Chapter 7

Relations

It is a melancholy truth that even great men have their poor relations.

- Charles Dickens, *Bleak House*.

In previous chapters we looked at grouping objects together into sets, as well as logics to reason about the elements in a set. We also studied functions $f : A \to B$ mapping elements in one set A to elements in another set B.

In this chapter we shall turn our attention towards more general *relationships* between elements of sets than simple mappings. Some everyday examples of such relationships are "parenthood" amongst the set of people *("A is a parent of B")* and "divisibility" amongst the set of integers *("x divides evenly into y")*. More generally, relationships can exist between elements of different sets, such as the "enrolment" relationship between the sets of students and courses *("student s takes course c")*. Relationships may even exist amongst elements of three or more sets, such as the "grade" relationship between students, courses and grades *("student s got a grade of g in course c")*.

7.1 Basic Definitions

We start by recalling that the *truth set* of a predicate such as

$$S(x, y, z) \quad = \quad \text{“}\textbf{student } x\textbf{, in course } y\textbf{, scored a grade of } z\text{”}$$

denotes a subset of a Cartesian product, in this case $S \times C \times G$, where S, C and G are the sets of students, courses, and grades, respectively. In this example, $S(s, c, g)$ is true if, and only if, s is a student who scored a grade of g in course c; and the truth set for this property is

$$\mathit{Grades} = \{ (s, c, g) : s \text{ is a student who scored a grade of } g \text{ in course } c \}.$$

An ***n*-ary relation** R is just such a subset of n-tuples. In the above example, the set *Grades* is a ternary (that is, a 3-ary) relation over $S \times C \times G$:

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Undergraduate Topics in Computer Science,
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$$Grades \subseteq S \times C \times G.$$

The most obvious use of n-ary relations is in representing databases. For example, the above relation *Grades* might represent a particular University's database of students' course grades.

Example 7.1

The Internet Movie Database (IMDb) `http://www.imdb.com` is a Web site which contains a massive, and ever-increasing, online database of films and TV shows, associating with each of these its actors and production crew personnel (directors, writers, producers, etc), as well as many other attributes such as year of release and genre.

For example, the table in Figure 7.1 represents the database of James Bond films, recording their title, year of release, starring actor, and director. This is a fraction of the information, presented in tabular form, delivered by IMDb as a result of a search on the term "James Bond". It can be viewed as a relation

$$\mathit{BondFilms} \subseteq \textsc{Titles} \times \mathbb{N} \times \textsc{Names} \times \textsc{Names}$$

over the sets

$$\begin{aligned} \textsc{Titles} &= \text{film titles;} \\ \mathbb{N} &= \text{natural numbers representing years;} \\ \textsc{Names} &= \text{names of people;} \end{aligned}$$

containing the 20 records (i.e., 4-tuples):

$$\begin{aligned} r01 &= (\text{Dr. No}, 1962, \text{Sean Connery}, \text{Terence Young}) \\ r02 &= (\text{Thunderball}, 1965, \text{Sean Connery}, \text{Terence Young}) \\ &\vdots \\ r20 &= (\text{Skyfall}, 2012, \text{Daniel Craig}, \text{Sam Mendes}). \end{aligned}$$

The main use to which such a database is put is for being queried. As an example, we may wish to query the database to find out which James Bond films star Roger Moore. The answer to this query would be a particular set of records:

$$\begin{aligned} Q &= \{ r \in \mathit{BondFilms} : r \text{ stars Roger Moore} \} \\ &= \{ r06, r07, r08, r10, r11 \}. \end{aligned}$$

Exercise 7.1 (Solution on page 435)

Express and answer the following queries about the above database of James Bond Films.

	Title	Year	Star	Director
$r01$	Dr. No	1962	Sean Connery	Terence Young
$r02$	Thunderball	1965	Sean Connery	Terence Young
$r03$	You Only Live Twice	1967	Sean Connery	Lewis Gilbert
$r04$	On Her Majesty's Secret Service	1969	George Lazenby	Peter R. Hunt
$r05$	Diamonds Are Forever	1971	Sean Connery	Guy Hamilton
$r06$	The Spy Who Loved Me	1977	Roger Moore	Lewis Gilbert
$r07$	Moonraker	1979	Roger Moore	Lewis Gilbert
$r08$	For Your Eyes Only	1981	Roger Moore	John Glen
$r09$	Never Say Never Again	1983	Sean Connery	Irvin Kershner
$r10$	Octopussy	1983	Roger Moore	John Glen
$r11$	A View to a Kill	1985	Roger Moore	John Glen
$r12$	The Living Daylights	1987	Timothy Dalton	John Glen
$r13$	Licence to Kill	1989	Timothy Dalton	John Glen
$r14$	Golden Eye	1995	Pierce Brosnan	Martin Campbell
$r15$	Tomorrow Never Dies	1997	Pierce Brosnan	Roger Spottiswoode
$r16$	The World Is Not Enough	1999	Pierce Brosnan	Michael Apted
$r17$	Die Another Day	2002	Pierce Brosnan	Lee Tamahori
$r18$	Casino Royale	2006	Daniel Craig	Martin Campbell
$r19$	Quantum of Solace	2008	Daniel Craig	Marc Forster
$r20$	Skyfall	2012	Daniel Craig	Sam Mendes

Figure 7.1: James Bond Films.

1. Which Bond films were directed by Lewis Gilbert?
2. Which Bond Films were released in the 1970s?

7.2 Binary Relations

Binary (that is, 2-ary) relations are the most common types of relations, and are of particular importance. Concepts such as

- order *("element a comes before element b")*,

- equivalence *("element a is the same as element b")*, and
- function *("input a results in output b")*

are all examples of binary relations, relating one thing a to another thing b. They are often written in ***infix*** style, so that we would write aRb rather than $(a,b) \in R$.

A binary relation $R \subseteq A \times B$ is thus just a set of ordered pairs, and is said to be a relation ***from*** the set A ***to*** the set B. The sets A and B are referred to as the ***source*** and ***target***, respectively, of R.

A binary relation $R \subseteq A \times A$ from a set A to itself is said to be a relation ***on*** A. In this case, the relation is said to be ***homogeneous***, whereas a relation $R \subseteq A \times B$ with $A \neq B$ is said to be ***heterogeneous***.

Example 7.2

As an example of a binary relation on the natural numbers $\mathbb{N}$ we can take the usual *less-than-or-equal-to* relation $\leq \ \subseteq \ \mathbb{N} \times \mathbb{N}$:

$$\begin{aligned} \leq \ &= \ \{ (x,y) \ : \ x \leq y \} \\ &= \ \{ (0,0),\ (0,1),\ (1,1),\ (0,2),\ (1,2),\ (2,2),\ \dots \}. \end{aligned}$$

As an example of a binary relation from the set H of humans to the natural numbers $\mathbb{N}$ we can take the relation $R \ \subseteq \ H \times \mathbb{N}$ given by:

$$R \ = \ \{ (x,n) \ \in \ H \times \mathbb{N} \ : \ x \text{ has } n \text{ children} \}.$$

As an example of a binary relation from the set C of cities to the set N of countries (nations) we can take the relation $R \ \subseteq \ C \times N$ given by:

$$R \ = \ \{ (c,n) \ \in \ C \times N \ : \ c \text{ is located in } n \}.$$

Example 7.3

Joel likes mint ice cream and coffee ice cream; Felix likes vanilla ice cream and cherry ice cream; Oskar likes vanilla ice cream and chocolate ice cream; and Amanda likes chocolate ice cream and mint ice cream. These properties can be related by the binary relation

$$\mathit{Likes} \ \subseteq \ \textsf{Children} \times \textsf{Flavours}$$

where

$$\begin{aligned} \textsf{Children} \ &= \ \{ \text{Joel, Felix, Oskar, Amanda} \} \quad \text{and} \\ \textsf{Flavours} \ &= \ \{ \text{Vanilla, Chocolate, Coffee, Cherry, Mint} \} \end{aligned}$$

consisting of the following ordered pairs:

$$\begin{aligned} \textit{Likes} \;=\; \{\, & (\text{Joel}, \text{Mint}),\ (\text{Joel}, \text{Coffee}), \\ & (\text{Felix}, \text{Vanilla}),\ (\text{Felix}, \text{Cherry}), \\ & (\text{Oskar}, \text{Vanilla}),\ (\text{Oskar}, \text{Chocolate}), \\ & (\text{Amanda}, \text{Chocolate}),\ (\text{Amanda}, \text{Mint}) \,\}. \end{aligned}$$

Thus,

$$\begin{aligned} \textit{Likes} \;=\; \{\, (c, f) \in \; & \mathsf{Children} \times \mathsf{Flavours} \;: \\ & \text{child } c \text{ likes ice cream flavour } f \,\}. \end{aligned}$$

Put differently, this relation is the truth set of the predicate L defined by

$$L(c, f) \;=\; \text{child } c \text{ likes ice cream flavour } f.$$

Exercise 7.3 (Solution on page 435)

Referring to the database of James Bond films in Example 7.1, give the binary relation $\textit{StarsIn} \subseteq \textsc{Names} \times \textsc{Titles}$ defined by

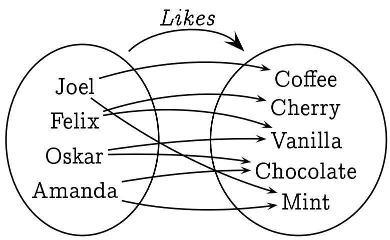
$$\textit{StarsIn} \;=\; \{\, (x, y) \;:\; x \text{ stars as James Bond in } y \,\}.$$

Binary relations can be visualised pictorially by drawing arrows connecting the related objects.

- A heterogeneous relation $R \subseteq A \times B$ from A to B would most naturally be depicted by drawing the two sets A and B side-by-side, and drawing an arrow from each element $a \in A$ in the first set to each of those elements $b \in B$ to which it is related; i.e., such that $(a, b) \in R$.
- A homogeneous relation $R \subseteq A \times A$ on A on the other hand might more naturally be depicted by simply laying out the elements of A in some natural fashion, and drawing an arrow from $a \in A$ to $b \in A$ whenever $(a, b) \in R$.

Example 7.4

The relation *Likes* of Example 7.3 is pictured as follows:



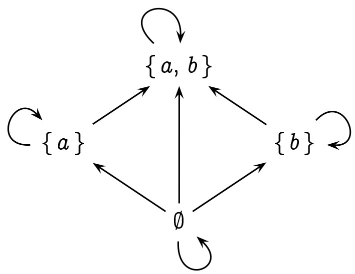
We have an arrow from a child $c \in$ Children to a flavour $f \in$ Flavours whenever $(c, f) \in \mathit{Likes}$.

Example 7.5

The subset relation $\subseteq$ on the powerset of $\{a, b\}$:

$$\mathcal{P}(\{a, b\}) = \Big\{ \emptyset, \{a\}, \{b\}, \{a, b\} \Big\}$$

is pictured as follows:



We have an arrow from one set A to another set B whenever $A \subseteq B$.

Exercise 7.5 (Solution on page 435)

Referring to the database of James Bond films in Example 7.1, let

$$\textsc{BondActors} \subseteq \textsc{Names}$$

be the set of six actors who have played the role of James Bond, and define the two binary relations *Before* and *FirstBefore* on BondActors as follows:

$$\mathit{Before} = \{ (x, y) \ : \ x \text{ stars as James Bond in an earlier film than one in which } y \text{ stars as James Bond} \};$$

$$\mathit{FirstBefore} = \{ (x, y) \ : \ x \text{ starred as James Bond before } y \text{ did} \}.$$

Present these relations pictorially as well as list out their elements.

(Be careful with this exercise. The way that the binary relation *Before* is defined allows each of two actors to appear before the other, and for one actor to appear before himself!)

Kinship relations are prime examples of binary relations. We all have an intuitive grasp of these and we can name a wide range of relationships, e.g. father, mother, sibling, great uncle. The English language is not even particularly rich in this respect. In Swedish, for example, you don't just refer to your aunt or your uncle, but more specifically to your *farbror* (father's brother), your *morbror* (mother's brother) your *faster* (father's sister), or your *moster* (mother's sister).

Example 7.6

The Duck family consists of the parents Hortense and Quackmore Duck, and their two children Della and Donald. Hortense has a brother Scrooge, and Della has three sons: Huey, Louis and Dewey. Let us consider the set of these eight Ducks:

$$\text{DUCKS} = \Big\{ \text{Quackmore, Hortense, Scrooge,} \\ \text{Donald, Della, Huey, Louis, Dewey} \Big\}.$$

There are a variety of kinship relations defined over DUCKS × DUCKS, such as the following:

$$\textit{Father} = \Big\{ (\text{Quackmore, Donald}), (\text{Quackmore, Della}) \Big\}.$$

$$\textit{Mother} = \Big\{ (\text{Hortense, Donald}), (\text{Hortense, Della}), \\ (\text{Della, Huey}), (\text{Della, Louis}), (\text{Della, Dewey}) \Big\}.$$

$$\textit{Parent} = \Big\{ (\text{Quackmore, Donald}), (\text{Quackmore, Della}), \\ (\text{Hortense, Donald}), (\text{Hortense, Della}), \\ (\text{Della, Huey}), (\text{Della, Louis}), (\text{Della, Dewey}) \Big\}.$$

$$\textit{Uncle} = \Big\{ (\text{Scrooge, Donald}), (\text{Scrooge, Della}), \\ (\text{Donald, Huey}), (\text{Donald, Louis}), (\text{Donald, Dewey}) \Big\}.$$

Exercise 7.6 (Solution on page 436)

Define the kinship relations *Child*, *Brother*, *Sister* and *Sibling* on the Duck family of Example 7.6, and present the *Child* relation pictorially.

7.2.1 Functions as Binary Relations

We have defined a function $f : A \to B$ to be an assignment of exactly one element of B to each element of A, and noted in Theorem 6.4 that such a function is completely determined by its graph:

$$\text{graph}(f) = \{ (a, b) \in A \times B \ : \ b = f(a) \}.$$

The graph of the function f is a binary relation from A to B satisfying the following special property: every element $a \in A$ is related to exactly one element $b \in B$.

Conversely, any binary relation $R \subseteq A \times B$ which satisfies this property defines a function $f_R : A \to B$.

Theorem 7.6

A binary relation $R \subseteq A \times B$ is the graph of a function from A to B if, and only if,

$$\forall\, a \in A\ \exists\,!\, b \in B\ \big((a, b) \in R\big) \qquad (\star)$$

Proof: If the relation $R \subseteq A \times B$ satisfies the property $(\star)$, then we can define a function $f_R : A \to B$ by mapping each $a \in A$ to the unique $b \in B$ such that $(a, b) \in R$. Clearly, $\mathsf{graph}(f_R) = R$, as given any $(a, b) \in A \times B$,

$$\begin{aligned} (a, b) \in \mathsf{graph}(f_R) &\Leftrightarrow f_R(a) = b && \textit{(by definition of graph}(f_R)\textit{)} \\ &\Leftrightarrow (a, b) \in R && \textit{(by definition of } f_R\textit{).} \end{aligned}$$

Conversely, if $R = \mathsf{graph}(f)$ for some function $f : A \to B$, then R must clearly satisfy $(\star)$, as the graph of any function must satisfy $(\star)$. □

7.3 Operations on Binary Relations

We have defined binary relations as certain sets; specifically, a binary relation from A to B is a subset of $A \times B$. With this view in mind, there are various operations which we can apply to binary relations to extract information from them, or to build further binary relations, typical of the sort employed by database queries.

7.3.1 Boolean Operations

As binary relations are sets (of pairs), the usual set operations can be applied to these, often quite usefully. In the above Duck family Example 7.6, for instance, the *Parent* relation is defined simply as the union of the *Father* and *Mother* relations:

$$Parent \;=\; Father \,\cup\, Mother.$$

This is intuitively clear, as x is a parent of y if, and only if, either x is the father of y, or x is the mother of y:

$$(x, y) \in Parent \text{ if, and only if, } (x, y) \in Father \text{ or } (x, y) \in Mother.$$

We can also express the *Father* relation in terms of the *Parent* and *Mother* relations, noting that a father is someone who is a parent but not a mother:

$$Father = Parent \setminus Mother.$$

Note that in order to apply set operations to binary relations, the relations being operated on must be defined over the same sets (in this case, DUCKS × DUCKS). It would not make much sense, for example, to take the union $Father \cup Before$ of the relation $Father \subseteq$ DUCKS × DUCKS from Example 7.6. and the relation $Before \subseteq$ NAMES × NAMES from Exercise 7.5.

Exercise 7.7 (Solution on page 437)

Let R_1, R_2 and R_3 represent the *less-than* relation $<$, the *equality* relation $=$, and the *less-than-or-equal-to* relation $\leq$, respectively, all on the set $\mathbb{N}$ of natural numbers:

$$R_1 = \{ (x, y) \in \mathbb{N}^2 \ : \ x < y \};$$
$$R_2 = \{ (x, y) \in \mathbb{N}^2 \ : \ x = y \};$$
$$R_3 = \{ (x, y) \in \mathbb{N}^2 \ : \ x \leq y \}.$$

What are the following relations?

1. $R_1 \cup R_2$
2. $R_3 \cap \overline{R_2}$
3. $R_3 \setminus R_1$

7.3.2 Inverting Relations

Given a binary relation, an obvious and natural thing to do is to turn it around, or invert it, and consider the converse relation. For example, the opposite, or inverse, of the *less-than-or-equal-to* relation $\leq$ is the *greater-than-or-equal-to* relation $\geq$ (as $x \leq y$ if, and only if, $y \geq x$); and the opposite, or inverse, of the *Parent* relation is the *Child* relation (as x is a parent of y if, and only if, y is a child of x).

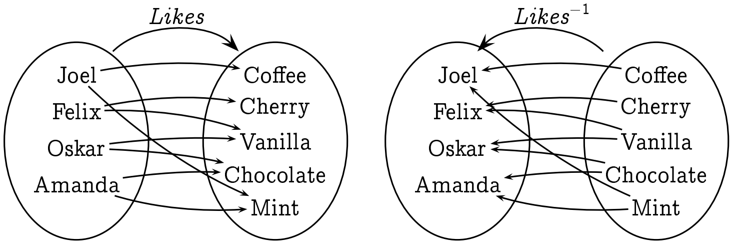
Given a binary relation $R \subseteq A \times B$ from a set A to a set B, the ***inverse*** relation $R^{-1} \subseteq B \times A$ from B to A is defined as

$$R^{-1} = \{ (b, a) \ : \ (a, b) \in R \}.$$

If we consider the pictorial representation of the relation R, we can derive the pictorial representation of R^{-1} simply by reversing the direction of all of the arrows, thus replacing each arrow from a to b where $(a, b) \in R$ by an arrow from b to a.

Example 7.7

The inverse of the relation $Likes \subseteq \text{Children} \times \text{Flavours}$ from Example 7.3 is the relation $Likes^{-1} \subseteq \text{Flavours} \times \text{Children}$ of *"is liked by"*:



For example, (Joel, Mint) $\in Likes$ indicates that Joel likes mint ice cream, while (Mint, Joel) $\in Likes^{-1}$ indicates that mint ice cream is liked by Joel.

Exercise 7.8 (Solution on page 437)

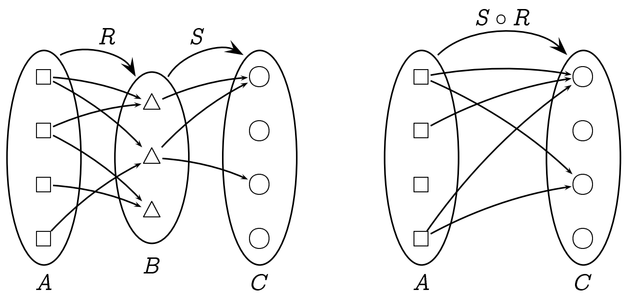
What is $Sibling^{-1}$, the inverse of the $Sibling$ relation?

7.3.3 Composing Relations

As well as turn relations around, another natural operation is to combine, or compose, two relations by following one with another. Given relations $R \subseteq A \times B$ from A to B and $S \subseteq B \times C$ from B to C, the ***composition*** of S and R is the relation $S \circ R \subseteq A \times C$ from A to C defined as

$$S \circ R \;=\; \{\, (a,c) \in A \times C \;:\; \exists b \in B \text{ such that } (a,b) \in R \text{ and } (b,c) \in S \,\}.$$

If we consider the pictorial representation of the relations R and S, we can derive the pictorial representation of $S \circ R$ simply by following an R-arrow by an S-arrow, as in the following example:



Note that the target of the relation R must be the same as the source of the relation S in order to form the composition. Also note carefully the order of the relations: the composition $S \circ R$ of the relations S and R first "applies" the relation R to its source before "applying" the relation S to the result. In this sense, the definition coincides with the composition of functions given in Definition 6.8.

Example 7.8

A grandfather is a father of a parent, and we can use this characterisation to define the *Grandfather* relation:

$$Grandfather \ = \ Parent \circ Father.$$

The order in which we write the two relations which are being composed is important. For example, a grandfather is a father of a parent, which is not the same thing as a parent of a father:

$$Father \circ Parent \ \neq \ Parent \circ Father.$$

Exercise 7.9 (Solution on page 437)

Define the relations *Uncle* and *Nephew* in terms of simpler relations, and derive these relations for the Duck family of Example 7.6.

7.3.4 The Domain and Range of a Relation

Given the relation $R \subseteq A \times B$ from A to B,

- the ***domain*** of R is the set

$$\mathsf{domain}(R) \ = \ \{\, a \in A \ : \ \exists\, b \in B \text{ such that } (a, b) \in R \,\};$$

- the ***range*** of R is the set

$$\mathsf{range}(R) \ = \ \{\, b \in B \ : \ \exists\, a \in A \text{ such that } (a, b) \in R \,\}.$$

That is to say, the domain of a relation consists of all elements of the source A of the relation which are related to something in the target B, and the range of a relation consists of all elements of the target B of the relation which are related to something in the source A.

Example 7.9

Consider the following relations on humans H:

$$Parent = \{\, (x,y) \;:\; x \text{ is a } \mathit{parent} \text{ of } y \,\}$$
$$Brother = \{\, (x,y) \;:\; x \text{ is a } \mathit{brother} \text{ of } y \,\}$$

Then

$$\begin{aligned} \mathsf{domain}(Parent) &= \text{the set of parents } (\textbf{\textit{not}} \text{ all of } H); \\ \mathsf{range}(Parent) &= \text{the set of children } (\textbf{\textit{all}} \text{ of } H); \\ \mathsf{domain}(Brother) &= \text{the set of brothers (males with siblings)}; \\ \mathsf{range}(Brother) &= \text{the set of humans with a brother.} \end{aligned}$$

Exercise 7.10 (Solution on page 438)

Prove that if $R \subseteq A \times B$ is the graph of a function $f : A \to B$, then domain$(R) = A$ (i.e., the domain of f) and range$(R) =$ range(f).

7.4 Properties of Binary Relations

There are various properties that a binary relation on a set A may or may not satisfy. Of particular interest are the properties of *reflexivity*, *symmetry* and *transitivity*, all of which we shall explore in this section.

7.4.1 Reflexive and Irreflexive Relations

The difference between the *less-than* relation $<$ and the *less-than-or-equal-to* relation $\leq$ on numbers is that any number is *less-than-or-equal-to* itself (since it is equal to itself), but no number is *less-than* itself. For example, $2 \leq 2$ is true but $2 < 2$ is not true. This motivates our first property.

Definition 7.10

A relation R on a set A is ***reflexive*** *if, and only if, every element of A is related to itself by R:*

$$\forall x \in A \; (xRx).$$

The relation is ***irreflexive*** *if, and only if, no element of A is related to itself:*

$$\forall x \in A \; \neg(xRx).$$

Thus, for example, the *less-than-or-equal-to* relation $\leq$ is reflexive, while the *less-than* relation $<$ is irreflexive. Note that irreflexive is not the same as non-reflexive: it is possible for a binary relation to relate some but not all elements to themselves, thus making the relation neither reflexive nor irreflexive.

Exercise 7.11 (Solution on page 438)

Is the relation *Before* from Exercise 7.5 reflexive, irreflexive, or neither? What about the relation *FirstBefore*?

7.4.2 Symmetric and Antisymmetric Relations

Equality between objects suggests – amongst other things – a certain symmetry between the objects, which is captured by the next property of interest.

Definition 7.11

A relation R on a set A is ***symmetric*** *if, and only if, y is related to x whenever x is related to y:*

$$\forall x, y \in A \ (xRy \Rightarrow yRx).$$

The relation is ***antisymmetric*** *if, and only if, y is never related to x whenever x is related to y, except possibly for when $x = y$:*

$$\forall x, y \in A \ \big((xRy \wedge yRx) \Rightarrow x = y\big).$$

Thus, for example, the relations $<$ and $\leq$ are both antisymmetric, while the relation $=$ is symmetric (as well as anytisymmetric).

Exercise 7.12 (Solution on page 438)

Is the relation *Before* from Exercise 7.5 symmetric, antisymmetric, or neither? What about the relation *FirstBefore*?

7.4.3 Transitive Relations

If one number is less than a second number which is itself less than a third number, then clearly the first number will also be less than the third number. This property of the *less-than* relation is embodied in the final property of interest.

Definition 7.12

A relation R on a set A is ***transitive*** *if, and only if, x is related to z whenever x is related to some y which is related to z:*

$$\forall x, y, z \in A \ \big((xRy \wedge yRz) \Rightarrow xRz\big).$$

Thus, for example, the relations $<$ and $\leq$ are both transitive.

Exercise 7.13 (Solution on page 438)

Is the relation *Before* from Exercise 7.5 transitive? What about the relation *FirstBefore*?

Example 7.13

Consider the sibling (brother or sister) relationship over people.

1. This is not reflexive, as you would not consider someone to be their own sibling. It is in fact irreflexive.
2. It is symmetric as anyone is obviously a sibling to each of their siblings. Clearly it is not antisymmetric.
3. Finally, it is not transitive, as this would imply that any person who has a sibling must be a sibling of themselves. Also, if we allow half-siblings, one person may be a sibling to a second person due to sharing a common father whilst having different mothers; and the second person may be a sibling to yet a third person due to sharing a common mother whilst having different fathers. In this scenario, the first and third children would not be siblings, as they do not share a common parent.

Exercise 7.14 (Solution on page 439)

Consider the relations *is-an-ancestor-of* and *is-married-to* defined over people. Indicate whether these are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive. Justify your answers.

7.4.4 Orderings Relations

Various common binary relations arrange the elements of their domain into some specific ordering. For example the *less-than-or-equal-to* relation $\leq$ orders the natural numbers into an increasing sequence: $0 \leq 1 \leq 2 \leq 3 \leq \cdots$. Note that this ordering is total in the sense that any two numbers a and b are related in one way or the other: either $a \leq b$ or $b \leq a$.

Whether or not a particular binary relation defined on a set orders the elements of that set depends on whether or not it satisfies certain of the properties defined above. Naturally, a *less-than-or-equal-to* relation should be:

- reflexive – any element should be *less-than-or-equal-to* itself;
- antisymmetric – if a is *less-than-or-equal-to* b and b is also *less-than-or-equal-to* a, then a and b should be equal.
- transitive – if a is *less-than-or-equal-to* b and b is *less-than-or-equal-to* c, then a should be *less-than-or-equal-to* c.

In fact, these three properties taken together indicate that a relation is an ordering relation as defined as follows.

Definition 7.15

A binary relation R on a set is a ***partial order*** *if, and only if, it is reflexive, antisymmetric, and transitive. It is a* ***total order*** *if, and only if, it is a partial order in which any two elements are related in one way or the other:*

$$\forall\, x, y \in A\, (xRy \vee yRx).$$

Example 7.15

- The *equality* relation $=$ on integers is a partial order, but it is not a total order.
- The *less-than-or-equal-to* relation $\leq$ on integers is a total order. However, the *less-than* relation $<$ on integers is not a (total or partial) order, as it is not reflexive.
- The *subset* relation $\subseteq$ on sets is a partial order but not a total order; for example, $\{\,1\,\} \nsubseteq \{\,2\,\}$ and $\{\,2\,\} \nsubseteq \{\,1\,\}$.

7.4.5 Equivalence Relations

A binary relation on a set may reflect a notion of *sameness* between elements of that set, defining when we might want to consider two elements of the set to be indistinguishable – that they are in some sense *equivalent*.

As with orderings, whether or not a particular relation over a set defines an *equivalence* between elements of that set depends on whether or not it satisfies certain of the properties defined above. Naturally, such a relation should be:

- reflexive – any element should be *the same as* itself;
- symmetric – if a is *the same as* b then b should be *the same as* a;
- transitive – if a is *the same as* b and b is *the same as* c, then a should be *the same as* c.

These three properties suffice to define a notion of *sameness*.

Definition 7.16

*A binary relation R on a set is an **equivalence relation** if, and only if, it is reflexive, symmetric, and transitive.*

Example 7.17

- The *equality* relation $=$ on integers is an equivalence.
- The *less-than-or-equal-to* relation $\leq$ on integers is not an equivalence relation, as it is not symmetric. Furthermore, the *less-than* relation $<$ on integers fails to be an equivalence relation for this same reason, as well as for not being reflexive.
- The *subset* relation $\subseteq$ on sets is not an equivalence relation, as it is not symmetric.

Example 7.18

Consider splitting up a set A of people into twelve groups depending on the month of their birthday; for example, one of the groups might consist of all those people in A whose birthday is in September. (There may actually be fewer than twelve groups, if there are months in which no one in A was born.) This naturally defines an equivalence relation R on A in which two people are related if, and only if, their birthdays are in the same month:

$$R = \{ (x, y) : x \text{ and } y \text{ have birthdays in the same month} \}.$$

Clearly this relation is reflexive, symmetric and transitive.

Exercise 7.18 (Solution on page 439)

Which of the following binary relations on $\mathbb{N}$ are partial orders? Which are total orders? Which are equivalences? Explain your answers.

1. The identity relation $I = \{ (n, n) : n \in \mathbb{N} \}$.
2. The universal relation $U = \{ (m, n) : m, n \in \mathbb{N} \}$.
3. The parity relation $P = \{ (m, n) : m = n \, (mod\, 2) \}$.

Exercise 7.19 (Solution on page 439)

Consider a set S of students who are each taking some number of courses chosen from a set C of courses. Define the following binary relations on S:

$$R_1 = \{ (s_1, s_2) \ : \ s_1 \text{ and } s_2 \text{ take all the same courses} \}.$$
$$R_2 = \{ (s_1, s_2) \ : \ s_1 \text{ and } s_2 \text{ take } \textit{some} \text{ course together} \}.$$

Are either of these an equivalence relation? Justify your answer.

7.4.6 Equivalence Classes and Partitions

Consider the equivalence relation R from Example 7.18 defined over some set A of people:

$$R = \{ (x, y) \ : \ x \text{ and } y \text{ have birthdays in the same month} \}.$$

We based this equivalence relation on a *partitioning* of the set A into disjoint sets. This idea is formalised in the following.

Definition 7.20

A ***partition*** *of a set A is a collection $\{ A_i \ : \ i \in I \}$ of disjoint non-empty subsets of A which together contain all of A. That is:*

1. $A_i \cap A_j = \emptyset$ *whenever* $i \neq j$*; and*
2. $\bigcup_{i \in I} A_i = A$.

The subsets A_i are called the ***blocks*** *of the partition. We say that one partition is a* refinement *of a second partition if, and only if, every block of the first is a subset of some block of the second.*

Example 7.20

We can refine the relation R from Example 7.18 by splitting the people of A not just according to the month of their birth, but according to sex as well, thus creating (up to) 24 groups; for example, one of the groups might consist of all females in A whose birthday is in September. This new partition of A is clearly a refinement of the original coarser partition defined only by birth month.

Exercise 7.21 (Solution on page 439)

What is the finest partition of a set A, in the sense that it cannot be refined into a different partition? What is the coarsest (i.e., least fine) partition?

Any partition of a set A naturally defines an equivalence relation, in just the way the partition of Example 7.18 gave rise to the equivalence relation R; two elements of A will be deemed equivalent if, and only if, they appear in the same block of the partition. Just as clearly, any equivalence relation partitions the elements over which it is defined into disjoint non-empty subsets, called *equivalence classes*.

Definition 7.21

Given an equivalence relation R on a set A, the ***equivalence class*** *of an element a of A with respect to R, denoted $[a]_R$, is the set of elements of A which are related to a by R:*

$$[a]_R = \{ x \in A : aRx \}.$$

Theorem 7.22

The collection of equivalence classes $\{ [a]_R : a \in A \}$ of an equivalence relation R is a partition of A.

Proof: To prove this we need to show the following:

1. Each $[a]_R$ is non-empty.

   This is true since $a \in [a]_R$.

2. The union of the equivalence classes is A.

   This is true since each $a \in A$ is in the equivalence class $[a]_R$.

3. The equivalence classes are disjoint; in other words, two non-disjoint equivalence classes must be equal.

   To see this, let us assume that $[a]_R$ and $[b]_R$ are not disjoint, that they contain a common element x; that is, aRx and bRx, which by symmetry means also that xRa, and thus by transitivity that bRa. Then

   $$\begin{aligned} y \in [a]_R &\Leftrightarrow aRy \\ &\Leftrightarrow bRy \ \text{(by transitivity, since } bRa \text{ and } aRy) \\ &\Leftrightarrow y \in [b]_R. \end{aligned}$$

   Thus we must have that $[a]_R = [b]_R$. □

Exercise 7.23 (Solution on page 440)

What are the equivalence relations defined by the finest and coarsest partitions of a set A identified in Exercise 7.21?

Exercise 7.24 (Solution on page 440)

Let the relation R on the set $A = \{ 1, 2, 3, \ldots, 29 \}$ of positive integers less than 30 be defined by:

$(x, y) \in R$ if, and only if, x and y have the same prime factors.

For example, $(12, 18) \in R$ since $12 = 2\times2\times3$ and $18 = 2\times3\times3$ have the same prime factors 2 and 3. Clearly this is an equivalence relation.

How many equivalence classes does R partition A into? List each of these equivalence classes.

7.5 Additional Exercises

1. Consider the following family members of Don Vito Corleone and his wife Carmella have four children: Santino, Federico, Michael and Constanzia. Santino is married to Sandra and they have four children: Santino Jr, Francesca, Kathryn and Frank. Michael is married to Kay and they have two children: Anthony and Mary. Constanzia is married to Carlo and they have two children: Victor and Michael Francis. Federico is not married and has no children.

   (a) List out the set CORLEONES of all persons mentioned above.

   (b) List out the relations *Father*, *Mother*, *Husband* and *Sibling*.

   (c) Define the relation *Father* in terms of *Mother* and *Husband*.

   (d) Define the relations *Parent*, *Wife* and *Spouse* in terms of the above relations, and list these out.

   (e) Define the relations *Father-In-Law*, *Mother-In-Law* and *Cousin* in terms of the above relations, and list these out.

2. Indicate which of the following relations defined over the integers $\mathbb{Z}$ are reflexive, which are irreflexive, which are symmetric, which are antisymmetric, and which are transitive. Justify your answers.

   (a) $R_1 = \{(a, b) : a = b \text{ or } a = -b\}$.

   (b) $R_2 = \{(a, b) : a = b-1\}$.

   (c) $R_3 = \{(a, b) : a+b \leq 10\}$.

   (d) $R_4 = \{(a, b) : a < 2b\}$.

3. Indicate which of the following relations defined over the positive integers are reflexive, which are irreflexive, which are symmetric, which are antisymmetric, and which are transitive. Justify your answers.

   (a) The *divisibility* relation $a \mid b$ which holds if, and only if, a divides evenly into b.

   (b) The *relatively prime* relation which holds between a and b if, and only if, their greatest common divisor is 1.

(c) The relation which holds between a and b if, and only if, their difference (i.e., the larger minus the smaller) is divisibly by 3.

4. What does a symmetric and transitive relation look like? Is it true that any binary relation which is symmetric and transitive must also be reflexive? Justify your answer.

5. Suppose R and S are symmetric relations on a set A. Which of the following must be a symmetric relation? Justify your answers.

   (a) $R \cup S$. (b) $R \cap S$. (c) $R \circ S$. (d) $\overline{R}$. (e) R^{-1}

6. Suppose R and S are transitive relations on a set A. Which of the following must be a transitive relation? Justify your answers.

   (a) $R \cup S$. (b) $R \cap S$. (c) $R \circ S$. (d) $\overline{R}$. (e) R^{-1}

7. Match the property of the binary relation R on A listed on the left to a characterisation of that property on the right:

1. reflexive	(a) $R \circ R \subseteq R$
2. irreflexive	(b) $\mathsf{id}_A \cap R = \emptyset$
3. symmetric	(c) $R = R^{-1}$
4. antisymmetric	(d) $\mathsf{id}_A \subseteq R$
5. transitive	(e) $R \cap R^{-1} \subseteq \mathsf{id}_A$

8. The ***reflexive closure*** of a relation R over a set A is the smallest reflexive relation that contains R. Similarly, the ***symmetric closure*** of a relation R over a set A is the smallest symmetric relation that contains R, and the ***transitive closure*** of a relation R over a set A is the smallest transitive relation that contains R.

   Compute the reflexive, symmetric and transitive closures of the binary relation $R = \{(0,1), (1,2), (3,4), (4,3)\}$ over the set $A = \{0,1,2,3,4\}$.

9. Prove that $R \cup \{ (a,a) : a \in R\}$ is the reflexive closure of R.

10. Prove that $R \cup R^{-1}$ is the symmetric closure of R.

11. Prove that

$$\{ (a_1, a_n) : \exists a_2, a_3, \ldots, a_{n-1} \text{ such that } (a_i, a_{i+1}) \in R \\ \text{for each } i = 1, 2, \ldots, n-1 \}$$

    is the transitive closure of R.

12. Let us say that two real numbers x and y are approximately equal, and write $x \approx y$, if, and only if, they differ by no more than 1/1000. Thus, the relation $\approx$ on $\mathbb{R}$ is defined as follows:

$$\approx \;=\; \{(x,y) : |x-y| < 1/1000\}.$$

Intuitively this ought to be an equivalence relation. Explain why this relation is – or is not – reflexive, symmetric and transitive.

13. Consider the relation $\leq$ defined on a Boolean algebra B as follows: for all $x, y \in B$, $x \leq y$ if, and only if, $x + y = y$.

    (a) Prove that $\leq$ is a partial order.
    (b) What does $\leq$ correspond to in the Boolean algebra of sets?
    (c) What does $\leq$ correspond to in the Boolean algebra of propositions?

14. Assuming that R is an equivalence relation on A, show directly from the definitions that the following statements about two elements a and b of A are equivalent:

    (a) aRb (b) $[a]_R = [b]_R$ (c) $[a]_R \cap [b]_R \neq \emptyset$