Chapter 9

Proofs by Induction

In the middle of a cloudy thing is another cloudy thing, and within that another cloudy thing, inside which is yet another cloudy thing...

... and in that is yet another cloudy thing, inside which is something perfectly clear and definite.

- Ancient Sufi saying.

One of the most common forms of reasoning used within the subject of Computer Science is ***inductive reasoning***. This is due to the fact, explored in the previous chapter, that Computer Science deals heavily with manipulating inductively-defined objects. Reasoning about such objects will naturally rely on exploiting the inductive nature of their definitions.

In Section 5.6 we explored the general technique for proving a property of the form $\forall x P(x)$, namely, to allow x to stand for an arbitrary value of the domain and to prove that $P(x)$ holds without making any assumptions about the value of x. Such a general approach is typically too weak to prove facts about natural numbers; we would like to be able to exploit the inductively-defined structure of natural numbers to arrive at our result. Such is the role of induction proofs.

9.1 Convincing but Inconclusive Evidence

Consider the following claim that the sum of the first n positive integers is $\frac{n(n+1)}{2}$:

Claim: For all $n \geq 0$, $1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

Note that the sum of the first zero natural numbers, which above is awkwardly written as $1+2+3+\cdots+0$, is naturally 0.

We can easily confirm this claim for various values of n:

F. Moller, G. Struth, *Modelling Computing Systems*,
Undergraduate Topics in Computer Science,
DOI 10.1007/978-1-84800-322-4_10,

$$
\begin{aligned}
& 0 = \frac{0(1)}{2}, && \text{so the claim is true when } n = 0. \\
1 &= 1 = \frac{1(2)}{2}, && \text{so the claim is true when } n = 1. \\
1+2 &= 3 = \frac{2(3)}{2}, && \text{so the claim is true when } n = 2. \\
1+2+3 &= 6 = \frac{3(4)}{2}, && \text{so the claim is true when } n = 3. \\
1+2+3+4 &= 10 = \frac{4(5)}{2}, && \text{so the claim is true when } n = 4. \\
1+2+3+4+5 &= 15 = \frac{5(6)}{2}, && \text{so the claim is true when } n = 5.
\end{aligned}
$$

Each instance of the claim which we verify to be true seems to lend support to the validity of the claim. However, no (finite) amount of checking of individual cases can confirm the validity of the claim for all values of n.

Now consider each of the following claims.

- ***Fermat's Last Theorem*** claims that for no integer $n>2$ does there exist a trio of positive integers x, y and z such that $x^n + y^n = z^n$. This claim went unproven for 350 years until Andrew Wiles' celebrated proof in the 1990s. By then, the conjecture was confirmed with the help of vast computer resources for all values of n up to 4 million. However, even if computers could have confirmed the truth of this conjecture for all values of n up to ten zillion, there would still be no reason why the conjecture should be true for ten zillion and one.

  Pierre de Fermat, after whom Fermat's Last Theorem is named, famously wrote the following about this Theorem in the margin of a textbook on arithmetic: *"Cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet."* ("I have a truly marvellous proof of this proposition which this margin is too narrow to contain.") It is universally believed that whatever argument he may have had in mind could not have been valid. This is partly due to the fact that no proof was ever found amongst his papers, and partly due to the extreme complexity of the only known proof by Wiles – which can be understood in its entirety by only a small number of mathematicians worldwide. It also partly due to the fact that Fermat believed many things which ultimately turned out to be false, such as the next example.

- ***Fermat numbers*** are integers of the form $F_n = 2^{2^n} + 1$. They are so called on account of the fact that Pierre de Fermat wrote, in a letter to Marin Mersenne on 25 December 1640, that: *"If I can determine the basic reason why*

  $$3, \quad 5, \quad 17, \quad 257, \quad 65{,}537, \quad \ldots$$

  are prime numbers, I feel that I would find very interesting results." Based on the properties of the first few numbers of this form, Fermat believed that they were all necessarily prime. Indeed the first few Fermat numbers listed by Fermat are prime:

$$\begin{aligned}
F_0 &= 2^{2^0}+1 = 2^1+1 = 3\\
F_1 &= 2^{2^1}+1 = 2^2+1 = 5\\
F_2 &= 2^{2^2}+1 = 2^4+1 = 17\\
F_3 &= 2^{2^3}+1 = 2^8+1 = 257\\
F_4 &= 2^{2^4}+1 = 2^{16}+1 = 65,537
\end{aligned}$$

Unfortunately for Fermat, his conjecture fails with the very next Fermat number:

$$F_5 = 2^{2^5}+1 = 2^{32}+1 = 4,294,967,297.$$

Fermat can be forgiven for not recognising this monstrosity to be a composite number. It was the great mathematician Leonhard Euler who first discovered in 1732 that this number can be factored as

$$641 \times 6,700,417.$$

Indeed, it is unknown whether *any* further Fermat numbers are prime (though it is known that a vast many are not).

- ***Goldbach's conjecture***, which states that every even number greater than 2 can be expressed as the sum of two prime numbers, has been confirmed, again with the help of vast computer resources, for all even numbers up to 10^{18} (i.e., $1,000,000,000,000,000,000$). But as far as anyone knows, there might be a yet larger even number which is not the sum of two primes. It worked out well for Fermat's Last Theorem, but this gives no reason for hope, as demonstrated by the next two examples.
- In 1919, the Hungarian mathematician George Pólya conjectured that most (i.e., more than 50%) of the natural numbers less than any given number have an *odd* number of prime factors. For example, every prime number has an odd number of prime factors, namely one, as does $12 = 2\times2\times3$ (three prime factors), while $14 = 2\times7$ has an even number (two) of prime factors. By the mid 1950's empirical evidence for Pólya's conjecture seemed clear: the conjecture was verified for all numbers up to 800,000. However, contrary to this ever-growing evidence, Pólya's conjecture was disproved in 1958 when C. Brian Haselgrove showed that it had to be false for some value around 2×10^{361} (that is, a 2 followed by 361 zeros). It has since been shown to fail already for $n = 906,150,257$.
- Consider the following claim:

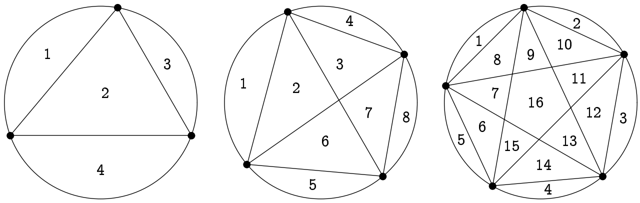

Figure 9.1: Counting regions in circles.
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$$\text{For all } n \geq 1, \quad 991n^2 + 1 \text{ is } \textit{not} \text{ a perfect square;}$$
$$\text{that is, } \sqrt{991n^2+1} \text{ is not an integer.}$$

We could confirm the validity of this claim for as many values of n as we have patience, but we could never conclude on the basis of the validity of a large number of cases that the claim is valid for all values of n. The claim is in fact false; however, the first value of n for which the claim fails is

$$n = 12,055,735,790,331,359,447,442,538,767.$$

We cannot be content with the mere experience of witnessing various instances of when a claim is true to lend reckless support to its universal truth. We cannot confidently lend any credence to Collatz's conjecture of Example 8.15 despite the comfort offered by the knowledge it holds for all values up to 2.22×10^{18}. Similarly, and more worrisome, a train may run perfectly for arbitrarily long – several years even – before a fault in its software control system contributes to a devastating crash.

Exercise 9.1 (Solution on page 444)

Some number of spots are placed randomly around the circumference of a circle, and every spot is connected to every other spot by a straight line. Assuming that no three lines intersect at a point inside the circle, we would like to know into how many regions is the circle divided?

For example, given 1, 2, 3, 4, or 5 spots, the circle is divided into 1, 2, 4, 8, or 16 regions, respectively; the final three of these are depicted in Figure 9.1.

How many regions are created by connecting six spots?

9.2 A Primary School Induction Argument

Suppose you wish to check that the formula

$$1+2+3+\cdots+n \;=\; \frac{n(n+1)}{2}$$

is true for the first 30 values of n, and you ask a classroom of 30 ten-year-olds to check this formula, each child checking if it is true for some value of n. For example, the 17th child will check that

$$1+2+3+\cdots+17 \;=\; \frac{17\times 18}{2}.$$

You watch each child working diligently on their individual problems and, as expected the first few children, working on confirming the formula for small values of n, are quick to report their success. Those working on larger values of n are taking longer. For example, it is taking a long while for the 17th child to add up the first 17 numbers to find they add up to 153, and then to compute $\frac{17\times 18}{2} = 153$ to discover that the claim is true for n=17. Some children are reporting failure before checking their work and finding errors in their calculations before ultimately reporting success.

Alone in the crowd is the 28th child, a little girl who is sitting quietly reading a novel instead of working away on her calculations. You ask her if she is done, and she says yes. You ask her if the formula is true for n=28 and she says she doesn't know – yet. Confused, you look at her sheet of paper and see the following calculation:

$$\begin{aligned} 1 + 2 + 3 + \cdots + 28 &= \underbrace{1 + 2 + 3 + \cdots + 27} + 28 \\ &= \frac{27\times 28}{2} + 28 \\ &= 28 \times \left(\frac{27}{2} + 1\right) \\ &= 28 \times \left(\frac{29}{2}\right) \\ &= \frac{28\times 29}{2} \end{aligned}$$

As you look over this calculation, the boy at the next desk announces that he has finished adding up the first 27 numbers and that they add up to $378 = \frac{27\times 28}{2}$ as expected: the formula is true for n=27. The little girl immediately responds to this by announcing that the formula is true for n=28.

What this precocious little girl realised was that she could leave most of the hard work of adding up the first 28 numbers to her friend beside her, the little boy who is busily adding up the first 27 numbers. Once he has done that, all she needs to do is add 28 to his total. Knowing *what* the first

27 numbers are *supposed* to add up to, namely $\frac{27\times28}{2}$, she doesn't wait for him to do his job, but rather goes to work under the assumption that her friend will confirm this expectation. This is the calculation that she carried out.

Having carried out this calculation, can she say that the first 28 numbers add up to $\frac{28\times29}{2}$? Not right away, as she made the assumption that the first 27 numbers add up to $\frac{27\times28}{2}$; once her friend, the 27th child, confirms this assumption, she can (and does) announce boldly that the formula is true for $n{=}28$.

There is nothing special about the number 28, just something special about this little girl. If she had the problem of checking the formula for any other number, she would have done the same thing. She was no doubt quietly wondering why her friend beside her was busily adding up all the first 27 numbers; and indeed why her other friend on her other side was busily adding up the first 29 numbers.

Exercise 9.2 (Solution on page 445)

What calculation would this little girl do if she was the 27th child?

Exercise 9.3 (Solution on page 445)

When he was ten years old, the great mathematician Carl Friedrich Gauss was reportedly set the problem of adding up the first 100 numbers. His teacher's intention was to keep the class busy and quiet for some time, but Gauss solved the problem almost immediately. What clever trick did young Gauss employ?

9.3 The Induction Argument

Just as we can inductively define functions over inductively-defined domains, we can exploit the structure of an inductive definition to reason about the objects it defines. For example, ***mathematical induction*** allows you to prove that a property $P(n)$ of natural numbers $n \in \mathbb{N}$ holds for all natural numbers if:

1. **(Base Case)** it holds for the value 0, that is, $P(0)$; and
2. **(Induction Step)** it holds for the value $k{+}1$ whenever it holds for k; that is,

$$P(k) \Rightarrow P(k{+}1).$$

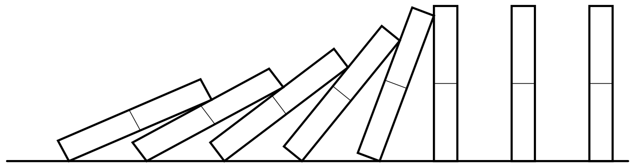


Figure 9.2: Dominoes inductively falling down.

$P(k)$ is referred to as the ***inductive hypothesis***, from which we want to deduce $P(k+1)$.

Clause 2 can be equally expressed as follows

2′. it holds for the value $k > 0$ whenever it holds for $k-1$; that is,

$$P(k-1) \;\Rightarrow\; P(k).$$

The little girl discussed above did precisely this type of reasoning in showing that the property $P(n)$, which states that the first n numbers add up to $\frac{n(n+1)}{2}$, holds for the value 28 assuming it holds for the value 27.

As an analogy, imagine a (possibly infinite) string of dominoes standing side-by-side as in Figure 9.2. If we can prove that the first domino falls (i.e., gets pushed over), and that if one domino falls, the next domino will fall (i.e., gets pushed over by the preceding domino), then this is enough to conclude that *all* of the dominoes will fall over.

We can think of induction as a method of extending our knowledge of the truth: we establish the claim for the first relevant value (typically 0). Next we show that if the claim is true for some value k then it must also be true for the next value $k+1$. The important thing to note here is that k is not given a specific value although it might have some conditions imposed on it (in this case $k \geq 0$). Now since we know the claim to be true for 0, it must also be true for 1; but then it must also be true for 2; but then it must also be true for 3; and continuing in this fashion, we realise that the claim must be true for any value $n \in \mathbb{N}$. In this way we are viewing induction proofs as a form of bootstrapping argument, as depicted in Figure 9.3.

Alternatively, we can think of induction as a proof by contradiction: if the claim is false – that is, if the property does *not* hold for *all* values of $n \in \mathbb{N}$ – then it must fail for some *smallest* value $n \geq 0$; that is, the claim holds for all values less than n but not for n itself. The question then is: what can n be? It cannot be 0, as the base case established that the claim

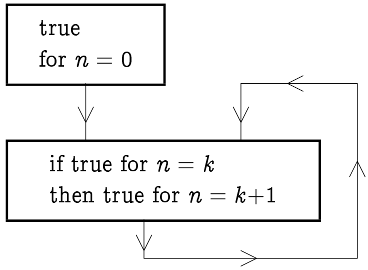


Figure 9.3: Induction as a boot-strapping operation.

holds for n=0. But then by the induction step, n cannot be 1 either; and hence not 2 either; and hence not 3 either; and hence not 4 either. We can carry on this reasoning indefinitely to show that n cannot be any value; for example, n cannot be $1,594$ since, being the smallest value for which the claim is false, the claim would be true for $1,593$, and thus by the induction step it must also be true for $1,594$. Continuing in this fashion, we realise our contradiction: the claim cannot actually fail for any value $n \in \mathbb{N}$.

Following this extensive discussion, we can finally offer the first formal proof by induction, as a model on which to base all other induction proofs.

Example 9.3

Fact: For all $n \geq 0$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof: By induction on n.

Base Case: We note that

$$1 + 2 + 3 + \cdots + 0 = 0 = \frac{0(0+1)}{2}.$$

Induction Step: We assume that, for *some* k,

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2},$$

and from this assumption (the inductive hypothesis) we prove that

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}.$$

That is, we demonstrate that if the statement of the theorem is true when $n = k$, then it must also be true when $n = k{+}1$.

By the inductive hypothesis we can rewrite the left-hand side of this equation that we want to prove true as

$$\frac{k(k+1)}{2} \ + \ (k+1).$$

We can then take out the common factor $(k+1)$ from these two terms, giving us

$$(k+1)\left(\frac{k}{2}+1\right),$$

which is the same as

$$(k+1)\left(\frac{k+2}{2}\right),$$

or in other words,

$$\frac{(k+1)(k+2)}{2},$$

which is the right-hand side that we desire.

In other words, we carried out the following equational derivation:

$$\begin{aligned} & 1 \ + \ 2 \ + \ 3 \ + \ \cdots \ + \ k \ + \ (k+1) \\ = \ & \frac{k(k+1)}{2} \ + \ (k+1) \qquad \textit{(by the inductive hypothesis)} \\ = \ & (k+1)\left(\frac{k}{2}+1\right) \\ = \ & (k+1)\left(\frac{k+2}{2}\right) \\ = \ & \frac{(k+1)(k+2)}{2}. \end{aligned}$$

At this point you should reflect on what the little girl in the Primary School problem from Section 9.2 did, and relate it to the induction step of the above argument. If her reasoning is clear, the following formulæ should be straightforward to verify.

Exercise 9.4 (Solution on page 446)

Show, by induction on n, that the following formulæ are true for all $n \geq 0$.

1. $1^2 \ + \ 2^2 \ + \ 3^2 \ + \ \cdots \ + \ n^2 \ = \ \dfrac{n(n+1)(2n+1)}{6}.$
2. $1 \ + \ 3 \ + \ 5 \ + \ \cdots \ + \ (2n-1) \ = \ n^2.$

3. $1{\cdot}2 \;+\; 2{\cdot}3 \;+\; 3{\cdot}4 \;+\; \cdots \;+\; n(n{+}1) \;=\; \frac{n(n{+}1)(n{+}2)}{3}.$

Exercise 9.5 (Solution on page 447)

Show, by induction on n, that for all $n \geq 0$:

$$F_0 \times F_1 \times \cdots \times F_n \;=\; F_{n+1} - 2$$

where $F_n = 2^{2^n} + 1$ are the Fermat numbers.

Induction is a very common technique for establishing mathematical formulæ such as the following.

Exercise 9.6 (Solution on page 448)

Show, by induction on n, that for any real number $r \neq 1$,

$$1 + r + r^2 + r^3 + \cdots + r^n \;=\; \frac{1 - r^{n+1}}{1 - r}$$

for all $n \geq 0$.

Note that if $-1 < r < 1$ then r^{n+1} approaches 0 as n approaches infinity; hence, as a corollary to the above, we can deduce that for any real r with $|r| < 1$,

$$1 + r + r^2 + r^3 + \cdots \;=\; \frac{1}{1 - r}.$$

So far we have used induction merely to prove simple formulæ. However, induction is more general than this, and the base case can be some value or values other than 0, as the next examples demonstrate.

Example 9.6

Fact: Any amount of postage of at least 8 pence can be made up from just 3-pence and 5-pence stamps.

Proof: By induction on n.

Base Case: A 3-pence stamp and a 5-pence stamp make up 8 pence.

Induction Step: Assume that we have a collection of such stamps adding up to a total of $n \geq 8$ pence.

- if there is a 5-pence stamp in this collection, remove it and replace it with two 3-pence stamps;
- If there are no 5-pence stamps, then there must be (at least) three 3-pence stamps in the collection; remove these and replace them with two 5-pence stamps.

In either case, we arrive at a collection of stamps adding up to $(n+1)$ pence. □

Example 9.7

Fact: The sum of the interior angles of a convex polygon with n sides is equal to $(n-2)180°$ for all $n \geq 3$. (A polygon is ***convex*** if every line joining two points of the polygon lies within the polygon.)

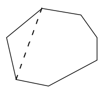
Proof: By induction on n.

Base Case: The sum of the interior angles of any triangle is 180°.

Induction Step: We assume that the theorem is true for some value $k \geq 3$: that the sum of the interior angles of any convex polygon with k sides is equal to $(k-2)180°$.

From this inductive hypothesis, we demonstrate that it must also be true for $k+1$: that the sum of the interior angles of any convex polygon with $k+1$ sides is equal to $(k-1)180°$.

Any $(k+1)$-gon can be decomposed into a triangle and a k-gon by connecting two non-adjacent vertices, as depicted in the diagram.



The sum of the interior angles of this $(k+1)$-gon is then the sum of the interior angles of the triangle, 180°, added to the sum of the interior angles of the k-gon which, by induction, is

$$180° + (k-2)180° \; = \; (k-1)180°.$$

□

Exercise 9.7 (Solution on page 449)

Suppose we draw n circles ($n \geq 1$) so that any two intersect at two points but no three intersect at any point. Prove, by induction on n, that these circles divide the plane into n^2-n+2 regions. Deduce from this that we cannot draw a Venn diagram for four or more sets with circles representing sets.

Induction is of immense importance in Computer Science where a great many of the objects under study are inductively defined. It is imperative that a Computer Scientist be comfortable with inductive reasoning in order to be successful with designing and understanding computing systems.

The following provides an example of reasoning inductively about a simple program.

Exercise 9.8 (Solution on page 449)

Consider the following piece of recursive program code:

function $f(n)$
  if $n{=}0$ **then return** 0
  else return $f(n{-}1) + 2n - 1$

This program code computes the following inductively-defined function:

$$f(n) = \begin{cases} 0, & \text{if } n{=}0 \\ f(n{-}1) + 2n - 1, & \text{if } n{>}0. \end{cases}$$

Show, by induction on n, that $f(n) = n^2$ for all $n \geq 0$.

9.4 Strong Induction

In a proof by induction we demonstrate that some property holds of some number based on the assumption that the property holds of the the previous number. Occasionally we may want to assume that the property holds of other smaller numbers, not just the previous number. An alternative form of induction which permits this is ***strong induction*** which allows you to prove that a property $P(n)$ of natural numbers holds of all natural numbers by demonstrating the following:

- $P(n)$ holds for n whenever it holds for all $k{<}n$; that is,

$$\big(\forall k{<}n : P(k)\big) \;\Rightarrow\; P(n).$$

You may well wonder at this point: what happened to the base case? In the case of $n{=}0$, the assumption that $P(k)$ holds for all values $k{<}n$ is vacuous, since there are no such values, and hence this one clause incorporates the base case of demonstrating that $P(0)$ holds under no assumption.

Example 9.8

Let

$$f(n) = \begin{cases} 0, & \text{if } n{=}0; \\ 2 \cdot f(n/2), & \text{if } n{>}0 \text{ even}; \\ f(n{-}1) + 1, & \text{if } n \text{ odd}. \end{cases}$$

Fact: $f(n) = n$ for every $n \geq 0$.

Proof: By (strong) induction on n, arguing by cases on the "structure" of n.

n=0: $f(0) = 0$.

n>0 even: $f(n) = 2 \cdot f(n/2)$

$= 2 \cdot (n/2) = n.$ *(By induction)*

n odd: $f(n) = f(n-1) + 1$

$= (n-1) + 1 = n.$ *(By induction)*

□

Exercise 9.9 (Solution on page 450)

Prove, by strong induction, that every integer $n>1$ is either prime or a product of primes.

This result, attributed first to Euclid over 2000 years ago, is referred to as the *Fundamental Theorem of Arithmetic*.

9.5 Induction Proofs from Inductive Definitions

We showed earlier how to define functions inductively, e.g., the Harmonic numbers H_n (Example 8.9) and the Fibonacci numbers (Example 8.10). Induction proofs are naturally used to reason about such inductively-defined functions, as evidenced by the following examples.

Example 9.9

Fact: For all $n \geq 0$,

$$H_1 + H_2 + H_3 + \cdots + H_n = (n+1)H_n - n.$$

Proof: By induction on n.

Base Case $(n = 0)$:

$$H_1 + H_2 + H_3 + \cdots + H_0 = 0 = (0+1)H_0 - 0.$$

Induction Step: $(n > 0)$:

$$H_1 + H_2 + H_3 + \cdots + H_n$$

$$= (H_1 + H_2 + H_3 + \cdots H_{n-1}) + H_n$$

$$= nH_{n-1} - (n-1) + H_n \qquad \textit{(by inductive hypothesis)}$$

$$= n(H_n - \tfrac{1}{n}) - (n-1) + H_n \qquad \textit{(since } H_n = H_{n-1} + \tfrac{1}{n}\textit{)}$$

$$= (n+1)H_n - n. \qquad \square$$

Exercise 9.10 (Solution on page 450)

Prove that for all $m \geq 1$ and all $n \geq m$, $H_n - H_m \geq \frac{n-m}{n}$.

Do this by assuming $m \geq 1$ and proving the result by induction on n.

Example 9.10

Fact: $f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$ for all $n \geq 0$.

Proof: By induction on n.

Base Case $(n = 0)$:

$$f_0 + f_1 + f_2 + \cdots + f_0 = f_0 = 0 = 1 - 1 = f_2 - 1.$$

Induction Step $(n > 0)$:

$$f_0 + f_1 + f_2 + \cdots + f_n + f_{n+1}$$

$$= (f_{n+2} - 1) + f_{n+1} \qquad \textit{(by the inductive hypothesis)}$$

$$= (f_{n+1} + f_{n+2}) - 1 = f_{n+3} - 1 \qquad \square$$

Exercise 9.11 (Solution on page 450)

Show, by induction on n, that

$$(f_0)^2 + (f_1)^2 + (f_2)^2 + \cdots (f_n)^2 = f_n f_{n+1}$$

for all $n \geq 0$.

We have seen that the base case may be some value n other than 0. There are also instances in which more than one base case is required. A simple example of this is provided by the following.

Example 9.11

Fact: For all $m \geq 2$ and for all $n \geq 1$, $f_{n+m-2} = f_n f_{m-1} + f_{n-1} f_{m-2}$.

Proof: We assume that $m \geq 2$ is fixed, and we prove the result by induction on n.

Base Case $(n = 1)$: $f_{1+m-2} = f_{m-1} = f_1 f_{m-1} + f_0 f_{m-2}$.

Base Case $(n = 2)$: $f_{2+m-2} = f_m = f_{m-1} + f_{m-2} = f_2 f_{m-1} + f_1 f_{m-2}$.

Induction Step: $(n > 2)$:

$$\begin{aligned} f_{n+m-2} &= f_{(n-1)+m-2} + f_{(n-2)+m-2} \\ &= (f_{n-1} f_{m-1} + f_{n-2} f_{m-2}) + (f_{n-2} f_{m-1} + f_{n-3} f_{m-2}) \\ &\qquad \text{(by inductive hypothesis, twice)} \\ &= (f_{n-1} + f_{n-2}) f_{m-1} + (f_{n-2} + f_{n-3}) f_{m-2} \\ &= f_n f_{m-1} + f_{n-1} f_{m-2} \end{aligned}$$

□

The above proof required two base cases, as the inductive hypothesis is invoked twice for the two values $n-1$ and $n-2$. If in the above proof we only do the base case for $n = 1$, and in the induction step we try to cater for all cases of $n > 1$ (in particular, $n = 2$), then the second invocation of the inductive hypothesis would be invalid in the particular instance where $n = 2$.

⋆ 9.6 Fun with Fibonacci Numbers

In this section we explore three extended induction arguments involving Fibonacci numbers.

9.6.1 A Fibonacci Number Test

Suppose we are given an arbitrary positive integer x and asked whether or not it is a Fibonacci number. For example, how might we determine whether or not the number 517 is a Fibonacci number? The only apparent way is to use the inductive definition to compute successive Fibonacci numbers until we reach (or – more likely – exceed) 517. This is, however, not necessary; we can instead use the following simple test:

> *A positive integer x is a Fibonacci number if, and only if, $5x^2 \pm 4$ is a perfect square.*

For example, $x=3$ is a Fibonacci number, and $5 \cdot 3^2 + 4 = 49 = 7^2$; and $x=5$ is a Fibonacci number, and $5 \cdot 5^2 - 4 = 121 = 11^2$. However, $x=4$ is not a Fibonacci number, and neither $5 \cdot 4^2 - 4 = 76$ nor $5 \cdot 4^2 + 4 = 84$ is a perfect square.

For our less-modest example $x = 517$ above, a few calculator keystrokes tells us that $5 \cdot 517^2 - 4 = 1336441$, and pressing the square root button gives us 1156.0454, so $5x^2 - 4$ is clearly not a perfect square; and $5 \cdot 517^2 + 4 = 1336449$, and pressing the square root button gives us 1156.0489, so $5x^2 + 4$ is also not a perfect square. Therefore, this test tells us that $x = 517$ is not a Fibonacci number. On the other hand, testing the value $x=610$, a few calculator keystrokes tells us that $5 \cdot 610^2 - 4 = 1860496$, and pressing the square root button gives us 1364; in this case $5x^2 - 4$ is a perfect square, meaning that the value $x=610$ is a Fibonacci number (indeed $f_{15} = 610$).

The following two exercises provide the basis for the argument that this test is valid.

Exercise 9.12 (Solution on page 451)

Show, by induction on n, that for all $n \geq 0$ the pair $(x, y) = (f_n, f_{n+1})$ satisfies the equation

$$y^2 - xy - x^2 = \pm 1.$$

Exercise 9.13 (Solution on page 451)

Show, by induction on $x+y$, that if the pair (x, y) of positive integers satisfies the equation

$$y^2 - xy - x^2 = \pm 1$$

then $(x, y) = (f_n, f_{n+1})$ for some $n \geq 0$. (Hint: For the induction step, show that the "smaller" positive integer pair $(y-x, x)$ also provides a solution.)

Theorem 9.13 Fibonacci Test

A positive integer x is a Fibonacci number if, and only if, $5x^2 \pm 4$ is a perfect square.

Proof: We start by recalling the ***quadratic formula*** which states that the quadratic equation

$$ay^2 + by + c = 0$$

is solved by the following values of y:

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In particular, for a given positive value of x, the quadratic equation

$$y^2 - xy - x^2 = \pm 1$$

is solved by the following positive value of y:

$$y = \frac{x + \sqrt{x^2 + 4(x^2 \pm 1)}}{2} = \frac{x + \sqrt{5x^2 \pm 4}}{2}.$$

By Exercise 9.12, if $x = f_n$, then the value of y given by this formula must be f_{n+1}, from which we can deduce that $5x^2 \pm 4$ must be a perfect square.

Conversely, if $5x^2 \pm 4$ is a perfect square for some positive integer x, then the value of y given by this formula, like x, must be a positive integer, in which case Exercise 9.13 tells us that x (as well as y) must be a Fibonacci number. □

9.6.2 A Carrollean Paradox

The following result is known as Cassini's Identity.

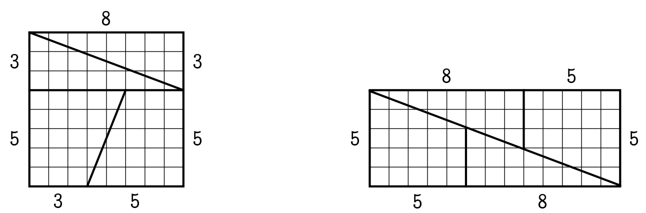
Exercise 9.14 (Solution on page 452)

Show, by induction on n, that $f_{n+1}^2 - f_n f_{n+2} = (-1)^n$ for all $n \geq 0$.

Cassini's Identity forms the basis of a famous puzzle devised by Lewis Carroll. The puzzle is described in the following exercise.

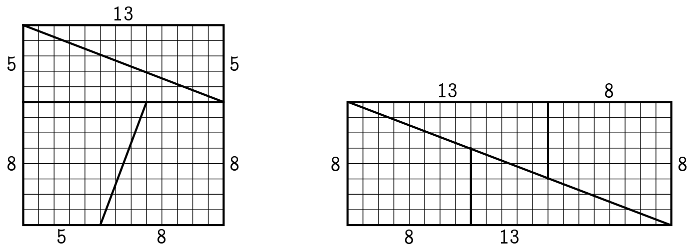
Exercise 9.15 (Solution on page 452)

Take a square whose sides are 8 units long, cut it into four sections (two triangles and two quadrilaterals), and rearrange these four sections into a rectangle whose sides are 5 units and 13 units long as shown here:



The area of the 8×8 square is 64 square units, but the area of the 5×13 rectangle is 65 square units! Where does the extra square unit come from?

This same phenomenon occurs with any square whose sides are of length taken from the Fibonacci numbers. For example consider the following 13×13 square cut up and rearranged into an 8×21 rectangle:



In this case, the area of the square is 169 square units, but the area of the rectangle is 168 square units, so this time we *lose* one square unit. Where did it go?

9.6.3 Fibonacci Decompositions

The Unique Prime Factorisation Theorem states that any positive integer n has a unique decomposition into the product of prime numbers. For example, the number $n{=}364$ decomposes uniquely into the product of primes as follows:

$$364 \;=\; 2\cdot 2\cdot 7\cdot 13.$$

The proof of this theorem is carried out by induction on n, and can be found in any but the most basic algebra reference book. Here, we present a similar decomposition result which we shall find useful later.

Example 9.15 Zeckendorf's Theorem

Fact: Every integer $N \geq 0$ can be expressed uniquely as

$$N \;=\; f_{k_1} \;+\; f_{k_2} \;+\; f_{k_3} \;+\; \cdots \;+\; f_{k_n}$$

where $0 \ll k_1 \ll k_2 \ll k_3 \ll \cdots \ll k_n$. (Here, $i \ll j$ means that $i \leq j{-}2$.)

For example, $100 \;=\; 3+8+89 \;=\; f_4+f_6+f_{11}$.

Proof: First we demonstrate, by induction on n, that for all $n \geq 1$,

$$f_{k_1} + f_{k_2} + f_{k_3} + \cdots + f_{k_n} \;<\; f_{k_n+1}$$

whenever $0 \ll k_1 \ll k_2 \ll k_3 \ll \cdots \ll k_n$.

Base Case $(n=1)$: $f_{k_1} < f_{k_1+1}$ as $k_1 \geq 2$ since $0 \ll k_1$.

Induction Step $(n>1)$:

$$
\begin{aligned}
& f_{k_1} + f_{k_2} + f_{k_3} + \cdots + f_{k_{n-1}} + f_{k_n} \\
& \quad < \ f_{k_{n-1}+1} + f_{k_n} && \textit{(by inductive hypothesis)} \\
& \quad \leq \ f_{k_n-1} + f_{k_n} && \textit{(since } k_{n-1} \ll k_n\textit{, so } k_{n-1}{+}1 \leq k_n{-}1\textit{)} \\
& \quad = \ f_{k_n+1} && \textit{(by definition)}
\end{aligned}
$$

Thus if $N = f_{k_1} + f_{k_2} + f_{k_3} + \cdots + f_{k_n}$ where $0 \ll k_1 \ll k_2 \ll \cdots \ll k_n$ then we must have that $f_{k_n} \leq N < f_{k_n+1}$.

The main result then follows by induction on $N \geq 0$.

Base Case $(N = 0)$: Trivially $0 = f_{k_1} + f_{k_2} + f_{k_3} + \cdots + f_{k_0}$.

Induction Step $(N > 0)$: Let k be such that $f_k \leq N < f_{k+1}$. Then

$$(N - f_k) \ < \ f_{k+1} - f_k \ = \ f_{k-1}.$$

If N is to be represented as required, then by the above result, f_k must be one (indeed the largest) of the summands.

But then by the inductive hypothesis, $(N - f_k) \geq 0$ can be expressed uniquely as

$$(N - f_k) \ = \ f_{k_1} + f_{k_2} + f_{k_3} + \cdots + f_{k_n}$$

where $0 \ll k_1 \ll k_2 \ll k_3 \ll \cdots \ll k_n$.

Furthermore, since $f_{k_n} \leq (N - f_k) < f_{k-1}$, we must have that $k_n{<}k{-}1$, i.e. that $k_n \ll k$.

Taking $k_{n+1} = k$, we thus get that N is expressed uniquely in the required form as

$$N \ = \ f_{k_1} + f_{k_2} + f_{k_3} + \cdots + f_{k_n} + f_{k_{n+1}}. \qquad \square$$

9.7 When Inductions Go Wrong

We give here a few examples illustrating common mis-applications and misconceptions of induction.

Example 9.16

Let $T : \mathbb{Z} \to \mathbb{Z}$ be the function which is defined by:

$$T(n) \ = \ \begin{cases} n{+}6, & \text{if } n \leq 0; \\ T(T(n{-}7)), & \text{otherwise.} \end{cases}$$

We can show that $T(n) = 6$ for all $n \geq 0$. To do this, it is tempting to use induction on n as follows.

Base Case $(n = 0)$: $T(0) = 0 + 6 = 6$.

Induction Step $(n > 0)$:
$$\begin{aligned} T(n) &= T(T(n-7)) \\ &= T(6) \qquad \textit{(by inductive hypothesis)} \\ &= 6 \qquad \textit{(by inductive hypothesis)} \end{aligned}$$

There are two errors in the above argument. First of all if $n < 7$ then $n-7 < 0$ and the first inductive hypothesis cannot be applied. Secondly the claim that $T(6) = 6$ certainly doesn't follow from the inductive hypothesis unless $n > 6$. These observations show that to make the induction work we need to verify a ***range*** of base cases, namely, $T(n) = 6$ for $0 \leq n \leq 6$.

Although the claim is true in the above Example, the argument presented demonstrates how easy it is to make illegitimate arguments to back up a claim. On the other hand, the following exercise demonstrates a blatantly false claim to be true through a seemingly innocuous induction argument.

Example 9.17 Sorites Paradox

Consider the following "proof" that sandpiles do not exist.

Claim: For each $n \geq 0$, n grains of sand do not make a sandpile.

Proof: By induction on n.

Base Case $(n = 0)$:

If there is no sand, then there can be no sandpile.

Induction Step: $(n > 0)$:

Suppose we have $(n+1)$ grains of sand which constitute a sandpile. Clearly taking away a single grain of sand from a sandpile will still leave us with a sandpile. However, we will only have n grains of sand left, which by induction does not constitute a sandpile. Hence our $n+1$ grains of sand cannot constitute a sandpile. $\square$

This is known as the ***sorites paradox*** or the ***heap paradox***. The name comes from the Greek word *soros* (σωρός) meaning "heap". It relies on the vagueness of words such as "heap" and "pile" and has many variations, each of which being a precise and accurate application of valid logical principles to arrive at a nonsensical conclusion.

- A man with only 1 hair is clearly bald.

- If a man with only 1 hair is bald,
  then a man with only 2 hairs is bald.
- If a man with only 2 hairs is bald,
  then a man with only 3 hairs is bald.

  ⋮
- If a man with only $9,999$ hairs is bald,
  then a man with only $10,000$ hairs is bald.

Each of these observations is precise and valid, yet chaining them all together allows us to conclude that a man with $10,000$ hairs on his head is bald, a wholly nonsense claim.

In reasoning about systems, it is imperative that we use great care to employ only concepts that are as rigorously defined and precise as the logical means we use to analyse them.

The sorites paradox provides a playground for philosophers wanting to debate the validity of inductive arguments, but relies heavily on ill-defined terms removed from the rigour of mathematics. However, in the next exercise we provide a subtle error hidden in an otherwise air-tight inductive argument which leads to a clearly false conclusion. Can you uncover this error?

Exercise 9.17 (Solution on page 452)

What is wrong with the following "proof" that all people are the same age?

We show, by induction on n, that for every collection S of $n \geq 0$ people, all people in S are the same age.

Base Case $(n = 0)$: Trivially the claim holds when S consists of 0 people.

Base Case $(n = 1)$: Trivially the claim holds when S consists of 1 person.

Inductive Step $(n > 1)$: Assuming that the claim holds for all collections of size less than n, we show that it holds for any collection of size n. Let S be a collection of n people. Let S' and S'' be two overlapping collections of people which together make up S: $S = S' \cup S''$. By the inductive assumption, all people in S' are the same age, and all people in S'' are the same age. As S' and S'' overlap, all people in S must be the same age.

9.8 Examples of Induction in Computer Science

The following example is typical of the type of analysis which arises in the study of algorithms.

Example 9.18

Consider the following recursive algorithm MinMax(A,p,q) for calculating (x,y) where x and y are, respectively, the minimum and maximum values appearing in the array $A[1\cdots n]$ between the indices p and q, inclusively (the intention is to initially call the algorithm with MinMax$(A,1,n)$).

```
MinMax(A, p, q)
  1  if p = q then return (A[p], A[p])
  2  else if p = q−1 then
  3     if A[p] < A[q] then return (A[p], A[q])
  4     else return (A[q], A[p])
  5  else
  6     (minL, maxL) := MinMax(A, p, p+1)
  7     (minR, maxR) := MinMax(A, p+2, q)
  8     return ( min(minL, minR), max(maxL, maxR) )
```

We are interested in calculating the number of comparisons which this algorithm makes, as an indication of how long it takes to execute (a comparison is made in line 3, and two are made in line 8 through the use of the functions min and max). A simple analysis gives us that the number $T(n)$ of comparisons made by a call to MinMax(A,p,q) with $n = q-p+1$ is as follows:

1. if $n = 1$, that is, if $p = q$, then the algorithm terminates on line 1 without making any comparisons. Thus $T(0) = 0$.
2. if $n = 2$, that is, if $p = q-1$, then the algorithm terminates on line 3 after making one comparison. Thus $T(2) = 1$.
3. if $n > 2$ then the algorithm makes

   (a) $T(2)$ comparisons on line 6; followed by
   (b) $T(n-2)$ comparisons on line 7; followed by
   (c) 2 comparisons on line 8

   before terminating. Thus $T(n) = T(2) + T(n-2) + 2$ for all $n > 2$.

The inductive definition of $T(n)$ is thus summarised as follows.

$$\begin{aligned} T(1) &= 0 \\ T(2) &= 1 \\ T(n) &= T(2) + T(n-2) + 2 \qquad \text{(for } n > 2\text{)} \end{aligned}$$

Fact $T(n) = \left\lceil \frac{3n}{2} \right\rceil - 2$ (where $\lceil x \rceil$ is x rounded up to the nearest integer.)

Proof: By induction on n.

Base Case ($n \leq 2$): Clearly the result is true when n=1 or n=2.

Induction Step ($n > 2$): Suppose the result is true for all values $k \leq n$ for some $n \geq 2$. In particular,

$$T(n-2) = \left\lceil \frac{3(n-2)}{2} \right\rceil - 2 = \left\lceil \frac{3n}{2} \right\rceil - 5.$$

Thus

$$\begin{aligned} T(n) &= T(2) + T(n-2) + 2 \\ &= 1 + \left(\left\lceil \frac{3n}{2} \right\rceil - 5\right) + 2 \quad \textit{(by inductive hypothesis)} \\ &= \left\lceil \frac{3n}{2} \right\rceil - 2. \end{aligned}$$

□

The next two examples describe the technique of ***structural induction***, which is arguably the most important variant of induction within computing.

Example 9.19

Let A be an alphabet containing (at least) two distinct characters a and b.

Fact $aw \neq wb$ for all words $w \in A^*$.

Proof: By induction on *length*(w).

Base Case (*length*(w) = 0): In this case, we must have that $w = \varepsilon$, so

$$aw = a \neq b = wb.$$

Induction Step (*length*(w) > 0): We consider two subcases, depending on whether w begins with the character a or with some other character c.

$\underline{w = au}$: Since $\mathit{length}(u) = \mathit{length}(w) - 1 < \mathit{length}(w)$, $au \neq ub$ by the inductive hypothesis. Hence

$$aw = aau \neq aub = wb.$$

$\underline{w = cu \text{ (where } c \neq a)}$: $aw = acu \neq cub = wb.$ □

The above is an example of a proof based on ***structural induction***: the inductive hypothesis assumes that the claim holds for all smaller structures (in this case, for all shorter words), and uses this assumption to establish that the claim holds for the structure in question. For this reason, such a proof is typically referred to as a proof by induction on the structure of words, and would more naturally be presented as follows.

Proof: By induction on the structure of words (that is, we prove the result for a word w under the inductive hypothesis that it is true for all smaller words), arguing by cases on the structure of w (that is, we consider in turn three possible forms of w, namely ε, au and cu where $c \neq a$).

$\underline{w = \varepsilon}$: $aw \;=\; a \;\neq\; b \;=\; wb.$

$\underline{w = au}$: By induction (since u is smaller than w), $au \neq ub$, so

$$aw \;=\; aau \;\neq\; aub \;=\; wb.$$

$\underline{w = cu \;\; \textbf{(where } c \neq a\textbf{)}}$: $aw \;=\; acu \;\neq\; cub \;=\; wb.$

We give one further example, without the excessive explanations.

Example 9.20

Fact: Every binary tree t has exactly one more leaf than internal node.

Proof: By induction on the structure of t, arguing by cases on the structure of t.

$\underline{t = \star}$: The tree $\star$ has 1 leaf and 0 internal nodes.

$\underline{t = N(t_1, t_2)}$: By induction, t_i (for $i = 1, 2$) must have n_i nodes and $n_i{+}1$ leaves, for some n_1, n_2. But then $N(t_1, t_2)$ must have $n_1{+}n_2{+}1$ nodes and $(n_1{+}1) + (n_2{+}1) = (n_1{+}n_2{+}1){+}1$ leaves. □

Exercise 9.20 (Solution on page 453)

Prove by induction that $\mathit{length}(L_1{+\!\!+}L_2) = \mathit{length}(L_1) + \mathit{length}(L_2)$ for all lists L_1 and L_2, using the inductive definition of the length of a list from Example 8.12, and your inductive definition of the append function from Exercise 8.13.

9.9 Additional Exercises

1. Prove the following hold for all $n \geq 0$, by induction on n.

(a) $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1+2+3+\cdots+n)^2$.
(This is known as *Nicomachus's Theorem.*)

(b) $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$.

(c) $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2-1)$.

(d) $1{\cdot}2{\cdot}3 + 2{\cdot}3{\cdot}4 + 3{\cdot}4{\cdot}5 + \cdots$
$\cdots + n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$.

(e) $1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n+1)! - 1$.

(f) $\dfrac{1}{4(1^2)-1} + \dfrac{1}{4(2^2)-1} + \dfrac{1}{4(3^2)-1} + \cdots + \dfrac{1}{4(n^2)-1} = \dfrac{n}{2n+1}$.

2. Show that every $n > 0$ can be expressed uniquely as

$$c_1(1!) + c_2(2!) + c_3(3!) + \cdots + c_n(n!)$$

where $0 \le c_j \le j$.

3. Prove, by induction on $(m+n)$, that for all $m, n \ge 0$:

$$\begin{aligned} &1{\cdot}2{\cdot}3{\cdot}\cdots{\cdot}m + 2{\cdot}3{\cdot}4{\cdot}\cdots{\cdot}(m+1) + 3{\cdot}4{\cdot}5{\cdot}\cdots{\cdot}(m+2) \\ &\qquad + \cdots + n(n+1)(n+2)\cdots(n+m-1) \\ &= \frac{n(n+1)(n+2)(n+3)\cdots(n+m)}{m+1}. \end{aligned}$$

4. Prove, by induction on n, that a finite set with n elements has 2^n subsets.

5. Define the sequence $\langle g_0, g_1, g_2, \ldots\rangle$ as follows: $g_0 = 0$; $g_1 = 1$; $g_2 = 1$; and for all $n \ge 2$,

$$g_{2n-1} = g_{n-1}^2 + g_n^2 \qquad \text{and} \qquad g_{2n} = g_{n+1}^2 - g_{n-1}^2.$$

Thus for example:

$n=2$:	$g_3 = g_1^2 + g_2^2$	$g_4 = g_3^2 - g_1^2$
$n=3$:	$g_5 = g_2^2 + g_3^2$	$g_6 = g_4^2 - g_2^2$
$n=4$:	$g_7 = g_3^2 + g_4^2$	$g_8 = g_5^2 - g_3^2$

Show, by induction on n, that $g_n = f_n$ for all $n \ge 0$.

6. Define the sequence $\langle x_0, x_1, x_2, \ldots\rangle$ as follows:

$$x_0 = 0; \qquad x_{n+1} = \frac{1}{1+x_n} \qquad (n \ge 0).$$

Show, by induction on n, that $x_n = \dfrac{f_n}{f_{n+1}}$ for all $n \ge 0$.

7. Provide a correct proof for the claim made in Example 9.16.

8. Suppose that in a particular country, every road is one-way, and every pair of cities is connected by exactly one direct road. Show, by induction on the number n of cities, that there exists a city which can be reached from every other city either directly or via only one other city.

9. Imagine drawing n straight lines in the plane (extending to infinity in both directions). The resulting configuration is to be coloured like a map, with no two bordering "countries" having the same colour (but two countries which meet at a single point may have the same colour).

   Show, by induction on n, that only two colours are needed.

   (Hint: Suppose you have such a coloured plane with n lines, and you draw a new line; clearly the colouring condition fails nowhere except across this new 'border'. How can you restore the colouring condition without altering the colours on one side of this border?)

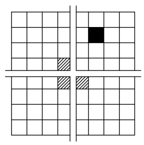
10. A collection of n circles drawn in the plane divide the plane into parts. Show that you can colour the parts with two colours so that no two parts with a common boundary line are coloured the same way.

    (Hint: Similar to the previous exercise.)

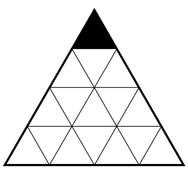
11. You are given a $2^n \times 2^n$ checkerboard with one black square arbitrarily placed on the board and the remaining $4^n - 1$ squares white. You are also given a supply of tiles which look like 2×2 checkerboards with one corner square removed. You want to tile the checkerboard so that each white square is covered exactly once, while the black square remains uncovered.

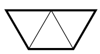
    Show, by induction on n, that the $2^n \times 2^n$ checkerboard can be so tiled, for all $n \geq 0$.

    (Hint: For the inductive step, place the first tile in the centre of the board with the gap in the quadrant containing the black square, and look at the four $2^{n-1} \times 2^{n-1}$ quadrants.)



12. You are given a checkerboard in the shape of an equilateral triangle with sides of length 2^n made up of smaller equilateral triangles with sides of unit length. The topmost equilateral triangle is black but all others are white. You are also given a supply of tiles in the form of bucket-shaped trapeziums made from three small equilateral triangles.





    You want to tile the large triangular-shaped checkerboard so that each white triangle is covered exactly once, while the black triangle remains uncovered.

Show, by induction on n, that the whole checkerboard can be so tiled, for all $n \geq 0$.

13. There are n identical cars on a circular track. Among all of them, they have just enough petrol for one car to complete a lap. Show that there is a car which can complete a lap by collecting petrol from the other cars on its way around.

    (Hint: For the induction step, first argue that there is a car A which can reach the next car B. Then consider removing B from the track, emptying its petrol into A.)

14. I put two cards on a table and tell two people that the cards have different positive integers written on their undersides. I tell them to take one card each at random and secretly look at the number written on their card. They are then put in a room with a clock which rings a bell every minute. They are not allowed to communicate in any way, but are instructed to wait in the room until one of them knows which card has the lower number and which has the higher number, and then to announce this fact the next time the clock rings.

    There seems to be no escape for these two people, as there seems to be no way for either of them to discover who has the larger number. Imagine being one of the two, sitting with a card with the number 26 written on it; how could you possibly determine whether the card held by the other person has a number which is smaller than this or greater than this? Paradoxically, it is doable.

    Prove, by induction on $n \geq 1$, that if n is the lower of the two numbers written on the two cards, then the person who has this card will announce that he has the card with the lower number after the bell rings n times.

15. What is wrong with the following "proof" that

    $$1 + 2 + 3 + \cdots + n = \frac{(n-1)(n+2)}{2}.$$

    Proof: By induction on n.

    $$\begin{aligned} 1 + 2 + 3 + \cdots + n & \\ &= (1 + 2 + 3 + \cdots + (n-1)) + n \\ &= \frac{(n-2)(n+1)}{2} + n \qquad \textit{(by inductive hypothesis)} \\ &= \frac{(n-1)(n+2)}{2}. \qquad \square \end{aligned}$$

16. What is wrong with the following "proof" that every natural number

is interesting[†].

Proof: By induction on n.

Base Case ($n = 0$): 0 is interesting as it is the smallest natural number.

Induction Step: ($n > 0$):

Suppose every number less than n is interesting. If n itself is interesting for some reason, then we are done. On the other hand, if there is nothing interesting about n, then it is in fact the first natural number which is not interesting, which makes it an interesting number indeed! □

17. What is wrong with the following "proof" that $x{=}2x$ for all real numbers $x{\geq}0$?

Proof: By induction on x.

Base Case ($x = 0$): $x = 0 = 2 \cdot 0 = 2x$.

Induction Step: ($x > 0$):

Suppose $y = 2y$ for every positive real number y less than x.
In particular, since $\frac{x}{2} < x$, $\frac{x}{2} = 2(\frac{x}{2}) = x$.
But then $x = 2(\frac{x}{2}) = 2x$ (by induction). □

18. Despite seeming more powerful, the principle of *strong* induction follows from *ordinary* induction, and hence provides added convenience but not added power. This can be demonstrated as follows.

Suppose, for a property $P(n)$ of natural numbers, the premise of strong induction holds:

$$\forall n \Big((\forall k{<}n\, P(k)) \Rightarrow P(n) \Big)$$

That is, $P(n)$ holds of a particular value n whenever it holds for *all* smaller values. We will show, by *ordinary* induction, that $\forall n\, P(n)$ is true. Let $Q(n)$ be the property $\forall k{<}n\, P(k)$.

(a) Show that $\forall n\, P(n) \;\Leftrightarrow\; \forall n\, Q(n)$ *without* using induction.
(b) Show that $\forall n\, Q(n)$ by *ordinary* induction. Thus, by part (a), $\forall n\, P(n)$.

[†] This proof is clearly wrong, as *no* number is interesting. Proof: Suppose some numbers are interesting; then there must be a *smallest* interesting number n. So what, who cares?