Chapter 9: Filter Circuits: Frequency Response, Bode Plots, and Fourier Transform

Overview

Prerequisites:
- Knowledge of complex arithmetic
- Knowledge of superposition principle for linear circuits (Chapter 3)
- Knowledge of harmonic voltage and current behavior (Chapter 8)
- Knowledge of phasor/impedance method for AC circuit analysis (Chapter 8)
- Knowledge of an operational amplifier with negative feedback (Chapter 5)

Objectives of Section 9.1:
- Establish the concept of a first-order analog filter as a two-port network
- Understand the difference between high-pass and low-pass filters
- Understand the effect of filter termination
- Become familiar with the fundamental filter characteristics including transfer function, break frequency, roll-off, and high-/low-frequency asymptotes
- Understand the construction of the Bode plot including decibels; become familiar with some of the jargon used by electrical engineers
- Establish the close agreement between first-order RC and RL filters; become familiar with the concept of cascaded filter networks

Objectives of Section 9.2:
- Establish the model for the open-loop gain of an operational amplifier as a function of frequency
- Understand the meaning of datasheet parameters such as unity-gain bandwidth and gain-bandwidth product
- Establish the model for the closed-loop gain of an operational amplifier as a function of frequency from first principles
- Find the frequency bandwidth for any practical operational amplifier circuit using the datasheet

S.N. Makarov et al., *Practical Electrical Engineering*, DOI 10.1007/978-3-319-21173-2_9

Objectives of Section 9.3:

- Obtain an introductory exposure to the continuous Fourier transform and be able to compute the transform for simple examples including the meaning of a sinc function
- Be able to relate continuous and discrete Fourier transform via the Riemann sum approximation
- Be able to define sampling points of the DFT in both time and frequency domain
- Understand the structure and ordering of the DFT frequency spectrum including its relation to negative frequencies
- Apply the DFT to a filter with a given transfer function and generate the discrete frequency spectrum of the output signal
- Apply the DFT to filter operation with input pulse or nonperiodic signals
- Apply the DFT (FFT and IFFT) in MATLAB

Application examples:

- Effect of a load connected to the filter
- Effect of next-stage filter load
- Finding bandwidth of an amplifier circuit using the datasheet
- Selection of an amplifier IC for proper frequency bandwidth
- Numerical differentiation via the FFT
- Filter operation for an input pulse signal
- Converting computational electromagnetic solution from frequency domain to time domain

Keywords:

Analog filter, RC filter, RL filter, Port, Two-port network, First-order high-pass filter, First-order low-pass filter, Filter termination, Amplitude transfer function, Phase transfer function, Power transfer function, Complex transfer function, Frequency response, Break frequency, Half-power frequency, 3-dB frequency, Corner frequency, Bode plot, Decibel, Roll-off, High-frequency asymptote, Low-frequency asymptote, Frequency band, Passband, Stopband, Decade, Octave, Power gain, Open-loop amplifier gain, Unity-gain bandwidth, Gain-bandwidth product, Internal compensation, Open-loop AC gain, Closed-loop AC gain, Amplifier circuit bandwidth, **Fourier transform continuous** (direct inverse Fourier spectrum, direct inverse Fourier spectrum, bandlimited spectrum, reversal property, sinc function, mathematical properties, amplitude-modulated signal, Parseval's theorem, energy spectral density), **Fourier transform discrete** (Fast digital signal processing (DSP), sampling points, sampling interval, sampling frequency, sampling theorem, Riemann sum approximation, rectangle rule, fundamental frequency, direct (DFT), inverse (IDFT), standard form, reversal property, structure of discrete spectrum, numerical differentiation, filter operation for pulse signals)

Section 9.1 First-Order Filter Circuits and Their Combinations

AC voltage divider circuits (either RC or RL) generally operate as analog *filters*. They pass certain voltage signals but stop or cut out other signals, depending on the signal's frequency content. The analog filters studied in this section are *first-order* filters since they may be described by first-order differential equations—we discussed them in Chapter 7. The phasor/impedance method is applied to solve the AC circuit, both in analytical and in numerical form. Although over the years the value of the numerical analysis has greatly increased in engineering, the analytical method remains important if we are interested in a parametric study such as a rigorous filter analysis. The analytical method involves (multiple) conversions of complex numbers or expressions from the rectangular to polar form and vice versa. Generally, division and multiplication are better carried out in polar form, whereas addition and subtraction require a rectangular form.

9.1.1 RC Voltage Divider as an Analog Filter

The RC voltage divider circuit shown in Fig. 9.1a is perhaps the oldest and best-known version of an *analog filter*. In order to understand its operation, we must obtain a general solution to the RC circuit in Fig. 9.1a. Even though the solution has to work at any frequency f or angular frequency ω of interest, it is not difficult to find.

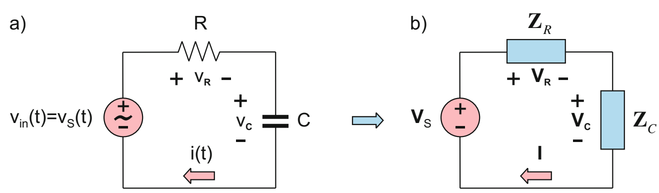


Fig. 9.1. RC voltage divider circuit and its solution by the phasor method. We note the phasors for the voltages $\mathbf{V}_S$, $\mathbf{V}_R$, $\mathbf{V}_C$ and the phasor for the circuit current $\mathbf{I}$.

What is an analog filter? The goal of the filter is to accept an AC voltage signal at its input and either pass the signal to the next circuit block or to stop (or "cut out") the signal, depending on its particular frequency. Imagine a human voice that is mixed with white noise with a spectrum extended over all frequencies. If the noise level is high, we will probably only hear screaming at high frequencies. However, if we only pass the voltage signals with frequencies below 3 kHz, where most of the voice power is concentrated, the resulting total signal will be much clearer for listening. The analog filter is an electric circuit which, in its simplest form, is identical to the circuit in Fig. 9.1.

General Solution

Let us first convert the circuit in Fig. 9.1a to a phasor form as shown in Fig. 9.1b. We assume that $v_S(t) = V_m \cos \omega t$; therefore $\mathbf{V}_S = V_m$. Next, we solve the resulting "DC circuit" in the complex domain. The voltage division yields

$$\mathbf{V}_C = \frac{\mathbf{Z}_C}{\mathbf{Z}_R + \mathbf{Z}_C} V_m = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} V_m = \frac{1}{1 + j\omega RC} V_m = \frac{1}{1 + j\omega\tau} V_m \ \ [\mathrm{V}] \tag{9.1a}$$

$$\mathbf{V}_R = \frac{\mathbf{Z}_R}{\mathbf{Z}_R + \mathbf{Z}_C} V_m = \frac{R}{R + \frac{1}{j\omega C}} V_m = \frac{j\omega RC}{1 + j\omega RC} V_m = \frac{j\omega\tau}{1 + j\omega\tau} V_m \ \ [\mathrm{V}] \tag{9.1b}$$

where $\tau = RC$ is exactly the *same* time constant that appears for transient circuits in Chapter 7. Converting Eq. (9.1a) and (9.1b) into polar form gives

$$\mathbf{V}_C = \frac{1}{\sqrt{1 + (\omega\tau)^2}} V_m \angle \varphi_C, \quad \varphi_C = -\tan^{-1}(\omega\tau) \tag{9.1c}$$

$$\mathbf{V}_R = \frac{\omega\tau}{\sqrt{1 + (\omega\tau)^2}} V_m \angle \varphi_R, \quad \varphi_R = \frac{\pi}{2} - \tan^{-1}(\omega\tau) \tag{9.1d}$$

After the polar form has been obtained, the real-valued voltages are found in the form

$$\begin{aligned} v_C(t) &= V_{mC} \cos(\omega t + \varphi_C) \ \ [\mathrm{V}], \quad V_{mC} = \frac{1}{\sqrt{1 + (\omega\tau)^2}} V_m \ \ [\mathrm{V}] \\ v_R(t) &= V_{mR} \cos(\omega t + \varphi_R) \ \ [\mathrm{V}], \quad V_{mR} = \frac{\omega\tau}{\sqrt{1 + (\omega\tau)^2}} V_m \ \ [\mathrm{V}] \end{aligned} \tag{9.1e}$$

The general solution of the RC circuit in Fig. 9.1a is now complete. The key observations are that the amplitudes of the resistor voltage and the capacitor voltage now become functions of frequency.

Qualitative Analysis

The circuit in Fig. 9.1 is a voltage divider. The supply voltage (or the input voltage to the filter) is divided between the capacitor and the resistor. Which voltage dominates at low frequencies and which at high frequencies? To answer those questions, we consider Eq. (9.1e). When $\omega \to 0$,

$$V_{\mathrm{mC}} = \frac{1}{\sqrt{1+(\omega\tau)^2}} V_{\mathrm{m}} \to V_{\mathrm{m}}$$
$$V_{\mathrm{mR}} = \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}} V_{\mathrm{m}} \to 0 \tag{9.2a}$$

Therefore, at *low frequencies*, the *capacitor voltage* dominates; it is approximately equal to the supply voltage. This fact is quite clear because the capacitor acts like an open circuit for DC, implying that the capacitor voltage "sees" nearly all the supply voltage. On the other hand, when $\omega \to \infty$,

$$V_{\mathrm{mC}} = \frac{1}{\sqrt{1+(\omega\tau)^2}} V_{\mathrm{m}} \to 0$$
$$V_{\mathrm{mR}} = \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}} V_{\mathrm{m}} \to V_{\mathrm{m}} \tag{9.2b}$$

Therefore, at *high frequencies*, the *resistor voltage* dominates; it approximately equals the supply voltage. This fact is also easy to understand because the capacitor acts like a *short circuit for a high-frequency AC*, $|\mathbf{Z}_{\mathrm{C}}| = 1/(\omega C) \to 0$, so that the capacitor voltage is nearly zero and all the supply voltage is "seen" by the resistor.

Filter Concept: Two-Port Network

Now, we will explore the concept of an *analog low-pass* RC filter. We consider the power supply AC voltage as the *input voltage* $v_{\mathrm{in}}(t)$ into the filter. We consider the capacitor voltage as the *output voltage* $v_{\mathrm{out}}(t)$ of the filter. According to Eq. (9.2a, 9.2b),

$$\begin{aligned} &v_{\mathrm{out}}(t) \approx v_{\mathrm{in}}(t) \quad && \text{at low frequencies} \\ &v_{\mathrm{out}}(t) \approx 0 && \text{at high frequencies} \end{aligned}$$

The circuit so constructed passes voltage signals with lower frequencies (like the human voice) but cuts out voltage signals with higher frequencies (like noise). Figure 9.2 on the left depicts the corresponding circuit transformation. This transformation implies that the input voltage is acquired from another circuit block and the output voltage is passed to another circuit block. The qualitative filter description is complete. You should note that both circuits on the right of Fig. 9.2 are called *two-port networks*. A *port* is nothing else but a pair of voltage terminals, either related to the input voltage or to the output voltage, respectively. Can we construct an RC filter that passes high frequencies but cuts out low frequencies? In other words, can we create a so-called *high-pass* filter? The solution is simple and elegant; the output voltage is now the resistor voltage, not the capacitor voltage. Figure 9.2b shows the corresponding circuit transformation.

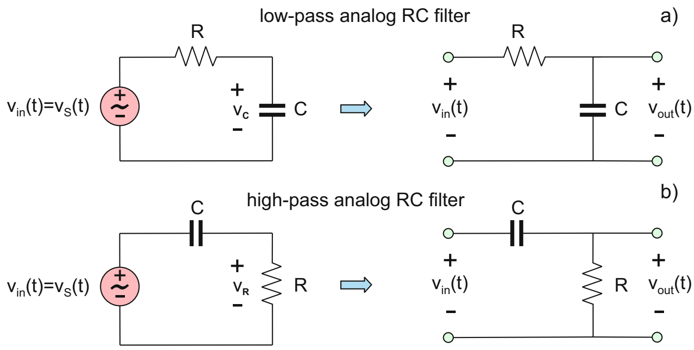


Fig. 9.2. (**a**) Transformation of the series RC circuit into the low-pass analog RC filter. (**b**) A similar transformation into the high-pass RC filter.

Example 9.1: The input voltage to the low-pass filter in Fig. 9.2a is a combination of three harmonics: $v_{\mathrm{in}}(t) = 10\cos\omega_1 t + 10\cos\omega_2 t\ + 10\cos\omega_3 t$ [V], each with an amplitude of 10 V. The filter has the following parameters: $C = 530\ \ \mathrm{nF}, \quad R = 100\ \ \Omega$. Determine the output voltage $v_{\mathrm{out}}(t)$ given that:

1. $f_1 = 20$ Hz (lower frequency of the acoustic range)
2. $f_2 = 3000$ Hz (frequency below which most of the acoustic power is present)
3. $f_3 = 20,000$ Hz (higher frequency of the acoustic range)

Solution: The key is the *superposition principle*, which is based on circuit *linearity*. Using the superposition principle, we apply Eq. (9.1e) (and Eq. (9.1d) reporting the phases) for the capacitor voltage (the output voltage to the filter) to each harmonic separately and then find the sum of three partial solutions. This will be the filter output voltage, which is given by

$$v_{\mathrm{out}}(t) = 10.00\cos(\omega_1 t - 0.4^\circ) + 7.07\cos(\omega_2 t - 45.0^\circ) + 1.48\cos(\omega_3 t - 81.5^\circ) \quad [\mathrm{V}] \tag{9.3}$$

The filter reduces the amplitudes of higher-frequency harmonics and simultaneously creates a certain phase shift. The high-pass filter operates in an opposite manner. The phase shift becomes positive.

Example 9.2: Solve Example 9.1 using MATLAB.

Solution: The text of the corresponding MATLAB is listed below. It is vectorized in the sense that *any* number of input harmonics may be taken into consideration:

Example 9.2 (cont.):

```
Vm      = [10 10 10];        %   input voltage amplitudes, V
f       = [20 3000 20000];   %   input voltage frequencies, Hz
omega   = 2*pi*f;            %   angular frequencies, rad/sec
R       = 100;               %   resistance, Ohm
C       = 530e-9;            %   capacitance, F
tau     = R*C;
VmC     = 1./sqrt(1+(omega*tau).^2).*Vm      %   output voltage ampl., V
phiC    = - atan(omega*tau)*180/pi           %   output phases in deg
```

Exercise 9.1: The input voltage to a high-pass filter circuit is a combination of two harmonics, $v_{in}(t) = 2\cos\omega_1 t + 2\cos\omega_2 t$, with the amplitude of 2 V each. The filter has the following parameters: $R = 100\ \text{k}\Omega$ and $C = 1.59\ \text{nF}$. Determine the output voltage $v_{out}(t)$ to the filter given that $f_1 = 100\ \text{Hz}$ and $f_2 = 100\ \text{kHz}$.

Answer: $v_{out}(t) = 1.99\cos(\omega_1 t - 5.7^\circ) + 0.02\cos(\omega_2 t - 89.4^\circ)\quad$ [V].

Application Example: Effect of a Load Connected to the Filter

The initial excitement about the simplicity of the theoretical filter model often fades quickly once we try to construct the filter circuit of Fig. 9.2a or Fig. 9.2b in the laboratory. And the circuit does not work. The major reason for this is the effect of a *load* connected

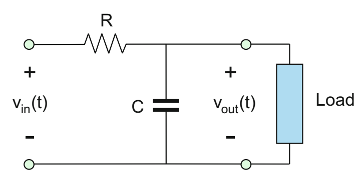


Fig. 9.3. A generic load connected to the low-pass RC filter.

to the filter. We consider the low-pass filter in Fig. 9.3.

To solve the circuit with the load, we need to apply the phasor method. The input voltage is now divided between the resistance R and the parallel combination of the capacitor impedance and the load resistance, R_L. Instead of Eq. (9.1c), we will have

$$\mathbf{V}_C = \frac{1}{\sqrt{(1 + R/R_L)^2 + (\omega\tau)^2}} V_m \angle\varphi_C \ \text{[V]}, \quad \varphi_C = -\tan^{-1}\left(\frac{\omega\tau}{1 + R/R_L}\right) \tag{9.4}$$

The proof of this result is suggested in Problems 9.5 and 9.6. The necessary condition for proper filter operation (both high pass or low pass) is that *the filter termination resistance* should be *much greater* than the filter's resistance R. Put in approximate mathematical terms: $R/R_L \ll 1$. The low-resistance load (e.g., a loudspeaker) would simply short out the capacitor! To avoid this effect, a buffer amplifier may have to be inserted between the load and the filter.

9.1.2 Half-Power Frequency and Amplitude Transfer Function

Low-Pass Filter

We are going to show how to construct a low-pass RC filter for a particular application. The design engineer needs to know at approximately which frequency the signal should be cut out. It is a common agreement to choose this frequency so that the amplitude of the output voltage is exactly $1/\sqrt{2} \approx 0.707$ of the input voltage amplitude V_m. In other words, the output filter power, which is proportional to the square of the output voltage, becomes exactly *half* of the input power. The corresponding frequency is called the *break frequency* or *half-power* frequency of the low-pass filter. According to Eq. (9.1e), the break frequency $\omega_b = f_b$ is found using the amplitude of the output (capacitor) voltage in the following way:

$$\frac{1}{\sqrt{1+(\omega_b\tau)^2}} = \frac{1}{\sqrt{2}} \Rightarrow \omega_b\tau = 1 \Rightarrow \omega_b = \frac{1}{\tau} \Rightarrow f_b = \frac{\omega_b}{2\pi} = \frac{1}{2\pi\tau} = \frac{1}{2\pi RC} \quad [\text{Hz}] \tag{9.5}$$

Expressed in terms of the break frequency, the amplitude of the output voltage to the voltage across the capacitor in Eq. (9.1e), has the form $V_m/\sqrt{1+(f/f_b)^2}$ since $\omega\tau = f/f_b$. With the input voltage amplitude to the filter being V_m, the ratio of the two amplitudes is the *amplitude transfer function* of the low-pass filter H_m. This transfer function is given by

$$H_m(f) = \frac{1}{\sqrt{1+(f/f_b)^2}} \le 1 \tag{9.6a}$$

We note that the transfer function is dimensionless (or has the units of V/V). For a given input voltage, the amplitude transfer function allows us to determine the output voltage amplitude. The behavior of Eq. (9.6a) is such that the amplitude transfer function is always less than one: the output voltage cannot exceed the input voltage.

High-Pass Filter

The break frequency, ω_b or f_b, of the high-pass filter has the meaning of reducing the voltage amplitude by a factor of $1/\sqrt{2}$ and reducing the signal power by the factor of ½. According to Eq. (9.1e) for the resistor voltage, it is found using the equality $\omega_b\tau/\sqrt{1+(\omega_b\tau)^2} = 1/\sqrt{2}$, which gives us exactly the *same* value as the break frequency for the low-pass filter; see Eq. (9.5). In other words, the definitions of the break frequency and the half-power frequency *coincide* for the first-order low-pass filter and the first-order high-pass filter, respectively. In terms of the break frequency, the amplitude of the output voltage to the high-pass filter, the voltage across the resistor in Eq. (9.1e), has the form $V_m f/f_b/\sqrt{1+(f/f_b)^2}$, whereas the input voltage amplitude to the filter is still V_m. The ratio of the two amplitudes is the *amplitude transfer function* of the high-pass filter, denoted here by the same letter H_m. This transfer function is given by

$$H_m(f) = \frac{f/f_b}{\sqrt{1+(f/f_b)^2}} \le 1 \tag{9.6b}$$

We note again that H_m cannot exceed 1. The implication is that the output voltage is always less than or equal to the input voltage; the filter cannot amplify the input.

Example 9.3: With the values of $C = 530\,\text{nF},\ R = 100\,\Omega$, determine the break frequency of both the low-pass RC filter and the high-pass RC filter, respectively.

Solution: We utilize the definition of Eq. (9.5), $f_b = \frac{1}{2\pi RC}$ [Hz], and obtain $f_b = 3.00$ kHz for either case. This is exactly why the particular signal at 3 kHz in Eq. (9.3) of Example 9.1 (the example uses the same parameters) was reduced by a factor of 0.707 at the output of the low-pass filter. If a high-pass filter were used, the corresponding output signal would have exactly the same form but with the phase shift of +45° instead of −45°.

Exercise 9.2: The input signal to a high-pass RC filter includes a 180-Hz component. Its amplitude is to be reduced by a factor of 10. What break frequency should the filter have?

Answer: 1791 Hz.

9.1.3 Bode Plot, Decibel, and Roll-Off

The *Bode plot* displays the amplitude transfer function defined by Eq. (9.6a, 9.6b) as a function of frequency on a logarithmic scale. It was first suggested by an electrical engineer and mathematician Hendrik Wade Bode (1905–1982), Bell Labs, NJ, USA. The advantage of the logarithmic scale is the ability to simultaneously observe the (very large) function variations at small and large frequencies. Furthermore, you can more clearly see the *roll-off* of the transfer function as a straight line (*asymptote*). This is impossible to see when using the linear scale. As the x-variable, we will always choose frequency f (*and avoid the angular frequency* ω). As the y-variable, we plot the logarithmic function

$$H_m(f)_{dB} = 20\log_{10} H_m(f) \quad [\text{dB}] \tag{9.7}$$

The *dimensionless* units for the amplitude transfer function in Eq. (9.7) are *decibels* or dB. Figure 9.4 shows the Bode plot for transfer function Eq. (9.6a) with $f_b = 100$ Hz. The selected values of the transfer function are given in Table 9.1 and where the last row is given in dB. The particular values of the resistance and capacitance are yet to be found; only their combination $\tau = RC = 1/(2\pi f_b) = 1.6$ ms is really important for the Bode plot. Despite the apparent simplicity of this operation, the Bode plot for an RC filter is a very likely question on the entrance exam for an industrial position in electrical engineering.

Table 9.1. Values of amplitude transfer function for a low-pass filter with $f_b = 100$ Hz.

f, Hz	1	10	**100**	1000	10^4	10^5	10^6	10^7
$H_m(f)$	1.000	0.995	**0.707**	1.0×10^{-1}	1.0×10^{-2}	1.0×10^{-3}	1.0×10^{-4}	1.0×10^{-5}
$20\log_{10}H_m(f)$	−0.0004	−0.0432	**−3.0103**	−20.043	−40.000	−60.000	−80.000	−100.00

Historical: The decibel is named in honor of Alexander Graham Bell (1847–1922), a Scottish scientist and inventor who later became a professor at Boston University, MA. Bell invented the first practical telephone at the age of 28 (US Patent 174,465) and very quickly became a millionaire. His father-in-law Gardiner Greene Hubbard founded the Bell Telephone Company in 1878, which subsequently transformed into American Telephone & Telegraph Company (AT&T).

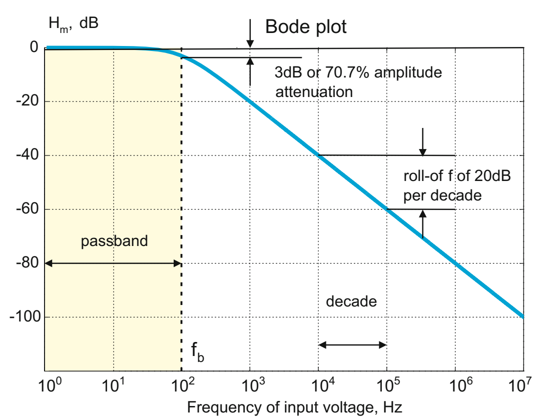


Fig. 9.4. Construction of a Bode plot for the amplitude transfer function of a low-pass RC filter with break frequency of $f_b = 100$ Hz.

A legitimate question to ask is what is the meaning of the factor 20 in Eq. (9.7)? The answer is based on the equality $20\log_{10}H_m(f) = 10\log_{10}H_m^2(f)$ where $H_m^2(f)$ is not the amplitude transfer function but rather attempts to represent *power*, which is proportional to voltage squared for a resistor. Therefore, Eq. (9.7) in fact attempts to plot the *power transfer function* even though the capacitor in Fig. 9.2a does not consume any power in the average sense, see Chapter 11. Also note that, when $f = f_b$, the transfer function in Table 9.1 is approximately -3 dB. Therefore, the break frequency is also called the *3-dB frequency* for obvious reasons. Another name, the *corner frequency*, will be explained shortly.

The interval on the Bode plot for which the frequencies differ by a factor of 10 is called a *decade*. Every division on the *x*-axis in Fig. 9.4 is *one decade*. The transfer function for *any* first-order low-pass filter decreases by 20 dB per decade or has the *20-dB-per-decade roll-off* as seen in Fig. 9.4. This not only occurs away from the break frequency, i.e., when $f \gg f_b$, but it is also approximately valid in the interval from f_b to $10f_b$; see Table 9.1. Note that an interval of frequencies is called the *frequency band*. The roll-off of 20 dB per decade (or equivalently, the slope of -20 dB per decade) means that the output amplitude of the filter decreases by a factor of 10 per decade (see Table 9.1), whereas the output power decreases by the factor of 100. Figure 9.4 shows a frequency band from 0 to f_b, which is the *passband* of the low-pass filter. The passband is the range of frequencies that are passed through a filter without being (significantly) attenuated. The opposite of the passband is the *stopband*. The required attenuation within the stopband may be specified between 20 and 120 dB as compared to the value of 0 dB, which means no attenuation. Besides the decade, the relative frequency interval of one *octave* is sometimes used. In this interval, the frequencies differ by the factor of 2, not 10. For example, a TV antenna that has the bandwidth of one octave (400–800 MHz) may be used to receive most of the (digital) commercial TV channels in the USA. It can be shown that the RC filter has a 6-dB-per-octave roll-off, away from the break frequency.

Historical: The career of Hendrik Wade Bode (1905–1982), a pioneer of modern control theory and electronic telecommunications, gives us an example of how important it is to have a comprehensive education in calculus and a solid background in electrical engineering. A graduate of the Ohio State University (BS in mathematics at the age of 19 and then MS in mathematics two years later), Hendrik Bode started his job at Bell Labs as a designer of electronic filters and invented the asymptotic plots we now call them Bode plots in 1938. These plots have proven to be extremely useful in feedback control theory. Today, any electrical engineer who works with amplifiers and their frequency responses is relying on Bode plots. Some consider Bode a pioneer of *robotics* as well, based on his invention of robotic antiaircraft artillery during WWII.

Exercise 9.3: The following values of the amplitude transfer function are given: $H_m(f) = 0.707,\ 0.0707,$ and 0.00707. Find the corresponding values of $H_m(f)_{dB}$.

Answer: -3.01 dB, -20.00 dB, and -40.00 dB.

Example 9.4: Design a medium-frequency-range RC low-pass filter (LPF) that has a break frequency of 1 kHz. The filter load has the resistance of $R = 100\ \text{k}\Omega$. Create the amplitude Bode plot in the range from 10 Hz to 100 kHz. Label the filter passband. Repeat the same task for the high-pass filter (HPF).

Solution: The condition $f_b = 1/(2\pi RC)$ yields $C = 1/(2\pi R f_b) = 1.6$ nF. The Bode plot may be generated by finding transfer function values for (at least) every decade and filling out a table similar to Table 9.1. The result is shown in Fig. 9.5a. The passband is the frequency band from 0 to f_b. For the high-pass filter, we repeat the same steps but replace the transfer function given by Eq. (9.6a) by the transfer function given by Eq. (9.6b). The result is shown in Fig. 9.5b. The passband extends from f_b to infinity and is only limited by the upper frequency of the Bode plot. Note that the Bode plot for the high-pass filter has the same form, but it is *mirror reflected* about the break frequency. This is another advantage of the logarithmic scale.

Figure 9.5 indicates that the amplitude response of both the low-pass filter and the high-pass filter follows two straight lines, which are known as *high-frequency and low-frequency asymptotes*. The corner between them is the break frequency, also called the *corner frequency*. Note that, for the high-pass filter, the meaning of high-frequency and low-frequency asymptotes is *interchanged* in Fig. 9.5b.

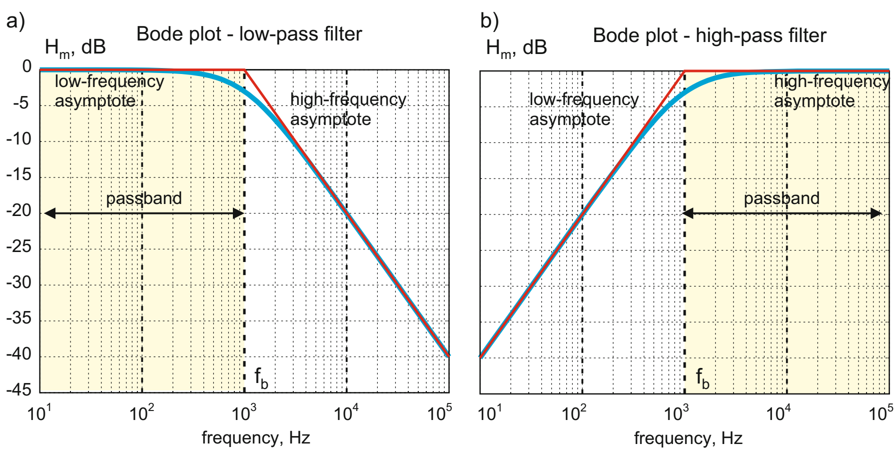


Fig. 9.5. (**a**) Bode plot for the amplitude transfer function of the low-pass RC filter with the break frequency $f_b = 1$ kHz. (**b**) The same Bode plot but for the high-pass RC filter. Note high-frequency and low-frequency asymptotes.

9.1.4 Phase Transfer Function and Its Bode Plot

According to Eq. (9.1e), it is not only the amplitude but also the phase of the input signal that undergoes a transformation when the signal is passed through the filter. The phase transformation is important since different frequencies (or harmonics) of the input signal may have a certain phase relation that is distorted by the filter. The *phase transfer function* is given by the phase variation of the filter's output voltage, which is either the capacitor voltage for the low-pass filter or the resistor voltage for the high-pass filter. From Eq. (9.1c) for the low-pass filter, the phase transfer function has the form

$$\varphi_{\mathrm{H}}(f) = -\tan^{-1}(\omega\tau) = -\tan^{-1}\left(\frac{f}{f_{\mathrm{b}}}\right) \quad \text{low-pass RC filter} \tag{9.8a}$$

From Eq. (9.1d) for the high-pass filter, the phase transfer function has the form

$$\varphi_{\mathrm{H}}(f) = \frac{\pi}{2} - \tan^{-1}(\omega\tau) = \frac{\pi}{2} - \tan^{-1}\left(\frac{f}{f_{\mathrm{b}}}\right) \quad \text{high-pass RC filter} \tag{9.8b}$$

where the break frequency is given by Eq. (9.5).

Example 9.5: Generate the phase Bode plots for the low-pass filter and the high-pass filter, respectively, with the same break frequency $f_{\mathrm{b}} = 1\,\mathrm{kHz}$. The frequency band is from 10 Hz to 100 kHz.

Solution: The phase Bode plots in Fig. 9.6 may be generated by calculating the phase transfer function according to Eq. (9.8a, 9.8b) for (at least) every decade. The result is shown in Fig. 9.6. You can see that the Bode plots only differ by a phase shift of 90°. Alternatively, a MATLAB script may be used:

```
f       = logspace(1, 5);       % frequency vector, Hz (from 10^1 to 10^5 Hz)
fb      = 1000;                 % break frequency, Hz
phiH1   = -atan(f/fb);          % low-pass filter phase transfer function
phiH2   = pi/2-atan(f/fb);      % high-pass filter phase transfer function
semilogx(f, phiH1/pi*180); grid on;
title('Bode plot'); ylabel('phase transfer function, deg'); xlabel('f, Hz')
```

9.1.5 Complex Transfer Function: Cascading Filter Circuits

The *complex transfer function* of the filter, $\mathbf{H}(f)$, is often called the *frequency response* of the filter. It describes not only the amplitude transformation but also the phase transformation. The transfer function now becomes a *complex expression*. It is equal to the ratio of two phasors; specifically, it denotes the ratio of the output phasor voltage to the input phasor voltage. The low-pass filter has the form of Fig. 9.1 with the input voltage equal to the supply voltage and the output voltage equal to the capacitor voltage. Its complex transfer function is given by Eq. (9.1c) divided by V_{m}. The high-pass filter also has the form of Fig. 9.1 with the input voltage equal to the supply voltage and the output voltage equal to the resistor voltage. Its complex transfer function is given by Eq. (9.1d) divided by V_{m}. Thus, we obtain

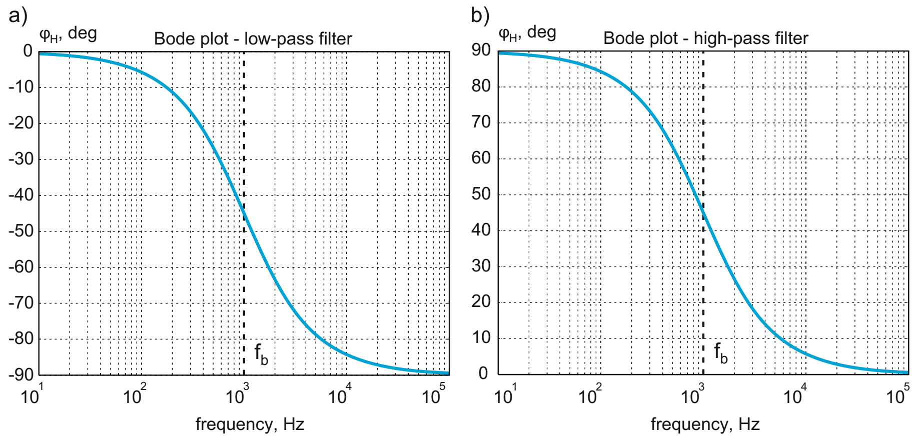


Fig. 9.6. Comparison of the phase Bode plots for (**a**) the low-pass-filter and (**b**) for the high-pass filter with the same break frequency $f_b = 1$ kHz. Both plots are identical to within a phase shift.

$$\mathbf{H}(f) \equiv H_m(f)\angle\varphi_H = \begin{cases} \dfrac{1}{1+j(f/f_b)} = \dfrac{1}{\sqrt{1+(f/f_b)^2}}\angle - \tan^{-1}\left(\dfrac{f}{f_b}\right); & \text{low-pass RC filter} \\[2ex] \dfrac{(f/f_b)}{1+j(f/f_b)} = \dfrac{f/f_b}{\sqrt{1+(f/f_b)^2}}\angle\dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{f}{f_b}\right); & \text{high-pass RC filter} \end{cases} \tag{9.9a}$$

This is consistent with Eqs. (9.6a, b) and (9.8a, b), respectively. Given the phasor of input voltage $\mathbf{V}_{in}$, the phasor of the output voltage is simply expressed by

$$\mathbf{V}_{out} = \mathbf{H}(f)\mathbf{V}_{in} \tag{9.9b}$$

Equation (9.9b), which is valid for any linear electronic filter and other linear systems, fully describes the filter operation and has great practical value.

Example 9.6: For a low-pass RC filter with the values $C = 530$ nF, $R = 100\ \Omega$, determine the output voltage in time domain when the input voltage is given by $v_{in}(t) = 1\cos(\omega t + 30^\circ)$ [V] where $\omega = 2\pi \times 3000$ rad/s.

Solution: The break frequency of the low-pass filter is $f_b = 3.00$ kHz, which coincides with the signal frequency in this particular case. According to the first Eq. (9.9a) and Fig. 9.6a, at that frequency, $\mathbf{H}(f) = \frac{1}{\sqrt{2}}\angle - 45^\circ$; therefore, the output voltage has the form $\mathbf{V}_{out} = \frac{1}{\sqrt{2}}\angle - 15^\circ$ or $v_{out}(t) = 0.71\cos(\omega t - 15^\circ)$ [V]. The same analysis may be applied at any frequency and phase of the input harmonic voltage signal.

Another advantage of the complex transfer function lies in the fact that the *series* or *cascade combination* of any number of filters (or two-port networks) shown in Fig. 9.7 has a transfer function that is simply the product of the corresponding transfer functions:

$$\mathbf{H}(f) = \mathbf{H}_1(f)\mathbf{H}_2(f) \Rightarrow H_\mathrm{m}(f) = H_{\mathrm{m}1}(f)H_{\mathrm{m}1}(f) \tag{9.10}$$

In this manner, a more advanced filter may be constructed from the individual filter blocks. To prove Eq. (9.10), we state that the phasor for the intermediate output voltage $v_{\mathrm{out}1}(t)$ in Fig. 9.7 is given by $\mathbf{V}_{\mathrm{out}1} = \mathbf{H}_1(f)\mathbf{V}_{\mathrm{in}}$. Hence, the phasor for the output voltage $v_{\mathrm{out}}(t)$ in Fig. 9.7 becomes $\mathbf{V}_{\mathrm{out}} = \mathbf{H}_1(f)\mathbf{V}_{\mathrm{out}1} = \mathbf{H}_1(f)\mathbf{H}_2(f)\mathbf{V}_{\mathrm{in}}$ which is equivalent to Eq. (9.10). Due to the logarithmic scale of the Bode plot, the product in Eq. (9.10) is replaced by the sum of two contributions when the decibel scale is used:

$$H_\mathrm{m}(f)_{\mathrm{dB}} = H_{\mathrm{m}1}(f)_{\mathrm{dB}} + H_{\mathrm{m}1}(f)_{\mathrm{dB}} \tag{9.11}$$

Thus, we simply add up two magnitude transfer functions in dB and obtain the resulting magnitude transfer function also in dB.

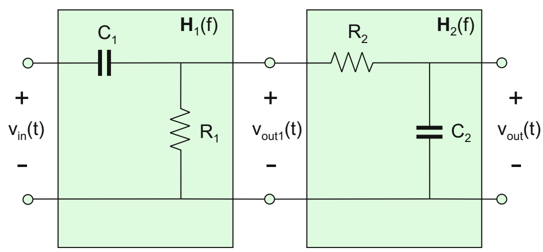


Fig. 9.7. Cascading a high-pass and a low-pass filter into a more complex filter structure.

Application Example: Effect of Next-Stage Filter Load

Equation (9.10) requires great care. For example, the equivalent impedance seen by the leftmost high-pass filter stage in Fig. 9.7 should be much greater than R_1; otherwise this stage will not operate as expected, and Eq. (9.10) will be inaccurate. In other words, a following filter stage should not appreciably *load* the previous one.

Example 9.7: For the combined circuit in Fig. 9.7, create the Bode plot for the transfer function of the cascade connection in the frequency band from 1 Hz to 1 MHz You are given $R_1 = 159.1\ \Omega$, $C_1 = 10\ \mu\mathrm{F}$ and $R_2 = 159.1\ \Omega$, $C_2 = 0.1\ \mu\mathrm{F}$.

Solution: The break frequency of the high-pass filter is calculated as 100.0 Hz, and the break frequency of the low-pass filter is found to be 10.0 kHz. The combined Bode plot is generated using Eqs. (9.6a, 9.6b) and (9.10). Alternatively, the transfer functions in dB, specified by Eq. (9.7), may be added. The result is a *band-pass* filter as shown in Fig. 9.8 by the solid curve. This result is expected to be accurate only if $|R_2 + Z_{C_2}| \gg R_1$. Though valid at low frequencies below 1 kHz, this inequality is violated above 1 kHz. The exact transfer function is obtained by solving the complete AC circuit in Fig. 9.7 with the open-circuited capacitor C_2. It is plotted in Fig. 9.8 by a dashed curve. There is clearly a significant deviation from the solution given by Eq. (9.10) at higher frequencies above 1 kHz. To avoid the loading effect seen in Fig. 9.8, a buffer amplifier may be inserted between the filter stages shown in Fig. 9.7.

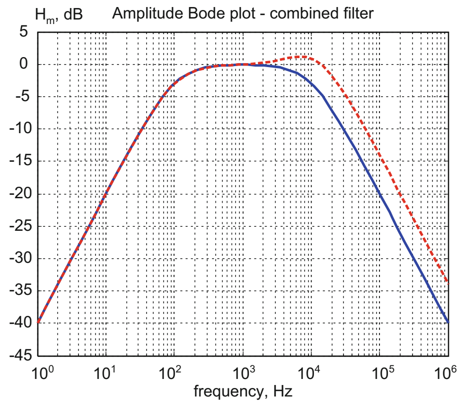


Fig. 9.8. *Solid curve*: Eq. (9.10) for the cascaded filters. *Dashed curve*: the exact solution with the open-circuited capacitor C_2.

In Fig. 9.8, the exact transfer function may exceed 0 dB. In other words, the voltage gain of the combined (still passive) filter may be greater than one. How is it possible? The answer is that, in contrast to the circuits in Fig. 9.2, the circuit in Fig. 9.7 is in fact already a *second-order* circuit. Second-order circuits may experience a resonance behavior where the circuit voltages across individual elements may (very considerably) exceed the original supply voltage. This effect, called *voltage multiplication*, is of great practical importance and will be considered in detail in Chapter 10 devoted to second-order AC circuits. Note that that the true *power gain* of a passive filter of any order and any topology is always less than one (less than 0 dB). Only electronic amplifiers may have a positive, and often high, power gain; this is discussed in the next section.

9.1.6 RL Filter Circuits

The RL circuits are used for the same filtering purposes as the RC circuits. Figure 9.9 depicts the concept. It may be demonstrated that the corresponding circuit theory and Eqs. (9.6a, 9.6b) for the transfer functions become equivalent to first-order RC filter circuits under the following conditions:

1. The time constant $\tau = RC$ is replaced by the time constant $\tau = L/R$, similar to the corresponding operation for the first-order transient circuits. The break frequency $f_b = 1/(2\pi\tau)$ remains the same.
2. The role of the capacitor and inductor are interchanged. For example, the RL circuit in Fig. 9.9a is a first-order high-pass filter because the inductor voltage, which is the output filter voltage, is exactly zero for a DC signal. However, it becomes a first-order low-pass filter if the inductor is replaced by a capacitor, as shown in Fig. 9.9a.

3. Similarly, the RL circuit in Fig. 9.9b is a first-order low-pass filter simply because the inductor becomes a short circuit at DC and the DC signal will pass through. However, it becomes a first-order high-pass filter if the inductor is replaced by a capacitor, as shown in Fig. 9.9b.

Furthermore, the filter specifications might require large inductance values, which lead to physically large inductor sizes.

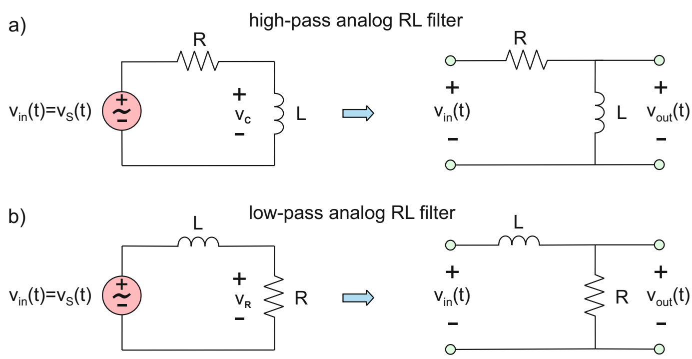


Fig. 9.9. (**a**) Transformation of a series RL circuit into a high-pass analog RL filter. (**b**) Similar transformation into the low-pass RL filter.

Example 9.8: For the two filter circuits in Fig. 9.9, create the amplitude Bode plots in the frequency band from 10 Hz to 100 kHz. You are given $R = 31.4\ \Omega$, $L = 5\,\text{mH}$.

Solution: The break frequencies of the high-pass filter and the low-pass filter in Fig. 9.9 coincide. In either case, we obtain $f_b = 1/(2\pi\tau)$, $\tau = L/R = 1.59\times\ 10^{-4}$ s. Thus, $f_b = 1.00$ kHz. The Bode plots may be generated by finding transfer function values based on Eq. (9.6) for (at least) every decade and filling out a table similar to Table 9.1. The result is shown in Fig. 9.10 along with high- and low-frequency asymptotes. We again observe the 20-dB roll-off per decade. The Bode plots given in Fig. 9.10 coincide with the Bode plots for RC filters having the same break frequency, see Fig. 9.5. However, given an identical component topology, the filter function is interchanged.

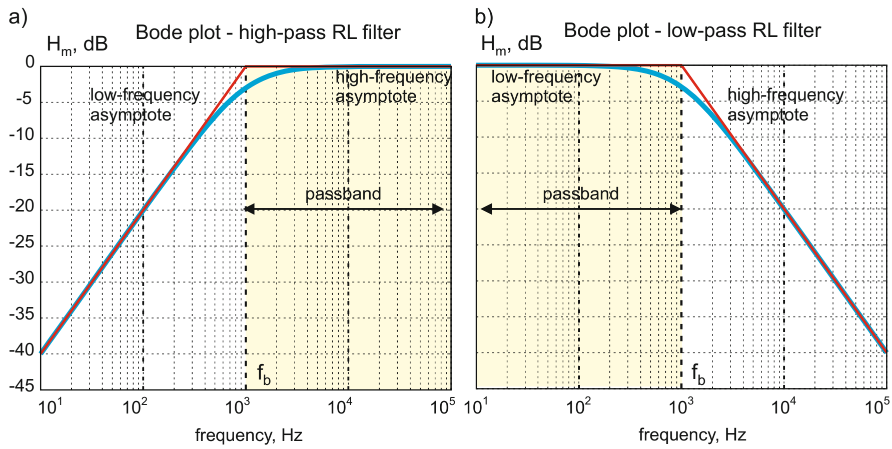


Fig. 9.10. (**a**) Bode plot for the amplitude transfer function of the high-pass RL filter with break frequency $f_b = 1\,\mathrm{kHz}$. (**b**) The same Bode plot but for the low-pass RL filter.

Exercise 9.4: An RL filter circuit in Fig. 9.9a has $R = 100\,\Omega$ and $L = 1\,\mathrm{mH}$. Establish the capacitance value of an equivalent RC filter, given that the resistances are the same in both cases.

Answer: 100 nF.

Section 9.2 Bandwidth of an Operational Amplifier

The operational amplifier circuits introduced earlier are implicitly assumed to operate equally well for any frequency of the input signal. In reality this is not true. An operational amplifier may operate only over a certain frequency band, and the associated *frequency bandwidth* is perhaps the most critical device parameter. Frequently we do not realize how severe this limitation can be and how difficult it is to build a high-frequency or radio-frequency amplifier. As an example, we should point out that none of the common amplifier ICs studied in introductory ECE classes can be used as a front-end amplifier for an AM radio receiver (520–1610 kHz), even if the noise levels were low. Indeed, high-frequency amplifiers with larger frequency bandwidths exist. A case in point is the accessible LM7171 chip. Key to understanding the amplifier frequency behavior is the theory of the first-order RC filters developed in the previous section.

9.2.1 Bode Plot of the Open-Loop Amplifier Gain

Open-Loop Amplifier Gain and Its Relation to the Previous Results

The (amplitude) *frequency response* of an operational amplifier is simply a plot of its gain magnitude versus frequency of the input AC voltage signal. This response is usually a Bode plot. The problem is that the gain of the amplifier (both open loop and closed loop) generally *decreases* with increasing frequency. We consider the *open-loop gain* (gain without the feedback loop) first. The open-loop gain magnitude will be denoted here by $A_{\mathrm{OL}} = A_{\mathrm{OL}}(f)$. Note that in Chapter 5 we have already introduced the *open-circuit gain*, A, of an amplifier at DC without the feedback loop. What is the relation between A_{OL} and A introduced previously? The answer is given by the equality $A = A_{\mathrm{OL}}(f = 0)$ as long as the amplifier is open circuited.

Open-Loop Gain Behavior

The open-loop gain decreases with increasing frequency of the input signal. Figure 9.11 shows the frequency response of an open-loop amplifier on a log-log scale. You may recall that the log-log scale used in this figure is simply the Bode plot introduced in the previous section. This figure is typical for the LM741 amplifier IC and similar general-purpose devices. Comparing the Bode plot in Fig. 9.11 with the Bode plot of the RC filter in Fig. 9.4 of the previous section, we discover that the amplifier's gain as a function of frequency is virtually identical to the transfer function of the RC filter for the same break frequency of 10 Hz, as seen in Fig. 9.11! In both cases, we have a roll-off of 20 dB per decade. Obviously, the scale is different. Why is this so? This occurs because the amplifier ICs are usually *internally compensated*, which means incorporating a simple RC filter network (in practice, it may be a single capacitor *C*) into the IC chip itself. This process is called *internal compensation* of the amplifier. The goal of such a modification is to ensure that the amplifier circuit will be stable. Stability refers to the amplifier's immunity to spontaneous oscillations. These undesired oscillations occur when the input frequency excites internal resonances, similar to a mechanical mass-spring system, that continue ad infinitum.

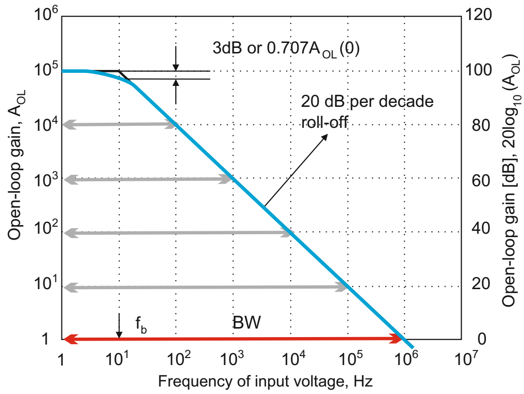


Fig. 9.11. Bode plot of the open-loop gain magnitude for the LM741-type amplifier IC. Note the logarithmic scale on the *left* and the corresponding scale in dB on the *right*. The frequency bandwidth given by the break frequency f_b is only 10 Hz.

9.2.2 Unity-Gain Bandwidth Versus Gain-Bandwidth Product

The amplifier gain in Fig. 9.11 decreases by a factor of 0.1 (or 20 dB) gain roll-off per frequency decade. The decay already starts at a relatively low break frequency of 10 Hz where the DC open-loop gain drops by the factor of 0.707 or $1/\sqrt{2}$. The corresponding value in dB is $20\log_{10} 1/\sqrt{2} = -3$ dB. The gain continues to decrease further and reaches unity at the frequency of 1 MHz. This frequency is equal to the *unity-gain bandwidth* (BW) of the amplifier, i.e., for the amplifier IC depicted in Fig. 9.11:

$$\mathrm{BW} = 1\ \ \mathrm{MHz} \tag{9.12}$$

A remarkable observation from Fig. 9.11 is that the *gain-bandwidth product* (sometimes denoted by GBW or GB in datasheets) remains constant over the band for *every* particular gain value. The gain-bandwidth product is equal to the length of every single arrow (in Hz) in Fig. 9.11 times the corresponding gain value (dimensionless), that is,

$$\begin{aligned}
f &= 10^2\mathrm{Hz} \Rightarrow \mathrm{GBW} = 10^2 \times 10^4 = 10^6\mathrm{Hz} = \mathrm{BW},\\
f &= 10^3\mathrm{Hz} \Rightarrow \mathrm{GBW} = 10^3 \times 10^3 = 10^6\mathrm{Hz} = \mathrm{BW},\\
f &= 10^4\mathrm{Hz} \Rightarrow \mathrm{GBW} = 10^4 \times 10^2 = 10^6\mathrm{Hz} = \mathrm{BW},
\end{aligned} \tag{9.13}$$

etc. Thus, the gain-bandwidth product is *exactly* equal to the unity-gain bandwidth BW; it is frequently specified in the manufacturer datasheet. In what follows, we will use the unity-gain bandwidth as the major parameter of interest. Note that instead of, or along with, the unity-gain bandwidth, the *rise time* of an amplifier may be specified in the datasheet. Approximately, we can state that $\mathrm{BW} = 0.35/\text{rise time}\ [\mathrm{Hz}]$.

9.2.3 Model of the Open-Loop AC Gain

The open-loop gain dependence on the frequency has the form of a low-pass filter. We could therefore describe the open-loop gain in a *complex form* that is identical to the complex transfer function of the low-pass filter given, for example, by Eqs. (9.9a, b) of the previous section. The *open-loop AC gain* in complex phasor form states

$$\mathbf{A}_{\mathrm{OL}}(f) = \frac{A_{\mathrm{OL}}(0)}{1 + j(f/f_{\mathrm{b}})}, \quad A_{\mathrm{OL}}(0) \text{ is the open-loop DC gain} \tag{9.14}$$

For example, $A_{\mathrm{OL}}(0) = 10^5$ in Fig. 9.11. According to Eq. (9.14), the open-loop AC gain is a complex-valued frequency-dependent transfer function. This circumstance is reflected in a phase difference between the output and input voltages. To be consistent with Fig. 9.11 and with the previous DC amplifier analysis, the magnitude of the complex gain function in Eq. (9.14) is denoted by the *same* symbol, A_{OL}, i.e.,

$$|\mathbf{A}_{\mathrm{OL}}| = A_{\mathrm{OL}}(f) = \frac{A_{\mathrm{OL}}(0)}{\sqrt{1 + (f/f_{\mathrm{b}})^2}} \tag{9.15}$$

The Bode plot applied to Eq. (9.15) will give us exactly the dependence shown in Fig. 9.11. According to Eq. (9.15), the unity-gain bandwidth satisfies the equality

$$1 = \frac{A_{\mathrm{OL}}(0)}{\sqrt{1 + (\mathrm{BW}/f_{\mathrm{b}})^2}} \tag{9.16}$$

Since $\mathrm{BW}/f_{\mathrm{b}} \gg 1$, one has $\sqrt{1 + (\mathrm{BW}/f_{\mathrm{b}})^2} \approx \mathrm{BW}/f_{\mathrm{b}}$ with a high degree of accuracy. Therefore, according to Eq. (9.16),

$$\mathrm{BW} = A_{\mathrm{OL}}(0) f_{\mathrm{b}} \tag{9.17}$$

Looking at Fig. 9.11, we observe a very significant decrease of the open-loop gain, even in the audio frequency range. For example, the open-loop gain decreases by a factor of 1000 in the audio range from 10 Hz to 10 kHz. Does it mean that the LM741 or any general-purpose amplifier cannot be used in this range? The general answer is that the operational amplifier is mostly used with a negative feedback loop. When the *closed-loop DC gain* is not very high (say 10), the corresponding *closed-loop AC gain* appears to be nearly constant over a much wider bandwidth (say up to 100 kHz). This critical result will be proved mathematically shortly.

Example 9.9: The internally compensated LM148-series amplifiers (LM148/248/348) have a unity-gain bandwidth BW of 1 MHz. The typical large-signal voltage gain at room temperature reported in the datasheet is 160 V/mV.

A. Find the open-loop DC gain in dB and the open-loop break frequency f_b.
B. Find the open-loop gain at 100 Hz, 1 kHz, and 10 kHz.

Solution: The open-loop DC gain is $A_{OL}(0) = 160,000$ or $20\log_{10}(160,000) = 104$ dB. The break frequency may be found from Eq. (9.17):

$$f_b = \frac{\mathrm{BW}}{160,000} = 6.25 \text{ Hz} \tag{9.18}$$

According to Eq. (9.15), the open-loop gain at 100 Hz, 1 kHz, and 10 kHz becomes 10^4, 10^3, and 100, which corresponds to 80 dB, 60 dB, and 40 dB.

Exercise 9.5: For an internally compensated amplifier IC, the open-loop DC gain is 120 dB. The break frequency is 100 Hz. Determine the unity-gain bandwidth.

Answer: BW = 100 MHz.

9.2.4 Model of the Closed-Loop AC Gain

Consider a negative feedback amplifier in an inverting configuration, as shown in Fig. 9.12. Since the open-loop gain significantly decreases with frequency, we can no longer apply the second summing-point constraint (the differential input voltage is zero), which was justified based on the condition of the very high (ideally infinite) open-loop gain. However, the first summing-point constraint of no current into the amplifier is still valid. Therefore, a direct theoretical derivation of the closed-loop gain can be performed.

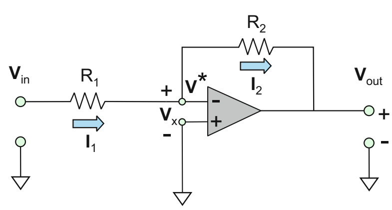


Fig. 9.12. Circuit configuration for deriving the frequency-dependent closed-loop gain.

We use the complex open-loop gain given by Eq. (9.14) and employ phasor voltages. Looking at Fig. 9.12, we conclude that $\mathbf{A}_{OL}\left(0\mathrm{V} - \mathbf{V}^*\right) = \mathbf{V}_{out}$, based on the amplifier definition. This definition is valid for either real (time-dependent) voltages or complex phasors. By KCL at the node associated with $\mathbf{V}^*$, we can develop

$$\frac{\mathbf{V}_{\text{in}} - \mathbf{V}^*}{R_1} = \frac{\mathbf{V}^* - \mathbf{V}_{\text{out}}}{R_2} \Rightarrow \frac{\mathbf{V}_{\text{in}} + \mathbf{V}_{\text{out}}/\mathbf{A}_{\text{OL}}}{R_1} = \frac{-\mathbf{V}_{\text{out}}/\mathbf{A}_{\text{OL}} - \mathbf{V}_{\text{out}}}{R_2} \Rightarrow$$
$$\frac{\mathbf{V}_{\text{in}}}{R_1} = \left(-\frac{1}{\mathbf{A}_{\text{OL}} R_1} - \frac{1}{\mathbf{A}_{\text{OL}} R_2} - \frac{1}{R_2}\right)\mathbf{V}_{\text{out}} \tag{9.19}$$

It follows from Eq. (9.19) that the output phasor voltage to the amplifier and the closed-loop amplifier phasor gain $\mathbf{A}_{\text{CL}}$ become

$$\mathbf{V}_{\text{out}} = -\frac{R_2}{R_1} \frac{\mathbf{V}_{\text{in}}}{1 + \frac{1}{\mathbf{A}_{\text{OL}}}\left(1 + \frac{R_2}{R_1}\right)} \Rightarrow \mathbf{A}_{\text{CL}} \equiv \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = -\frac{R_2}{R_1} \frac{1}{1 + \frac{1}{\mathbf{A}_{\text{OL}}}\left(1 + \frac{R_2}{R_1}\right)} \tag{9.20}$$

Next, we substitute Eq. (9.15) into Eq. (9.20) and rearrange terms to obtain the form

$$\mathbf{A}_{\text{CL}}(f) = -\frac{R_2}{R_1} \frac{1}{\left[1 + \frac{1}{A_{\text{OL}}(0)}\left(1 + \frac{R_2}{R_1}\right)\right] + \frac{1}{A_{\text{OL}}(0)}\left(1 + \frac{R_2}{R_1}\right) j\frac{f}{f_{\text{b}}}} \tag{9.21}$$

The first term in the denominator on the right-hand side of Eq. (9.20) is one with a high degree of accuracy since $A_{\text{OL}}(0) \approx 10^5 - 10^8$. This approximation is valid for any realistic resistor values. Therefore, we again arrive at the first-order low-pass filter response:

$$\mathbf{A}_{\text{CL}}(f) = \frac{A_{\text{CL}}(0)}{1 + j\left(f/f_{\text{b}}^{\text{closed loop}}\right)},$$
$$A_{\text{CL}}(0) = -\frac{R_2}{R_1}, \quad f_{\text{b}}^{\text{closed loop}} = \frac{A_{\text{OL}}(0) f_{\text{b}}}{1 + R_2/R_1} = \frac{\text{BW}}{1 + R_2/R_1} \tag{9.22}$$

but with a very different break frequency $f_{\text{b}}^{\text{closed loop}}$. A similar treatment holds for the *non-inverting amplifier configuration*. The result is *identical* to Eq. (9.22); however, the closed-loop DC gain $A_{\text{CL}}(0)$ is now given by

$$A_{\text{CL}}(0) = 1 + \frac{R_2}{R_1} \tag{9.23}$$

9.2.5 Application Example: Finding Bandwidth of an Amplifier Circuit

The relation reported in Eq. (9.22) is perhaps the most important single result with regard to the AC behavior of operational amplifiers. It reveals that the *closed-loop AC gain* has conceptually the *same* RC filter response as the open-loop gain; see Eq. (9.15). However, the corresponding break frequency $f_b^{\text{closed loop}}$ is *much larger*, namely, by a factor of $A_{OL}(0)/(1+R_2/R_1)$. This implies that the frequency response remains *flat* up to a very high frequency. The *amplifier bandwidth* in the closed-loop configuration *coincides* with the break frequency $f_b^{\text{closed loop}}$ determined by Eq. (9.22). Therefore, the bandwidth is directly proportional to the unity-gain bandwidth BW reported in the datasheet and inversely proportional to the factor $1+R_2/R_1$, which is straightforwardly calculated using the known values of the feedback resistances.

Example 9.10: An amplifier with the open-loop gain of Fig. 9.11 ($A_{OL}(0)=10^5$, $f_b = 10\,\text{Hz}$) is used in the closed-loop inverting configuration with $R_2/R_1 = 9$ (the DC inverting gain is -9). Create the Bode plot for the gain magnitude $A_{CL}(f)$, compare this result with the open-loop gain, and determine the bandwidth of the amplifier.

Solution: According to Eq. (9.22), the gain magnitude is given by

$$A_{CL}(f) = \frac{R_2}{R_1}\frac{1}{\sqrt{1+\left(f/f_b^{\text{closed loop}}\right)^2}}, \quad f_b^{\text{closed loop}} = \frac{10^5}{10}\times 10 \ \text{Hz} = 100 \ \text{kHz} \tag{9.24}$$

In Fig. 9.13, we plot the closed-loop gain versus the open-loop gain given by Eq. (9.15). The amplifier bandwidth in the closed-loop configuration is now a respectable 100 kHz.

Exercise 9.6: The unity-gain bandwidth of an amplifier IC is 1 MHz. Determine the bandwidth of the non-inverting amplifier circuit with a gain of 200.

Answer: 5 kHz.

9.2.6 Application Example: Selection of an Amplifier IC for Proper Frequency Bandwidth

The required bandwidth and closed-loop gain usually are known to the circuit designer. Using Eq. (9.22), we can estimate whether or not a specific amplifier IC will meet those requirements. There is clearly a trade-off between the closed-loop gain and bandwidth according to Fig. 9.13 and Eq. (9.22). For a given amplifier IC, the lower the closed-loop gain requirement, the wider the achievable bandwidth.

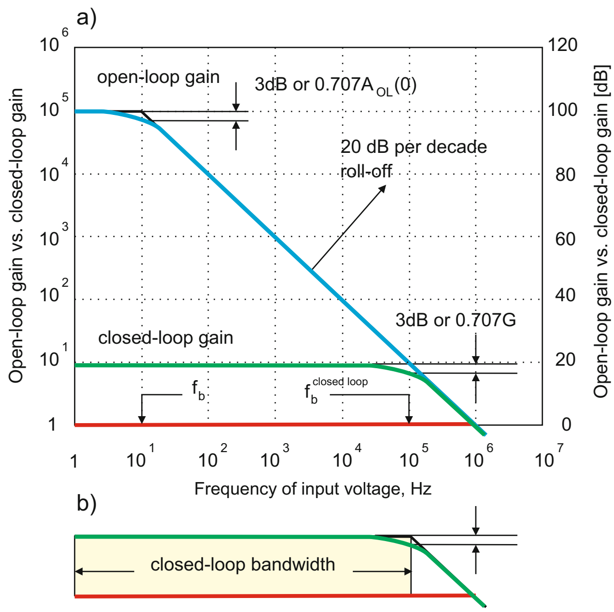


Fig. 9.13. Closed-loop AC gain $A_{CL}(f)$ (*lower curve*) versus open-loop AC gain $A_{OL}(f)$ (*upper curve*) for an inverting amplifier with $A_{OL}(0) = 10^5$ and $1 + R_2/R_1 = 10$ (the amplifier DC gain is -9).

Example 9.11: An inverting amplifier with a gain of -20 and bandwidth of at least 20 kHz is needed. Is the LM348 chip appropriate for this purpose?

Solution: From the LM348 datasheet, we obtain $\mathrm{BW} = 1$ MHz. Because the inverting gain is -20, we should use a ratio of $R_2/R_1 = 20$. According to Eq. (9.22), this gives $f_b^{\text{closed loop}} = 47.6$ kHz. The closed-loop 3-dB bandwidth of the amplifier coincides with this value. Therefore, the LM348 chip is sufficient for our purposes. However, if its gain is forced to a higher value, say to 100, then the useful bandwidth reduces to 10 kHz.

Exercise 9.7: A non-inverting amplifier with a gain of 31 and a bandwidth of at least 90 kHz is needed. Is an LM741-based amplifier IC appropriate for this circuit?

Answer: No.

Section 9.3 Introduction to Continuous and Discrete Fourier Transform

9.3.1 Meaning and Definition of Fourier Transform

Consider a phasor as introduced in the previous chapter. This phasor is in fact a *transform*. It converts a harmonic sinusoidal time-domain signal into a complex number for easier, algebraic computation of circuit values. After determining the phasor value of a voltage or current signal, we transform it back to the time-domain expression. What if we do not have a pure sinusoidal tone, but an arbitrary voltage pulse $v(t)$ in the time domain? Another important example is a bit stream of arbitrary data, which can also be described by a certain voltage function $f(t)$. Could we still introduce a "phasor" for an arbitrary signal $f(t)$ in the time domain? The answer is yes; however, instead of a single complex number, we will have an entire complex function $F(\omega)$ of angular frequency ω. This function essentially consists of individual phasors, corresponding to all possible harmonic signals, which form the time-domain signal $f(t)$. Mathematically, the *direct Fourier transform* (from time domain to frequency domain) is given by

$$F(\omega) \equiv \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-j\omega t}dt, \tag{9.25a}$$

whereas the *inverse Fourier transform* (from frequency domain to time domain) is given by a similar integral

$$f(t) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{j\omega t}d\omega \tag{9.25b}$$

The pair of integrals in Eqs. (9.25a, 9.25b) completely describes the Fourier transform. Function $F(\omega)$ is called the *Fourier spectrum* (or simply the *spectrum*) of the signal $f(t)$. This function is generally complex; however, in contrast to the previous convention, we will not use boldface here in order to preserve the most common mathematical notations.

Exercise 9.8: Establish a relation between $F(-\omega)$ and $F(\omega)$ for a real signal $f(t)$, which is called a *reversal property* of the Fourier transform.

Answer:

$$F(-\omega) = F^*(\omega) \tag{9.26}$$

where the star denotes complex conjugate.

The spectrum is said to be *bandlimited* if $F(\omega)$ is zero above a certain angular frequency $\omega_{\max}$. According to Eq. (9.26), this simultaneously means that $F(\omega)$ is zero below $-\omega_{\max}$. Many useful signals are approximately bandlimited.

Example 9.12: Derive the Fourier transform of a rectangular pulse in the form of one bit of data shown in Fig. 9.14a.

Solution: The integral in Eq. (9.25a) is reduced to

$$F(\omega) = V_{\mathrm{m}} \int_{-T/2}^{T/2} \mathrm{e}^{-j\omega t} dt = \frac{V_{\mathrm{m}}}{-j\omega} \mathrm{e}^{-j\omega t} \Big|_{-T/2}^{T/2} = V_{\mathrm{m}} T \frac{\sin \omega T/2}{\omega T/2} \tag{9.27}$$

The function $\mathrm{sinc}(x) \equiv \sin \pi x / (\pi x)$ is called a *sinc function*. Using its definition, the final result for the spectrum has the form

$$F(\omega) = V_{\mathrm{m}} T \mathrm{sinc}\left(\frac{\omega T}{2\pi}\right) \tag{9.28}$$

and is plotted in Fig. 9.14b using a few lines of MATLAB code:

```
Vm      = 1;                              %  input voltage amplitude, V
T       = 1e-6;                           %  pulse duration, s
omega   = linspace(-12*pi/T, 12*pi/T);    %  angular frequency, rad/s
F       = Vm*T*sinc(omega*T/(2*pi));      %  inverse Fourier transform
plot(omega, F); grid on;
```

In contrast to the original signal, the pulse spectrum is not bounded and extends to infinity. This is due to the fact that the original pulse has sharp edges, which are described by higher-frequency harmonics. The pulse spectrum in the form of a sinc function is famous in communications theory. Figure 9.15 shows the shape of the sinc function depicted on an electronics store in Silicon Valley, CA.

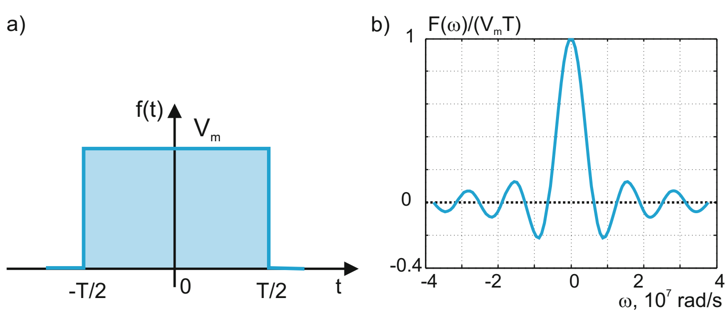


Fig. 9.14. (**a**) Rectangular pulse *f*(*t*) of duration $T = 1$ μs and (**b**) its Fourier spectrum in the form of a sinc function.

Fig. 9.15. Fry's Electronics store in Sunnyvale, Silicon Valley, with an emblem depicting the sinc function.

9.3.2 Mathematical Properties of Fourier Transform

Major *mathematical properties* of the Fourier transform follow from its definition and are listed in Table 9.2.

Table 9.2. Major mathematical properties of Fourier transform.

$f(t)$	$F(\omega)$
$Kf(t)$	$KF(\omega)$
$f_1(t) - f_2(t) + f_3(t)$	$F_1(\omega) - F_2(\omega) + F_3(\omega)$
$d^n f(t)/dt^n$	$(j\omega)^n F(\omega)$
$\int_{-\infty}^{t} f(\tau)d\tau, \quad \int_{-\infty}^{\infty} f(t)dt = 0$	$\frac{1}{j\omega}F(\omega)$
$f(at)$	$(1/a)F(\omega/a), \quad a > 0$
$f(t-a)$	$e^{-j\omega a}F(\omega)$

The two first properties follow from Fourier transform linearity. Multiplication of $f(t)$ by a constant corresponds to multiplying $F(\omega)$ by the same constant. Also, addition (subtraction) in the time domain corresponds to addition (subtraction) in the frequency domain. The next two properties (differentiation and integration) make the Fourier transform useful for solving ODEs since the time-domain derivatives and integrals will correspond to multiplication and division by $j\omega$ in the frequency domain. The two last properties (scaling and translation) directly follow from Eq. (9.25a).

Exercise 9.9: The Fourier transform of $f(t)$ is $F(\omega)$. What is the Fourier transform of $df(t)/dt + 5f(t)$?

Answer: $(5 + j\omega)F(\omega)$.

We emphasize that the properties listed in Table 9.2 also apply to the discrete Fourier transform studied below, but the corresponding indexing of discrete frequencies has to be carefully arranged.

9.3.3 Discrete Fourier Transform and Its Implementation

Direct Discrete Fourier Transform

Present and future demands are such that we must process continuous signals by discrete methods. Perhaps the most important method is the *discrete Fourier transform* (DFT) and its fast versions: *fast Fourier transform* (FFT) and *inverse fast Fourier transform* (IFFT). Let $f(t)$ be a continuous pulse signal which is the source of the data. We assume that $f(t)$ is zero outside of the interval $0 \le t < T$. Let $f(t_n), \quad n = 0, \ldots, N-1$ be its values at N uniformly distributed *sampling points* $t_n = \Delta T n, \quad n = 0, \ldots, N-1$ within the interval of interest. Here

$$\Delta T = \frac{T}{N} \tag{9.29}$$

is the *sampling interval*. Then, the integral of the direct Fourier transform in Eq. (9.25a) may be found using the *rectangle rule* (or the *Riemann sum approximation*)

$$F(\omega) = \Delta T \sum_{n=0}^{N-1} \mathrm{e}^{-j\omega n \Delta T} f(t_n) \tag{9.30}$$

We could in principle evaluate this expression at any value of ω. However, with *only N* data points to start with, *only N* final outputs will be significant. We choose those N uniformly distributed frequency sampling points as $\omega_m = \omega_0 m, \quad m = 0, \ldots, N-1$, where

$$\omega_0 = \frac{2\pi}{T} \tag{9.31}$$

is the *fundamental frequency* (with one period over the interval T). Let $F(\omega_m), m = 0, \ldots, N-1$ be the values of $F(\omega)$ at the frequency sampling points. Then, Eq. (9.30) gives

$$F(\omega_m) = \Delta T \sum_{n=0}^{N-1} \mathrm{e}^{-j\frac{2\pi}{N}mn} f(t_n), \quad m = 0, \ldots, N-1 \tag{9.32}$$

Inverse Discrete Fourier Transform (IDFT)

A very similar operation is applied to the integral of the inverse Fourier transform given by Eq. (9.25b). We first assume that $F(\omega)$ is zero outside of the interval $0 \le \omega < N\omega_0$; in other words, it is *bandlimited*. Then, the corresponding integral in Eq. (9.25b) is again approximated using the rectangle rule so that the final result has the form

$$f(t_n) = \frac{1}{N\Delta T}\sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}mn}F(\omega_m), \quad n = 0, \ldots, N-1 \tag{9.33}$$

Definition of Discrete Fourier Transform

It is rather inconvenient to keep the factor ΔT in both Eqs. (9.32) and (9.33), respectively. Therefore, we may introduce the notation

$$f[n] \equiv \Delta T f(t_n), \quad F[m] \equiv F(\omega_m) \tag{9.34}$$

and obtain the *standard form* of the discrete Fourier transform

$$F[m] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}mn} f[n], \quad m = 0, \ldots, N-1 \tag{9.35a}$$

$$f[n] = \frac{1}{N}\sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}mn} F[m], \quad n = 0, \ldots, N-1 \tag{9.35b}$$

Here, $f[n]$ may be treated as an impulse having the area of $\Delta T f(t_n)$.

Exercise 9.10: Establish a relation between $F[N-m]$ and $F[m]$ for a real signal $f(t)$, which is called a *reversal property* of the discrete Fourier transform.

Answer:

$$F^*[N-m] = F[m] \tag{9.36}$$

where the star again denotes complex conjugate.

Example 9.13: It is possible to very significantly minimize the actual number of multiplications necessary to compute a given DFT in Eqs. (9.35a, b). The DFT so constructed is the *fast Fourier transform* (FFT) and *inverse fast Fourier transform* (IFFT). It works best when N is a power of two. For a pulse $f(t) = \exp\left(-2(t-5)^2\right)$, $0\ \text{s} \le t < 10\ \text{s}$, compute its FFT and then the IFFT and finally compare the end result with the original pulse form given that $N = 64$.

Solution: The solution is conveniently programmed using a few lines of a self-explanatory MATLAB code, which uses Eq. (9.29) and plots two final curves:

Example 9.13 (cont.):

```
T = 10; N = 64;
dT = T/N; t = dT*(0:N-1);
f0 = exp(-2*(t-5).^2);
F = fft(f0); f = ifft(F);
plot(t, f, t, f0, '*');
```

Both curves are virtually identical: the relative integral error (integral of signal difference magnitude over the integral of signal magnitude) does not exceed 10^{-16}.

Structure of Discrete Fourier Spectrum

The set of spectrum values $F[m]$, $m = 0, \ldots, N-1$, of the DFT has an important redundancy property illustrated in the following example.

Example 9.14: Express all discrete Fourier spectrum values $F[m]$ present in Eq. (9.35a) through $N/2$ first values of $F[m]$ only. *Hint:* Use Eq. (9.36).

Solution:

$$\begin{aligned}&F[0], F[1], \ldots, F\left[\frac{N}{2}-1\right], F\left[\frac{N}{2}\right], F\left[\frac{N}{2}+1\right], \ldots, F[N-1] = \\ &F[0], F[1], \ldots, F\left[\frac{N}{2}-1\right], F\left[\frac{N}{2}\right], F^*\left[\frac{N}{2}-1\right], \ldots, F^*[1]\end{aligned} \tag{9.37}$$

Equation (9.37) demonstrates how the output of the DFT (and of the FFT, in particular in MATLAB) is arranged in reality. It is a symmetric conjugate about $m = N/2$. Equation (9.37) is a key to finding derivatives and arbitrary filter transformations of the input signal with the FFT. Only a frequency with $m \leq N/2$ is considered to be *valid*; its mirror reflection about $m = N/2$ is a higher "*aliasing frequency*." We emphasize that, according to Eq. (9.26), the complex conjugates may be replaced by spectrum values at a negative frequency, i.e., $F^*[1] = F[-1]$. Thus, the spectrum above $m = N/2$ corresponds to negative frequencies with $m > -N/2$.

9.3.4 Sampling Theorem

It follows from Example 9.14 that only frequency samples with $\omega_m \leq \frac{N}{2}\omega_0$ are really needed. This fact is a consequence of the *sampling theorem*, which states that any signal bandlimited to $\omega_{\max}$ can be reproduced *exactly* using the discrete Fourier transform if

$$\omega_{\max} \leq \frac{N}{2}\omega_0 \tag{9.38a}$$

Accordingly, the *maximum possible* sampling interval may be found from inequality

$$\Delta T \le \frac{1}{2}\frac{1}{f_{\max}}, \quad f_{\max} = \frac{\omega_{\max}}{2\pi} \tag{9.38b}$$

Exercise 9.11: Examples of the maximum frequency of interest for some biomedical signals are:

1. Electrocardiogram (ECG) where $f_{\max} \approx 250\,\text{Hz}$
2. Blood flow where $f_{\max} \approx 25\,\text{Hz}$
3. Respiratory rate where $f_{\max} \approx 10\,\text{Hz}$
4. Electromyogram where $f_{\max} \approx 10\,\text{kHz}$

Establish the maximum possible sampling interval of the DFT and the minimum possible *sampling frequency*, which is equal to $1/\Delta T$.

Answer: (1) 2 ms and 500 Hz; (2) 20 ms and 50 Hz; (3) 50 ms and 20 Hz; (4) 50 μs and 20 kHz.

9.3.5 Applications of Discrete Fourier Transform

The DFT is one of the most important tools in *digital signal processing* (DSP). In particular, the DFT can calculate a signal's frequency spectrum. This is a direct examination of information encoded in the frequency, phase, and amplitude of the component sinusoids. For example, human speech and hearing use signals with this type of encoding. Second, the DFT or rather its variation, the *discrete cosine transform*, is used in sound compression; the MP3 format is one such example. The DFT is also an important image processing tool which is used to decompose an image into its sine and cosine components. The output of the transformation represents the image in the Fourier or frequency domain, while the input image is the spatial domain equivalent. In the Fourier domain image, each point represents a particular frequency contained in the spatial domain image. In particular, the JPEG format is using a modification of the DFT for image compression; the DFT is also used for image filtering and reconstruction. Along with this, the DFT is used widely in bioinformatics/computational biology to analyze DNA sequences. Last but not least, many computational modeling tools, such as antenna and high-speed circuit simulators, typically operate at one particular signal frequency (in the frequency domain). Collecting the solutions at many such frequencies makes it possible to establish evolution of an arbitrary signal or wave field in time.

9.3.6 Application Example: Numerical Differentiation via the FFT

We have established that a filter is characterized by its transfer function $\mathbf{H}(f)$ or $\mathbf{H}(\omega)$ and found this transfer function for simple cases. Given the input sinusoidal signal, we have also shown how to evaluate the filter's output when its transfer function is known. But what if the input signal is an arbitrary pulse? How could the solution for the output pulse be obtained? The answer relies upon an observation that the transfer function given by Eq. (9.9b) may be applied to every harmonic component of the input signal $f_{in}(t)$ *separately*. Those harmonics are all described by the Fourier spectrum of the pulse, $F(\omega)$. Therefore, the output Fourier pulse spectrum is given by

$$F_{out}(\omega) = \mathbf{H}(\omega) F_{in}(\omega) \tag{9.39}$$

The remaining part is to find the output pulse itself, which is clearly the inverse Fourier transform:

$$f_{out}(t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} F_{out}(\omega) e^{j\omega t} d\omega \tag{9.40}$$

When moving from continuous toward discrete Fourier transform and toward digital signal processing, Eq. (9.39) becomes a somewhat tricky operation. According to Eq. (9.37), the discrete version of Eq. (9.39) must have the form

$$\begin{aligned} \mathbf{H}F \to \mathbf{H}[0]F[0],\ \mathbf{H}[1]F[1],\ \ldots,\ \mathbf{H}\left[\frac{N}{2}-1\right]F\left[\frac{N}{2}-1\right],\ \mathbf{H}\left[\frac{N}{2}\right]F\left[\frac{N}{2}\right], \\ \mathbf{H}^*\left[\frac{N}{2}-1\right]F\left[\frac{N}{2}+1\right], \ldots\ \mathbf{H}^*[1]F[N-1] \end{aligned} \tag{9.41}$$

This version corresponds to the full list of monotonic frequency data $\omega_m = \omega_0 m, \quad m = 0, \ldots, N-1$. Also note that, in all realistic linear systems,

$$\mathbf{H}^*(m) = \mathbf{H}(-m) \tag{9.42}$$

Therefore, Eq. (9.41) simultaneously describes a set of data for the following non-monotonic frequency list $0,\ \omega_m, \ldots, \frac{N}{2}\omega_0, \left(1-\frac{N}{2}\right)\omega_0, \left(2-\frac{N}{2}\right), \ldots, -\omega_0$, which also includes the negative frequencies.

Example 9.15 (numerical differentiation via the FFT): Prove Eq. (9.41) for a pulse $f(t) = \exp\left(-2(t-5)^2\right), \quad 0\ \mathrm{s} \le t < 10\ \mathrm{s}$ and for $\mathbf{H}(\omega) = j\omega$. Such a transfer function corresponds to numerical differentiation via the FFT. Use the FFT and IFFT with $N = 64$.

Solution: The solution is conveniently programmed in a self-explanatory MATLAB code, which uses Eq. (9.41) and plots two final results in Fig. 9.16, the numerical pulse derivative and the analytical derivative, respectively:

Example 9.15 (numerical differentiation via the FFT) (cont.):

```
T = 10; N = 64;
dT = T/N; t = dT*(0:N-1);
f = exp(-2*(t-5).^2);                    %   input pulse
omega = (2*pi/T)*[0:N/2];                %   non-aliasing frequencies
H       = j*omega;                       %   H at non-aliasing frequencies
F       = fft(f);                        %   FFT spectrum
HF      = F.*[H, conj(H(end-1:-1:2))];   %   HF according to Eq. (9.40)
fder    = real(ifft(HF));                %   numerical derivative
fder0   = -4*(t-5).*f;                   %   analytical derivative
plot(t, fder0, t, fder, 'd');            %   compare both derivatives
```

Both curves are virtually identical: the relative integral error (integral of signal difference magnitude over the integral of analytical signal magnitude) does not exceed 1.3×10^{-15}.

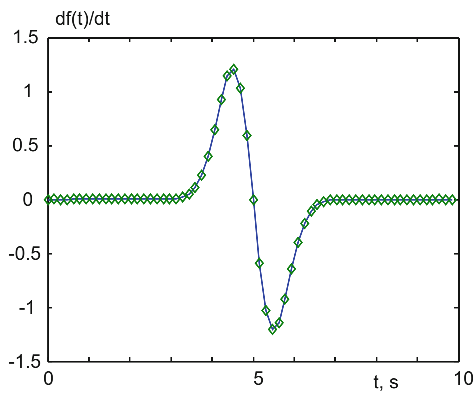


Fig. 9.16. Analytical (*solid curve*) and numerical (*diamonds*) differentiation of the original Gaussian pulse.

9.3.7 Application Example: Filter Operation for an Input Pulse Signal

The filter operation for an input pulse signal exactly follows Example 9.15 but with a different transfer function $\mathbf{H}(\omega)$.

Example 9.16: A pulse $f(t) = \exp\left(-2(t-5)^2\right), \quad 0\ \text{s} \le t < 10\ \text{s}$ is an input to a first-order high-pass filter. Find the filter output when its (angular) break frequency is given by a) $\omega_0 = 1$ rad/s and b) $\omega_0 = 10$ rad/s. Use the FFT and IFFT with $N = 64$.

Solution: The solution is performed and programmed exactly described in the previous example, but the transfer function is now given by Eq. (9.1b):

```
H = j*omega/omega0./(1+j*omega/omega0);
```

Example 9.16 (cont.):
Figure 9.17 plots two output pulse forms corresponding to two different values of the break frequency. Note that the value $\omega_0 = 10$ rad/s approximately corresponds to a mean value for non-aliasing frequencies of the FFT. Also note that when the break frequency becomes sufficiently high, the HPF behaves as an ideal differentiator but with a significant amplitude decay.

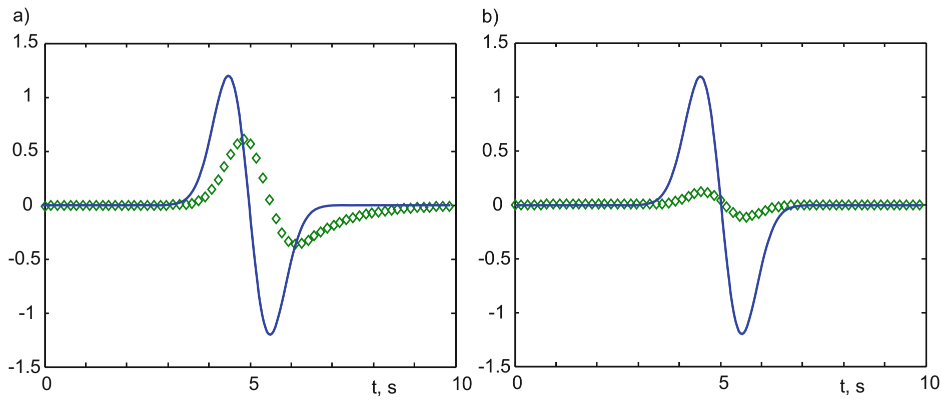


Fig. 9.17. *Diamonds*: HPF output for (**a**) $\omega_0 = 1$ rad/s and (**b**) $\omega_0 = 10$ rad/s, respectively. *Solid curve*: analytical derivative of the input Gaussian pulse.

9.3.8 Application Example: Converting Computational Electromagnetic Solution from Frequency Domain to Time Domain

Many computational modeling tools operate at one particular signal frequency or at a set of those (in frequency domain). To obtain the solution for an arbitrary pulse at an arbitrary point in space, we can again use the method of the transfer function and the FFT described previously. As an example, we consider a TMS (*transcranial magnetic stimulation*) coil above the head of a computational human phantom in Fig. 9.18a. Once a current pulse is applied to the coil, an electric field will be excited in the brain according to Faraday's law of induction. This field may help to reestablish some neuron connections lost, for example, in Parkinson's disease. For safety considerations, the field at arbitrary locations within the body needs to be evaluated, let's say at node 2 in Fig. 9.18a. In order to do so, the problem is first solved for about 40 single-frequency excitations, which will presumably cover the spectrum content of the desired TMS pulse in Fig. 9.18b well. The ratio of the electric field phasor at the observation point to the coil current phasor is the transfer function value at a desired frequency, $\mathbf{H}(\omega)$. This ratio does not depend on particular amplitude of the coil current. Next, we introduce the DFT of size N for the original pulse shown in Fig. 9.18b, interpolate the transfer function over $N/2 + 1$ required frequency points, and apply the method of Examples 91.5 and 9.16 with this new transfer function. The result is an electric field pulse at node 2 shown in Fig. 9.18c (the dominant z-component has been plotted), which is excited by the coil current shown in Fig. 9.18b.

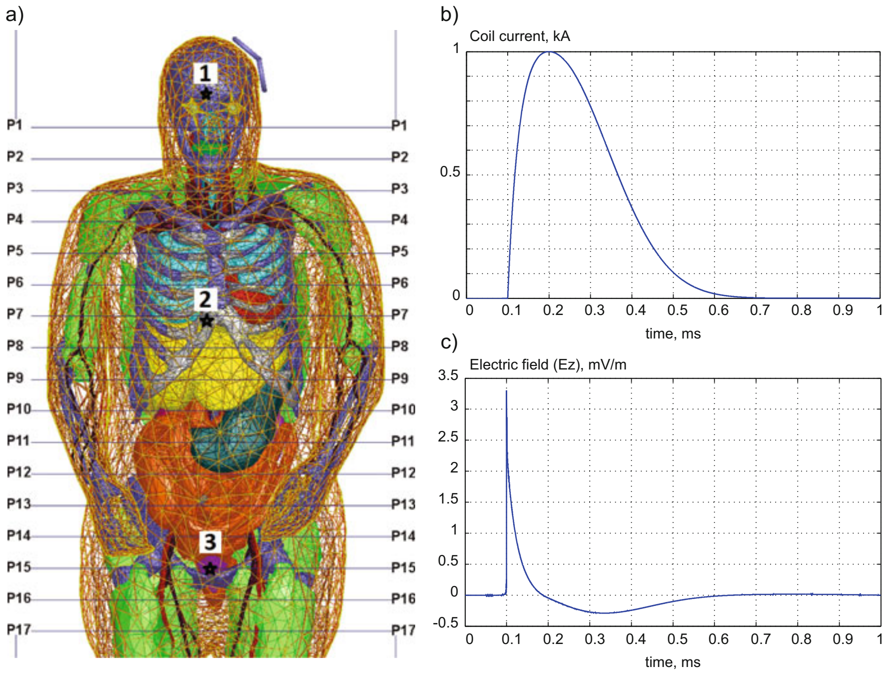
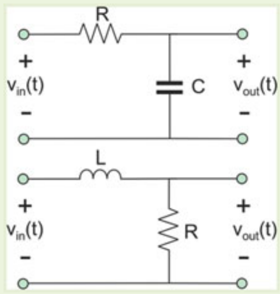
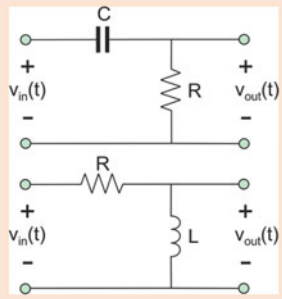
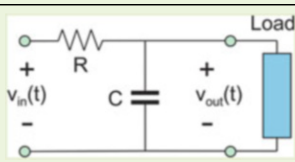
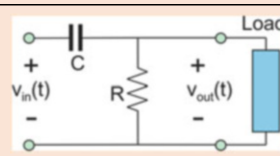
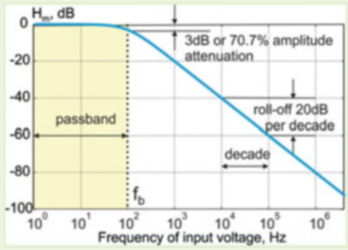
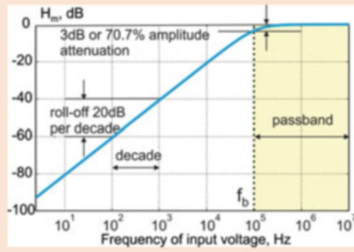
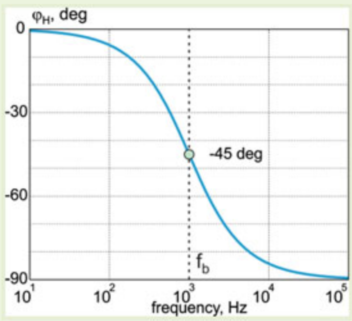
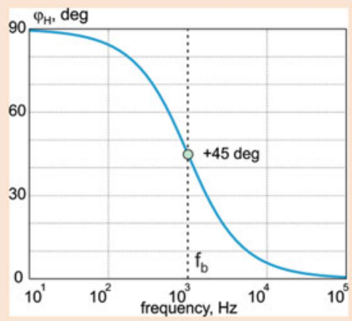
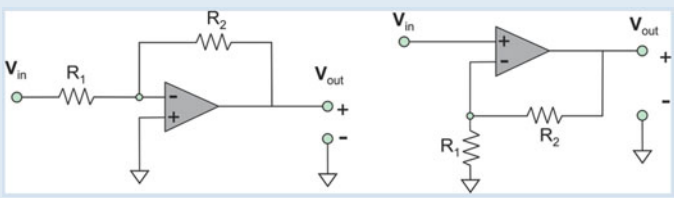


Fig. 9.18. Time-domain computational solution for the induced electric field within a human body obtained from the frequency-domain data via the FFT.

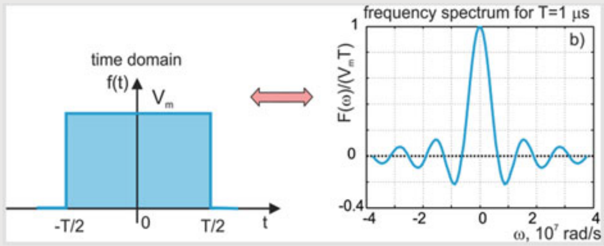
Summary

Property	**First-order low-pass filter**	**First-order high-pass filter**
Circuit schematic		
Transmission at $f = 0$ (DC)	**1** (DC path through the resistor)	**0** (No DC path)
Transmission at $f \to \infty$	**0** (Inductor is an open circuit at $f \to \infty$)	**1** (DC path through the resistor)
Transfer function $\mathbf{H}(f)$	$\dfrac{1}{1 + j(f/f_b)}$	$\dfrac{(f/f_b)}{1 + j(f/f_b)}$
Decibels of $H = \lvert\mathbf{H}\rvert$	$20\log_{10}H$ [dB]	$20\log_{10}H$ [dB]
Decibels of 1 and 0.1	0 dB and −20 dB	0 dB and −20 dB
Transfer function magnitude $H_m(f)$	$\dfrac{1}{\sqrt{1 + (f/f_b)^2}}$	$\dfrac{f/f_b}{\sqrt{1 + (f/f_b)^2}}$
Transfer function phase $\angle\varphi_H$	$\angle - \tan^{-1}\left(\dfrac{f}{f_b}\right)$	$\dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{f}{f_b}\right)$
Break frequency, (half-power frequency, 3-dB frequency, corner frequency)	$f_b = \dfrac{1}{2\pi\tau}$ [Hz] $\tau = RC$ or $\dfrac{L}{R}$ [s]	$f_b = \dfrac{1}{2\pi\tau}$ [Hz] $\tau = RC$ or $\dfrac{L}{R}$ [s]
Passband (3 dB bandwidth), Hz	From 0 to f_b	From f_b to ∞
Filter with a resistive load R_L		

(continued)

Transfer function with the load $\mathbf{H}(f)$	$\mathbf{H}(f) = \dfrac{1}{1 + R/R_L + j(f/f_b)}$ $f_b = \dfrac{1}{2\pi\tau}, \quad \tau = RC$	$\mathbf{H}(f) = \dfrac{(f/f_b)}{1 + j(f/f_b)}$ $f_b = \dfrac{1}{2\pi\tau}, \quad \tau = (R\|\|R_L)C$
Amplitude Bode plots		
Phase Bode plots		
Meaning of the transfer function for harmonic signals represented by phasors	$\mathbf{V}_{out} = \mathbf{H}(f)\mathbf{V}_{in}$ where $\mathbf{V}_{in}$ is the input voltage phasor and $\mathbf{V}_{out}$ is the output voltage phasor	
Cascading filters and linear systems (series combination)	$\mathbf{H}(f) = \mathbf{H}_1(f)\mathbf{H}_2(f)$ only if the loading effect of individual blocks is minimized	
3-dB bandwidth of an operational amplifier circuit (inverting or non-inverting amplifier configuration)	 From 0 to $f_b^{\text{closed loop}}$ (closed-loop 3-dB frequency) where $f_b^{\text{closed loop}} = \dfrac{\text{BW}}{1 + R_2/R_1}$ BW is the unity-gain bandwidth reported in the datasheet	

(continued)

Property	**Continuous and discrete Fourier transform**
Fourier transform definition	$F(\omega)\equiv\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \quad f(t)\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$ $F(-\omega)=F^*(\omega)$ if $f(t)$ is real
Fourier transform of a rectangular pulse	 $F(\omega)=V_mT\ \mathrm{sinc}\left(\frac{\omega T}{2\pi}\right)$
Fourier transform of a Gaussian pulse	$f(t)=e^{-at^2}\Leftrightarrow F(\omega)=\sqrt{\frac{\pi}{a}e^{-\omega^2/(4a)}}$
Major properties of Fourier transform	see table below
Definition of sampling points: discrete Fourier transform	$t_n=\Delta T n,\ \ n=0,\ldots,N-1,\ T=N\Delta T$ $\omega_m=\omega_0 m,\ \ m=0,\ldots,N-1,\ \omega_0=\frac{2\pi}{T}$
Definition of samples: discrete Fourier transform	$f[n]\equiv\Delta T f(t_n),\quad n=0,\ldots,N-1$ $F[m]\equiv F(\omega_m),\quad m=0,\ldots,N-1$
Discrete/fast Fourier transform	$F[m]=\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}mn} f[n],\quad m=0,\ldots,N-1$ $f[n]=\frac{1}{N}\sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}mn} F[m],\quad n=0,\ldots,N-1$

Major properties of Fourier transform:

$f(t)$	$F(\omega)$
$Kf(t)$	$KF(\omega)$
$f_1(t)-f_2(t)+f_3(t)$	$F_1(\omega)-F_2(\omega)+F_3(\omega)$
$d^n f(t)/dt^n$	$(j\omega)^n F(\omega)$
$\int_{-\infty}^{t} f(\tau)d\tau,\quad \int_{-\infty}^{\infty} f(t)dt=0$	$\frac{1}{j\omega}F(\omega)$
$f(at)$	$(1/a)F(\omega/a),\quad a>0$
$f(t-a)$	$e^{-j\omega a}F(\omega)$

(continued)

Sampling theorem	1. Any signal bandlimited to $\omega_{\max}$ can be reproduced exactly using the discrete Fourier transform if $\omega_{\max} \le \frac{N}{2}\omega_0$ 2. Alternatively, the sampling interval must satisfy inequality $$\Delta T \le \frac{1}{2}\frac{1}{f_{\max}}, \quad f_{\max} = \frac{\omega_{\max}}{2\pi}$$
Structure of discrete Fourier spectrum	$$F^*[N-m] = F[m]$$ $$\Downarrow$$ $$F[0], F[1], \ldots, F\left[\frac{N}{2}-1\right], F\left[\frac{N}{2}\right], F\left[\frac{N}{2}+1\right], \ldots, F[N-1] =$$ $$F[0], F[1], \ldots, F\left[\frac{N}{2}-1\right], F\left[\frac{N}{2}\right], F^*\left[\frac{N}{2}-1\right], \ldots, F^*[1]$$
Equivalent frequency samples for negative frequencies	$$0,\ \omega_m,\ \ldots,\ \frac{N}{2}\omega_0, \left(1-\frac{N}{2}\right)\omega_0, \left(2-\frac{N}{2}\right), \ldots, -\omega_0$$
Transfer function multiplication	$$\mathbf{H}F \to \mathbf{H}[0]F[0],\ \mathbf{H}[1]F[1],\ \ldots,\ \mathbf{H}\left[\frac{N}{2}-1\right]F\left[\frac{N}{2}-1\right],\ \mathbf{H}\left[\frac{N}{2}\right]F\left[\frac{N}{2}\right],$$ $$\mathbf{H}^*\left[\frac{N}{2}-1\right]F\left[\frac{N}{2}+1\right],\ \ldots,\ \mathbf{H}^*[1]F[N-1]$$

Problems

9.1 First-Order Filter Circuits and Their Combinations

9.1.1 RC Voltage Divider as an Analog Filter

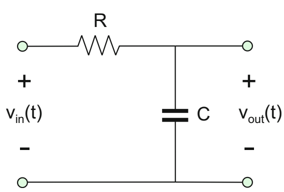
Problem 9.1

A. Explain the function of an analog RC filter.
B. Write the capacitor and resistor voltages $v_R(t)$ and $v_C(t)$ of the series RC circuits in the general form, as functions of the AC angular frequency.
C. Which circuit element (or which voltage) dominates at low frequencies? At high frequencies?

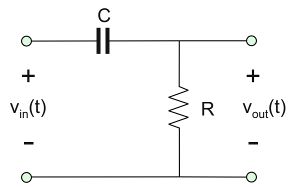
Problem 9.2

A. Draw a schematic of the low-pass analog RC filter. Show the input and output ports.
B. Repeat the same task for the high-pass analog RC filter.

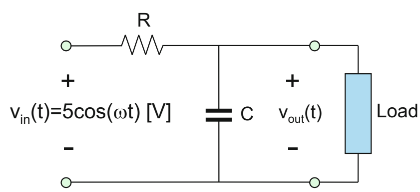
Problem 9.3. The input voltage to the filter circuit shown in the following figure is a combination of two harmonics, $v_{in}(t) = 1\cos\omega_1 t + 1\cos\omega_2 t$, with the amplitude of 1 V each. The filter has the following parameters: $R = 100$ kΩ and $C = 1.59$ nF. Determine the output voltage $v_{out}(t)$ to the filter given that $f_1 = 100$ Hz and $f_2 = 100$ kHz. Express all phase angles in degrees.



Problem 9.4. Repeat the previous problem for the filter circuit shown in the following figure. All other parameters remain the same.

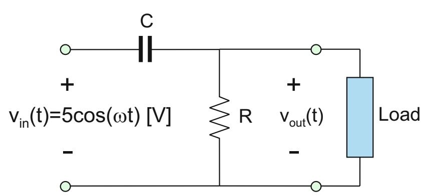


Problem 9.5. The input voltage to the RC filter circuit shown in the figure is $V_{in}(t) = 5\cos\omega t$ [V]. The filter has the following parameters: $C = 1$ μF and $R = 100$ Ω. The filter operates in the frequency band from 100 Hz to 50 kHz. The filter is connected to a load with the load resistance of 1 MΩ. By solving the corresponding AC circuit, determine the output voltage amplitude across the load (and its percentage versus the input voltage amplitude) *with and without the load* at $f = 100$ Hz, $f = 1592$ Hz, and $f = 50$ kHz.



Problem 9.6. Repeat the previous problem when the load resistance changes from 1 MΩ to 100 Ω (decreases).

Problem 9.7. Repeat Problem 9.5 for the filter circuit shown in the following figure. Assume the load resistance of 100 Ω.



9.1.2 Half-Power Frequency and Amplitude Transfer Function

9.1.3 Bode Plot, Decibel, and Roll-off

Problem 9.8.

A. Describe the physical meaning of the (half-power) break frequency in your own words.
B. Give the expression for the break frequency in terms of circuit parameters of an RC filter. Is it different for low-pass and high-pass filters?

Problem 9.9. Given $R = 100\ \text{k}\Omega$ and $C = 1.59$ nF, determine the break frequency of the low-pass RC filter and of the high-pass RC filter, respectively.

Problem 9.10. List all possible alternative names for the break frequency.

Problem 9.11. Write the amplitude transfer function for the low-pass RC filter. Repeat for the high-pass RC filter. Indicate units (if any).

Problem 9.12. The input signal to a high-pass RC filter includes a 60-Hz component. Its amplitude is to be reduced by a factor of 10. What break frequency should the filter have?

Problem 9.13. The input signal to a low-pass RC filter includes a 10-kHz component. Its amplitude is to be reduced by a factor of 5. What break frequency should the filter have?

Problem 9.14. Describe the meaning of the Bode plot in your own words.

Problem 9.15. It is known that $H_m(f)_{dB} = 0, \ -6, \ -20$ [dB]. Find the corresponding values of $H_m(f)$.

Problem 9.16. The following values are given $H_m(f) = 1, \ 0.707, \ 0.1,$ and 100. Find the corresponding values of $H_m(f)_{dB}$.

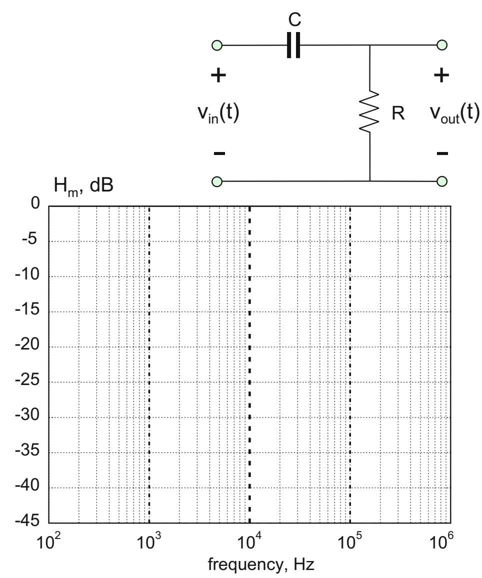
Problem 9.17

A. When the ratio of the amplitudes of two signals is $\sqrt{2}$, what is the difference between the two corresponding decibel measures in dB?
B. When the ratio of the amplitudes of two signals is $1/\sqrt{2}$, what is the difference between the two corresponding decibel measures in dB?
C. When the ratio of the amplitudes of two signals is $\sqrt{20}$, what is the difference between the two corresponding decibel measures in dB?
D. When the ratio of the powers of two signals is 1000, what is the difference between the two corresponding decibel measures in dB?

Problem 9.18. What do engineers mean by one decade? One octave?

Problem 9.19. For the filter circuit shown in the following figure, given that $R = 100\ \text{k}\Omega$ and $C = 159$ pF:

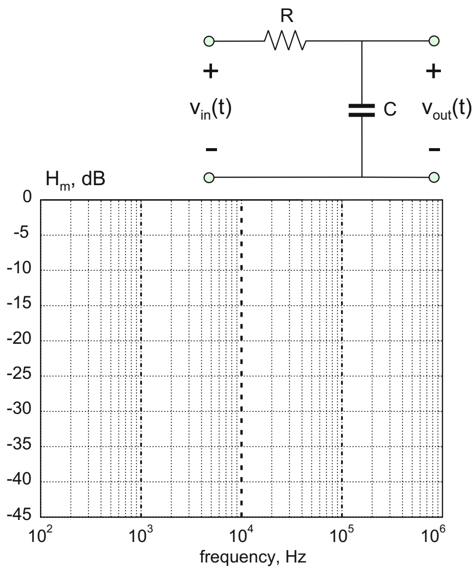
A. Create the amplitude Bode plot by finding transfer function values for (at least) every decade.
B. Label the break frequency.
C. Label the filter passband.



Problem 9.20. Repeat the previous problem with $R = 100\ \text{k}\Omega$ and $C = 53$ pF.

Problem 9.21. For the filter circuit shown in the following figure, assume the values $R = 10\ \text{k}\Omega$ and $C = 1.59\ \text{nF}$.

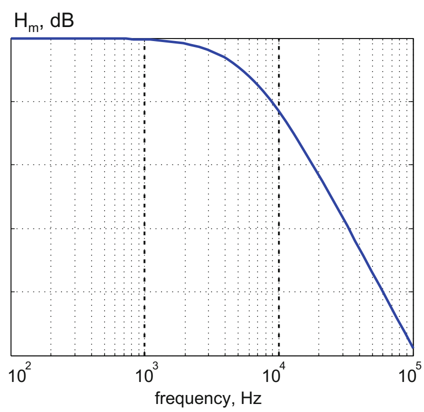
A. Create the amplitude Bode plot by finding the transfer function values for (at least) every decade.
B. Label the break frequency.
C. Label the filter passband.



Problem 9.22. Repeat the previous problem with $R = 100\ \text{k}\Omega$ and $C = 53\ \text{pF}$.

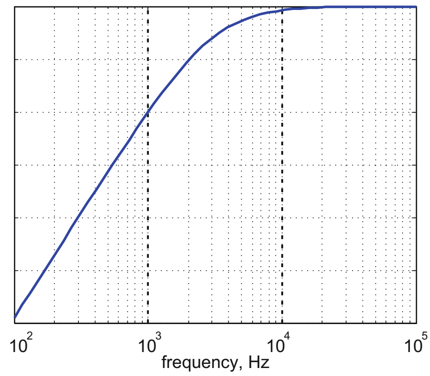
Problem 9.23. An amplitude Bode plot for a certain RC filter is shown in the figure below.

A. Approximately determine the filter's resistance R if it is known that $C = 265\ \text{pF}$. Describe each step of your approach.
B. Suggest a way to verify your solution.



Problem 9.24. An amplitude Bode plot for a certain RC filter is shown in the figure below.

A. Approximately determine the filter's capacitance, C, for a given $R = 100\ \text{k}\Omega$. Describe each step of your approach.
B. Suggest a way to verify your solution.



Problem 9.25. Prove analytically that the amplitude transfer functions of the low-pass filter and the high-pass filter are the mirror reflections of each other about the break frequency in the Bode plot.

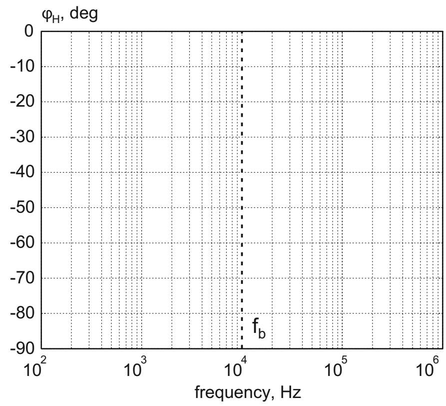
9.1.4 Phase Transfer Function and Its Bode Plot

Problem 9.26. Write the phase transfer function for the low-pass RC filter. Repeat for the high-pass RC filter. Show units.

Problem 9.27. The input voltage to a low-pass RC filter has a zero phase. At what frequency in terms of the break frequency f_b is the phase shift at the output equal to $-1°$, $-45°$, and $-89°$?

Problem 9.28. The input voltage to a high-pass RC filter has a zero phase. At what frequency in terms of the break frequency f_b is the phase shift at the output equal to $5°$, $45°$, and $85°$?

Problem 9.29. A low-pass RC filter has the break frequency of 10 kHz. Create the phase Bode plot by finding the transfer function values for (at least) every decade.

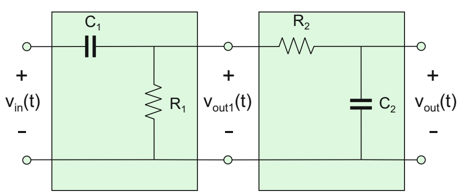


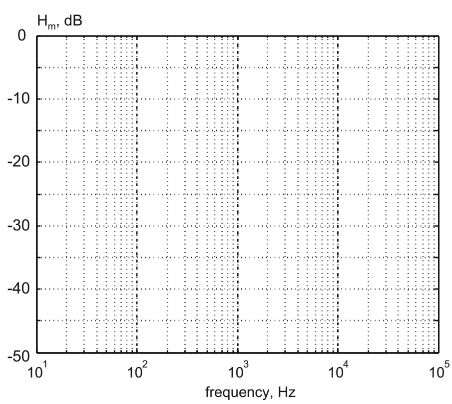
Problem 9.30. Repeat the previous problem for a high-pass RC filter with the same break frequency.

9.1.5 Complex Transfer Function: Cascading Filter Circuits

9.1.6 RL Filter Circuits

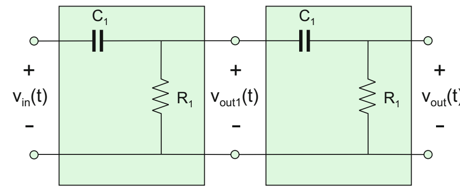
Problem 9.31. For the filter circuit shown in the figure below, create the amplitude response of the Bode plot by finding transfer function values for (at least) every decade. The two individual filter blocks both have the break frequency of 1 kHz. Assume that the loading effect of the filter stages is negligibly small; in practice, a buffer amplifier stage could be used.

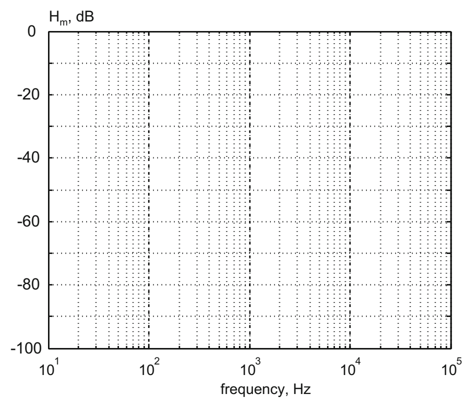




Problem 9.32

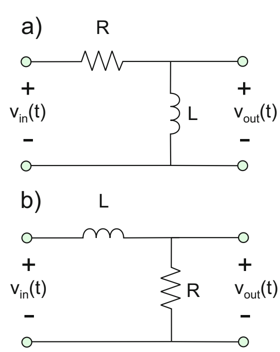
A. Repeat the previous problem for the filter circuit shown in the figure below.
B. Analytically determine the roll-off per decade in dB.





Problem 9.33. For two RL filter circuits with $R = 31.4\ \Omega$ and $L = 1\ \text{mH}$ shown in the figure below:

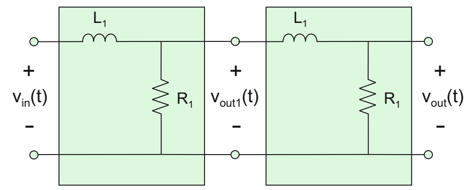
A. Determine the break frequency.
B. Draw the corresponding RC counterpart.
C. Establish the capacitance values of the RC filters, which assure the equivalent transfer functions, given that the resistances of the RC filters are 100 kΩ in both cases.

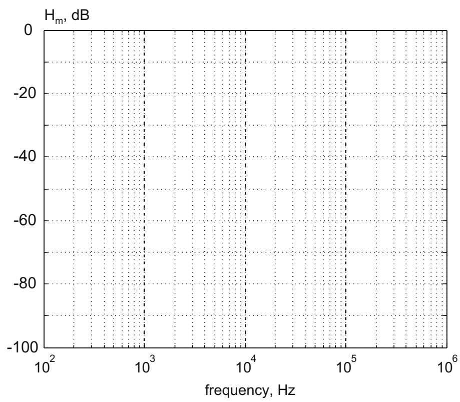


Problem 9.34. For the filter circuit shown in the figure below, assume the values $R_1 = 628$ Ω and $L_1 = 10$ mH.

A. Create the amplitude Bode plot by finding transfer function values for (at least) every decade.
B. Determine the roll-off per decade in dB.

Assume that the loading effect of the filter stages is negligibly small (e.g., a buffer amplifier stage is used).





Problem 9.35. The transfer function of a filter circuit is given by $H(f) = \frac{1+j(f/1000)}{1+(f/1000)^2}$. Create its amplitude and phase Bode plots in the frequency band from 10 Hz to 100 kHz by finding transfer function values for (at least) every decade.

9.2 Bandwidth of an Operational Amplifier

9.2.1 Bode Plot of the Open-loop Amplifier Gain

9.2.2 Unity-gain Bandwidth Versus Gain-Bandwidth Product

Problem 9.36. An amplifier has the unity-gain bandwidth BW of 5 MHz. What exactly does this mean? Explain and provide equations.

Problem 9.37. Using a manufacturing company's website (usually it is a more accurate frequently updated source) or the corresponding datasheet, find the unity-gain bandwidth for the following amplifier ICs:

A. TL082
B. LM741
C. LM7171

Problem 9.38. Frequency response of an amplifier is characterized by the open-loop DC gain $A_{OL}(0) = 1.41 \times 10^6$ and the break frequency of $f_b = 20$ Hz. Numerically calculate the gain-bandwidth product for the amplifier at:

A. 20 Hz,
B. 2 kHz,
C. 2 MHz.

9.2.3 Model of the Open-Loop AC Gain

Problem 9.39. Frequency response of an amplifier is characterized by the open-loop DC gain $A_{OL}(0) = 10^6$ and the break frequency of $f_b = 20$ Hz. Plot the open-loop gain magnitude in dB over the range of frequencies (the frequency band) from 1 Hz to 10 MHz on the log-log scale (the Bode plot) and label the axes.

Problem 9.40. In the previous problem, find the unity-gain bandwidth BW of the amplifier.

Problem 9.41. Internally compensated LM358-series amplifiers have the unity-gain bandwidth (BW) of 1 MHz. The typical large-signal DC voltage gain at room temperature is 100 V/mV.

A. Find the open-loop DC gain in dB and the open-loop break frequency f_b.
B. Find the open-loop gain at 100 Hz, 1 kHz, and 10 kHz.

Problem 9.42. The open-loop gain magnitude of an internally compensated high-frequency amplifier has been given as

$$A_{OL}(100\ \text{Hz}) = 0.9 \times 10^6,$$
$$A_{OL}(1\ \text{MHz}) = 1.0 \times 10^2$$

at room temperature. Determine:

A. 3-dB break frequency,
B. DC open-loop gain,
C. Unity-gain bandwidth BW of the amplifier.

Problem 9.43. Repeat the previous problem for

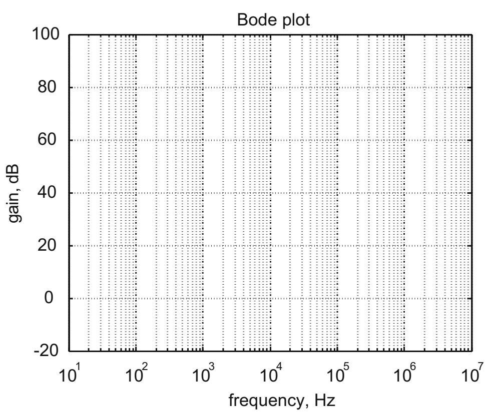
$$A_{OL}(100\ \text{Hz}) = 0.5 \times 10^6,$$
$$A_{OL}(1\ \text{MHz}) = 1.0 \times 10^2.$$

9.2.4 Model of the Closed-loop AC Gain

9.2.5 Application Example: Finding Bandwidth of an Amplifier Circuit

Problem 9.44. An amplifier with the open-loop gain described by the first-order RC circuit response with $A_{OL}(0) = 10^5$ and $f_b = 20$ Hz is used in a closed-loop inverting configuration with $R_2/R_1 = 9$ and $R_2/R_1 = 99$, respectively.

A. Using the template that follows, create the Bode plots for the corresponding frequency response (closed-loop gain), $G(f)$, in the band from 10 Hz to 10 MHz on the same graph. Plot the gain values for (at least) every decade.
B. Also on the same graph, plot the open-loop gain as a function of frequency.
C. Determine the bandwidth of the closed-loop amplifier so constructed in every case.



Problem 9.45. The unity-gain bandwidth of an amplifier IC is 1 MHz. Determine the bandwidth of the following amplifier circuits:

A. An inverting amplifier with the gain of -1,
B. An inverting amplifier with the gain of -10,
C. A non-inverting amplifier with the gain of 100,
D. A voltage follower (buffer amplifier).

constructed using the same IC.

9.2.6 Application Example: Selection of an Amplifier IC for Proper Frequency Bandwidth

Problem 9.46. An inverting amplifier with a gain of -20 and a bandwidth of at least 200 kHz is needed. Which amplifier chip is appropriate for this circuit (and which is not)?

A. LM358
B. TL082
C. LM741
D. LM7171
E. LM8272

Problem 9.47. A non-inverting amplifier with a gain of 31 and a bandwidth of at least 90 kHz is needed. Which amplifier chip is appropriate for this circuit (and which is not)?

A. TL082
B. LM7171
C. LM8272

9.3 Introduction to Continuous and Discrete Fourier Transform

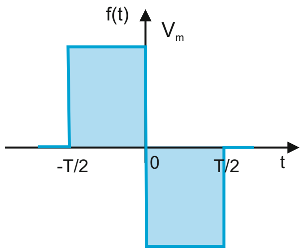
9.3.1 Meaning and Definition of Fourier Transform

Problem 9.48. Establish all values of the angular frequency ω in Fig. 9.14b at which the Fourier spectrum $F(\omega)$ of a rectangular pulse crosses the frequency axis (becomes zero). Express your result in terms of pulse duration T.

Problem 9.49

A. Establish the value of the Fourier transform $F(\omega)$ for the pulse shown in the following figure at $\omega = 0$.

B. Establish the complete pulse spectrum $F(\omega)$ at all values of angular frequency ω.



Problem 9.50
Establish the Fourier transform $F(\omega)$ for the following voltage signals in time domain:

A. $f(t) = A\sin\left(\frac{\pi}{2}t\right), \quad -2 \le t < 2$
$f(t) = 0, \quad$ otherwise

B. $f(t) = A\cos\left(\frac{\pi}{2}t\right), \quad -2 \le t < 2$
$f(t) = 0, \quad$ otherwise

Problem 9.51
Show that for an arbitrary real voltage signal $f(t)$:

A. The real part of $F(\omega)$ is an even function of angular frequency ω.
B. The imaginary part of $F(\omega)$ is an odd function of angular frequency ω.
C. The magnitude of $F(\omega)$ is an even function of angular frequency ω.
D. Replacing ω by $-\omega$ generates the complex conjugate of $F(\omega)$; in other words, $F(-\omega) = F^*(\omega)$.

9.3.2 Mathematical Properties of Fourier Transform

Problem 9.52. The Fourier transform of $f(t)$ is $F(\omega)$. What is the Fourier transform of $d^2f(t)/dt^2 - 2\int_{-\infty}^{t} f(\tau)d\tau$?

Problem 9.53. The Fourier transform of $f(t)$ is $F(\omega)$. What is the Fourier transform of $f(-t)$?

Problem 9.54. The function $f(t)\cos\omega_0 t$ is an *amplitude-modulated signal*: a high-frequency carrier $\cos\omega_0 t$, which is transmitted wirelessly, has a low-frequency envelope $f(t)$, which carries information and is being demodulated at the receiver. If the Fourier transform of $f(t)$ is $F(\omega)$, what is the Fourier transform of $f(t)\cos\omega_0 t$?

Problem 9.55. If $f(t)$ represents the voltage across a 1-Ω load, then $f^2(t)$ is the power delivered to the load and $\int_{-\infty}^{\infty} f^2(t)dt$ is the total energy delivered to the load. Prove *Parseval's theorem*,

$$\int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega,$$

which relates the total energy to an integral of the *energy spectral density*, $|F(\omega)|^2 = F(\omega)F^*(\omega)$, of the signal.
Hint: Use the reversal property of the Fourier transform given by Eq. (9.26).

Problem 9.56. Based on Parseval's theorem established in the previous problem, find the value of the integral $\int_{-\infty}^{\infty} \mathrm{sinc}^2(t)dt$.

9.3.3 Discrete Fourier Transform and Its Implementation

Problem 9.57. You are using the discrete Fourier transform of length 8 ($N = 8$) for a signal $f(t) = \sin t$ over a time interval from 0 to 2π s.

A. Compute all sampling points in the time domain.
B. Compute all sampling points in the frequency domain.

C. Compute equivalent frequency samples using negative frequencies.
D. Compute all discrete samples $f[n]$.
E. Compute all discrete samples $F[m]$ using the definition of the discrete Fourier transform. Explain the physical meaning of their values.
F. Repeat the previous step using function `fft` of MATLAB. Compare both sets of $F[m]$.
G. Restore all discrete samples $f[n]$ using the definition of the inverse discrete Fourier transform. Compare them with the exact function values.
H. Repeat the previous step using function `ifft` of MATLAB. Compare both sets of $f[n]$.

Problem 9.58. Repeat the previous problem for the signal $f(t) = \cos t$. All other parameters remain the same.

Problem 9.59. For Problem 9.57, establish and prove a discrete version of Parseval's theorem formulated in Problem 9.55.

Problem 9.60. An input signal to a filter has a discrete frequency spectrum $F[m]$, $m = 0, \ldots, N-1$ computed via the FFT. You are given filter transfer function $\mathbf{H}$ computed at $\frac{N}{2}+1$ frequency points of the FFT, $\mathbf{H}[m]$, $m = 0, \ldots, N/2$. Compute the discrete spectrum of the filter's output to be fed into the IFFT.

9.3.6 Application Example: Numerical Differentiation via the FFT

9.3.7 Application Example: Filter Operation for an Input Pulse Signal

Problem 9.61*. Present the text of a MATLAB script that numerically differentiates the input signal $f(t) = \sin t$ over the time interval from 0 to 4π s using the FFT with 4096 sampling points and plot the resulting signal derivative.

Problem 9.62. Repeat the previous problem for the signal $f(t) = \exp\left(-(t-2\pi)^2\right)$. All other parameters remain the same.

Problem 9.63. A monopolar pulse $f(t) = \exp\left(-2(t-5)^2\right)$, $0 \le t < 10$ s is an input to a series combination of two identical first-order high-pass filters. Find the output of the filter combination when the (angular) break frequency is given by:
A. $\omega_0 = 0.5$ rad/s
B. $\omega_0 = 10$ rad/s

Use the FFT and IFFT with $N = 64$. Plot the filter output and explain the output signal behavior in every case.

Problem 9.64. A bipolar pulse $f(t) = (5-t)\exp\left(-2(t-5)^2\right)$, $0 \le t < 10$ s is an input to a first-order low-pass filter. Find the filter output when its (angular) break frequency is given by:
A. $\omega_0 = 0.5$ rad/s
B. $\omega_0 = 5$ rad/s

Use the FFT and IFFT with $N = 64$. Plot the filter output along with the input signal on the same graph and explain the output signal behavior in both cases.