

Chapter 2

Sets

I refuse to join any club that would have me as a member.

- Groucho Marx.

Propositional logic allows us to reason about the world by inferring new facts from facts that we already know. However, we also need to structure our knowledge by grouping things together and by relating such collections of things with each other. In the parlance of Computer Science, we don't only need algorithms that process information, but also data structures that collect and store it.

There are many words in English for describing a collection of things (especially animals) such as: a *pack* (of wolves), a *school* (of fish), a *gaggle* (of geese), a *host* (of angels), a *den* (of thieves), a *crowd* (of onlookers), or a *fleet* (of cars). The idea of regarding a collection of things as a single entity is fundamental in mathematics as well as in everyday parlance. However, mathematics usually restricts itself to using a single collective noun: *set*.

2.1

Set Notation

A *set* is a collection of objects which typically share a property. The objects belonging to the collection are individually referred to as its *elements*, or *members*. The number of objects in a set A is referred to as its *cardinality*, and is written $|A|$. If there are not too many elements in the set, then it is most typically described by writing its elements in a comma-separated list between curly braces, as in the following four examples of sets:

- { false, true };
- { 3, 7, 14 };
- { red, blue, yellow };
- { Joel, Felix, Oskar, Amanda }.
- { Aberystwyth, Bangor, Cardiff, Lampeter, Newport, Swansea };

The above sets all contain a small number of elements – their cardinalities are 2, 3, 3, 4 and 6, respectively – and as such are easily written out. Larger sets which aren't so easily written out explicitly are often informally described using an *ellipsis* "...", as in the following three examples:

- $\{1, 3, 5, \dots, 99\}$ (*the set of 50 odd positive integers below 100*);
- $\{a, b, c, \dots, z\}$ (*the set of 26 letters of the alphabet*);
- $\{2, 3, 5, 7, 11, 13, 17, \dots\}$ (*the infinite set of prime numbers*).

Though we shall freely use this notation, it is generally inadequate. For example, how confident are you that the final set above denotes the set of prime numbers? Having an infinite number of elements, it would be impossible to list them all inside curly braces, so we would have to stop somewhere. But perhaps the next element we have in mind in the sequence after 17 is 21. Perhaps it isn't even a number; perhaps the next element in the sequence is Groucho Marx!

To avoid any ambiguity, sets are typically describe not by explicitly listing the elements between curly braces, but rather by describing the property that the elements share. In general, we shall describe sets using the following *set-builder notation*:

$$\{x : x \text{ has property } P\}.$$

That is, this set consists of exactly those objects x which satisfy the property P . We may, of course, use a more appropriate variable than x .

Example 2.1

The following are all examples of sets:

1. The collection of all beaches on the Gower Peninsula:

$$\{b : b \text{ is a beach on the Gower Peninsula}\}.$$

2. The collection of all people who have climbed Mount Kailash:

$$\{p : p \text{ has climbed Mount Kailash}\}.$$

3. The collection of all prime numbers:

$$\{n : n \text{ is a prime number}\}.$$

4. The collection of all sets of people who have a common grandmother:

$$\{A : A \text{ is a set of people who share a common grandmother}\}.$$

The first set is finite, and its members can be explicitly listed by referring to a map of the Gower Peninsula. The second set – as far as we know – has no members. The third set has infinitely many members, and so could not be explicitly listed. The members of the fourth set are themselves sets.

You will likely be familiar with many standard mathematical sets such as the following.

$\emptyset = \{ \}$	(the empty set)
$\mathbb{B} = \{ 0, 1 \}$	(the binary digits, or bits)
$\mathbb{N} = \{ 0, 1, 2, 3, \dots \}$	(the natural numbers)
$\mathbb{Z} = \{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots \}$	(the integers)
$\mathbb{Q} = \{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \}$	(the rational numbers)
$\mathbb{R} = \{ x : x \text{ is a real number} \}$	(the real numbers)

Note that \emptyset and $\{ \emptyset \}$ are *different* sets; the set \emptyset contains *no* elements, while the set $\{ \emptyset \}$ contains *one* element, namely the set \emptyset itself, and hence is not the same as the empty set \emptyset .

Also note that each set in the above list is bigger than the one above it, in the sense that it includes all of the elements of the set above it plus other elements not in the set above.

Exercise 2.1 (Solution on page 416)

Write out the following sets explicitly, by listing their elements within curly braces.

- $\{ x : x \text{ is an odd integer with } 0 < x < 8 \}$.
- $\{ x : x \text{ is a day of the week not containing the letter n} \}$.
- $\{ x : x \text{ was a wife of Henry VIII} \}$.
- $\{ x : x \text{ starred as James Bond in the official series of films} \}$.

2.2 Membership, Equality and Inclusion

A set is defined solely by its members, so clearly the most basic question we can pose is to ask if an object x is a member of a set A . *Membership* is denoted by \in , pronounced “*is an element (or a member) of*”, as for example in

$$7 \in \{ 3, 7, 14 \} \text{ (“7 is an element of the set } \{ 3, 7, 14 \} \text{ ”),}$$

or

$$\text{Felix} \in \{\text{Joel, Felix, Oskar}\},$$

whilst non-membership is denoted by \notin , as for example in

$$8 \notin \{3, 7, 14\} \text{ ("8 is not an element of the set } \{3, 7, 14\} \text{ ")},$$

or

$$\text{Amanda} \notin \{\text{Joel, Felix, Oskar}\}.$$

That is, $x \notin A$ is the same as $\neg(x \in A)$.

Exercise 2.2 (Solution on page 416)

Write out the following sets explicitly, by listing their elements within curly braces.

1. $\{x : x \text{ is an integer with } x = 2y \text{ where } y \in \{1, 2, 3, 4, 5\}\}$.
2. $\{x : x \text{ is an integer with } 2x = y \text{ where } y \in \{1, 2, 3, 4, 5\}\}$.

Exercise 2.3 (Solution on page 416)

Which of the following propositions are true?

1. $2 \in \{1, 2, 3\}$.
2. $\{2\} \in \{1, 2, 3\}$.
3. $\{2\} \in \{\{1\}, \{2\}, \{3\}\}$.
4. $\emptyset \in \{\}$.
5. $\emptyset \in \{\emptyset\}$.

Since a set is defined solely by its members, two sets are *equal* if, and only if, they have the same elements. So when you list the elements of a set, the order in which you list them, and the number of times you list each element, doesn't matter. Thus, for example,

$$\{3, 7, 14\} = \{7, 14, 3, 7, 3\}$$

while

$$\{\text{Joel, Felix, Oskar}\} \neq \{\text{Joel, Felix, Oskar, Amanda}\}.$$

If you want to show that two sets are different, it suffices to find a witness to this fact; that is, an element of one set which is not in the other.

Exercise 2.4 (Solution on page 416)

Which of the following sets are equal?

- $A = \{1, \{1, 2\}\}$
- $B = \{1, \{2\}\}$
- $C = \{1, \{1\}\}$
- $D = \{\{1, 1\}, 1\}$
- $E = \{\{2, 1\}, 1\}$

One set A is a *subset* of another set B if, and only if, each element of A is an element of B ; in such a case we write $A \subseteq B$. We also say that A is *included*, or *contained*, in B ; or that B is a *superset* of A , written $B \supseteq A$; or that B *includes*, or *contains*, A . Reflecting on the description of equality of sets above, two sets A and B are thus equal, $A = B$, if, and only if, each is included in the other:

$$A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A;$$

that is, if any element of one is an element of the other.

As further notation, we write $A \not\subseteq B$ to denote that A is *not* a subset of B , that is, if there is an element of A which is *not* an element of B . In other words, $A \not\subseteq B$ is the same as $\neg(A \subseteq B)$. Finally, we write $A \subset B$ if A is a *proper* subset of B , that is, if $A \subseteq B$ but $A \neq B$.

Example 2.4

As already noted above, the binary digits form a proper subset of the natural numbers; the natural numbers form a proper subset of the integers; the integers form a proper subset of the rational numbers; and the rational numbers form a proper subset of the real numbers:

$$\emptyset \subset \mathbb{B} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

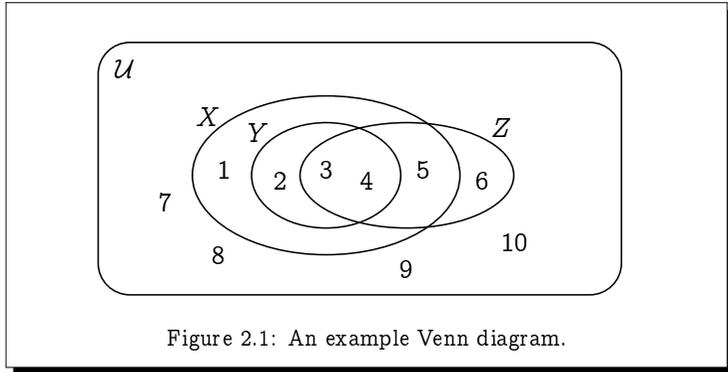
A useful graphical way of depicting sets, and in particular the relationship between them, is by so-called *Venn diagrams*. Such a diagram is obtained by laying out the elements of a set on a piece of paper and then encircling them. For example, we can depict the sets

$$X = \{1, 2, 3, 4, 5\}$$

$$Y = \{2, 3, 4\}$$

$$Z = \{3, 4, 5, 6\}$$

by the Venn diagram in Figure 2.1. The rectangle represents some understood universal set \mathcal{U} , referred to as the *universe of discourse* consisting of all elements under consideration, which in this example we take to be the integers from 1 to 10:



$$\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\},$$

and the sets X , Y and Z are represented by encircling the relevant elements, depicted in the Venn diagram in Figure 2.1. The diagram clearly shows that $Y \subseteq X$, and indeed $Y \subset X$, since $1 \in X$ but $1 \notin Y$; whereas Z is incomparable to both X and Y : $X \not\subseteq Z$ and $Z \not\subseteq X$; and $Y \not\subseteq Z$ and $Z \not\subseteq Y$.

Furthermore, it is clear that for any set A : $\emptyset \subseteq A$ and $A \subseteq \mathcal{U}$; and that for any sets A , B and C : if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Exercise 2.5 (Solution on page 416)

Which of the following propositions are true?

1. $\{2\} \subseteq \{1, 2, 3\}$.
2. $\{1, 2, 3\} \subseteq \{\{1\}, \{2\}, \{3\}\}$.
3. $\{\{1, 2\}\} \subseteq \{\{1, 2, 3\}\}$.

As a final observation, we can note the following special properties of the subset relation, all of which are obvious using Venn diagrams.

1. It is *reflexive*, meaning that $A \subseteq A$ holds for every set A .
2. It is *antisymmetric*, meaning that if $A \subseteq B$ and $B \subseteq A$ then $A = B$.
3. It is *transitive*, meaning that if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Moreover, the empty set is the least set with respect to inclusion; that is, it is contained in any other set: $\emptyset \subseteq A$ holds for each set A .

2.3 Sets and Properties

We have already seen that listing elements is not appropriate for defining sets with infinitely many elements. Instead of writing

$$\text{Primes} = \{2, 3, 5, 7, 11, 13, 17, 19, \dots\}$$

for the set of all prime numbers, we use the set-builder notation

$$\text{Primes} = \{x : x \text{ is a prime number}\}.$$

to define Primes as the set of all objects x such that x is a prime number. More generally, we use the notation

$$\{x : x \text{ has the property } P\}$$

to indicate that we are building (defining) the set of all objects x which satisfy the property P .

This set-builder notation is typically used to define a subset B of an existing set A , in which case we write:

$$B = \{x \in A : x \text{ has the property } P\}$$

instead of

$$B = \{x : x \in A \text{ and } x \text{ has the property } P\}$$

The set-builder notation used in this way *separates* the objects in set A which satisfy a given property from those that do not.

Example 2.5

Philosophers have classified humans as rational animals (albeit a reasonable rationality criterion might be to disagree with this classification). Accordingly, the property of being rational separates humans from all other animals; it holds of all humans, and of no other animals. Letting *Animals* denote the set of all animals and *Humans* the set of all humans, we can write

$$\text{Humans} = \{x \in \text{Animals} : x \text{ is rational}\}.$$

Thus, $x \in \text{Humans}$ if, and only if, $x \in \text{Animals}$ and x is rational.

Example 2.6

Given two real numbers $a, b \in \mathbb{R}$ the following four intervals frequently occur in mathematics:

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\};$$

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\};$$

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\};$$

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

Given two integers $m, n \in \mathbb{Z}$ the interval between them is defined as

$$[m..n] = \{k \in \mathbb{Z} : m \leq k \leq n\}.$$

In all of the above intervals, if the first (left-hand) value is greater than the second (right-hand) value, then the interval defined is the empty set \emptyset .

Example 2.7

Obviously, x is in the set

$$\{\text{Joel, Felix, Oskar, Amanda}\}$$

if, and only if,

$$x = \text{Joel} \text{ or } x = \text{Felix} \text{ or } x = \text{Oskar} \text{ or } x = \text{Amanda},$$

which is a property of x . Therefore, the above set can be rewritten, somewhat tediously, using set-builder notation as:

$$\{x : x = \text{Joel} \text{ or } x = \text{Felix} \text{ or } x = \text{Oskar} \text{ or } x = \text{Amanda}\}.$$

2.3.1 Russell's Paradox

The set-builder notation is very powerful; however, it must be used with some care.

We have seen that sets can contain any type of object, including sets themselves. Normally a set will not be a member of itself, but there is nothing to preclude us considering abnormal sets that *are* elements of themselves. Consider, then, the set of normal sets: those sets that are not elements of themselves; this set, which we call R , can be defined using the set-builder notation as follows:

$$R = \{A : A \notin A\}.$$

We can then ask: is R itself a normal set? That is, do we have $R \in R$? Or do we have $R \notin R$? Certainly one of these two must be true: either R is a normal set, or it isn't.

- Suppose that $R \in R$. Then R must satisfy the property required of being an element of R , namely we must have that $R \notin R$.

- Suppose that $R \notin R$. Then R must *fail* to satisfy the property required of being an element of R , namely we must *not* have that $R \notin R$; that is, we must have that $R \in R$.

By the Law of the Excluded Middle, one of the above two cases must hold. This means that we must have *both* $R \in R$ and $R \notin R$; that is, R is *both* a normal set and an abnormal set. This is a contradiction, and as such cannot be true.

This anomaly is known as *Russell's Paradox*, after the philosopher Bertrand Russell who devised it to demonstrate the need to be vigilant in how you define sets. In particular, it should not be possible to speak of the set of all sets, as such circularity leads directly to contradictions. Fortunately, this anomaly cannot arise as long as we restrict the use of the set-builder notation to the restricted form

$$\{x \in A : x \text{ has the property } P\}$$

in which we define the set as a subset of another given set which has been previously defined. We also need not worry about using the general set-builder notation if we have an implicit underlying universe of discourse.

Exercise 2.7 (Solution on page 416)

Let A be any set, and define the set R by

$$R = \{X \in A : X \notin X\}.$$

Do we now have $R \in R$? Or do we have $R \notin R$? Why is Russell's Paradox not a problem here?

2.4 Operations on Sets

In the previous sections we have seen that sets can be constructed directly by putting curly braces around a listing of its elements, or indirectly using the set-builder notation. In this section we will consider a variety of operations which can be used to construct new sets from old.

2.4.1 Union

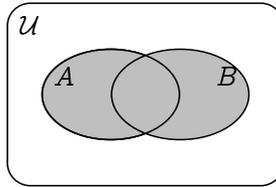
The *union* $A \cup B$ of two sets A and B consists of exactly those elements of the universe of discourse which are in either A or B (or both):

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Thus,

$$x \in A \cup B \Leftrightarrow x \in A \vee x \in B.$$

This is depicted by the following Venn diagram, where the gray area represents $A \cup B$.



Example 2.8

$$\{1, 2, 3, 4, 5\} \cup \{2, 4, 6, 8, 10\} = \{1, 2, 3, 4, 5, 6, 8, 10\}$$

Example 2.9

The union of the set of people who can speak English and the set of people living in France is the set of people who can either speak English or who live in France (or both).

2.4.2 Intersection

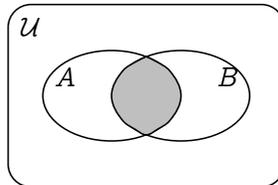
The *intersection* $A \cap B$ of two sets A and B consists of exactly those elements of the universe of discourse which are in both A and B :

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Thus,

$$x \in A \cap B \Leftrightarrow x \in A \wedge x \in B.$$

This is depicted by the following Venn diagram, where the gray area represents $A \cap B$.



Example 2.10

$$\{1, 2, 3, 4, 5\} \cap \{2, 4, 6, 8, 10\} = \{2, 4\}$$

Example 2.11

The intersection of the set of people who can speak English and the set of people living in France is the set of people living in France who can speak English.

Two sets A and B are said to be *disjoint* if they have no elements in common; that is to say, if their intersection is empty: $A \cap B = \emptyset$. In terms of Venn diagrams, this means that the regions depicting A and B do not overlap.

There will typically be fewer elements in the union of two finite sets A and B , $|A \cup B|$, than $|A| + |B|$; the whole will generally be less than the sum of the parts. This is due to the fact that $|A| + |B|$ counts the members of the intersection $A \cap B$ twice. To balance this, we have the the following principle.

Theorem 2.11 Inclusion-Exclusion Principle

For finite sets A , B and C : $|A \cup B| = |A| + |B| - |A \cap B|$.

2.4.3 Difference

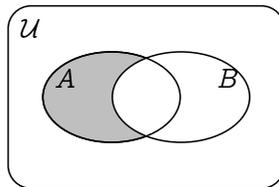
The *difference* $A \setminus B$ of two sets A and B consists of exactly those elements of the universe of discourse which are in A but not in B :

$$A \setminus B = \{x \in A : x \notin B\}.$$

Thus,

$$x \in A \setminus B \Leftrightarrow x \in A \wedge x \notin B.$$

This is depicted by the following Venn diagram, where the gray area represents $A \setminus B$.

**Example 2.12**

$$\{1, 2, 3, 4, 5\} \setminus \{2, 4, 6, 8, 10\} = \{1, 3, 5\},$$

and

$$\{2, 4, 6, 8, 10\} \setminus \{1, 2, 3, 4, 5\} = \{6, 8, 10\}.$$

Example 2.13

The difference of the set of people who can speak English and the set of people living in France is the set of English-speaking people who do not live in France.

Conversely, the difference of the set of people living in France and the set of people who can speak English is the set of non-English-speaking people living in France.

2.4.4 Complement

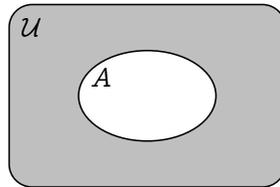
The *complement* \bar{A} of a set A is the set consisting of exactly those elements of the universe of discourse which are *not* elements of A :

$$\bar{A} = \{x : x \notin A\}.$$

Thus,

$$x \in \bar{A} \Leftrightarrow x \notin A.$$

The set \bar{A} is thus the same as $\mathcal{U} \setminus A$, and is depicted by the following Venn diagram, where the gray area represents \bar{A} .

**Example 2.14**

Assuming the universe of discourse is $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,

$$\overline{\{1, 2, 3, 4, 5\}} = \{6, 7, 8, 9, 10\},$$

and

$$\overline{\{2, 4, 6, 8, 10\}} = \{1, 3, 5, 7, 9\}.$$

Example 2.15

Assuming the universe of discourse is the set of people in the world, the complement of the set of people who can speak English is the set of non-English-speaking people; and the complement of the set of people living in France is the set of people who do not live in France.

Exercise 2.15 (Solution on page 416)

Consider the following sets:

$$\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, \quad (\text{the universe of discourse})$$

$$A = \{1, 3, 5, 7, 9\},$$

$$B = \{3, 4, 5\},$$

$$C = \{5, 6, 7, 8, 9\}.$$

Draw a Venn diagram depicting these sets, and compute the following sets:

1. $A \cap C$.
2. $(A \cap B) \cup C$
3. $A \cap (B \cup C)$
4. $(A \cup B) \setminus C$.
5. $\overline{(A \cup B)} \cap C$.

Exercise 2.16 (Solution on page 417)

Let A , B and C be sets.

1. If $A \subseteq B$, what can you say about $A \cup B$ and $A \cap B$?
2. If $A \subseteq B$, what can you say about \overline{A} and \overline{B} ?
3. What is $\overline{\overline{A}}$, the complement of the complement of A ?
4. If $C \subseteq A$ and $C \subseteq B$, how is C related to $A \cap B$?
5. If $A \subseteq C$ and $B \subseteq C$, how is C related to $A \cup B$?

2.4.5 Powerset

The *powerset* $\mathcal{P}(A)$ of a set A is the set consisting of all subsets of A :

$$\mathcal{P}(A) = \{X : X \subseteq A\}.$$

Thus,

$$x \in \mathcal{P}(A) \Leftrightarrow x \subseteq A.$$

In particular, $\emptyset \in \mathcal{P}(A)$ and $A \in \mathcal{P}(A)$.

We might only be interested in *finite* subsets. In this case we shall denote by $\mathcal{P}_{\text{fin}}(A)$ the set consisting of all *finite* subsets of A :

$$\mathcal{P}_{\text{fin}}(A) = \{X : X \subseteq A \text{ and } X \text{ is finite}\}.$$

Example 2.16

1. The set $\{0, 1\}$ has four subsets:

$$\mathcal{P}(\{0, 1\}) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}.$$

More specifically, there are the following subsets:

- one subset with no elements (the empty set);
- two singleton subsets (one for each element in the set); and
- one subset with two elements (the whole set itself).

2. The set $\{\text{cola}, \text{fanta}, \text{sprite}\}$ has eight subsets:

$$\begin{aligned} \mathcal{P}(\{\text{cola}, \text{fanta}, \text{sprite}\}) \\ = \{ \emptyset, \\ \quad \{\text{cola}\}, \{\text{fanta}\}, \{\text{sprite}\}, \\ \quad \{\text{cola}, \text{fanta}\}, \{\text{cola}, \text{sprite}\}, \{\text{fanta}, \text{sprite}\}, \\ \quad \{\text{cola}, \text{fanta}, \text{sprite}\} \}. \end{aligned}$$

More specifically, there are the following subsets:

- one subset with no elements (the empty set);
- three singleton subsets (one for each element in the set);
- three subsets with two elements (one for each element left out);
and
- one set with three elements (the whole set itself).

3. The set $\{\text{Joel}, \text{Felix}, \text{Oskar}, \text{Amanda}\}$ has 16 subsets:

$$\begin{aligned} \mathcal{P}(\{\text{Joel}, \text{Felix}, \text{Oskar}, \text{Amanda}\}) \\ = \{ \emptyset, \\ \quad \{\text{Joel}\}, \{\text{Felix}\}, \{\text{Oskar}\}, \{\text{Amanda}\}, \\ \quad \{\text{Joel}, \text{Felix}\}, \{\text{Joel}, \text{Oskar}\}, \{\text{Joel}, \text{Amanda}\}, \\ \quad \{\text{Felix}, \text{Oskar}\}, \{\text{Felix}, \text{Amanda}\}, \{\text{Oskar}, \text{Amanda}\}, \\ \quad \{\text{Joel}, \text{Felix}, \text{Oskar}\}, \{\text{Joel}, \text{Felix}, \text{Amanda}\}, \\ \quad \{\text{Joel}, \text{Oskar}, \text{Amanda}\}, \{\text{Felix}, \text{Oskar}, \text{Amanda}\}, \\ \quad \{\text{Joel}, \text{Felix}, \text{Oskar}, \text{Amanda}\} \}. \end{aligned}$$

More specifically, there are the following subsets:

- one subset with no elements (the empty set);
- four singleton subsets (one for each element in the set);
- six subsets with two elements (one for each pair);

- four subsets with three elements (one for each element left out);
and
 - one set with four elements (the whole set itself).
4. In general, if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$: a set with n elements has 2^n different subsets.

Example 2.17

Amanda has invited the following six friends to her birthday party: Daniel, Ella, Mia, Rhodri and Zoe. However, some of them might not show up. If we let

$$\text{Friends} = \{\text{Daniel, Ella, Mia, Rhodri, Zoe}\}$$

then the collection of combinations of friends that might come to Amanda's party is given by $\mathcal{P}(\text{Friends})$. For example, perhaps Ella and Rhodri are busy that day, but the others all come; then the set of friends that come to Amanda's party is:

$$\{\text{Daniel, Mia, Zoe}\} \in \mathcal{P}(\text{Friends}).$$

Exercise 2.17 (Solution on page 417)

List the elements of $\mathcal{P}(\text{Friends})$, where Friends is the set defined in Example 2.17 above. How many sets of each size are there?

Exercise 2.18 (Solution on page 418)

Form the following sets from the empty set \emptyset :

1. the set $A = \mathcal{P}(\emptyset)$;
2. the set $B = \mathcal{P}(A)$;
3. the set $C = \mathcal{P}(B)$.

How many elements are in each of these sets?

Exercise 2.19 (Solution on page 418)

Given an arbitrary set A , what are $\mathcal{P}(A) \cap \emptyset$ and $\mathcal{P}(A) \cap \{\emptyset\}$?

★ 2.4.6 Generalised Union and Intersection

It makes perfect sense to take the union or intersection of any number of sets, not just two. For example, we can consider the union

$$A \cup B \cup C$$

of three sets A , B and C , meaning the set whose elements are those objects which are members of *any* of the sets A , B or C ; or the intersection

$$A \cap B \cap C \cap D \cap E$$

of five sets A , B , C , D and E , meaning the set whose elements are those objects which are members of *all* of the sets A , B , C , D and E . We don't have to worry about which order we take the sets; for example, the set $A \cup (B \cup C)$ is clearly the same as $(C \cup A) \cup B$. This is because the union and intersection operations are associative:

$$A \cup (B \cup C) = (A \cup B) \cup C \text{ and } A \cap (B \cap C) = (A \cap B) \cap C;$$

and commutative:

$$A \cup B = B \cup A \text{ and } A \cap B = B \cap A.$$

In fact, we can extend union and intersection to apply to arbitrary families (sets) of sets: if \mathcal{F} is a set of sets, then

$$\bigcup \mathcal{F} = \{x : x \in A \text{ for some } A \in \mathcal{F}\}$$

$$\bigcap \mathcal{F} = \{x : x \in A \text{ for all } A \in \mathcal{F}\}$$

In particular, $A \cup B = \bigcup \{A, B\}$ and $A \cap B = \bigcap \{A, B\}$. With a little thought, the following identities become apparent:

1. $A = \bigcup \{A\}$ and $A = \bigcap \{A\}$.
2. $A = \bigcup \mathcal{P}(A)$ and $\emptyset = \bigcap \mathcal{P}(A)$.
3. $\emptyset = \bigcup \emptyset$ and $\mathcal{U} = \bigcap \emptyset$, where \mathcal{U} is the universe of discourse.

The final two identities are worth further explanation. By definition, $x \in \bigcup \emptyset$ if, and only if, $x \in A$ for some $A \in \emptyset$; but since there can be no such $A \in \emptyset$, there can be no such $x \in A$.

Similarly, by definition, $x \in \bigcap \emptyset$ if, and only if, $x \in A$ for all $A \in \emptyset$; but since there can be no such $A \in \emptyset$, it is vacuously true that $x \in A$ for each of these $A \in \emptyset$.

Example 2.19

Suppose, e.g., that CS101 is the set of all students enrolled on the course *Computer Science 101*, and that ClassLists is the set of all class lists, so that, for example, $\text{CS101} \in \text{ClassLists}$. Then the set Students of all students, that is, all people who are enrolled on some course, would be

$$\text{Students} = \bigcup \text{ClassLists}.$$

The set $\bigcap \text{ClassLists}$ would *likely* be empty, as it would contain those students who are enrolled on *all* courses.

Exercise 2.20 (Solution on page 418)

Given an arbitrary set A , what are $\bigcap \mathcal{P}_{\text{fin}}(A)$ and $\bigcup \mathcal{P}_{\text{fin}}(A)$?

2.5 Ordered Pairs and Cartesian Products

An *ordered pair* is simply a pair of objects (a, b) with *first coordinate* a and *second coordinate* b . For example, points in the xy -plane are denoted by ordered pairs; the ordered pair $(4, 9)$, for example, denotes the point with x -coordinate 4 and y -coordinate 9. The ordered pair (a, b) is different from the set $\{a, b\}$ in that it is ordered; $(a, b) \neq (b, a)$ (unless, of course, $a=b$), whereas $\{a, b\} = \{b, a\}$. More precisely,

$$(a, b) = (c, d) \text{ if, and only if, } a = c \text{ and } b = d.$$

The *Cartesian product* $A \times B$ of two sets A and B is the set of all ordered pairs in which the first coordinate a is an element of A and the second coordinate b is an element of B .

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Thus,

$$(a, b) \in A \times B \Leftrightarrow a \in A \wedge b \in B.$$

For example, $\mathbb{R} \times \mathbb{R}$, typically written as \mathbb{R}^2 , denotes the set of points in the xy -plane.

Example 2.20

The Cartesian product $[1..m] \times [1..n]$ of the intervals $[1..m]$ and $[1..n]$ can model a finite grid, such as the points of an LCD screen or the squares on a chess board.

$$\begin{array}{cccc} (1, 1) & (1, 2) & \cdots & (1, n) \\ (2, 1) & (2, 2) & \cdots & (1, n) \\ \vdots & \vdots & \ddots & \vdots \\ (m, 1) & (m, 2) & \cdots & (m, n) \end{array}$$

Example 2.21

Many programming languages offer abstract data types that allow you to store and retrieve data using *key-value pairs*. These data types have different names in different programming languages, such as associative array, dictionary, map, or table. But key-value pairs are always ordered pairs from a Cartesian product $\text{Keys} \times \text{Values}$, where Keys is a set of keys and Values is a set of values. The values are the pieces of information that are stored in the data type, and the keys – which are unique for each value – allow you to retrieve the value.

As an example, we may have a national database in which each person is assigned a unique identification number. In this case, names serve as keys and the values are the identification numbers associated with each name:

$$\begin{aligned} \text{IDNumbers} = \{ & (\text{Joel}, 7613), \\ & (\text{Felix}, 8217), \\ & (\text{Oskar}, 6457), \\ & (\text{Amanda}, 9601), \\ & \dots \end{aligned}$$

As another example, a correspondence can be made between countries and their capital cities:

$$\begin{aligned} \text{CapitalCities} = \{ & (\text{France}, \text{Paris}), \\ & (\text{Peru}, \text{Lima}), \\ & (\text{Japan}, \text{Tokyo}), \\ & (\text{Mali}, \text{Bamako}), \\ & \dots \end{aligned}$$

We can form the Cartesian product of *any* number $n \in \mathbb{N}$ of sets, whose elements are n -tuples. For example

$$A \times B \times C = \{(a, b, c) : a \in A, b \in B \text{ and } c \in C\}$$

represents the set of triples (a, b, c) in which the first coordinate a is an element of A , the second coordinate b is an element of B , and the third coordinate c is an element of C . In general, we write A^n to denote $A \times A \times \dots \times A$, that is, the Cartesian product of n copies of the set A . Three-dimensional space, thus, is defined by $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Note that the number of elements in a product is the product of the number of elements in the individual sets. In particular, for any set A ,

$$A \times \emptyset = \emptyset = \emptyset \times A.$$

Example 2.22

Let S represent all students, C represent all courses, and G represent possible grades. Then $S \times C \times G$ represents all triples (s, c, g) where $s \in S$ is a student, $c \in C$ is a course and $g \in G$ is a grade. A University student database would be represented as a subset of this set, recording the grades for all students registered in each course.

Example 2.23

A *pixel* is a point on a computer screen, and these are laid out in a rectangular grid $[1..h] \times [1..v]$ as in Example 2.20, with the number of pixels dependent on the size and resolution of the screen.

Each pixel is displayed as a dot of a certain colour. In the *RGB model*, a colour is specified by a triple

$$(r, g, b) \in [0, 1]^3 \text{ where } [0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$$

representing an intensity of *red*, *green* and *blue*, respectively, with 0 being no intensity at all and 1 being maximum intensity. For example, black is represented by $(0, 0, 0)$ (no colours) while white is represented by $(1, 1, 1)$ (maximum intensity of all colours); and *red*, *green* and *blue* are obviously represented by $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, respectively. We can thus define the following two sets:

$$\text{Pixel} = [1..h] \times [1..v] \text{ and}$$

$$\text{Colour} = [0, 1]^3,$$

and use them to define a point on the screen as a member of the set

$$\text{Point} = \text{Pixel} \times \text{Colour}$$

which assigns a colour to a pixel. Each point is therefore represented by an ordered pair $((x, y), (r, g, b))$ whose first coordinate is the ordered pair (x, y) , and whose second coordinate is the ordered triple (r, g, b) .

Exercise 2.23 (Solution on page 418)

Every rational number can be represented as an ordered pair of integers. The number $3/4$, for example, corresponds to the ordered pair $(3, 4)$. Define the operations of addition and multiplication on ordered pairs of integers such that they correspond to the standard operations on fractions.

2.6 Modelling with Sets

As the fundamental data structures of mathematics, sets inevitably occur in the specifications of systems. In many cases, sets capture system properties more concisely than propositional logic. In this section, we explore a number of examples, starting with revisiting Amos Judd.

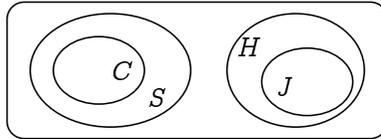
Example 2.24

Consider the following three assumptions:

1. All candy has sugar.
2. John eats only healthy foods.
3. No healthy food contains sugar.

We can reason about these assumptions by introducing the sets H , S , J and C to represent, respectively, the set of healthy foods, the set of sugary foods, the set of foods that John eats, and the set of candy. The above assumptions can be expressed, equationally and with a Venn diagram, as follows:

1. $C \subseteq S$
2. $J \subseteq H$
3. $S \cap H = \emptyset$



From this picture it is clear that no candy is healthy, and as such that John doesn't eat candy.

Exercise 2.24 (Solution on page 418)

Recall the situation regarding Amos Judd from Exercise 1.14 (page 33), in which the fact that Amos Judd loves cold mutton could be inferred from the following assumptions:

1. All the policemen on this beat sup with our cook.
2. No man with long hair can fail to be a poet.
3. Amos Judd has never been in prison.
4. Our cook's cousins all love cold mutton.
5. None but policemen on this beat are poets.
6. None but her cousins ever sup with our cook.
7. Men with short hair have all been in prison.

Demonstrate how to solve this problem by reasoning about appropriately-defined sets.

Exercise 2.25 (Solution on page 419)

Another one of Lewis Carroll's famous puzzles has the following premises:

All babies are illogical.

Nobody is despised who can manage a crocodile.

Illogical persons are despised.

Use an appropriate Venn diagram to deduce from these premises that no baby can manage a crocodile.

Exercise 2.26 (Solution on page 420)

Use an appropriate Venn diagram to determine whether or not the following argument is valid.

All oceans are full of water.

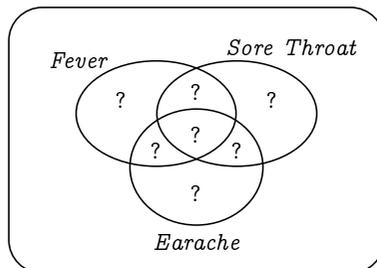
No ponds are oceans.

Therefore no ponds are full of water.

Example 2.26

At a certain hospital, 40 patients each have at least one of the following symptoms: a fever, a sore throat, or an earache. 18 of them have an earache and 25 of them have a sore throat, while eight of them have both an earache and a sore throat. Of the fever sufferers, 11 of them have sore throats, nine have earaches, and two have both a sore throat and an earache. How many fever sufferers are there?

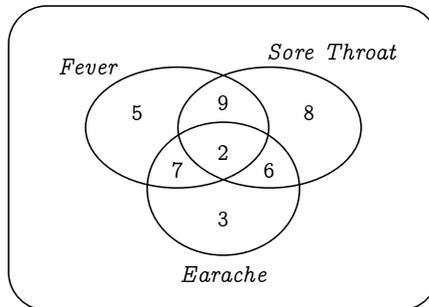
We can use a Venn diagram to solve this problem, by drawing the three sets of patients as follows:



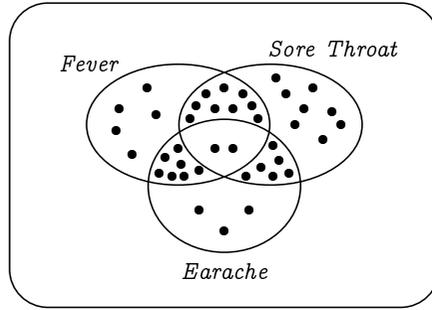
The question marks represent the numbers of patients in the relevant subsets, and these numbers must add up to 40. We merely need to replace these question marks with the relevant numbers based on the information given in the problem, which we can do by working from the inside out.

- We first put a 2 in the intersection of all three sets, depicting the two patients who are suffering from all three symptoms.
- Since eight patients are suffering from both a sore throat and an earache, six of these must not have fever, so we can put a 6 in the relevant place in the diagram.
- Next, there are 11 fever sufferers who have a sore throat; we know that two of these also have an earache, so nine of these must not have an earache, so we can thus put a 9 in the relevant place in the diagram.
- Also, there are nine fever sufferers who have an earache; we know that two of these also have a sore throat, so seven of these must not have a sore throat, so we can thus put a 7 in the relevant place in the diagram.
- There are 18 patients with earaches, 15 of which have other symptoms; thus three have no other symptoms, so we can put a 3 in the relevant place in the diagram.
- There are 25 patients with sore throats, 17 of which have other symptoms; thus eight have no other symptoms, so we can put an 8 in the relevant place in the diagram.
- As there are 40 patients in total, and 35 are accounted for as having either a sore throat or an earache, there are five patients who only suffer from fever, so we can put a 5 in the relevant place in the diagram.

The Venn diagram thus looks as follows:



Note that this is *not* a Venn diagram in the usual sense: the elements of the universe are not the numbers $\{2, 3, 5, 6, 7, 8, 9\}$. Rather, the 5, for example, represents five elements of the set of patients who are suffering only from fever. The Venn diagram would more rightly look something like the following:



Here, each dot represents a distinct patient. However, the original Venn diagram, with just the numbers, is far easier to read.

With this, a final simple count tells us that 23 patients suffer from fever.

★ 2.7 Algebraic Laws for Set Identities

We can often represent the same set in a variety of ways; for example, we've already noted that it doesn't matter whether we write $A \cup B$ or $B \cup A$ as these give the same set. In this section we list a variety of identities, which will allow us to reason algebraically about sets. All of the laws presented can be verified informally by considering the appropriate Venn diagrams.

Commutativity Laws

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associativity Laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Idempotence Laws

$$A \cup A = A$$

$$A \cap A = A$$

Distributivity Laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

De Morgan's Laws

$$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$$

$$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$$

Double Complement Law

$$\overline{\overline{A}} = A$$

Universe Laws

$$A \cup \mathcal{U} = \mathcal{U}$$

$$A \cap \mathcal{U} = A$$

Empty Set Laws

$$A \cup \emptyset = A$$

$$A \cap \emptyset = \emptyset$$

Complement Laws

$$A \cup \overline{A} = \mathcal{U}$$

$$A \cap \overline{A} = \emptyset$$

Absorption Laws

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

You can (and should!) convince yourself of all of the above identities by constructing appropriate Venn diagrams.

Exercise 2.27 (Solution on page 420)

Draw the Venn diagrams which justify the two Distributive Laws.

We can use the above identities to derive even more identities, bypassing the need to construct Venn diagrams to justify them.

Example 2.27

We can derive the identity $A \cup (\overline{A} \cap B) = A \cup B$ using the following sequence of steps:

$$\begin{aligned} A \cup (\overline{A} \cap B) &= (A \cup \overline{A}) \cap (A \cup B) && \text{(Distributivity)} \\ &= \mathcal{U} \cap (A \cup B) && \text{(Complement)} \\ &= (A \cup B) \cap \mathcal{U} && \text{(Commutativity)} \\ &= A \cup B && \text{(Universe)} \end{aligned}$$

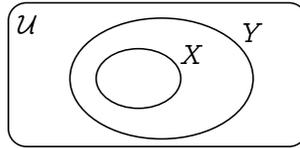
Exercise 2.28 (Solution on page 420)

Give a derivation of the identity $A \cap (\overline{A} \cup B) = A \cap B$.

The above laws allow us to reason about set inclusions as well as identities, by observing first that the set inclusion $X \subseteq Y$ can be expressed as a set identity, in any of the following ways:

$$X \cup Y = Y, \quad X \cap Y = X, \quad X \setminus Y = \emptyset, \quad \overline{X} \cup Y = \mathcal{U}.$$

That each of the above are equivalent to the proposition that $X \subseteq Y$ can be readily be checked by considering the appropriate Venn diagram:


Example 2.28

We can derive the new law $A \subseteq A \cup B$ as follows:

- By Associativity and Idempotence, $A \cup (A \cup B) = A \cup B$.
- Letting $X = A$ and $Y = A \cup B$, this says that $X \cup Y = Y$.
- By the above, this means that $X \subseteq Y$; that is, that $A \subseteq A \cup B$.

Exercise 2.29 (Solution on page 421)

Derive the law $A \cap B \subseteq A$.

★ 2.8 Logical Equivalences versus Set Identities

The astute reader will have noticed that there is a direct correspondence between the Equivalence Laws for Propositional Logic from Section 1.7 and the Set Identities from the previous Section 2.7. For convenience, these laws are listed once again here, side-by-side.

Commutativity Laws

$$P \vee Q \Leftrightarrow Q \vee P$$

$$P \wedge Q \Leftrightarrow Q \wedge P$$

Commutativity Laws

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associativity Laws

$$P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$$

$$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$$

Associativity Laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Idempotence Laws

$$P \vee P \Leftrightarrow P$$

Idempotence Laws

$$A \cup A = A$$

$$P \wedge P \Leftrightarrow P$$

$$A \cap A = A$$

Distributivity Laws

$$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$$

Distributivity Laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

De Morgan's Laws

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$$

$$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$$

De Morgan's Laws

$$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$$

$$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$$

Double Negation Law

$$\neg\neg P \Leftrightarrow P$$

Double Complement Law

$$\overline{\bar{A}} = A$$

Tautology Laws

$$P \vee \text{true} \Leftrightarrow \text{true}$$

$$P \wedge \text{true} \Leftrightarrow P$$

Universe Laws

$$A \cup \mathcal{U} = \mathcal{U}$$

$$A \cap \mathcal{U} = A$$

Contradiction Laws

$$P \vee \text{false} \Leftrightarrow P$$

$$P \wedge \text{false} \Leftrightarrow \text{false}$$

Empty Set Laws

$$A \cup \emptyset = A$$

$$A \cap \emptyset = \emptyset$$

Excluded Middle Laws

$$P \vee \neg P \Leftrightarrow \text{true}$$

$$P \wedge \neg P \Leftrightarrow \text{false}$$

Complement Laws

$$A \cup \bar{A} = \mathcal{U}$$

$$A \cap \bar{A} = \emptyset$$

Absorption Laws

$$P \vee (P \wedge Q) \Leftrightarrow P$$

$$P \wedge (P \vee Q) \Leftrightarrow P$$

Absorption Laws

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Each law of equivalence for propositions gives rise to a set identity by replacing \vee by \cup , \wedge by \cap , and \neg by $\bar{}$ (as well as false by \emptyset and true by \mathcal{U}). This exploits a tight analogy between logical equivalence $P \Leftrightarrow Q$ and equality of sets $A = B$, which can be extended to logical implication $P \Rightarrow Q$ and subset inclusion $A \subseteq B$ as in the following example.

Example 2.29

The *Implication Law* from Section 1.7:

$$P \Rightarrow Q \Leftrightarrow \neg P \vee Q$$

gives rise to the following property of sets:

$$A \subseteq B \text{ if, and only if, } \overline{A} \cup B = U.$$

This property is arrived at by translating $P \Rightarrow Q$ into $A \subseteq B$, and expressing $\neg P \vee Q$ as $\neg P \vee Q \Leftrightarrow \text{true}$ before translating it in the above fashion. (The equivalence symbol itself is translated merely into English.)

Exercise 2.30 (Solution on page 421)

Find properties of sets corresponding to the following laws for propositions taken from Section 1.7.

1. *Contrapositive Law*: $P \Rightarrow Q \Leftrightarrow \neg Q \Rightarrow \neg P$.
2. *Equivalence Law*: $P \Leftrightarrow Q \Leftrightarrow (P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

Although the analogy between propositions and sets is tight, care must be taken when trying to use it. You should always check the validity of a property of sets which is so derived, for example by considering the relevant Venn diagrams.

Exercise 2.31 (Solution on page 421)

What property of sets is suggested by the following law for propositions:

$$\neg(P \Rightarrow Q) \Leftrightarrow P \wedge \neg Q$$

If you do this exercise carefully, you may well arrive at a property which is generally *not* true of sets. This exercise thus serves to point out that it is dangerous to rely on informal, intuitively-correct arguments.

2.9 Additional Exercises

1. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8, 9\}$, and $C = \{2, 4, 6, 8\}$.

What are the following sets?

- (a) $A \cup B \cup C$.
- (b) $A \cap B \cap C$.

- (c) $(A \cap B) \cup C$.
 (d) $A \cap (B \cup C)$.
2. What sets are defined by $\{x : x \neq x\}$ and $\{x \in A : x = x\}$?
3. Draw Venn diagrams to justify the two De Morgan Laws $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
4. Draw the Venn diagrams which justify the following laws.
- (a) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.
 (b) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.
 (c) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.
 (d) $(A \setminus B) \setminus C = A \setminus (B \cup C)$.
 (e) $A \cup (B \setminus C) = A \cup ((A \cup B) \setminus (A \cup C))$.
5. What can you say about the sets A and B if we know the following to be true?
- (a) $A \cup B = A$.
 (b) $A \cap B = A$.
 (c) $A \setminus B = A$.
 (d) $A \setminus B = B \setminus A$.
6. Form the following sets from the set $A = \{a\}$:
- (a) the set $B = \mathcal{P}(A)$;
 (b) the set $C = \mathcal{P}(B)$;
 (c) the set $D = \mathcal{P}(C)$.
7. Let $A = \{1, \{2, 3\}, \{4, 5, \{6\}\}\}$.
- (a) What is $\mathcal{P}(A)$?
 (b) State whether the following are true or false.
- i. $\emptyset \in A$.
 ii. $1 \in A$.
 iii. $\{2, 3\} \subseteq A$.
 iv. $\{\{2, 3\}\} \subseteq A$.
 v. $\{4, 5, \{6\}\} \subseteq A$.
8. The *symmetric difference* of two sets A and B , denoted $A \oplus B$ is the set which contains those elements which are in A or B but not in both A and B .
- (a) Draw a Venn diagram depicting $A \oplus B$.
 (b) Draw Venn diagrams to justify the following laws.
- i. $A \oplus B = (A \setminus B) \cup (B \setminus A)$.

$$\text{ii. } A \oplus B = (A \cup B) \setminus (A \cap B).$$

(c) What propositional connective does \oplus correspond to?

9. Use the Inclusion-Exclusion Principle of Fact 2.11 to show the following three-set version: for finite sets A , B and C ,

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |A \cap C| - |B \cap C| \\ &\quad + |A \cap B \cap C|. \end{aligned}$$

Explain informally why this principle holds.

10. Use the three-set version of the Inclusion-Exclusion Principle from the previous exercise to solve the hospital problem of Example 2.26.
11. Felix, Oskar and Amanda play a game to see who can list the most countries in five minutes. They each make a list, and after five minutes they compare these lists, crossing off any countries that are on more than one list.

Felix had listed the most countries, 29, but they were mostly common countries that the other two got: in fact, 23 of them were on Oskar's list, and 12 of these 23 were also on Amanda's list.

Amanda had listed the fewest countries, 22, but – with more than a little help from Joel – she had come up with many obscure countries: she had listed seven countries that were not on Felix's list, and nine countries that were not on Oskar's list.

After crossing out all of the duplicated countries, they were left with a total of 13 countries on their lists.

Who won the game?

12. The ordered pair (x, y) can be defined as the set $\{\{x\}, \{x, y\}\}$.
- (a) With this definition, show that $(x, y) = (u, v)$ if, and only if, $x=u$ and $y=v$.
- (b) Why can we not define the ordered pair as $(x, y) = \{x, \{y\}\}$?
13. In a certain town lives a barber, who is a man, who shaves every man in the town who does not shave himself.
- The question is: Who shaves the barber? Explain your answer.
14. An adjective is *autological* if it describes itself. For example, “short” is autological since it is short; and “pentasyllabic” is autological since it is pentasyllabic; that is, it has five syllables. Any adjective that is *not* autological is said to be *heterological*. For example, “long” and “monosyllabic” are heterological.

The question is: Is “heterological” autological, or is it heterological? Explain your answer. What about “autological”?