Chapter 3
Mathematical Description of Physical Phenomena

Abstract The chapter provides an overview of the conservation principles governing fluid flow, heat and mass transfer, and other related transport phenomena of interest in this book. The physical laws controlling the conservation principles are translated into mathematical relations, written in the form of partial differential equations, representing the needed vehicle for their simulations. First the continuity, momentum, and energy equations (collectively known as the Navier-Stokes equations) expressing the principles of conservation of mass, momentum, and total energy, respectively, are derived. This is followed by the development of a typical conservation equation for a general scalar, vector, or tensor quantity. The mathematical properties of the various terms in these equations are also examined. Moreover, the common practice of writing the conservation equations in a non-dimensional form using dimensionless quantities is explained and some of the dimensionless groups resulting from the application of this procedure, which are very useful for performing parametric studies of engineering problems, are discussed.

3.1 Introduction

Researchers and practitioners of computational fluid dynamics encounter and work with the Navier-Stokes equations [1, 2] almost on daily basis. Many do not realize that these equations are over one hundred seventy years old. Whereas the name Navier-Stokes initially referred to the conservation equation of linear momentum, it is used nowadays to denote collectively the conservation equations of mass, momentum, and energy. These equations can be used to model a wide range of fluid flow configurations, whether it is the flow in a hurricane or in a turbomachine, around an airplane or a submarine, in arteries or in lungs, in pumps or in compressors, the Navier-Stokes equations can describe all these phenomena.

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3.2 Classification of Fluid Flows

Fluids, which denote liquids and gases are substances that do not permanently change under a large stress (force per unit area). Whereas a solid resists an applied shear or tangential stress by deforming, a fluid cannot and a shear stress applied to a fluid puts it to motion. Moreover, unlike solids which have well-defined shapes, fluids do not have a definite shape. While gases are fluids that completely fill their domains, liquids are fluids that form a free surface in the presence of a gravitational field.

In analyzing fluid flow phenomena [3–6], attention is focused on what happens at the macroscopic rather than the microscopic scale. It is also assumed that the fluid is a continuum, so that its physical and flow properties are defined at every point in space. Within this assumption, fluid flow behavior can be categorized as either Newtonian or non-Newtonian. Newtonian fluids are characterized by a linear relationship between the shear stress and the shear rate, with the molecular viscosity μ, which is a measure of the ability of a fluid subjected to a stress to resist deformation, representing the slope of the linear function. On the other hand, for non-Newtonian fluids this relationship is nonlinear. Similarly fluid flow can be classified into various classes, such as one-dimensional or multi-dimensional, single phase or multi-phase, steady or unsteady, real (viscous) or ideal (inviscid), compressible or incompressible, turbulent or laminar, and rotational or irrotational, among others. The purpose of these classifications is to simplify the process of analysis and modeling of fluid flow phenomena.

Flows are also classified mathematically according to the partial differential equations describing them. Second order partial differential equations in two independent variables, for example, are categorized as hyperbolic, parabolic, or elliptic. In these equations information travels along two characteristic lines, which may be real and distinct, real and coincident, or complex depending on whether they are of the hyperbolic, parabolic, or elliptic type, respectively. This variation in the nature of the equations necessitates different solution methodologies that should also be recognized by any numerical method used to solve them.

As will be shown in this chapter, fluid flows are governed by the Navier-Stokes equations, which are highly nonlinear second order partial differential equations in four independent variables since, in general, flows are unsteady and three dimensional. Therefore, the above classification does not really apply to them. Nevertheless, the same terminology is used in their categorization as they share many of the properties characterizing second order equations in two independent variables. Transient and supersonic flows are hyperbolic, boundary layer flows are parabolic, and recirculating flows are elliptic. As flows may be subsonic in a certain part of the domain and supersonic in other parts (e.g., flow in a converging-diverging nozzle), or viscous dominated close to walls and essentially inviscid in the core region, it is hard to describe a flow as falling under one of the above three types and in general it is of the mixed type. This categorization is numerically translated into the following: parabolic flows that are affected by upstream locations only, elliptic flows by both upstream and downstream locations, and hyperbolic flows supporting discontinuities in the solution, e.g., shock waves.

3.3 Eulerian and Lagrangian Description of Conservation Laws

The principle of conservation states that for an isolated system certain physical measurable quantities are conserved over a local region. This conservation principle or conservation law is an axiom that cannot be proven mathematically but can be expressed by a mathematical relation. Laws of this type govern several physical quantities such as mass, momentum, and energy (i.e., the Navier-Stokes equations).

The conservation laws involving fluid flow and related transfer phenomena can be mathematically formulated following either a Lagrangian (material volume, MV) or an Eulerian (control volume) approach [7]. In the Lagrangian specification of the flow field (Fig. 3.1a), the fluid is subdivided into fluid parcels and every fluid parcel is followed as it moves through space and time. These parcels are tagged using a time-independent position vector field $\mathbf{x}_0$, usually selected to be the parcels' centre of mass at some initial time t_0, and the flow is described by a function $\mathbf{x}(t, \mathbf{x}_0)$. The path line described by a fluid parcel (Fig. 3.1a) is obtained as the collection of positions occupied at different times.

On the other hand, the Eulerian approach (Fig. 3.1b) focuses on specific locations in the flow region as time passes. Thus the flow variables are functions of position $\mathbf{x}$ and time t and the flow velocity is represented by $\mathbf{v}(t, \mathbf{x})$. As the derivative of the position of a fluid parcel $\mathbf{x}_0$ with respect to time represents its velocity, the two specifications are related by

$$\mathbf{v}(t, \mathbf{x}(\mathbf{x}_0, t)) = \frac{\partial}{\partial t}\mathbf{x}(t, \mathbf{x}_0) \tag{3.1}$$

Based on the above description, changes in the properties of a moving fluid can be measured either on a fixed point in space while fluid particles are crossing it (Eulerian), or by following a fluid parcel along its path (Lagrangian).

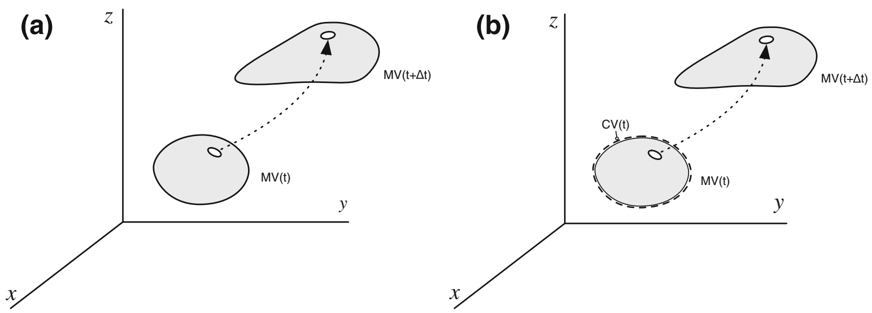


Fig. 3.1 **a** Lagrangian and **b** Eulerian specification of the flow field

3.3.1 Substantial Versus Local Derivative

The derivative (rate of change) of a field variable $\phi(t, \mathbf{x}(t))$, which may be a scalar or a vector quantity representing density, velocity, temperature, etc., with respect to a fixed position in space is called the Eulerian derivative $(\partial\phi/\partial t)$ while the derivative following a moving fluid parcel is called the Lagrangian, substantial, or material derivative and is denoted by $(D\phi/Dt)$. The substantial derivative of variable ϕ, which can be derived through application of the chain rule to account for changes induced by all independent variables along the path, is given by

$$\begin{aligned}\frac{D\phi}{Dt} &= \frac{\partial\phi}{\partial t}\frac{dt}{dt} + \frac{\partial\phi}{\partial x}\underbrace{\frac{dx}{dt}}_{u} + \frac{\partial\phi}{\partial y}\underbrace{\frac{dy}{dt}}_{v} + \frac{\partial\phi}{\partial z}\underbrace{\frac{dz}{dt}}_{w} \\ &= \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + w\frac{\partial\phi}{\partial z} \\ &= \underbrace{\frac{\partial\phi}{\partial t}}_{\substack{\textit{local rate}\\ \textit{of change}}} + \underbrace{\mathbf{v}\cdot\nabla\phi}_{\substack{\textit{convective rate}\\ \textit{of change}}}\end{aligned} \tag{3.2}$$

where $\mathbf{v}$ is the velocity vector and ∇ is the "del" or "gradient" operator defined earlier [6–8].

Equation (3.2) shows that the total rate of change of the function ϕ as a fluid parcel moves through a flow field described by its Eulerian specification $\mathbf{v}$ from position $\mathbf{x}$ at time t to position $\mathbf{x} + \mathbf{v}\delta t$ at time $t + \delta t$ (Fig. 3.2) is equal to the sum of the local and convective rates of change of ϕ.

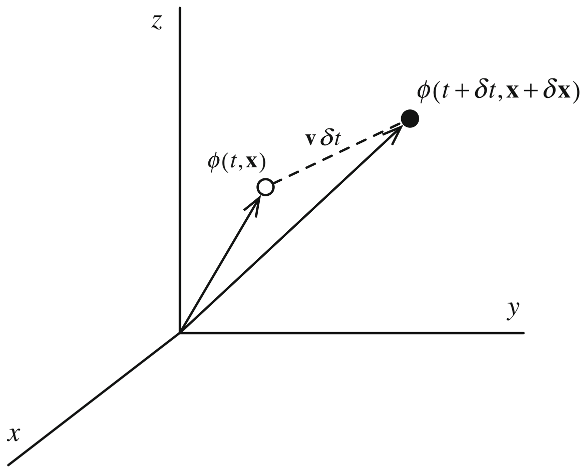


Fig. 3.2 Total rate of change of the field variable ϕ between time t and $t + \delta t$

An important example of a material derivative is $D\mathbf{v}/Dt$, the rate of change of velocity following the flow, which is the acceleration vector given by

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \tag{3.3}$$

In this book, the conservation laws are described following an Eulerian formulation where the focus is on the flow within a specified region in space, called control volume. This choice is based on the fact that the Eulerian approach follows a field (system) rather than a particle approach, it abandons the tedious and often unnecessary task of tracking individual particles, and focuses attention on what happens at a fixed point (or volume) as different particles go by. Moreover, a critical shortcoming of the Lagrangian approach is its inability to control the domain of interest since fluid parcels travel to where the flow takes them, which may not be the region of interest. This limits the usefulness of the approach as in most fluid flow applications fluid properties in a fixed region are required, e.g., the shear stress on the surface of a moving train, and not the properties of moving material volumes. Nevertheless it should be mentioned that the Eulerian approach introduces into the conservation equations the local effect of transport by the fluid flow through the advective rate of change term, $\mathbf{v} \cdot \nabla\phi$, which represents the product of an unknown velocity field and the gradient of an unknown variable field. This nonlinearity leads to the most interesting and most challenging phenomena of fluid flows.

3.3.2 Reynolds Transport Theorem

The conservation laws mentioned above apply to moving material volumes of fluids (Fig. 3.1), and not to fixed points or control volumes. In order to express these laws following an Eulerian approach, there is a need to know the Eulerian equivalent of an integral taken over a moving material volume of fluid. This is provided through the Reynolds transport theorem [9].

The conversion formula differs slightly according to whether the control volume is fixed, moving, or deformable. To derive the formula, let B be any property of the fluid (mass, momentum, energy, etc.) and let $b = dB/dm$ be the intensive value of B (amount of B per unit mass) in any small element of the fluid.

For the arbitrary moving and deformable control volume shown in Fig. 3.1, the instantaneous total change of B in the material volume (MV) is equal to the instantaneous total change of B within the control volume (V) plus the net flow of B into and out of the control volume through its control surface (S). Let ρ denotes the density of the fluid, $\mathbf{n}$ the outward normal to the control volume surface, $\mathbf{v}(t, \mathbf{x})$ the velocity of the fluid, $\mathbf{v}_s(t, \mathbf{x})$ the velocity of the deforming control volume surface, and $\mathbf{v}_r(t, \mathbf{x})$ the relative velocity by which the fluid enters/leaves the control volume [i.e., $\mathbf{v}_r = \mathbf{v}(t, \mathbf{x}) - \mathbf{v}_s(t, \mathbf{x})$], then the Reynolds transport theorem gives

$$\left(\frac{dB}{dt}\right)_{MV} = \frac{d}{dt}\left(\int_{V(t)} b\rho dV\right) + \int_{S(t)} b\rho \mathbf{v}_r \cdot \mathbf{n}\, dS \tag{3.4}$$

For a fixed control volume, $\mathbf{v}_s = 0$ and the geometry is independent of time implying that the time derivative term on the right hand side of Eq. (3.4) can be written using Leibniz rule as

$$\frac{d}{dt}\left(\int_{V} b\rho\, dV\right) = \int_{V} \frac{\partial}{\partial t}(b\rho) dV \tag{3.5}$$

Therefore Eq. (3.4) simplifies to

$$\left(\frac{dB}{dt}\right)_{MV} = \int_{V} \frac{\partial}{\partial t}(b\rho) dV + \int_{S} b\rho \mathbf{v} \cdot \mathbf{n}\, dS \tag{3.6}$$

Applying the divergence theorem to transform the surface integral into a volume integral, Eq. (3.6) becomes

$$\left(\frac{dB}{dt}\right)_{MV} = \int_{V} \left[\frac{\partial}{\partial t}(\rho b) + \nabla \cdot (\rho \mathbf{v} b)\right] dV \tag{3.7}$$

An alternative form of Eq. (3.7) can be obtained by expanding the second term in the square bracket and using the substantial derivative to get

$$\left(\frac{dB}{dt}\right)_{MV} = \int_{V} \left[\frac{D}{Dt}(\rho b) + \rho b \nabla \cdot \mathbf{v}\right] dV \tag{3.8}$$

Equation (3.7) or (3.8) can be used to derive the Eulerian form of the conservation laws in fixed regions.

3.4 Conservation of Mass (Continuity Equation)

The principle of conservation of mass [6, 10] indicates that in the absence of mass sources and sinks, a region will conserve its mass on a local level.

Considering the material volume of fluid shown in Fig. 3.3 of mass m, density ρ, and velocity $\mathbf{v}$, conservation of mass in material (Lagrangian) coordinate system can be written as

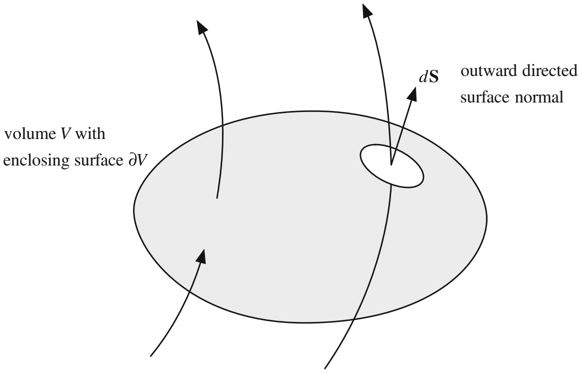


Fig. 3.3 Conservation of mass for a material volume of a fluid of mass m

$$\left(\frac{dm}{dt}\right)_{MV} = 0 \tag{3.9}$$

For $B = m$ the corresponding intensive quantity is $b = 1$, and based on Eq. (3.8) the equivalent expression of mass conservation in an Eulerian coordinate system is

$$\int_V \left[\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v}\right] dV = 0 \tag{3.10}$$

For the integral given in Eq. (3.10) to be true for any control volume V, the integrand should be equal to zero, giving the differential form of the mass conservation or continuity equation as

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0 \tag{3.11}$$

The flux form of the continuity equation can be derived using Eq. (3.7) and leading to

$$\int_V \left(\frac{\partial\rho}{\partial t} + \nabla\cdot[\rho\mathbf{v}]\right) dV = 0 \tag{3.12}$$

Again for the integral in Eq. (3.12) to be true for any control volume V, the integrand should be equal to zero, giving the flux form of the mass conservation or continuity equation as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot[\rho\mathbf{v}] = 0 \tag{3.13}$$

In the absence of any significant absolute pressure or temperature changes, it is acceptable to assume that the flow is incompressible; that is, the pressure changes do not have significant effects on density. This is almost invariably the case in liquids, and is a good approximation in gases at speeds much less than that of sound. (Note that sound waves are compressible phenomena.) The most important consequence in fluid dynamics is that the mass conservation (continuity) equation can no longer be used to compute the density.

The incompressibility condition indicates that ρ does not change with the flow, which mathematically can be expressed as $D\rho/Dt = 0$. Using the mass conservation equation given by Eq. (3.11), this is equivalent to saying that the continuity equation for incompressible flow is given by

$$\nabla \cdot \mathbf{v} = 0 \tag{3.14}$$

or in integral form as

$$\int_S (\mathbf{v} \cdot \mathbf{n}) dS = 0 \tag{3.15}$$

Equation (3.15) states that for incompressible flows the net flow across any control volume is zero, i.e., "flow out" = "flow in".

Note also that $D\rho/Dt = 0$ does not imply that ρ is the same everywhere (although this happens to be the case in many hydraulic applications), but that ρ does not change along a streamline. To be more accurate, the incompressibility approximation means that each fluid element keeps its original density as it moves. In practice, density differences are commonly encountered in water due to variation in salt concentration and in air due to temperature differences resulting in important buoyancy forces.

3.5 Conservation of Linear Momentum

The principle of conservation of linear momentum [6, 10] indicates that in the absence of any external force acting on a body, the body retains its total momentum, i.e., the product of its mass and velocity vector. Since momentum is a vector quantity, its components in any direction will also be conserved.

For the material volume of a substance, Newton's Second Law of motion asserts that the momentum of this specific volume can change only in the presence of a net force acting on it, which could include both surface forces and body forces. Therefore, by considering the material volume of fluid shown in Fig. 3.4 of mass m, density ρ, and velocity $\mathbf{v}$, Newton's law in Lagrangian coordinates can be written as

$$\left(\frac{d(m\mathbf{v})}{dt}\right)_{MV} = \left(\int_V \mathbf{f}\, dV\right)_{MV} \tag{3.16}$$

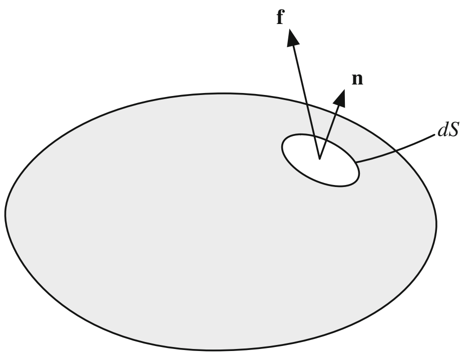


Fig. 3.4 Conservation of linear momentum for a material volume of a fluid of mass m

where $\mathbf{f}$ is the external force per unit volume acting on the material volume. The term on the right hand side of Eq. (3.16) is a volume integral over material coordinates performed over the volume occupied instantaneously by the moving fluid, thus

$$\left(\int_V \mathbf{f}\, dV\right)_{MV} = \int_V \mathbf{f}\, dV \tag{3.17}$$

The equivalent expression of Eq. (3.16) in Eulerian coordinates can be written in two different ways known as the conservative and non-conservative forms.

3.5.1 Non-Conservative Form

Noticing that in this case $b = \mathbf{v}$, the non-conservative form is obtained by using Eq. (3.8) in the derivation yielding

$$\int_V \left[\frac{D}{Dt}[\rho\mathbf{v}] + [\rho\mathbf{v}\nabla\cdot\mathbf{v}] - \mathbf{f}\right] dV = 0 \tag{3.18}$$

Again for the integral to be zero over any control volume, the integrand has to be zero. Thus,

$$\frac{D}{Dt}[\rho\mathbf{v}] + [\rho\mathbf{v}\nabla\cdot\mathbf{v}] = \mathbf{f} \tag{3.19}$$

Expanding the material derivative of the momentum term and regrouping, the non-conservative form is obtained as

$$\rho\frac{D\mathbf{v}}{Dt} + \mathbf{v}\underbrace{\left(\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v}\right)}_{Continuity} = \mathbf{f} \tag{3.20}$$

Applying the continuity constraint and expanding the material derivative, the non-conservative form of the momentum equation reduces to

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right]=\mathbf{f} \tag{3.21}$$

3.5.2 Conservative Form

The conservative (or flux) version is obtained by applying the form of the Reynolds transport theorem given by Eq. (3.7) and is written as

$$\int_V \left[\frac{\partial}{\partial t}[\rho\mathbf{v}]+\nabla\cdot\{\rho\mathbf{v}\mathbf{v}\}-\mathbf{f}\right]dV=0 \tag{3.22}$$

By setting the integrand to zero for the integral to be zero for any volume V, the conservative form of the momentum equation is obtained as

$$\frac{\partial}{\partial t}[\rho\mathbf{v}]+\nabla\cdot\{\rho\mathbf{v}\mathbf{v}\}=\mathbf{f} \tag{3.23}$$

where $\rho\mathbf{v}\mathbf{v}$ is the dyadic product, described in Chap. 2, which is a special case of tensor product with its divergence being a vector.

Both forms will be used in this book for better describing the discretization concepts and for showing actual implementation details. In the derivations to follow the conservative form will be used. The non-conservative form can be easily obtained from the conservative form at any step by invoking the continuity constraint as explained above.

The full form of the momentum equation is obtained once the external surface and body forces acting on the control volume are specified. The force $\mathbf{f}$ is split into two parts one denoted by $\mathbf{f}_s$ representing the surface forces and the second by $\mathbf{f}_b$ representing the body forces such that

$$\mathbf{f}=\mathbf{f}_s+\mathbf{f}_b \tag{3.24}$$

The details of these forces are given next.

3.5.3 Surface Forces

For the arbitrary macroscopic volume element depicted in Fig. 3.4, the forces acting on its surface are due to pressure and viscous stresses which can be expressed in

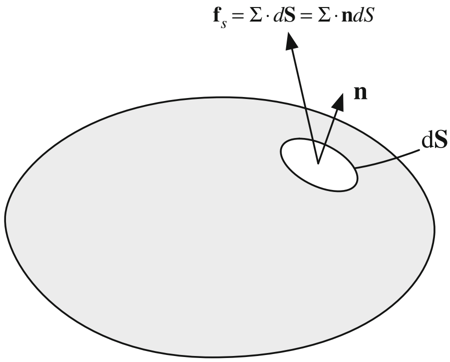


Fig. 3.5 The surface forces acting on a differential surface element expressed in terms of the stress tensor

term of the total stress tensor $\boldsymbol{\Sigma}$, as shown in Fig. 3.5. In general there are nine components of stress at any given point; one normal component and two shear components (parallel to the surface that receives the stress) in each coordinate plane. Thus in Cartesian coordinates the stress tensor is given by

$$\Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} & \Sigma_{xz} \\ \Sigma_{yx} & \Sigma_{yy} & \Sigma_{yz} \\ \Sigma_{zx} & \Sigma_{zy} & \Sigma_{zz} \end{pmatrix} \tag{3.25}$$

where terms of the form Σ_{ii} represent normal stresses and Σ_{ij} shear stresses. A normal stress can be either a compression, if $\Sigma_{ii} \leq 0$, or a tension, if $\Sigma_{ii} \geq 0$. The most important compressive normal stress is usually due to pressure rather than to viscous effects. The component Σ_{ij} represents the stress acting on face i in the j direction with the direction of face i being positive if the outward normal to the face is in the positive direction.

In practice the stress tensor is split into two terms such that

$$\boldsymbol{\Sigma} = -\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + \begin{pmatrix} \overbrace{\Sigma_{xx+p}}^{\tau_{xx}} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \overbrace{\Sigma_{yy+p}}^{\tau_{yy}} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \overbrace{\Sigma_{zz+p}}^{\tau_{zz}} \end{pmatrix} = -p\mathbf{I} + \boldsymbol{\tau} \tag{3.26}$$

where $\mathbf{I}$ is the identity tensor of size (3×3), p is the pressure, and $\boldsymbol{\tau}$ is the deviatoric or viscous stress tensor. The pressure is the negative of the mean of the normal stresses and is given by

$$p = -\frac{1}{3}\left(\Sigma_{xx} + \Sigma_{yy} + \Sigma_{zz}\right) \tag{3.27}$$

The surface force acting on a differential surface element of area dS and orientation $\mathbf{n}$, as illustrated in Fig. 3.5, is $(\boldsymbol{\Sigma} \cdot \mathbf{n})dS$. Applying the divergence theorem, the total surface force acting on the control volume is given by

$$\int_V \mathbf{f}_s \, dV = \int_S \boldsymbol{\Sigma} \cdot \mathbf{n} \, dS = \int_V \nabla \cdot \boldsymbol{\Sigma} \, dV \Rightarrow \mathbf{f}_s = [\nabla \cdot \boldsymbol{\Sigma}] = -\nabla p + [\nabla \cdot \boldsymbol{\tau}] \quad (3.28)$$

3.5.4 Body Forces

Body forces, which are presented as forces per unit volume, may also arise due to a variety of effects. There are plenty of examples, but the predominant ones are given next.

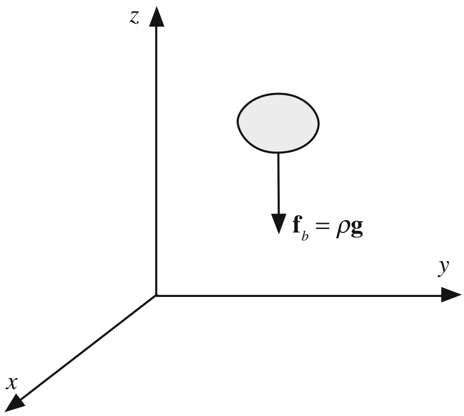
3.5.4.1 Gravitational Forces

The force representing the weight of the material volume per unit volume in the presence of a gravitational field is denoted by gravitational force (Fig. 3.6) and given by

$$\mathbf{f}_b = \rho \mathbf{g} \quad (3.29)$$

where $\mathbf{g}$ is the gravitational acceleration vector.

Fig. 3.6 Body forces acting on a differential element



3.5.4.2 System Rotation

When solving fluid flow problems in a rotating frame of reference, the forces arising as a result of the rigid body rotation of the reference frame should be accounted for. These can be viewed as body forces of the form

$$\mathbf{f}_b = \underbrace{-2\rho[\varpi \times \mathbf{v}]}_{\textit{Coriolis forces}} - \underbrace{\rho[\varpi \times [\varpi \times \mathbf{r}]]}_{\textit{Centrifugal forces}} \tag{3.30}$$

where ϖ is the angular velocity of the rotating reference frame and $\mathbf{r}$ is the position vector (Fig. 3.7). Note that gravitational and centrifugal forces are dependent on position but not on velocity. Thus they can be absorbed into a modified pressure and hence effectively ignored as a separate entity unless they appear in boundary conditions. Coriolis forces however have to be treated explicitly. Other forces, such as magnetic and electric, may be added depending on the particular situation. Due to the many possible types of body forces, no specific type will be adopted in the equations to follow and the generic $\mathbf{f}_b$ force is retained.

Substituting the external force $\mathbf{f}$ in Eq. (3.23) by its equivalent expression, the general conservative form of the momentum equation is obtained as

$$\frac{\partial}{\partial t}[\rho\mathbf{v}] + \nabla \cdot \{\rho\mathbf{v}\mathbf{v}\} = -\nabla p + [\nabla \cdot \tau] + \mathbf{f}_b \tag{3.31}$$

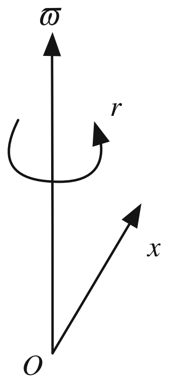


Fig. 3.7 Body forces due to a rigid body rotation in a rotating frame of reference

3.5.5 Stress Tensor and the Momentum Equation for Newtonian Fluids

To proceed further with the momentum equation, the type of fluid should be known in order to relate the stress tensor $\boldsymbol{\tau}$ to the flow variables. For a Newtonian fluid, the stress tensor is a linear function of the strain rate [2] and is given by

$$\boldsymbol{\tau} = \mu\left\{\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right\} + \lambda(\nabla\cdot\mathbf{v})\mathbf{I} \tag{3.32}$$

where μ is the molecular viscosity coefficient, λ the bulk viscosity coefficient usually set equal to $-(2/3)\mu\,(\lambda = -(2/3)\mu)$, the superscript T refers to the transpose of $\nabla\mathbf{v}$, and $\mathbf{I}$ is the unit or identity tensor of size (3 × 3) defined as

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.33}$$

The expanded form of the stress tensor in a three-dimensional Cartesian coordinate system can be written as

$$\boldsymbol{\tau} = \begin{bmatrix} 2\mu\dfrac{\partial u}{\partial x} + \lambda\nabla\cdot\mathbf{v} & \mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & 2\mu\dfrac{\partial v}{\partial y} + \lambda\nabla\cdot\mathbf{v} & \mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) \\ \mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) & \mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & 2\mu\dfrac{\partial w}{\partial z} + \lambda\nabla\cdot\mathbf{v} \end{bmatrix} \tag{3.34}$$

The divergence of the stress tensor is a vector that can be expressed as

$$\begin{aligned}[\nabla\cdot\boldsymbol{\tau}] &= \nabla\cdot\left[\mu\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right)\right] + \nabla(\lambda\nabla\cdot\mathbf{v}) \\ &= \begin{bmatrix} \dfrac{\partial}{\partial x}\left[2\mu\dfrac{\partial u}{\partial x} + \lambda\nabla\cdot\mathbf{v}\right] + \dfrac{\partial}{\partial y}\left[\mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right)\right] + \dfrac{\partial}{\partial z}\left[\mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right)\right] \\ \dfrac{\partial}{\partial x}\left[\mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right)\right] + \dfrac{\partial}{\partial y}\left[2\mu\dfrac{\partial v}{\partial y} + \lambda\nabla\cdot\mathbf{v}\right] + \dfrac{\partial}{\partial z}\left[\mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right)\right] \\ \dfrac{\partial}{\partial x}\left[\mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right)\right] + \dfrac{\partial}{\partial y}\left[\mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right)\right] + \dfrac{\partial}{\partial z}\left[2\mu\dfrac{\partial w}{\partial z} + \lambda\nabla\cdot\mathbf{v}\right] \end{bmatrix}\end{aligned} \tag{3.35}$$

Substituting into Eq. (3.31), the final conservative form of the momentum equation for Newtonian fluids becomes

$$\frac{\partial}{\partial t}[\rho\mathbf{v}] + \nabla\cdot\{\rho\mathbf{v}\mathbf{v}\} = -\nabla p + \nabla\cdot\left\{\mu\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right]\right\} + \nabla(\lambda\nabla\cdot\mathbf{v}) + \mathbf{f}_b \tag{3.36}$$

For later reference the momentum equation is expanded into

$$\frac{\partial}{\partial t}[\rho\mathbf{v}] + \nabla\cdot\{\rho\mathbf{v}\mathbf{v}\} = \nabla\cdot\{\mu\nabla\mathbf{v}\} - \nabla p + \underbrace{\nabla\cdot\left\{\mu(\nabla\mathbf{v})^{\mathrm{T}}\right\} + \nabla(\lambda\nabla\cdot\mathbf{v}) + \mathbf{f}_b}_{\mathbf{Q}^{\mathrm{v}}} \tag{3.37}$$

and rewritten as

$$\frac{\partial}{\partial t}[\rho\mathbf{v}] + \nabla \cdot \{\rho\mathbf{v}\mathbf{v}\} = \nabla \cdot \{\mu\nabla\mathbf{v}\} - \nabla p + \mathbf{Q}^{\mathbf{v}} \tag{3.38}$$

For incompressible flows, the divergence of the velocity vector is zero, i.e., $\nabla \cdot \mathbf{v} = 0$, and the momentum equation reduces to

$$\frac{\partial}{\partial t}[\rho\mathbf{v}] + \nabla \cdot \{\rho\mathbf{v}\mathbf{v}\} = -\nabla p + \nabla \cdot \left\{\mu\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right]\right\} + \mathbf{f}_b \tag{3.39}$$

If the viscosity is constant, the momentum equation can be further simplified. Taking just the first component of the vector equation [Eq. (3.35)], and assuming μ is constant, the following can be written:

$$\begin{aligned}
&\mu\frac{\partial}{\partial x}\left[2\frac{\partial u}{\partial x}\right] + \mu\frac{\partial}{\partial y}\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \mu\frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] \\
&= \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial yx} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial zx}\right] \\
&= \mu\left[\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial yx} + \frac{\partial^2 w}{\partial zx}\right] \\
&= \mu\left[\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]
\end{aligned} \tag{3.40}$$

Substitution in Eq. (3.37) yields after simplification

$$\frac{\partial}{\partial t}[\rho\mathbf{v}] + \nabla \cdot \{\rho\mathbf{v}\mathbf{v}\} = -\nabla p + \mu\nabla^2\mathbf{v} + \mathbf{f}_b \tag{3.41}$$

For inviscid flows the viscosity is zero and the momentum equation for incompressible and compressible inviscid flows becomes

$$\frac{\partial}{\partial t}[\rho\mathbf{v}] + \nabla \cdot \{\rho\mathbf{v}\mathbf{v}\} = -\nabla p + \mathbf{f}_b \tag{3.42}$$

3.6 Conservation of Energy

The conservation of energy [6, 10] is governed by the first law of thermodynamics which states that energy can be neither created nor destroyed during a process; it can only change from one form (mechanical, kinetic, chemical, etc.) into another. Consequently, the sum of all forms of energy in an isolated system remains constant.

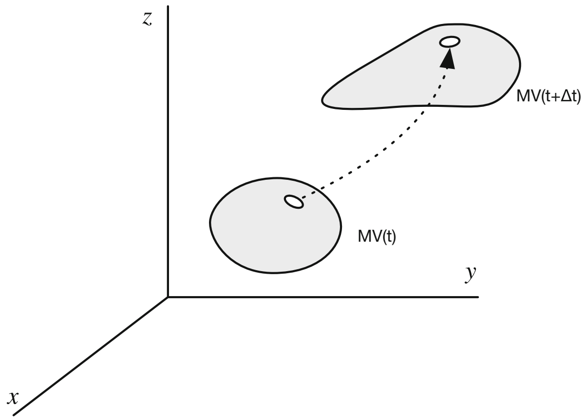


Fig. 3.8 A material volume moves with the particles it encloses

Considering the material volume shown in Fig. 3.8, of mass m, density ρ, and moving with a velocity $\mathbf{v}$. Defining the total energy E of the material volume at time t as the sum of its internal and kinetic energies, then E can be written as

$$E = m\left(\hat{u} + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) \tag{3.43}$$

where $\hat{u}$ is the fluid specific internal energy (internal energy per unit mass). The first law of classical thermodynamics applied to the material volume states that the rate of change of the total energy of the material volume is equal to the rate of heat addition and work extraction through its boundaries. Mathematically this is given by

$$\left(\frac{dE}{dt}\right)_{MV} = \dot{Q} - \dot{W} \tag{3.44}$$

The adopted sign convention is such that heat added to the material volume and work done by the material volume are positive. To apply the Reynolds transport theorem on the material volume, B is set equal to E and b to e (the total energy per unit mass) such that

$$B = E \Rightarrow b = \frac{dE}{dm} = \hat{u} + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} = e \tag{3.45}$$

The net rate of heat transferred to the material element $\dot{Q}$ is the sum of two components. The first component is the rate transferred across the surface of the element $\dot{Q}_S$ and the second generated/destroyed (e.g., due to a chemical reaction) within the material volume $\dot{Q}_V$. Moreover, the net rate of work done by the material volume $\dot{W}$ is due to the rate of work done by the surface forces $\dot{W}_S$ and the rate of work done by the body forces $\dot{W}_b$. Thus the first law can be written as

$$\left(\frac{dE}{dt}\right)_{MV} = \dot{Q}_V + \dot{Q}_S - \dot{W}_b - \dot{W}_S \tag{3.46}$$

By definition, work is due to a force acting through a distance and power is the rate at which work is done. Therefore the rate of work done by body and surface forces can be represented by

$$\begin{aligned}\dot{W}_b &= -\int_V (\mathbf{f}_b \cdot \mathbf{v})dV \\ \dot{W}_S &= -\int_S (\mathbf{f}_S \cdot \mathbf{v})dS\end{aligned} \tag{3.47}$$

The rate of work due to surface forces can be expanded by replacing $\mathbf{f}_S$ by its equivalent expression as given in Eq. (3.26) through (3.28). This leads to

$$\dot{W}_S = -\int_S [\Sigma \cdot \mathbf{v}] \cdot \mathbf{n}\, dS = -\int_V \nabla \cdot [\Sigma \cdot \mathbf{v}]dV = -\int_V \nabla \cdot [(-p\mathbf{I} + \boldsymbol{\tau}) \cdot \mathbf{v}]dV \tag{3.48}$$

After manipulation, $\dot{W}_S$ can be rewritten as

$$\dot{W}_S = -\int_V (-\nabla \cdot [p\mathbf{v}] + \nabla \cdot [\boldsymbol{\tau} \cdot \mathbf{v}])dV \tag{3.49}$$

If $\dot{q}_V$ represents the rate of heat source or sink within the material volume per unit volume and $\dot{q}_S$ the rate of heat transfer per unit area across the surface area of the material element, then $\dot{Q}_V$ and $\dot{Q}_S$ can be written as

$$\dot{Q}_V = \int_V \dot{q}_V dV \quad \dot{Q}_S = -\int_S \dot{q}_s \cdot \mathbf{n}\, dS = -\int_V \nabla \cdot \dot{q}_s dV \tag{3.50}$$

Applying the Reynolds transport theorem and substituting the rate of work and heat terms by their equivalent expressions, Eq. (3.46) becomes

$$\begin{aligned}\left(\frac{dE}{dt}\right)_{MV} &= \int_V \left[\frac{\partial}{\partial t}(\rho e) + \nabla \cdot [\rho \mathbf{v} e]\right] dV \\ &= -\int_V \nabla \cdot \dot{q}_s dV + \int_V (-\nabla \cdot [p\mathbf{v}] + \nabla \cdot [\boldsymbol{\tau} \cdot \mathbf{v}])dV + \int_V (\mathbf{f}_b \cdot \mathbf{v})dV + \int_V \dot{q}_V dV\end{aligned} \tag{3.51}$$

Collecting terms together, the above equation is transformed to

$$\int\limits_V \left[\frac{\partial}{\partial t}(\rho e) + \nabla \cdot [\rho \mathbf{v} e] + \nabla \cdot \dot{q}_s + \nabla \cdot [p\mathbf{v}] - \nabla \cdot [\boldsymbol{\tau} \cdot \mathbf{v}] - \mathbf{f}_b \cdot \mathbf{v} - \dot{q}_V\right] dV = 0 \tag{3.52}$$

For the volume integral in Eq. (3.52) to be true for any control volume, the integrand has to be zero. Thus,

$$\frac{\partial}{\partial t}(\rho e) + \nabla \cdot [\rho \mathbf{v} e] = -\nabla \cdot \dot{q}_s - \nabla \cdot [p\mathbf{v}] + \nabla \cdot [\boldsymbol{\tau} \cdot \mathbf{v}] + \mathbf{f}_b \cdot \mathbf{v} + \dot{q}_V \tag{3.53}$$

which represents the mathematical description of energy conservation or simply the energy equation written in terms of specific total energy. The energy equation may also be written in terms of specific internal energy, specific static enthalpy (or simply specific enthalpy), specific total enthalpy, and under special conditions in terms of temperature.

3.6.1 Conservation of Energy in Terms of Specific Internal Energy

To rewrite the energy equation [Eq. (3.53)] in terms of specific internal energy, the dot product of the momentum equation [Eq. (3.23)] with the velocity vector is performed resulting in

$$\left[\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot \{\rho \mathbf{v}\mathbf{v}\}\right] \cdot \mathbf{v} = \mathbf{f} \cdot \mathbf{v} \tag{3.54}$$

After some manipulations Eq. (3.54) becomes

$$\frac{\partial}{\partial t}(\rho \mathbf{v} \cdot \mathbf{v}) - \rho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot [\rho(\mathbf{v} \cdot \mathbf{v})\mathbf{v}] - \rho \mathbf{v} \cdot [(\mathbf{v} \cdot \nabla)\mathbf{v}] = \mathbf{f} \cdot \mathbf{v} \tag{3.55}$$

Rearranging and collecting terms the following is obtained:

$$\frac{\partial}{\partial t}(\rho \mathbf{v} \cdot \mathbf{v}) + \nabla \cdot [\rho(\mathbf{v} \cdot \mathbf{v})\mathbf{v}] - \mathbf{v} \cdot \underbrace{\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right]}_{\substack{= \mathbf{f} \\ Eq.\,(3.21)}} = \mathbf{f} \cdot \mathbf{v} \tag{3.56}$$

Noticing that the third term on the left side is $\mathbf{v} \cdot \mathbf{f}$ and replacing $\mathbf{f}$ by its equivalent expression, an equation for the flow kinetic energy is obtained as

$$\frac{\partial}{\partial t}\left(\rho\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)+\nabla\cdot\left[\rho\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\right]=-\mathbf{v}\cdot\nabla p+\mathbf{v}\cdot[\nabla\cdot\boldsymbol{\tau}]+\mathbf{f}_b\cdot\mathbf{v} \tag{3.57}$$

This equation can be modified and rewritten in the following form:

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\rho\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)+\nabla\cdot\left[\rho\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\right]\\&\quad=-\nabla\cdot[p\mathbf{v}]+p\nabla\cdot\mathbf{v}+\nabla\cdot[\boldsymbol{\tau}\cdot\mathbf{v}]-(\boldsymbol{\tau}:\nabla\mathbf{v})+\mathbf{f}_b\cdot\mathbf{v}\end{aligned} \tag{3.58}$$

Subtracting Eq. (3.58) from Eq. (3.53), the energy equation with specific internal energy as its main variable is obtained as

$$\frac{\partial}{\partial t}(\rho\hat{u})+\nabla\cdot[\rho\mathbf{v}\hat{u}]=-\nabla\cdot\dot{q}_s-p\nabla\cdot\mathbf{v}+(\boldsymbol{\tau}:\nabla\mathbf{v})+\dot{q}_V \tag{3.59}$$

3.6.2 Conservation of Energy in Terms of Specific Enthalpy

Rewriting the energy equation in terms of specific enthalpy is straightforward and follows directly from its definition according to which the specific internal energy and specific enthalpy are related by

$$\hat{u}=\hat{h}-\frac{p}{\rho} \tag{3.60}$$

Substituting $\left(\hat{h}-p/\rho\right)$ for $\hat{u}$ in Eq. (3.59) and performing some algebraic manipulations, the energy equation in terms of specific enthalpy evolves as

$$\frac{\partial}{\partial t}\left(\rho\hat{h}\right)+\nabla\cdot\left[\rho\mathbf{v}\hat{\mathrm{h}}\right]=-\nabla\cdot\dot{q}_s+\frac{Dp}{Dt}+(\boldsymbol{\tau}:\nabla\mathbf{v})+\dot{q}_V \tag{3.61}$$

3.6.3 Conservation of Energy in Terms of Specific Total Enthalpy

The energy equation in terms of specific total enthalpy can be derived by expressing e in terms of $\hat{h}_0$ to get

$$e=\hat{u}+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}=\hat{h}-\frac{p}{\rho}+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}=\hat{h}_0-\frac{p}{\rho} \tag{3.62}$$

Then by substituting $\left(\hat{h} - p/\rho\right)$ for e in Eq. (3.53) and performing some algebraic manipulations, the energy equation in terms of specific total enthalpy is obtained as

$$\frac{\partial}{\partial t}\left(\rho\hat{h}_0\right) + \nabla\cdot\left[\rho\mathbf{v}\hat{h}_0\right] = -\nabla\cdot\dot{q}_s + \frac{\partial p}{\partial t} + \nabla\cdot[\boldsymbol{\tau}\cdot\mathbf{v}] + \mathbf{f}_b\cdot\mathbf{v} + \dot{q}_V \tag{3.63}$$

All forms of the energy equation presented so far are general and applicable to Newtonian and non-Newtonian fluids. The only limitation is that they are applicable to a fixed control volume.

3.6.4 Conservation of Energy in Terms of Temperature

To be able to write the energy equation with temperature as the main variable some constraints have to be imposed. Assuming $\hat{h}$ to be a function of p and T, the fluid is expected to be Newtonian. Therefore the derivations to follow are applicable to Newtonian fluids only. If $\hat{h} = \hat{h}(p, T)$, then $d\hat{h}$ can be written as

$$d\hat{h} = \left(\frac{\partial\hat{h}}{\partial T}\right)_p dT + \left(\frac{\partial\hat{h}}{\partial p}\right)_T dp \tag{3.64}$$

Using the following ordinary equilibrium thermodynamics relation:

$$\left(\frac{\partial\hat{h}}{\partial p}\right)_T = \hat{V} - T\left(\frac{\partial\hat{V}}{\partial T}\right)_p \tag{3.65}$$

where $\hat{V}$ is the specific volume, the expression for $d\hat{h}$ can be modified to

$$d\hat{h} = c_p dT + \left[\hat{V} - T\left(\frac{\partial\hat{V}}{\partial T}\right)_p\right]dp \tag{3.66}$$

The left side of the specific enthalpy [Eq. (3.61)], with $d\hat{h}$ given by Eq. (3.66), can be rewritten in terms of T as

$$\begin{aligned}\frac{\partial}{\partial t}\left(\rho\hat{h}\right) + \nabla\cdot\left[\rho\mathbf{v}\hat{h}\right] = \rho\frac{D\hat{h}}{Dt} &= \rho c_p\frac{DT}{Dt} + \rho\left[\hat{V} - T\left(\frac{\partial\hat{V}}{\partial T}\right)_p\right]\frac{DP}{Dt}\\ &= \rho c_p\frac{DT}{Dt} + \rho\left[\frac{1}{\rho} - T\left(\frac{\partial(1/\rho)}{\partial T}\right)_p\right]\frac{DP}{Dt}\\ &= \rho c_p\frac{DT}{Dt} + \left[1 + \left(\frac{\partial(Ln\rho)}{\partial(LnT)}\right)_p\right]\frac{DP}{Dt}\end{aligned} \tag{3.67}$$

Substituting Eq. (3.67) into Eq. (3.61) gives the energy equation with T as its main variable as

$$\rho c_p \frac{DT}{Dt} = -\nabla \cdot \dot{q}_s - \left(\frac{\partial(Ln\rho)}{\partial(LnT)}\right)_p \frac{Dp}{Dt} + (\boldsymbol{\tau} : \nabla \mathbf{v}) + \dot{q}_V \tag{3.68}$$

The above equation is equivalently given by

$$c_p \left[\frac{\partial}{\partial t}(\rho T) + \nabla \cdot [\rho \mathbf{v} T]\right] = -\nabla \cdot \dot{q}_s - \left(\frac{\partial(Ln\rho)}{\partial(LnT)}\right)_p \frac{Dp}{Dt} + (\boldsymbol{\tau} : \nabla \mathbf{v}) + \dot{q}_V \tag{3.69}$$

The heat flux $\dot{q}_S$ appearing in all forms of the energy equation represents heat transfer by diffusion, which is a phenomenon occurring at the molecular level and is governed by Fourier's law according to

$$\dot{q}_s = -[k \nabla T] \tag{3.70}$$

where k is the thermal conductivity of the substance. The above equation states that heat flows in the direction of temperature gradient and assumes that the material has no preferred direction for heat transfer with the same thermal conductivity in all directions, i.e., the medium is isotropic. However some solids are anisotropic for which Eq. (3.70) is replaced by

$$\dot{q}_s = -[\boldsymbol{\kappa} \cdot \nabla T] \tag{3.71}$$

where $\boldsymbol{\kappa}$ is a second order symmetric tensor called the thermal conductivity tensor. Consequently, the heat flux in anisotropic medium is not in the direction of the temperature gradient. In the derivations to follow the medium is assumed to be isotropic and Eq. (3.70) is applicable. Replacing $\dot{q}_s$ using Fourier's law, the energy equation, Eq. (3.69), becomes

$$c_p \left[\frac{\partial}{\partial t}(\rho T) + \nabla \cdot [\rho \mathbf{v} T]\right] = \nabla \cdot [k \nabla T] - \left(\frac{\partial(Ln\rho)}{\partial(LnT)}\right)_p \frac{Dp}{Dt} + (\boldsymbol{\tau} : \nabla \mathbf{v}) + \dot{q}_V \tag{3.72}$$

The expression for $(\boldsymbol{\tau} : \nabla \mathbf{v})$ in terms of the flow variables in a three-dimensional Cartesian coordinate system is given by

$$(\tau : \nabla \mathbf{v}) = \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 + \mu \begin{pmatrix} 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 \\ + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 \end{pmatrix} \tag{3.73}$$

Defining Ψ and Φ as

$$\Psi = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 \tag{3.74}$$

$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 \tag{3.75}$$

The energy equation in terms of temperature reduces to

$$c_p\left[\frac{\partial}{\partial t}(\rho T) + \nabla \cdot [\rho \mathbf{v} T]\right] = \nabla \cdot [k\nabla T] - \left(\frac{\partial(Ln\,\rho)}{\partial(Ln\,T)}\right)_p \frac{Dp}{Dt} + \lambda\Psi + \mu\Phi + \dot{q}_V \tag{3.76}$$

For later reference the energy equation is expanded to

$$\frac{\partial}{\partial t}\left(\rho c_p T\right) + \nabla \cdot \left[\rho c_p \mathbf{v} T\right] = \nabla \cdot [k\nabla T] + \underbrace{\rho T\frac{Dc_p}{Dt} - \left(\frac{\partial(Ln\,\rho)}{\partial(Ln\,T)}\right)_p \frac{Dp}{Dt} + \lambda\Psi + \mu\Phi + \dot{q}_V}_{Q^T} \tag{3.77}$$

and rewritten as

$$\frac{\partial}{\partial t}\left(\rho c_p T\right) + \nabla \cdot \left[\rho c_p \mathbf{v} T\right] = \nabla \cdot [k\nabla T] + Q^T \tag{3.78}$$

The energy equation is rarely solved in its full form and depending on the physical situation several simplified versions can be developed. The dissipation term Φ has negligible values except for large velocity gradients at supersonic speeds. Moreover, for incompressible fluids the continuity equation implies that $\Psi = 0$ and because the density is constant it follows that $(\partial(Ln\rho)/\partial(LnT)) = 0$. Therefore the energy equation [Eq. (3.77)] for incompressible fluid flow is simplified to

$$\frac{\partial}{\partial t}\left(\rho c_p T\right) + \nabla \cdot \left[\rho c_p \mathbf{v} T\right] = \nabla \cdot [k\nabla T] + \underbrace{\dot{q}_V + \rho T\frac{Dc_p}{Dt}}_{Q^T} \tag{3.79}$$

Equation (3.79) is also applicable for a fluid flowing in a constant pressure system. For the case of a solid, the density is constant, the velocity is zero, and if changes in temperature are not large then the thermal conductivity may be considered constant, in which case the energy equation becomes

$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T + \dot{q}_V \tag{3.80}$$

For ideal gases $(\partial(Ln\rho)/\partial(LnT)) = -1$ and the energy equation for compressible flow of ideal gases reduces to

$$c_p\left[\frac{\partial}{\partial t}(\rho T) + \nabla\cdot[\rho \mathbf{v} T]\right] = \nabla\cdot[k\nabla T] + \frac{Dp}{Dt} + \lambda\Psi + \mu\Phi + \dot{q}_V \tag{3.81}$$

If viscosity is neglected (i.e. the flow is inviscid), Eq. (3.81) is further simplified to

$$c_p\left[\frac{\partial}{\partial t}(\rho T) + \nabla\cdot[\rho \mathbf{v} T]\right] = \nabla\cdot[k\nabla T] + \frac{Dp}{Dt} + \dot{q}_V \tag{3.82}$$

3.7 General Conservation Equation

From the above, the governing equations describing the conservation of mass, momentum, and energy are written in terms of *specific* quantities or *intensive* properties, i.e., quantities expressed on a per unit mass basis. The momentum equation, for example, expressed the principle of conservation of linear momentum in terms of the momentum per unit mass, i.e., velocity. The same type of conservation equation may be applied to any intensive property ϕ, e.g., concentration of salt in a solution or the mass fraction of a chemical species. The variation of ϕ in the control volume over time can be expressed as a balance equation of the form

Change of of ϕ over time Δt within the material volume (MV)	=	Surface flux of ϕ over time Δt across the control volume	+	Source/sink of ϕ over time Δt within control volume
Term I		Term II		Term III

For the fixed control volume shown in Fig. 3.9, the change of ϕ over time within the material volume can be written using the Reynolds transport theorem as

$$\text{Term I} = \frac{d}{dt}\left(\int\limits_{MV} (\rho\phi)dV\right) = \int\limits_{V}\left[\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot(\rho\mathbf{v}\phi)\right]dV \tag{3.83}$$

where ρ is the fluid density and V the volume of the control volume of surface area S. The term $\rho\mathbf{v}\phi$ represents the transport of ϕ by the flow field and is denoted by the convective flux, i.e.,

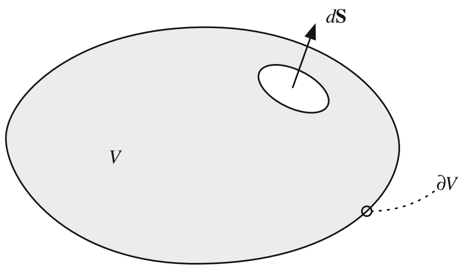


Fig. 3.9 Arbitrary fixed control volume

$$\mathbf{J}^{\phi}_{convection} = \rho \mathbf{v} \phi \tag{3.84}$$

The second term represents variation of ϕ due to physical phenomena occurring across the control volume surface. For the physical phenomena of interest in this book, the mechanism causing the influx/out flux of ϕ is due to diffusion, which is produced by molecular collision and is designated by $\mathbf{J}^{\phi}_{diffusion}$. Denoting the diffusion coefficient of ϕ by Γ^{ϕ}, the diffusion flux may be written as

$$\mathbf{J}^{\phi}_{diffusion} = -\Gamma^{\phi} \nabla \phi \tag{3.85}$$

and Term II becomes

$$\text{Term II} = -\int\limits_{S} \mathbf{J}^{\phi}_{diffusion} \cdot \mathbf{n}\, dS = -\int\limits_{V} \nabla \cdot \mathbf{J}^{\phi}_{diffusion} dV = \int\limits_{V} \nabla \cdot \left(\Gamma^{\phi} \nabla \phi\right) dV \tag{3.86}$$

where **n** is the outward unit vector normal to the surface and the negative sign is due to the adopted sign convention (i.e., inward flux is positive). Term III can be written as

$$\text{Term III} = \int\limits_{V} Q^{\phi} dV \tag{3.87}$$

where Q^{ϕ} is the generation/destruction of ϕ within the control volume per unit volume, which is also called the source term. Thus the conservation equation can be expressed as

$$\int\limits_{V} \left[\frac{\partial}{\partial t}(\rho\phi) + \nabla \cdot (\rho \phi \mathbf{v})\right] dV = \int\limits_{V} \nabla \cdot \left(\Gamma^{\phi} \nabla \phi\right) dV + \int\limits_{V} Q^{\phi} dV \tag{3.88}$$

which can be rearranged into

$$\int_V \left[\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot(\rho\mathbf{v}\phi) - \nabla\cdot\left(\Gamma^{\phi}\nabla\phi\right) - Q^{\phi}\right] dV = 0 \tag{3.89}$$

For the integral to be zero for any control volume, the integrand has to be zero giving the conservation equation in differential form as

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot(\rho\mathbf{v}\phi) - \nabla\cdot\left(\Gamma^{\phi}\nabla\phi\right) - Q^{\phi} = 0 \tag{3.90}$$

For later reference the above equation may be rewritten as

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot\mathbf{J}^{\phi} - Q^{\phi} = 0 \tag{3.91}$$

where the total flux $\mathbf{J}^{\phi}$ is the sum of the convective and diffusive fluxes given by

$$\mathbf{J}^{\phi} = \mathbf{J}^{\phi,C} + \mathbf{J}^{\phi,D} = \rho\mathbf{v}\phi - \Gamma^{\phi}\nabla\phi \tag{3.92}$$

The final form of the general conservation equation, Eq. (3.90), for the transport of a property ϕ is expressed as

$$\underbrace{\frac{\partial}{\partial t}(\rho\phi)}_{\textit{unsteady term}} + \underbrace{\nabla\cdot(\rho\mathbf{v}\phi)}_{\textit{convection term}} = \underbrace{\nabla\cdot\left(\Gamma^{\phi}\nabla\phi\right)}_{\textit{diffusion term}} + \underbrace{Q^{\phi}}_{\textit{source term}} \tag{3.93}$$

By comparing Eq. (3.93) to the various conservation equations derived earlier, it can be easily inferred that by assigning the right values for ϕ, Γ^{ϕ}, and Q^{ϕ}, Eq. (3.93) is a general equation that can represent any of the conservation equations. This is a very important observation that will reduce the necessary developments of the numerical techniques in the coming chapters by concentrating on the general equation [Eq. (3.93)] rather than the individual conservation equations.

3.8 Non-dimensionalization Procedure

The differential equations representing conservation laws are rarely solved using dimensional variables. The common practice is to write these equations in a non-dimensional form using dimensionless quantities that are obtained through the use of proper characteristic scales. The use of non-dimensional variables has several advantages. It allows reducing the number of appropriate parameters for the problem considered and helps revealing the relative magnitude of the various terms in the conservation equation and consequently those that can be neglected.

This simplifies the equation to be solved and leaves only terms of similar order of magnitude, which results in better numerical accuracy. In addition, the generated solution will be applicable to all dynamically similar problems.

A dimensional variable is transformed into a non-dimensional one by dividing the variable by a quantity (composed of one or more physical properties) that has the same dimension as the original variable. For example spatial coordinates can be divided by a characteristic length; velocity can be divided by a characteristic velocity or a combination of quantities $\left(\mu_{ref}/\rho_{ref}l_{ref}\right)$ that have collectively the same units as velocity (m/s); pressure is usually divided by a reference dynamic pressure $\left(\rho_{ref}\left|\mathbf{v}_{ref}\right|^2\right)$; time can be divided by the ratio of a characteristic length to a reference velocity $\left(l_{ref}/\left|\mathbf{v}_{ref}\right|\right)$, and so on. The best way to fully understand how to write equations in non-dimensional form is through an example. For that purpose, an incompressible viscous flow with constant viscosity and thermal conductivity, and with body forces acting in the y-direction (i.e., the gravitational acceleration is given by $\mathbf{g} = (0, -g, 0)$) is considered. The equations governing conservation of mass, momentum, and energy in a three-dimensional Cartesian coordinate system with no heat generation are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3.94}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho vu) + \frac{\partial}{\partial z}(\rho wu) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{3.95}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho vv) + \frac{\partial}{\partial z}(\rho wv) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) - \rho g \tag{3.96}$$

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho uw) + \frac{\partial}{\partial y}(\rho vw) + \frac{\partial}{\partial z}(\rho ww) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{3.97}$$

$$c_p\left[\frac{\partial}{\partial t}(\rho T) + \frac{\partial}{\partial x}(\rho uT) + \frac{\partial}{\partial y}(\rho vT) + \frac{\partial}{\partial z}(\rho wT)\right] = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{3.98}$$

When body forces are of negligible magnitude in comparison with other forces, the term $\rho\mathbf{g}$ can be set to zero and removed from the equations. In such situation, the flow field is independent of the temperature field and solution for the velocity field can be established separately followed by the solution to the temperature field. However for a flow to exist the fluid should be forced through the domain.

Therefore the fluid should possess an inlet velocity. This velocity becomes an important parameter (the characteristic velocity) when writing the equations in non-dimensional forms and the heat transfer mechanism, if any, is said to occur by forced convection.

On the other hand, when a flow field is naturally established due to a temperature difference in the domain, body forces cannot be neglected. In such situations, variations in temperature cause variations in density (as mentioned earlier), which give rise to buoyancy forces that drive the flow. In this case the transfer of heat is stated to happen by natural convection. As the flow is initiated naturally, a characteristic velocity is not apparent and cannot be part of the dimensionless number since its scale is not known. Therefore in order to write the velocity in a non-dimensional form, a combination of physical quantities that has the same dimension as a velocity should be used. The following discussion assumes a natural convection problem.

If the difference in temperature $\Delta T = T - T_\infty$ (where T_∞ is a reference temperature between the minimum and maximum temperature in the domain, usually taken as the average value) is small such that terms of order ΔT^2 or higher can be neglected, then the value of density at any temperature T can be written as a function of its value at the reference temperature T_∞ using a truncated Taylor series expansion as

$$\rho = \rho|_{T=T_\infty} + \left.\frac{d\rho}{dT}\right|_{T=T_\infty}(T - T_\infty) \tag{3.99}$$

where terms of order ΔT^2 or higher are omitted. Introducing the coefficient of volume expansion β defined as

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \tag{3.100}$$

the equation for density (or equation of state) becomes

$$\rho = \rho_\infty[1 - \beta(T - T_\infty)] \tag{3.101}$$

which is known in the literature by the Boussinesq approximation [11]. Using this expression for ρ in the body force term only and denoting the constant density value by ρ to simplify the notation, the y-momentum equation is transformed to

$$\begin{aligned} &\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}(\rho v v) + \frac{\partial}{\partial z}(\rho w v) \\ &= -\frac{\partial}{\partial y}(p + \rho g y) + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho g \beta (T - T_\infty) \end{aligned} \tag{3.102}$$

This clearly shows that in solving natural convection problems the momentum and energy equations are coupled together necessitating a simultaneous solution of both equations.

The non-dimensional forms of the conservation equations are obtained by defining the following dimensionless parameters:

$$\begin{aligned}&\hat{x}=\frac{x}{L},\hat{y}=\frac{y}{L},\hat{z}=\frac{z}{L},\hat{u}=\frac{u}{\mu/(\rho L)},\hat{v}=\frac{v}{\mu/(\rho L)},\hat{w}=\frac{w}{\mu/(\rho L)}\\&\hat{t}=\frac{t}{\rho L^2/\mu},\hat{p}=\frac{p+\rho g y}{\mu^2/(\rho L^2)},\hat{T}=\frac{T-T_\infty}{T_{\max}-T_\infty}\end{aligned} \tag{3.103}$$

where L is a characteristic length, μ the dynamic viscosity of the fluid, T_{max} the maximum temperature in the domain, and the over $\wedge$ is used to designate non-dimensional quantities. The various expressions in the conservation equations are written in terms of the new variables as described next. The procedure is explained by considering a typical term from each category.

Typical term in the continuity equation:

$$\frac{\partial u}{\partial x}=\frac{\partial[\mu\hat{u}/(\rho L)]}{\partial(L\hat{x})}=\frac{\mu/(\rho L)}{L}\frac{\partial\hat{u}}{\partial\hat{x}}=\frac{\mu}{\rho L^2}\frac{\partial\hat{u}}{\partial\hat{x}} \tag{3.104}$$

Typical terms in the momentum equations:

$$\frac{\partial}{\partial t}(\rho u)=\frac{\partial(\mu\hat{u}/L)}{\partial(\rho L^2\hat{t}/\mu)}=\frac{\mu/L}{\rho L^2/\mu}\frac{\partial\hat{u}}{\partial\hat{t}}=\frac{\mu^2}{\rho L^3}\frac{\partial\hat{u}}{\partial\hat{t}} \tag{3.105}$$

$$\frac{\partial}{\partial t}(\rho uu)=\frac{\partial[\mu^2/(\rho L^2)\hat{u}\hat{u}]}{\partial(L\hat{x})}=\frac{\mu^2/(\rho L^2)}{L}\frac{\partial}{\partial\hat{x}}(\hat{u}\hat{u})=\frac{\mu^2}{\rho L^3}\frac{\partial}{\partial\hat{x}}(\hat{u}\hat{u}) \tag{3.106}$$

$$\begin{aligned}\hat{p}=\frac{p+\rho g y}{\mu^2/(\rho L^2)}\Rightarrow\frac{\partial\hat{p}}{\partial\hat{x}}&=\frac{\partial\{(p+\rho g y)/[\mu^2/(\rho L^2)]\}}{\partial(x/L)}\\&=\frac{\rho L^3}{\mu^2}\frac{\partial p}{\partial x}\Rightarrow\frac{\partial p}{\partial x}=\frac{\mu^2}{\rho L^3}\frac{\partial\hat{p}}{\partial\hat{x}}\end{aligned} \tag{3.107}$$

$$\mu\frac{\partial^2 u}{\partial x^2}=\mu\frac{\partial^2[\mu\hat{u}/(\rho L)]}{\partial(L\hat{x})^2}=\mu\frac{\mu/(\rho L)}{L^2}\frac{\partial^2\hat{u}}{\partial\hat{x}^2}=\frac{\mu^2}{\rho L^3}\frac{\partial^2\hat{u}}{\partial\hat{x}^2} \tag{3.108}$$

$$\rho g\beta(T-T_\infty)=\rho g\beta(T_{\max}-T_\infty)\hat{T}=\rho g\beta(\Delta T)\hat{T} \tag{3.109}$$

Typical terms in the energy equation:

$$\frac{\partial}{\partial t}(\rho T)=\frac{\partial\left[\rho\left(T_\infty+\Delta T\hat{T}\right)\right]}{\partial(\rho L^2\hat{t}/\mu)}=\frac{\mu\Delta T}{L^2}\frac{\partial\hat{T}}{\partial\hat{t}} \tag{3.110}$$

$$\frac{\partial}{\partial x}(\rho u T) = \frac{\partial\left(\mu\hat{u}\left(T_\infty + \Delta T\hat{T}\right)/L\right)}{\partial(L\hat{x})} = \frac{\mu T_\infty}{L^2}\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\mu\Delta T}{L^2}\frac{\partial}{\partial \hat{x}}\left(\hat{u}\hat{T}\right) \tag{3.111}$$

$$k\frac{\partial^2 T}{\partial x^2} = k\frac{\partial^2\left(T_\infty + \Delta T\hat{T}\right)}{\partial(L\hat{x})^2} = \frac{k\Delta T}{L^2}\frac{\partial^2 \hat{T}}{\partial \hat{x}^2} \tag{3.112}$$

Substituting terms by their equivalent expressions, the non-dimensional forms of the continuity, momentum, and energy equations are obtained as

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} = 0 \tag{3.113}$$

$$\frac{\partial \hat{u}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{u}\hat{u}) + \frac{\partial}{\partial \hat{y}}(\hat{v}\hat{u}) + \frac{\partial}{\partial \hat{z}}(\hat{w}\hat{u}) = -\frac{\partial \hat{p}}{\partial \hat{x}} + \left(\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{z}^2}\right) \tag{3.114}$$

$$\frac{\partial \hat{v}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{u}\hat{v}) + \frac{\partial}{\partial \hat{y}}(\hat{v}\hat{v}) + \frac{\partial}{\partial \hat{z}}(\hat{w}\hat{v}) = -\frac{\partial \hat{p}}{\partial \hat{y}} + \left(\frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{z}^2}\right) + Gr\hat{T} \tag{3.115}$$

$$\frac{\partial \hat{w}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{u}\hat{w}) + \frac{\partial}{\partial \hat{y}}(\hat{v}\hat{w}) + \frac{\partial}{\partial \hat{z}}(\hat{w}\hat{w}) = -\frac{\partial \hat{p}}{\partial \hat{z}} + \left(\frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{w}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{w}}{\partial \hat{z}^2}\right) \tag{3.116}$$

$$\frac{\partial \hat{T}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}\left(\hat{u}\hat{T}\right) + \frac{\partial}{\partial \hat{y}}\left(\hat{v}\hat{T}\right) + \frac{\partial}{\partial \hat{z}}\left(\hat{w}\hat{T}\right) = \frac{1}{Pr}\left(\frac{\partial^2 \hat{T}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{T}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{T}}{\partial \hat{z}^2}\right) \tag{3.117}$$

where Gr is the Grashof number, Pr is the Prandtl number, and ν the kinematic viscosity defined as

$$Gr = \frac{g\beta\Delta T L^3}{\nu^2} \quad Pr = \frac{\mu c_p}{k} \quad \nu = \frac{\mu}{\rho} \tag{3.118}$$

The Grashof and Prandtl numbers [12–14] are dimensionless groups formed of a combination of the involved physical properties. Therefore the number of parameters affecting the solution was reduced to two and solutions can be generated for different values of these two parameters. Moreover any single solution will be valid for many combinations of the physical properties of which these two numbers are composed, as long as these combinations result in the Gr and Pr values for which the solution was obtained. The physical significance of these two dimensionless numbers and others that may arise when writing the conservation equations in non-dimensional forms using other dimensionless parameters under different conditions is discussed next.

3.9 Dimensionless Numbers

Writing the conservation equations in non-dimensional forms, results in dimensionless numbers that are very useful for performing parametric studies of engineering problems. For incompressible viscous flow, the dimensionless parameters governing natural convection heat transfer were reduced to the two dimensionless numbers *Gr* [12–14] and *Pr* [12–14]. Under different conditions (e.g., compressible flows, Porous flows, etc.) other types of fluid forces and dissipation terms may be included in the governing equations resulting in different non-dimensional groups. For flow in porous media, for example, Darcy number (Da) [15, 16] emerges as an important parameter, for a free surface flow the Weber number (We) [17, 18], for an open channel flow the Froude number (Fr) [19], for a compressible flow the Mach number (M) [20], and so on. Some of the most important dimensionless groups are discussed below.

3.9.1 Reynolds Number

The Reynolds number (Re) [12, 13] is defined as

$$Re = \frac{\rho U L}{\mu} \tag{3.119}$$

and may be interpreted as a measure of the relative importance of advection (inertia) to diffusion (viscous) momentum fluxes. If the momentum fluxes are in the same direction then the Reynolds number reveals the boundary layer characteristics of the flow. If the fluxes are defined such that the diffusion is in the cross stream direction, then as shown in Fig. 3.10 *Re* conveys the flow regime (i.e. laminar, transitional, or turbulent).

An example showing the flow field for different values of Reynolds number is depicted in Fig. 3.11. It represents a driven flow in a square cavity of side L generated by the velocity U imparted to its top wall. The streamlines shown in Fig. 3.11 indicates that the strength of the flow increases as $Re = \rho UL/\mu$ increases.

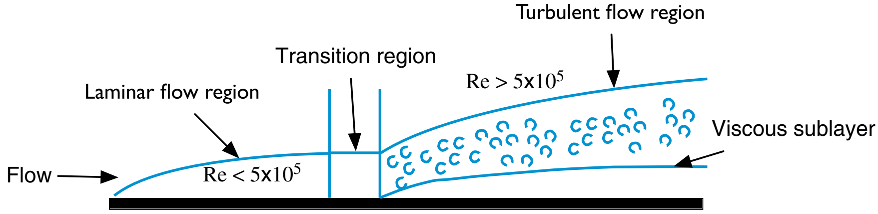


Fig. 3.10 Schematic of the flow over a flat plate showing the laminar, transitional, and turbulent flow regimes based on the value of *Re*

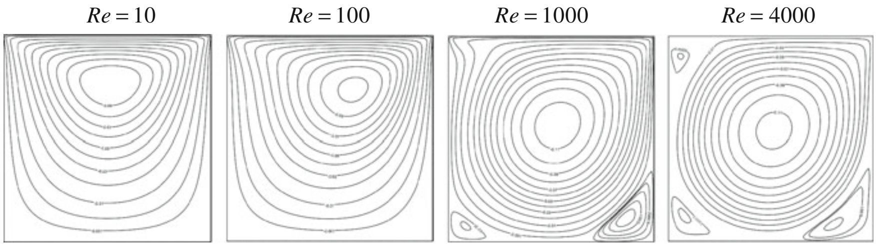


Fig. 3.11 Streamlines at increasing values of Reynolds number for driven flow in a square cavity

3.9.2 Grashof Number

As derived above the Grashof number [12–14] is given by

$$Gr = \frac{g\beta\Delta TL^3}{\nu^2} \tag{3.120}$$

The Grashof number represents the ratio of buoyant to viscous forces. It plays in natural convection the same role played by the Reynolds number in forced convection. An example showing the effect of Grashof number is depicted in Fig. 3.12. The physical situation represents natural convection heat transfer in the annulus between a hot circular cylinder and its cold square enclosure. Isotherms displayed in the figure are seen to become more distorted at higher values of Gr due to higher natural convection effects caused by a stronger flow field.

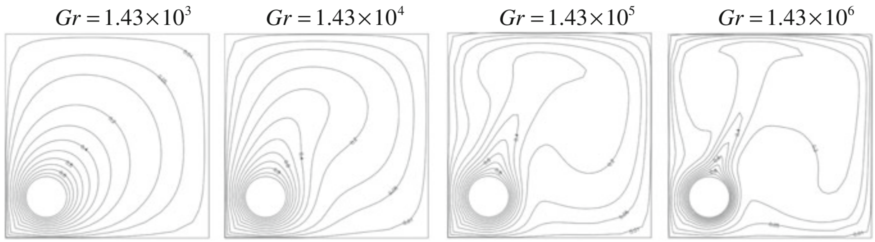


Fig. 3.12 Isotherms at increasing values of Grashof number for natural convection in the annulus between eccentric horizontal hot circular and cold square cylinders

3.9.3 Prandtl Number

The Prandtl number [12–14] is defined as the ratio of momentum diffusivity (kinematic viscosity ν) to thermal diffusivity (α), i.e.,

$$Pr = \frac{\mu c_p}{k} = \frac{\mu/\rho}{k/\rho c_p} = \frac{\nu}{\alpha} \tag{3.121}$$

The Prandtl number represents the ratio of hydrodynamic boundary layer to thermal boundary layer. As displayed in Fig. 3.13, the thermal boundary layer is larger than the hydrodynamic boundary layer for $Pr < 1$ (Fig. 3.13a) and the opposite is true for $Pr > 1$ (Fig. 3.13b). Both layers coincide for $Pr = 1$.

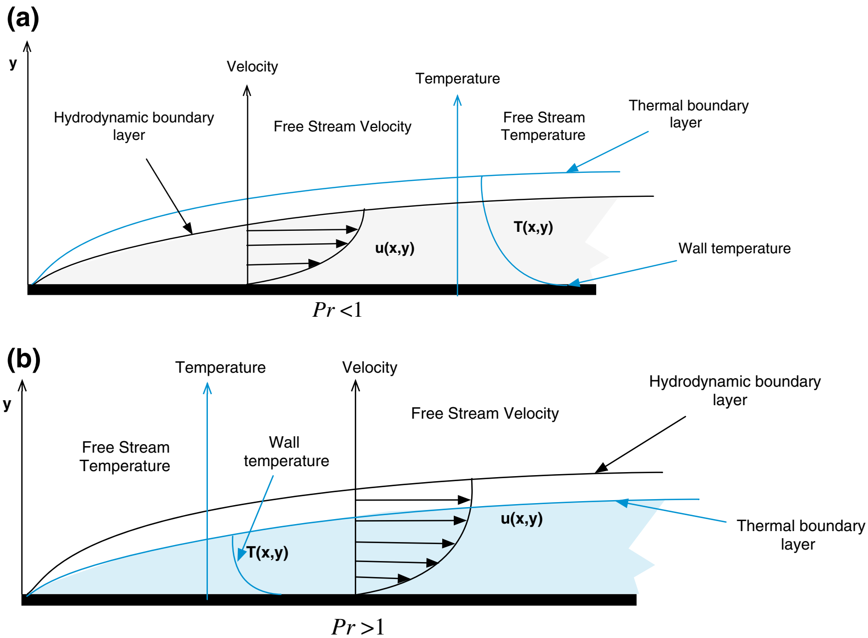


Fig. 3.13 The thermal and hydrodynamic boundary layer thicknesses for **a** $Pr < 1$ and **b** $Pr > 1$

For the driven flow in a square cavity problem presented above, isotherms over the domain are depicted in Fig. 3.14 for different values of Pr while holding Re constant at 100. The increase in convection over conduction as Pr increases can be easily inferred from the plots.

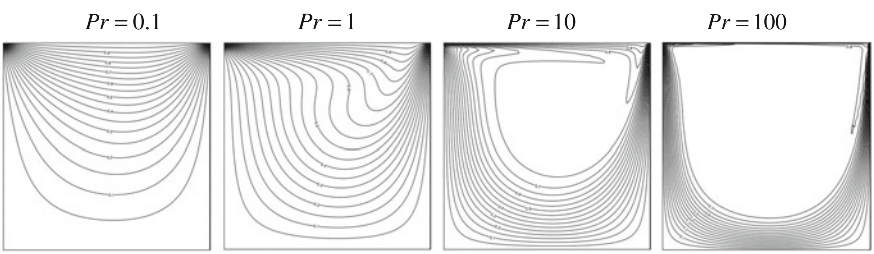


Fig. 3.14 Isotherms at increasing values of Prandtl number for driven flow in a square cavity ($Re = 100$)

3.9.4 Péclet Number

The Péclet number [5] is defined as the ratio of the advective transport rate of a physical quantity to its diffusive transport rate. For the case of heat transfer, the Péclet number is given by

$$Pe = \frac{\rho U L c_p}{k} = \frac{UL}{\alpha} = Re^{*} Pr \tag{3.122}$$

In this situation the Pe is equivalent to the product of the Reynolds number and the Prandtl number. An example of the effects of Pe is shown in Fig. 3.15, where isotherms over a flat hot plate are displayed at different values of Péclet number. Heat transfer is seen to be dominated by conduction at low values of Pe with convection gaining increasing importance as Pe increases to become clearly the dominant heat transfer mode at $Pe = 1000$.

Fig. 3.15 Isotherms at increasing values of Péclet number for fluid flow over a flat plate maintained at a hot uniform temperature

For mass transport, the Péclet number is given by

$$Pe = \frac{UL}{D} = Re^{*} Sc \tag{3.123}$$

where D is the mass diffusivity and Sc the Schmidt number. In this case Pe is equivalent to the product of the Reynolds number and the Schmidt number.

A large Péclet number indicates low dependence of the flow on downstream locations and high dependence on upstream locations. Therefore simpler computational models can be adopted for simulating situations with high Péclet numbers.

3.9.5 Schmidt Number

The Schmidt number [14] is defined as

$$Sc = \frac{\nu}{D} \tag{3.124}$$

The Schmidt number in mass transfer is the counterpart of the Prandtl number in heat transfer. It represents the ratio of the momentum diffusivity (ν) to mass diffusivity (D). Physically, the Sc relates the thicknesses of the hydrodynamic and mass transfer boundary layers. An example showing the effect of Schmidt number is shown in Fig. 3.16.

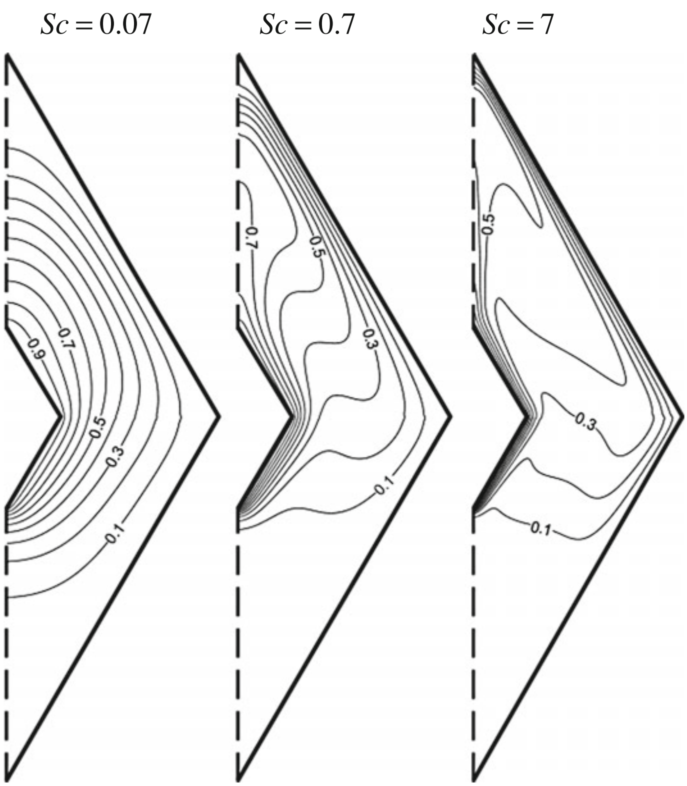


Fig. 3.16 Iso-concentrations at increasing values of Schmidt number (other parameters held fixed) for natural convection mass transfer in the annulus between concentric horizontal cylinders of rhombic cross sections with larger solute concentration on the inner wall

The figure above represents natural convection mass transfer in the annulus between two horizontal pipes of rhombic cross sections. The solute concentration is higher along the inner wall of the enclosure. The concentration non-uniformity causes variations in density establishing a flow field. The strength of the flow increases with increasing Sc values as manifested by the higher distortion of iso-concentration lines that indicates an increase in convection mass transfer over diffusion mass transfer, which dominates at low Sc values.

3.9.6 Nusselt Number

The Nusselt number [12–14] expressed as

$$Nu = \frac{hL}{k} \tag{3.125}$$

is the dimensionless form of the convection heat transfer coefficient h and provides a measure of the convection heat transfer at a solid surface. The Nusselt number does not arise as a dimensionless group when writing the conservation equations in non-dimensional forms; rather, it is widely used to report convection heat transfer data.

3.9.7 Mach Number

The Mach number (M) [20] is defined as the ratio of speed of an object moving through a fluid and the local speed of sound [20]. Mathematically it is written as

$$M = \frac{|\mathbf{v}|}{a} \tag{3.126}$$

where $|\mathbf{v}|$ is the local magnitude of the fluid velocity relative to the medium in which it is flowing and a is the speed of sound. The general equation for the speed of sound is given by

$$a = \sqrt{\gamma \left(\frac{\partial p}{\partial \rho}\right)_T} \tag{3.127}$$

For an ideal gas, it reduces to

$$a = \sqrt{\gamma R T} \tag{3.128}$$

where γ is the ratio of specific heat at constant pressure to specific heat at constant volume $\left(c_p/c_v\right)$ and R is the gas constant.

Flows for which the Mach number is less than 0.2 can be treated as incompressible. For $M < 1$ the flow is called subsonic, for $M = 1$ sonic, for $1 < M < 5$ supersonic, and for $M > 5$ hypersonic. Moreover, a flow accelerating from subsonic to supersonic is called a transonic flow. The value of Mach number (less than 1 or greater than 1) at the boundaries of a domain dictates the number of required boundary conditions there.

Examples of subsonic, transonic, and supersonic flow fields are presented in Fig. 3.17 via Mach contours. The physical situation represents a fluid flowing over

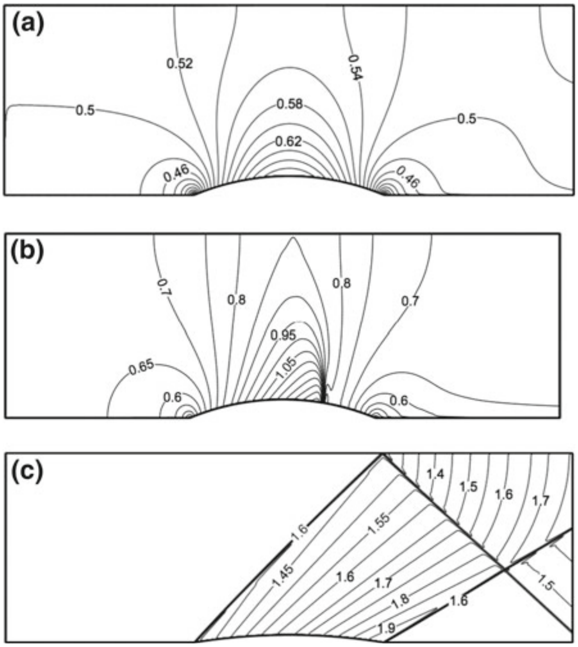


Fig. 3.17 Mach contours for the flow over a circular arc bump at **a** subsonic, **b** transonic, and **c** supersonic speeds

a circular arc bump with a maximum curvature of 10 % the channel height in the subsonic and transonic cases and of 4 % in the supersonic case. The change in the flow type from elliptic to hyperbolic (with discontinuities in the form of shock waves) as the Mach number increases from subsonic $(M < 1)$ to supersonic $(M > 1)$ is apparent.

3.9.8 Eckert Number

The Eckert number (Ec) [21] is a dimensionless number relating the kinetic energy of the flow to its enthalpy and is computed as

$$Ec = \frac{\mathbf{v} \cdot \mathbf{v}}{c_p \Delta T} \tag{3.129}$$

where ΔT is a characteristic temperature difference. This dimensionless number appears as a factor multiplying the viscous dissipation term Φ, when non-dimensionalizing the compressible energy equation. A large value of Ec indicates high viscous dissipation occurring at high speed of the flow (high kinetic energy). For small Eckert number $(Ec \ll 1)$ several terms in the energy equation become negligible (e.g., viscous dissipation, body forces, etc.). This reduces the energy equation to its incompressible form (i.e., a balance between conduction and convection).

3.9.9 Froude Number

The Froude number (Fr) [19] is a dimensionless number defined as the ratio of a characteristic velocity (U) to a gravitational wave velocity $(\sqrt{gL})$ as

$$Fr = \frac{U}{\sqrt{gL}} \tag{3.130}$$

It is a measure of the resistance of partially immersed objects moving through fluids, with higher Fr values indicating higher fluid resistance.

For the free-surface flow shown in Fig. 3.18, the nature of the flow is dictated by the value of Froude number. For $Fr > 1$ the flow is supercritical and for $Fr < 1$ it is subcritical. The flow at the interface between the two regions, known as the “hydraulic jump”, is just critical and is characterized by a Froude number value of 1.

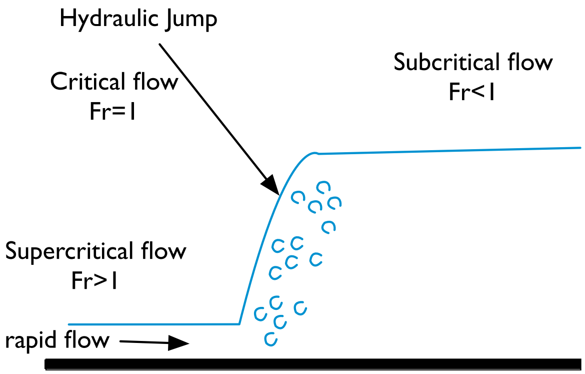


Fig. 3.18 A free surface flow showing the supercritical, critical, and subcritical regions

3.9.10 Weber Number

The dimensionless Weber number (We) [17, 18] is defined as

$$We = \frac{\rho U^2 L}{\sigma} \tag{3.131}$$

where U (m/s) and L (m) are the characteristic velocity and length, respectively, and σ the surface tension (N/m). The Weber number, which represents the ratio of inertia to surface tension forces, is helpful in analyzing multiphase flows involving interfaces between two different fluids, with curved surfaces such as droplets and bubbles.

3.10 Closure

This chapter has shown that many physical phenomena can be modeled through conservation equations. These equations are derived from first principles by writing balances over a finite volume. It was also shown that the conservation equations governing the transport of mass, momentum, energy, and other specific quantities have a common form embodied in the general scalar transport equation. This equation has transient, convection, diffusion and source terms. Each term brings a characteristic contribution to the equation that needs to be reproduced by the discretization procedure.

3.11 Exercises

Exercise 1

By comparing the continuity, momentum, and energy equations with the general scalar transport equation, derive expressions for ϕ, Γ^{ϕ} and S^{ϕ}.

Exercise 2

Show that for an incompressible flow of constant viscosity the following holds:

$$\nabla \cdot \left\{ \mu \left[\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}} \right] \right\} = \mu \nabla^2 \mathbf{v}$$

Exercise 3

A steady incompressible flow field is defined by the following velocity vector:

$$\mathbf{v} = (x + y)\mathbf{i} + (y + z)\mathbf{j} + 2(x - z)\mathbf{k}$$

(a) Verify that it satisfies the continuity equation.
(b) Assuming constant viscosity μ, calculate the viscous stress tensor $\boldsymbol{\tau}$.
(c) Denoting the fluid density by ρ and neglecting body forces, develop an equation for the pressure gradient.

Exercise 4

The vorticity ω of a flow field is defined as the curl of the velocity vector, i.e.,

$$\omega = \nabla \times \mathbf{v}$$

Using the above definition of vorticity, show that for an incompressible fluid, the following relation between the velocity and vorticity vectors holds:

$$\nabla \cdot [(\mathbf{v} \cdot \nabla)\mathbf{v}] = 0.5\nabla^2(\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \cdot \left[\nabla^2 \mathbf{v}\right] - \omega \cdot \omega$$

Exercise 5
A flow is said to be irrotational if its vorticity (defined in Exercise 4 above) is zero, i.e., $\omega = 0$. Show that for a steady two dimensional incompressible irrotational flow the velocity field satisfies the following Laplace equation

$$\nabla^2 \mathbf{v} = 0$$

Exercise 6
Consider a two-dimensional square enclosure of side L. The enclosure is filled with an incompressible fluid of viscosity μ and density ρ. The top side of the enclosure is covered with an infinite horizontal wall moving with a constant velocity U. The other sides are fixed in place. Due to the motion of the top wall a flow field is established within the enclosure. Using appropriate dimensionless variables, write the simplified momentum equation for the flow field in dimensionless form showing that the Reynolds number $(Re = \rho UL/\mu)$ is the only dimensionless group affecting the flow.

Exercise 7
Consider the steady two dimensional mixed convection heat transfer in a vertical rectangular channel of width W. A cold fluid of density ρ_{in} and temperature T_{in} enters the channel with a velocity V_{in}. As it flows vertically upward, the fluid is heated by the duct walls, which are maintained at the uniform hot temperature T_w. Taking buoyancy forces into consideration through the Boussinesq approximation and using the following dimensionless variables

$$x^* = \frac{x}{W}, y^* = \frac{y}{W}, u^* = \frac{u}{V_{in}}, v^* = \frac{v}{V_{in}}, \theta = \frac{T - T_{in}}{T_w - T_{in}}, p^* = \frac{p + \rho_{in} g y}{\rho_{in} V_{in}^2}$$

write the conservation equations of mass, momentum, and energy in dimensionless forms. What are the dimensionless groups governing the flow and heat transfer in the channel? What does each of them represent?

Exercise 8
Estimates the Reynolds number of the following flows:

(a) Water flowing at a speed of 15 km/hr over a whale 10 m long.
(b) Air flowing at a speed of 800 km/hr over the wing of an F16 airplane of mean chord length 3.450336 m.
(c) Glycerine of dynamic viscosity 0.96 kg/ms and density 1258 kg/m^3 flowing at a speed of 2.8 m/s in a pipe inclined at 25° to the horizontal and of diameter 250 mm.

Exercise 9 Starting from the incompressible version of the Navier-Stokes equations derive simplified equations based on the following assumptions:

(a) Viscous effects are much more significant than any effects of fluid acceleration, i.e.,

$$\frac{\partial}{\partial t}(\mathbf{v}) + \nabla \cdot [\mathbf{v}\mathbf{v}] \ll \nabla \cdot [\mu \nabla \mathbf{v}]$$

which corresponds to $Re = \rho UL/\mu \ll 1$ (Stokes Equations).

(b) Inertial effects dominate and viscous effects are considered to be negligible throughout the flow domain, i.e.,

$$\frac{\partial}{\partial t}(\mathbf{v}) + \nabla \cdot [\mathbf{v}\mathbf{v}] \gg \nabla \cdot [\mu \nabla \mathbf{v}]$$

which corresponds to $Re = \rho UL/\mu \gg 1$ (Euler equations).

(c) Derive the Bernoulli equation from momentum conservation with the following hypothesis: one dimensional steady state conditions of a frictionless fluid $\mu = 0$.

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