sits in the front seat (again, assuming the universe of discourse is the set of the four children).

Again, the variable x in $\exists x\, \mathit{Front}(x)$ is a bound variable, bound by the quantifier "$\exists x$"; and like universal quantification, existential quantification is assumed to bind more strongly than all of the propositional connectives.

Example 4.6

The statement

"Someone didn't do the homework"

is expressed as:

$$\exists x\, \neg H(x)$$

where $H(x)$ = *"x did the homework"*.

The universe of discourse is again (assumed to be) the set of students who were assigned the homework to do.

This proposition states that $\neg H(x)$ holds of *some* student: perhaps no one did the homework (as expressed by the proposition given in Example 4.3); or perhaps several did the homework while several others didn't; or perhaps all but one person did the homework. This proposition doesn't distinguish between these possibilities; it merely notes that at least one element of the universe of discourse satisfies the predicate, that is, at least one person did not do the homework.

Example 4.7

The statement

"If some dog that has stayed in the kennel has been in contact with a dog with rabies, then every dog that has stayed in the kennel will have to go into quarantine"

is expressed as:

$$\exists x \left(K(x) \wedge \exists y \, (C(x,y) \wedge R(y)) \right) \Rightarrow \forall x \left(K(x) \Rightarrow Q(x) \right)$$

where
$$\begin{aligned} K(x) &= \text{“}x \text{ has stayed in the kennel”} \\ R(x) &= \text{“}x \text{ has rabies”} \\ C(x,y) &= \text{“}x \text{ and } y \text{ have been in contact”} \\ Q(x) &= \text{“}x \text{ will have to go into quarantine”}. \end{aligned}$$

Exercise 4.7 (Solution on page 424)

Assuming the universe of discourse is the set of human beings, consider the following predicates

$$\begin{aligned} Male(x) &= \text{“}x \text{ is male”} \\ Female(x) &= \text{“}x \text{ is female”} \\ Parent(x,y) &= \text{“}x \text{ is a parent of } y\text{”} \\ Father(x,y) &= \text{“}x \text{ is the father of } y\text{”} \\ Mother(x,y) &= \text{“}x \text{ is the mother of } y\text{”} \\ Sibling(x,y) &= \text{“}x \text{ and } y \text{ are siblings”} \\ Cousin(x,y) &= \text{“}x \text{ and } y \text{ are cousins”} \end{aligned}$$

Using these predicates, express the following properties in predicate logic.

1. Every human is either male or female, but no human is both.
2. Mothers are female parents.
3. Every human has exactly one mother and exactly one father.
4. Siblings have the same parents.
5. Cousins each have a parent who are siblings.

Exercise 4.8 (Solution on page 425)

Using the following predicates:

$$\begin{aligned} Horse(h) &= \text{“}h \text{ is a horse”} \\ Animal(a) &= \text{“}a \text{ is an animal”} \\ Head(x,y) &= \text{“}x \text{ is the head of } y\text{”} \end{aligned}$$

formalise the following argument in predicate logic:

All horses are animals.

Therefore, all horse heads are animal heads.

Explain why the argument is valid.

4.2.3 Bounded Quantifications

There are two forms of ***bounded quantification*** which we use for convenience. These restrict the range of the variables being quantified.

Firstly, to declare that the predicate $P(x)$ is true of every element of the set A, we write

$$\forall\, x{\in}A\ P(x)$$

which is pronounced as

"for all values x in A, $P(x)$".

This is logically equivalent to

$$\forall x \left(\, x{\in}A \;\Rightarrow\; P(x) \,\right).$$

Similarly, to declare that the predicate $P(x)$ is true of some element of the set A, we write

$$\exists\, x{\in}A\ P(x)$$

which is pronounced as

"there is some value x in A such that $P(x)$".

This is logically equivalent to

$$\exists x \left(\, x{\in}A \;\wedge\; P(x) \,\right).$$

One further useful restriction for the existential quantifier is declare that exactly one value x satisfies $P(x)$. This is written

$$\exists! x\, P(x)$$

which is pronounced as

"there is exactly one value x such that $P(x)$".

This is logically equivalent to

$$\exists x \left(\, P(x) \;\wedge\; \neg\exists y\,(P(y) \wedge y \neq x) \,\right).$$

This says that there is a value x such that $P(x)$, but there is not a different value $y{\neq}x$ such that $P(y)$. For example, if the predicate $\mathit{Front}(x)$ denotes that child x sits in the front seat of the car, where, again, the universe of discourse is the set of the four children, then

$$\exists! x\ \mathit{Front}(x)$$

states that exactly one of the children sits in the front seat.

Note that you can combine the last two constructions to declare that exactly one value from a set A satisfies $P(x)$: $\exists! x \in A\, P(x)$. Also note that x is of course bound by the quantifiers in each case.

Example 4.8

You may be aware that $\sqrt{2}$ is irrational: that it cannot be expressed as a fraction p/q. (We shall justify this claim in Example 5.6, page 139.) In fact, *any* nonnegative integer is either a perfect square, such as $25 = 5^2$, or its square root is irrational. We can express this fact as follows:

$$\forall n \in \mathbb{Z}\, (\, \exists k \in \mathbb{Z}\, (n = k^2) \quad \vee \quad \neg \exists q \in \mathbb{Q}\, (n = q^2)\,).$$

This says that for all integers n, either there exists another integer k such that $n = k^2$ (that is, n is a perfect square with square root k), or there does *not* exist a rational number q such that $n = q^2$ (that is, it does not have a rational square root).

Example 4.9

Recall the following puzzle from Exercise 1.16 (page 35). Joel, Felix and Oskar each write their name on a piece of paper, and then exchange the pieces of paper so that no one has the piece with their own name on it. They then hold these pieces of paper so that Amanda can't see what's on them, but tell her that each has the name of one of the others, and they challenge her to figure out who is holding each name. She is allowed to look at the name written on any one piece of paper. She decides to look at Joel's piece, and finds "Oskar" written on it.

Let $\textsc{Boys} = \{\,\text{Joel, Felix, Oskar}\,\}$ be the set of three boys; and let $\textsc{Papers} = \{\, J, F, O\,\}$ be the set of three pieces of paper with names written on them: J is the piece with "Joel" written on it; F is the piece with "Felix" written on it; and O is the piece with "Oskar" written on it. Furthermore, let $Holds(b,p)$ be the predicate which says that boy b holds the piece of paper p. Then we can formulate the conditions describe in this problem as follows:

1. Each boy holds precisely one piece of paper:

$$\forall b \in \textsc{Boys}\ \exists! p \in \textsc{Papers}\ Holds(b,p).$$

2. Each piece of paper is held by precisely one boy:

$$\forall p \in \textsc{Papers}\ \exists! b \in \textsc{Boys}\ Holds(b,p).$$

3. No piece of paper is being held by the boy whose name is on the paper:

$$\neg Holds(\text{Joel}, J) \ \wedge \ \neg Holds(\text{Felix}, F) \ \wedge \ \neg Holds(\text{Oskar}, O).$$

4. Joel's piece of paper has "Oskar" written on it:

$$Holds(\text{Joel}, O).$$

Exercise 4.9 (Solution on page 425)

Let $T(s, c)$ stand for the predicate *"student s takes course c."* Express the following statements in predicate logic.

1. *Alice and Bob take exactly one course together."*
2. *Alice and Bob take exactly two courses together."*

4.3 Rules for Quantification

If it is not the case that the predicate $P(x)$ is true for *all* values of x, then this must mean that $P(x)$ is not true for *some* value of x; that is,

$$\neg\forall x\, P(x) \quad \Leftrightarrow \quad \exists x\, \neg P(x).$$

Equally, if it is not the case that the predicate $P(x)$ is true for *some* value of x, then this must mean that $P(x)$ is not true for *all* values of x; that is,

$$\neg\exists x\, P(x) \quad \Leftrightarrow \quad \forall x\, \neg P(x).$$

These two laws coincide with De Morgan's Laws:

$$\neg(P \wedge Q) \quad \Leftrightarrow \quad \neg P \vee \neg Q$$
$$\neg(P \vee Q) \quad \Leftrightarrow \quad \neg P \wedge \neg Q$$

if we consider universal quantification as a (potentially) infinite conjunction, and existential quantification as a (potentially) infinite disjunction. Suppose that the universe of discourse is $\mathcal{U} = \{\, a, b, c, \ldots \}$. Then

$$\begin{aligned} \neg\forall x P(x) &\Leftrightarrow \neg(P(a) \wedge P(b) \wedge P(c) \wedge \cdots) \\ &\Leftrightarrow \neg P(a) \vee \neg P(b) \vee \neg P(c) \vee \cdots \quad \textit{(De Morgan's Law)} \\ &\Leftrightarrow \exists x \neg P(x); \end{aligned}$$

and

$$\begin{aligned} \neg\exists x P(x) &\Leftrightarrow \neg(P(a) \vee P(b) \vee P(c) \vee \cdots) \\ &\Leftrightarrow \neg P(a) \wedge \neg P(b) \wedge \neg P(c) \wedge \cdots \quad \textit{(De Morgan's Law)} \\ &\Leftrightarrow \forall x \neg P(x). \end{aligned}$$

Example 4.10

Recall from the Example in Section 4.2 that Joel, Felix, Oskar and Amanda must all brush their teeth before going to school in the morning; that is, that the proposition

$$\forall x\ \mathit{Teeth}(x)$$

is true, where – as before – we use $\mathit{Teeth}(x)$ to denote the statement that child x has brushed their teeth, and we continue to take the universe of discourse to consist of the set of four children in question:

$$\text{Children} = \{\text{Joel, Felix, Oskar, Amanda}\}.$$

On a particular day, it may be discovered that this statement is *not* true. For example, perhaps Joel, Oskar and Amanda have all brushed their teeth, but Felix has not. This is the reason that $\forall x\ \mathit{Teeth}(x)$ is false, i.e., that

$$\neg\forall x\ \mathit{Teeth}(x)$$

is true: that there is someone (namely Felix) who has *not* brushed their teeth:

$$\exists x \neg \mathit{Teeth}(x)$$

This is an example of the general law that

$$\neg\forall x\, P(x) \Leftrightarrow \exists x \neg P(x).$$

We have also earlier noted that, when driving to school, one of the children must sit in the front seat of the car: that is, that the statement

$$\exists x\, \mathit{front}(x)$$

must be true, where – as before – we use $\mathit{front}(x)$ to denote the statement that child x sits in the front seat. For this statement to be false, it would have to mean that *none* of the children are sitting in the front seat, or in other words that *all* of them are *not* sitting in the front seat:

$$\forall x \neg \mathit{front}(x).$$

This is an example of the general law that

$$\neg\exists x\, P(x) \Leftrightarrow \forall x \neg P(x).$$

Exercise 4.10 (Solution on page 425)

For each of the following statements, identify which of the options provided correctly expresses its negation. Translate each statement into predicate logic to confirm your choices.

1. Some people like mathematics.
   (a) Some people dislike mathematics.
   (b) Everybody dislikes mathematics.
   (c) Everybody likes mathematics.
2. All cats have fur and a tail.
   (a) No cat has fur and a tail.
   (b) Some cats are bald and tailless.
   (c) Some cats are bald or tailless.
3. Everyone who had not been vaccinated got sick.
   (a) Everyone who had been vaccinated did not get sick.
   (b) Some people who had been vaccinated got sick.
   (c) Some people who had not been vaccinated did not get sick.

Having established how quantifiers interact with negation, we next consider how they interact with conjunction and disjunction. Specifically, we may wonder which of the following is true:

1. $\forall x(P(x) \wedge Q(x)) \;\overset{?}{\Leftrightarrow}\; \forall x P(x) \;\wedge\; \forall x Q(x)$.
2. $\exists x(P(x) \wedge Q(x)) \;\overset{?}{\Leftrightarrow}\; \exists x P(x) \;\wedge\; \exists x Q(x)$.
3. $\forall x(P(x) \vee Q(x)) \;\overset{?}{\Leftrightarrow}\; \forall x P(x) \;\vee\; \forall x Q(x)$.
4. $\exists x(P(x) \vee Q(x)) \;\overset{?}{\Leftrightarrow}\; \exists x P(x) \;\vee\; \exists x Q(x)$.

We carefully consider each of these in turn.

1. This property is valid.

   If $P(x) \wedge Q(x)$ is true of every object x, then certainly $P(x)$ must be true of every object x and $Q(x)$ must be true of every object x.

   Equally, if $P(x)$ is true of every object x and $Q(x)$ is true of every object x, then $P(x) \wedge Q(x)$ must be true of every object x.

2. This property is *not* valid.

   If $P(x) \wedge Q(x)$ is true of some object x, then $P(x)$ must be true of that object x and $Q(x)$ must be true of that object x.

   However, $P(x)$ may be true of some object, and $Q(x)$ may be true of some *different* object, while $P(x) \wedge Q(x)$ may never be true of the same object x.

   For example, it is true that prime numbers and perfect squares exist:

   $\exists x\, \mathit{Prime}(x) \;\wedge\; \exists x\, \mathit{Square}(x)$ is true.

For instance $Prime(17)$ is true and $Square(25)$ is true. However, no number can be both prime *and* a perfect square at the same time:

$$\exists x(Prime(x) \wedge Square(x)) \text{ is false.}$$

We have, however, established the weaker property:

$$2'.\ \exists x(P(x) \wedge Q(x)) \;\Rightarrow\; \exists x P(x) \;\wedge\; \exists x Q(x).$$

3. This property is *not* valid.

   If $P(x)$ is true for all objects x, then certainly $P(x) \vee Q(x)$ must be true of all objects x; Equally, if $Q(x)$ is true for all objects x, then $P(x) \vee Q(x)$ must be true of all objects x.

   However, $P(x) \vee Q(x)$ may be true of all objects x without it being the case that $P(x)$ is true of all objects x, nor that $Q(x)$ is true of all objects x.

   For example, it is true that all integers are either even or odd:

   $$\forall x\,(Even(x) \vee Odd(x)) \text{ is true.}$$

   However, not every integer is even, and not every integer is odd:

   $$\forall x\, Even(x) \;\vee\; \forall x\, Odd(x) \text{ is false.}$$

   We have, however, established the weaker property:

   $$3'.\ \forall x(P(x) \vee Q(x)) \;\Leftarrow\; \forall x P(x) \;\vee\; \forall x Q(x).$$

4. This property is valid.

   If $P(x) \vee Q(x)$ is true of some object x, then either $P(x)$ must be true of that object x or $Q(x)$ must be true of that object x.

   Equally, if $P(x)$ is true of some object x or $Q(x)$ is true of some object x, then $P(x) \wedge Q(x)$ must be true of that object x.

As a final note, the following are clearly valid properties:

1. $\forall x\, \forall y\, P(x,y) \;\Leftrightarrow\; \forall y\, \forall x\, P(x,y)$;
2. $\exists x\, \exists y\, P(x,y) \;\Leftrightarrow\; \exists y\, \exists x\, P(x,y)$.

That is, we can rearrange the order in which universal quantifications are applied, as well as the order in which existential quantifications are applied. It is common practice to write these as $\forall x, y\, P(x,y)$ and $\exists x, y\, P(x,y)$, respectively. However, as we see in the following example, we cannot rearrange different quantifiers:

$$\forall x \exists y\, P(x,y) \;\not\Leftrightarrow\; \exists y \forall x\, P(x,y).$$

Example 4.11

A certain mathematics textbook has an exercise which asks its reader to translate the following sentence into predicate logic:

> *"Every real number is smaller than some integer."*

This informal English sentence can be interpreted in (at least) the following two different ways:

1. $\forall r{\in}\mathbb{R}\ \exists n{\in}\mathbb{Z}\ (r < n)$

   Given any real number r, we can find a larger integer n.

2. $\exists n{\in}\mathbb{Z}\ \forall r{\in}\mathbb{R}\ (r < n)$

   There is an integer which is larger than every real number.

The first of these statements is true – and is undoubtedly the interpretation intended by the author – while the second statement is blatantly false. The author of this mathematics textbook was trying to state a basic fact about numbers, but the ambiguity of English complicated this task.

4.4 Modelling in Predicate Logic

The language of predicate logic gives us tools on top of propositional logic and set theory with which to model scenarios. In this section we we present a few examples.

Example 4.12

Recall the Carrollean puzzle from Exercise 2.25, where we are given the three premises:

> ***All babies are illogical.***
>
> ***Nobody is despised who can manage a crocodile.***
>
> ***Illogical persons are despised.***

from which we are to deduce that no baby can manage a crocodile. Let us introduce the following predicates:

$$\begin{aligned} B(x) &= \text{“}x \textit{ is a baby”} \\ I(x) &= \text{“}x \textit{ is illogical”} \\ D(x) &= \text{“}x \textit{ is despised”} \\ M(x) &= \text{“}x \textit{ can manage a crocodile”} \end{aligned}$$

Then the above three premises translate into the following propositions:

1. $\forall x(B(x) \Rightarrow I(x))$
2. $\forall x(M(x) \Rightarrow \neg D(x))$ or, equivalently, $\forall x(D(x) \Rightarrow \neg M(x))$
3. $\forall x(I(x) \Rightarrow D(x))$

and the conclusion translates into $\forall x(B(x) \Rightarrow \neg M(x))$.

However, for any x such that $B(x)$ is true (if x is a baby), by first premise, $I(x)$ is true (x is illogical); and thus by the third premise, $D(x)$ is true (x is despised); and therefore by the second premise, $\neg M(x)$ is true (x cannot manage a crocodile).

Hence the conclusion does indeed follow from the premises.

Exercise 4.12 (Solution on page 426)

Formalise the following two arguments in predicate logic:

1. *Everybody loves somebody.*

   Therefore somebody is loved by everybody.
2. *Somebody loves everybody.*

   Therefore everybody is loved by somebody.

In each case, discuss any ambiguities that you identify in the English statements, but use what you consider to be the intended interpretations.

Are these arguments valid?

Example 4.13

Figure 4.1 presents an example Sudoku puzzle which consists of a 9×9 grid with numbers entered into some of the squares. The objective is to completely fill in the grid so that each column, each row, and each of the nine 3×3 blocks contains the digits from 1 to 9 exactly once. Properly set, the initial numbers will allow for only one valid solution.

This is a classic logic-style puzzle, and as such is perfectly suited for modelling in predicate logic. If you struggle with solving the puzzle given in Figure 4.1, an Internet search engine will find any number of Web sites which will solve it for you; and the means by which these Web sites' software does this will inevitably work on the following formal representation (or something very similar).

We start by defining the universe of discourse to be the interval $I = [1..9]$ of integers from 1 to 9. This reflects the fact that there are:

- 9 rows, listed from top to bottom as row 1 through to row 9;

							6	7
4				9				
3			2			9	8	
				2	3	6		
	2			6			5	
		1	7	4				
	3	4			7			9
				1				6
9	8							

Figure 4.1: A Sudoku puzzle.

- 9 columns, listed from left to right as column 1 through to column 9;
- 9 blocks, listed from left to right and top to bottom as block 1 through to block 9;
- 9 values, 1 through 9, to be inserted into the squares.

We then define the following predicate:

$$V(i, j, k) \;=\; \text{“square } (i, j) \text{ holds the value } k\text{.”}$$

That is, the number k is in the square located in row i and column j. Thus, for the example puzzle in Figure 4.1, the following propositions are true:

$$V(1,8,6) \qquad V(1,9,7)$$
$$V(2,1,4) \qquad V(2,5,9)$$
$$V(3,1,3) \qquad V(3,4,2) \qquad V(3,7,9) \qquad V(3,8,8)$$
$$V(4,5,2) \qquad V(4,6,3) \qquad V(4,7,6)$$
$$V(5,2,2) \qquad V(5,5,6) \qquad V(5,8,5)$$
$$V(6,3,1) \qquad V(6,4,7) \qquad V(6,5,4)$$
$$V(7,2,3) \qquad V(7,3,4) \qquad V(7,6,7) \qquad V(7,9,9)$$
$$V(8,5,1) \qquad V(8,9,6)$$
$$V(9,1,9) \qquad V(9,2,8)$$

Next we define the following predicate:

$$B(i,j,b) \;=\; \text{“square } (i,j) \text{ is in block } b.\text{”}$$

This property is represented by the following nine propositions (one for each block):

$$\begin{aligned}
B(i,j,1) &\Leftrightarrow (i,j) \in [1..3] \times [1..3]\\
B(i,j,2) &\Leftrightarrow (i,j) \in [1..3] \times [4..6]\\
B(i,j,3) &\Leftrightarrow (i,j) \in [1..3] \times [7..9]\\
B(i,j,4) &\Leftrightarrow (i,j) \in [4..6] \times [1..3]\\
B(i,j,5) &\Leftrightarrow (i,j) \in [4..6] \times [4..6]\\
B(i,j,6) &\Leftrightarrow (i,j) \in [4..6] \times [7..9]\\
B(i,j,7) &\Leftrightarrow (i,j) \in [7..9] \times [1..3]\\
B(i,j,8) &\Leftrightarrow (i,j) \in [7..9] \times [4..6]\\
B(i,j,9) &\Leftrightarrow (i,j) \in [7..9] \times [7..9]
\end{aligned}$$

Finally, we are ready to represent the properties satisfied by a valid solution to the puzzle.

1. Every square (i,j) holds exactly one value k: $\forall i\,\forall j\,\exists! k\,V(i,j,k)$.
2. Every row i contains every value k: $\forall i\,\forall k\,\exists j\,V(i,j,k)$.
3. Every column j contains every value k: $\forall j\,\forall k\,\exists i\,V(i,j,k)$.
4. Every block b contains every value k: $\forall b\,\forall k\,\exists i\,\exists j\,V(i,j,k) \wedge B(i,j,b)$.

All that is required now is to deduce truth values of the predicates $V(i,j,k)$ which satisfy these properties. This is a non-trivial and tedious task to do by hand, but is the sort of thing that computers can do very well (and very rapidly).

Exercise 4.13 (Solution on page 426)

Solve the Sudoku puzzle in Figure 4.1.

4.5 Additional Exercises

1. Let $V(x)$ stand for the predicate "x visits his parents every weekend", where the domain of discourse is the set of students in your class. Express each of the following quantifications in English:

   (a) $\exists x V(x)$

   (b) $\forall x V(x)$

(c) $\exists! x \neg V(x)$
(d) $\forall x \neg V(x)$

2. Using the predicates:

$$B(x) = \text{"}x \textit{ is a bee"}$$
$$F(x) = \text{"}x \textit{ is a flower"}$$
$$L(x,y) = \text{"}x \textit{ likes } y\text{"}$$

write each of the following statements in predicate logic.

(a) All bees like some flowers.
(b) No bee likes only flowers.
(c) No bee hates (that is, does not like) all flowers.

3. Express the negation of each of the statements in the previous question, both in English as well as in predicate logic.

4. Let $T(s,c)$ stand for the predicate *"student s takes course c."* Express the following statements in predicate logic.

(a) *"Alice and Bob take all the same courses."*
(b) *"Alice and Bob do not take any courses together."*

5. Express the following properties in predicate logic, using only the usual operations of addition and multiplication as well as the less than relation $<$ between numbers.

(a) x is a divisor of y.
(b) x and y have no common divisors.
(c) x is a prime number.
(d) Every integer greater than one has a unique smallest prime divisor.
(e) *(Goldbach's Conjecture)* Every even integer greater than two can be written as the sum of two primes.

6. Express in English what each of the following propositions is saying about the set of real numbers $\mathbb{R}$, and determine whether they are true or false.

(a) $\forall x\, \exists y\, (x+y = x)$.
(b) $\exists y\, \forall x\, (x+y = x)$.
(c) $\forall x\, \exists y\, (x^2 = y)$.
(d) $\forall y\, \exists x\, (x^2 = y)$.
(e) $\forall x\, \forall y\, (x<y \;\vee\; y<x)$.

7. Express the following in predicate logic.
   (a) At least three items have property P.
   (b) At most 3 items have property P.
   (c) Exactly three items have property P.

8. A particular jazz standard recorded by Doris Day has the following title and lyrics:

   Everybody loves my baby

   But my baby don't love nobody but me

   Express the above in predicate logic. What can you deduce from these two statements about who "my baby" is?

9. Samuel Goldwyn, on being told by a friend told him that he had named his son Sam, exclaimed, *"Why did you do that? Every Tom, Dick and Harry is named Sam!"* Assuming Goldwyn was right, and assuming he was restricting his attention to first names, how many Sams, Dicks and Harrys are there? Formulate your answer in predicate logic, including the assertion that every person has exactly one first name.

10. Lewis Carroll made the following argument.

    Everybody who is sane can do logic.

    No lunatics are fit to serve on a jury.

    None of your sons can do logic.

    Therefore, none of your sons is fit to serve on a jury.

    Formulate the four claims in predicate logic. Do you consider this a valid argument?

11. Lewis Carroll also made the following three claims.

    No professor is ignorant.

    All ignorant people are vain.

    No professor is vain.

    Formulate these three claims in predicate logic. Do any of them follow from the other two?

12. What is wrong with the following argument:

    A ham sandwich is better than nothing.

    Nothing is better than eternal happiness.

    Therefore, a ham sandwich is better than eternal happiness.