Chapter 10: Second-Order RLC Circuits

Overview

Prerequisites:

- Knowledge of complex arithmetic
- Knowledge of phasor/impedance method for AC circuit analysis (Chapter 8)

Objectives of Section 10.1:

- Learn the concept of a resonant circuit and its relation to other engineering disciplines
- Understand the internal dynamics of the series/parallel RLC resonator including voltage and current behavior near the resonant frequency
- Establish the meaning and be able to calculate resonant frequency, quality factor, and bandwidth of the second-order resonant circuits
- Establish and quantify the duality between series and parallel RLC resonators

Objectives of Section 10.2:

- Construct four major types of the second-order RLC filters
- Relate all filter concepts to the corresponding circuit diagrams
- Specify two filter design parameters: the undamped resonant frequency and the quality factor
- Realize the advantages of the second-order filters versus the first-order filters

Objectives of Section 10.3:

- Become familiar with the concept of the near-field wireless link
- Apply the theory of the series resonant RLC circuit to the basic design of the near-field wireless transmitter and receiver
- Understand the operation of proximity sensors based on resonant RLC circuits

Application examples:

- Near-field wireless link in undergraduate laboratory
- Proximity sensors

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Keywords:
Self-oscillating LC circuit, Series resonant RLC circuit, Parallel resonant RLC circuit, Series RLC tank circuit, Parallel RLC tank circuit, Undamped resonant frequency, Resonant frequency, Quality factor of the series resonant RLC circuit, Quality factor of the parallel resonant RLC circuit, Quality factor (general definition, interpretation, mechanical analogy, trade-off between Q-factor and inductance value), Bandwidth of the series resonant RLC circuit, Bandwidth of the parallel resonant RLC circuit, Half-power bandwidth, Upper half-power frequency, Lower half-power frequency, Duality of series/parallel RLC circuits, Ideal filter, Cutoff frequency, Second-order band-pass RLC filter, Second-order low-pass RLC filter, Second-order band-reject (or band-stop or notch) RLC filter, Second-order high-pass RLC filter, Quality factor of the filter circuit, Center frequency of the band-pass filter, Lower and upper half-power frequencies, Butterworth response, Quality factor of the nonideal inductor, Voltage multiplier circuit, Voltage multiplication, Near-field wireless link, Horseshoe coil

Section 10.1 Theory of Second-Order Resonant RLC Circuits

In this section, we study the last group of standard AC circuits—the resonators. They are second-order AC circuits in LC or LCR configuration. The term second order means that the circuits will be described by second-order differential equations if we work in the time domain. The value of a resonator circuit in electronics cannot be overstated. In order to proceed with any type of wireless communication, we first need to create a high-frequency AC signal as part of a resonator circuit. Beyond high-frequency circuits, resonators are often used in power electronics and as sensors. In this section, we apply the phasor/impedance method to analyze resonator circuits. We will discover that the most important characteristic is the *resonant frequency*. Another important parameter is the *quality factor*, which also determines the *resonator bandwidth*.

10.1.1 Self-Oscillating Ideal LC Circuit

The circuit shown in Fig. 10.1a includes an inductor and a capacitor and there is no power source connected to the circuit. The circuit is also *ideal*, which means that there is no resistance. In other words, the parasitic resistance of the inductor, parasitic resistance of the capacitor, and the wire resistance are all neglected. We assume that the power supply (voltage or current) was disconnected at $t = 0$, after the resonator was excited. The steady-state alternating current and the AC voltages across the circuit elements are sought once the oscillation process has been stabilized, i.e., at $t \to \infty$.

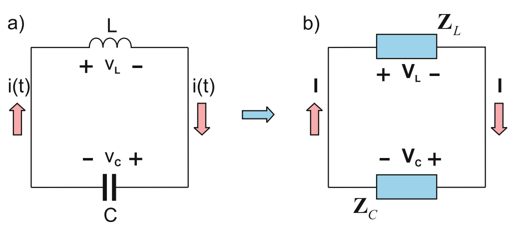


Fig. 10.1. Self-oscillating ideal LC circuit and its phasor representation.

When we apply the phasor/impedance method to the circuit in Fig. 10.1a, we obtain the circuit shown in Fig. 10.1b. KVL in phasor form yields (note the passive reference configuration)

$$\mathbf{V}_\mathrm{L} + \mathbf{V}_\mathrm{C} = 0 \Rightarrow \mathbf{Z}_\mathrm{L}\mathbf{I} + \mathbf{Z}_\mathrm{C}\mathbf{I} = 0 \Rightarrow (\mathbf{Z}_\mathrm{L} + \mathbf{Z}_\mathrm{C})\mathbf{I} = 0 \tag{10.1}$$

Generally, Eq. (10.1) requires the phasor current $\mathbf{I}$ to be zero. Obviously, if the phasor current is zero, then the real current is also zero and so are the voltages across the inductor and the capacitor. The circuit is not functioning. However, you should note that, if

$$\mathbf{Z}_L + \mathbf{Z}_C = 0 \tag{10.2}$$

in Eq. (10.1), the phasor current does not have to be zero and may have *any* value depending on the initial excitation. Equation (10.2) is satisfied at only *one* single frequency f_0:

$$\begin{aligned} & j\omega_0 L + \frac{1}{j\omega_0 C} = 0 \Rightarrow \text{ (multiply by } j) \Rightarrow -\omega_0 L + \frac{1}{\omega_0 C} = 0 \Rightarrow \\ & \omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow f_0 = \frac{1}{2\pi\sqrt{LC}} \end{aligned} \tag{10.3}$$

which is the *undamped resonant frequency* of the LC circuit. Equation (10.3) is perhaps the most important result of resonator theory. Once Eq. (10.3) is satisfied, the solution for the circuit current is obtained in the form:

$$\mathbf{I} = \mathbf{I}_0 \Rightarrow i(t) = I_m \cos \omega_0 t \tag{10.4a}$$

The current amplitude I_m may be arbitrary; it is determined by the initial excitation. The voltages are found accordingly:

$$\begin{aligned} \mathbf{V}_L &= \mathbf{Z}_L \mathbf{I} \Rightarrow v_L(t) = \omega_0 L I_m \cos(\omega_0 t + 90^\circ) \\ \mathbf{V}_C &= \mathbf{Z}_C \mathbf{I} \Rightarrow v_C(t) = 1/(\omega_0 C) I_m \cos(\omega_0 t - 90^\circ) \end{aligned} \tag{10.4b}$$

The ideal *self-oscillating LC circuit* in Fig. 10.1 can oscillate indefinitely long. What is the physical basis of self-oscillations in an LC circuit? To answer this question, let us take a closer look at Eqs. (10.4). When the circuit current is at its maximum, the magnetic field energy stored in the inductor also has reached its maximum. Since the voltages are shifted by $\pm\pi/2$ versus the current, they are exactly zero at that time instance. The zero capacitor voltage means that no energy of the electric field is stored in the capacitor. All of the energy stored in the circuit is concentrated in the inductor. When the circuit current reaches zero, the situation becomes the opposite: the capacitor stores the entire circuit energy, and the inductor does not have any stored energy. As time progresses, the process continues so that the current flows back and forth in the circuit charging and discharging the capacitor (and in certain sense the inductor) periodically. Figure 10.2 shows the ideal mechanical counterpart of the circuit in Fig. 10.1. A massive wheel with a rotational inertia represents the inductor and the flexible membrane, the capacitor. The fluid flows back and forth either rotating the wheel (increasing its rotational energy) or bending the membrane (increasing its release energy).

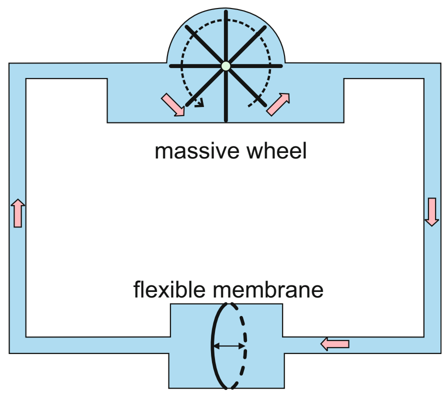


Fig. 10.2. Self-oscillating mechanical counterpart of the LC circuit shown in Fig. 10.1.

Example 10.1: An LC circuit in Fig. 10.1 has the circuit parameters $L = 1\ \mu\text{H}$, $C = 1\ \mu\text{F}$. Determine its resonant frequency, also known as the self-oscillation frequency.

Solution: Equation (10.3) is applied, which gives $\omega_0 = 10^6\ \text{rad/s} \Rightarrow f_0 = 159$ kHz. For practical reasons, the resonant frequency is most often measured and reported in Hz, instead of rad/s.

10.1.2 Series Resonant Ideal LC Circuit

What if an alternating pressure pump is connected to the oscillating mechanical system in Fig. 10.2, which will add a small pressure "push" at every period of oscillation? Since there is no friction, the oscillations may grow up indefinitely. This phenomenon is known as *resonance*. The corresponding electrical counterpart is the circuit shown in Fig. 10.3a. The circuit in Fig 10.3a is solved using the phasor method, see Fig. 10.3b. We assume $v_S(t) = V_m \cos \omega t$. The equivalent impedance is given by,

$$\mathbf{Z}_{eq} = \mathbf{Z}_L + \mathbf{Z}_C = j\omega L + \frac{1}{j\omega C} = j\left(\omega L - \frac{1}{\omega C}\right) = -j\frac{1 - LC\omega^2}{\omega C} \quad [\Omega] \tag{10.5a}$$

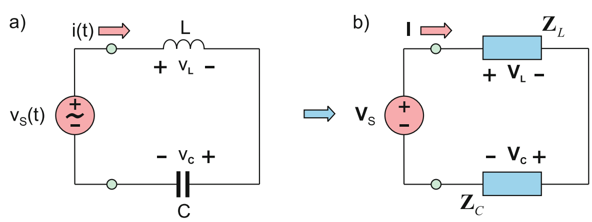


Fig. 10.3. Series resonant ideal LC circuit and its phasor representation.

The phasor voltages and phasor current become

$$\mathbf{I} = V_{\mathrm{m}}/\mathbf{Z}_{\mathrm{eq}} = \frac{V_{\mathrm{m}}\omega C}{1 - LC\omega^2}\angle 90^\circ, \quad \mathbf{V}_{\mathrm{L}} = \mathbf{Z}_{\mathrm{L}}\mathbf{I} = \frac{V_{\mathrm{m}}LC\omega^2}{1 - LC\omega^2}\angle 180^\circ,$$
$$\mathbf{V}_{\mathrm{C}} = \mathbf{Z}_{\mathrm{C}}\mathbf{I} = \frac{V_{\mathrm{m}}}{1 - LC\omega^2}\angle 0^\circ \tag{10.5b}$$

The real-valued circuit parameters are given by

$$i(t) = \frac{V_{\mathrm{m}}\omega C}{1 - LC\omega^2}\cos(\omega t + 90^\circ), \quad \upsilon_{\mathrm{L}}(t) = \frac{V_{\mathrm{m}}LC\omega^2}{1 - LC\omega^2}\cos(\omega t + 180^\circ),$$
$$\upsilon_{\mathrm{C}}(t) = \frac{V_{\mathrm{m}}}{1 - LC\omega^2}\cos(\omega t) \tag{10.5c}$$

The solution remains finite at any source frequency except the undamped resonant frequency $\omega_0 = 1/\sqrt{LC}$ or $f_0 = \omega_0/(2\pi)$. The closer the source frequency approaches the undamped resonant frequency, the higher the circuit current, capacitor voltage, and the inductor voltage become. Eventually, at the exact undamped resonant frequency, they all become infinitely high! The denominator in Eq. (10.5c) approaches zero and the circuit starts to "resonate." At the undamped resonant frequency, the impedances of the inductor and capacitor cancel out and their combination is a short circuit: an ideal wire of zero resistance. Moreover, the voltage source is shorted out. Note that the resonant frequency of an LC circuit was first derived by James Clerk Maxwell in 1868. A young man at this point, he spent a night working over this problem, which arose from an experiment of Sir William Grove, and wrote a report to him the next morning.

Example 10.2: Find the sum of the real-valued voltages $\upsilon_{\mathrm{L}}(t)$, $\upsilon_{\mathrm{C}}(t)$ in Fig. 10.3a.

Solution: The capacitor and inductor voltages are in *antiphase* (the phases differ by 180°). Therefore, they largely cancel out. According to Eq. (10.5c), the sum of the voltages is exactly the supply voltage $\upsilon_{\mathrm{S}}(t)$, irrespective of how high the individual voltages are.

Exercise 10.1: For the ideal series resonant ideal LC circuit in Fig. 10.3, determine the phasor voltages across the inductor and capacitor given that $V_{\mathrm{m}} = 1$ V and $\omega^2 = 0.9\omega_0^2$.

Answer: $\mathbf{V}_{\mathrm{L}} = 9\angle 180^\circ$ [V], $\mathbf{V}_{\mathrm{C}} = 10\angle 0^\circ$ [V]

10.1.3 Series Resonant RLC Circuit: Resonance Condition

The ideal LC circuit shown in Fig. 10.1a, or the series LC resonator shown in Fig. 10.3a, never exists in practice. Internal power supply resistance, wire resistance, or parasitic resistances of realistic capacitors and inductors lead to the realistic resonant RLC circuit model shown in Fig. 10.4a. Here, the resistance R models the combined resistances present in the circuit. The resistance reduces the resonant effect and leads to finite voltages/currents at the resonance.

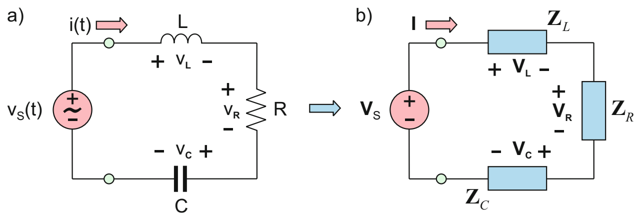


Fig. 10.4. Realistic series resonant RLC circuit and its phasor representation.

The circuit in Fig 10.4a is solved using the phasor method, see Fig. 10.4b. We again assume $v_S(t) = V_m \cos \omega t$. The equivalent impedance is given by

$$\mathbf{Z}_{eq} = \mathbf{Z}_R + \mathbf{Z}_L + \mathbf{Z}_C = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right) \quad [\Omega] \tag{10.6a}$$

The *resonance condition* for *any* AC circuit, and not necessarily the circuit shown in Fig. 10.4, states that the imaginary part (the *reactance* X) of the equivalent circuit impedance *seen by the power source* must be equal to zero:

$$\text{Im}\left(\mathbf{Z}_{eq}\right) = X = 0 \quad [\Omega] \tag{10.6b}$$

When applied to Eq. (10.6a), this condition defines the circuit's *resonant frequency* in the form $\omega_0 = 1/\sqrt{LC}$, $f_0 = 1/\left(2\pi\sqrt{LC}\right)$. For the series RLC circuit, the resonant frequency clearly coincides with the undamped resonant frequency of the ideal LC circuit. Unfortunately, this is not always the case for general RLC circuits. This important question is addressed in the homework problems.

At the resonant frequency, the impedances of the inductor and capacitor in Eq. (10.6a) cancel out; their combination is a short circuit since only the resistance R remains. The real-valued circuit current and the real-valued voltages are given by

$$i(t) = \frac{V_m}{R}\cos(\omega_0 t), \quad v_L(t) = \frac{V_m}{R}\omega_0 L \cos(\omega_0 t + 90°), \quad v_C(t) = \frac{V_m}{R}\frac{1}{\omega_0 C}\cos(\omega_0 t - 90°) \tag{10.6c}$$

at resonance. Those are the *highest* amplitudes of the circuit current and the individual voltages that could be achieved in the series RLC circuit. If the frequency deviates from the resonant frequency, smaller amplitude values are obtained. When the circuit resistance is small, large circuit current and large capacitor and inductor voltages may be achieved at the resonance. You have to be aware of the fact that it is not uncommon to measure voltage amplitudes of 50–500 V across the individual elements in the laboratory, whereas the driving source voltage may only have an amplitude of 10 V. The circuit in Fig. 10.4 is also called *the series RLC tank circuit*.

Exercise 10.2: In the series resonant RLC circuit shown in Fig. 10.4, $V_m = 10$ V, $L = 50$ μH, $C = 0.5$ nF, $R = 50$ Ω. Determine the real-valued circuit current and the inductor/capacitor voltages at the resonance.

Answer:

$$i(t) = 0.2\cos(\omega t)\ \text{[A]},\quad v_L(t) = 63.3\cos(\omega t + 90°)\ \text{[V]},$$
$$v_C(t) = 63.3\cos(\omega t - 90°)\ \text{[V]}. \tag{10.7}$$

Could we increase the resonant voltage amplitudes of the series RLC circuit in Fig. 10.4a [see Eq. (10.6c)] while keeping the voltage source and the circuit resistance unaltered? Yes we can. However, one more concept is required for this and similar problems: the concept of the *quality factor* of a resonator.

10.1.4 Quality Factor Q of the Series Resonant RLC Circuit

Multiple factors in front of resonant voltages and currents expressions can be reduced to one single factor. Using the definition of the resonant frequency ω_0, Eq. (10.6c) at the resonance may be rewritten in the simple form

$$i(t) = \frac{V_m}{R}\cos(\omega_0 t),\quad v_L(t) = QV_m\cos(\omega_0 t + 90°),\quad v_C(t) = QV_m\cos(\omega_0 t - 90°) \tag{10.8}$$

where the dimensionless constant

$$Q = \frac{\sqrt{L/R}}{\sqrt{RC}} = \frac{\sqrt{L/C}}{R} \tag{10.9}$$

is called the *quality factor of the series resonant RLC circuit*. The equivalent forms are

$$Q = \frac{1}{\omega_0 RC} = \omega_0\frac{L}{R} \tag{10.10}$$

Thus, in order to increase the resonant voltage amplitudes in Eq. (10.8), we should simply increase the quality factor of the resonator. Even if the circuit resistance remains the same, we can still improve Q by increasing the ratio of L/C in Eq. (10.9). This observation provides one physical interpretation of the quality factor: it determines the *maximum amplitude* of the resonant oscillations. A higher *Q-factor* results in larger amplitudes. Yet another, perhaps even more important, interpretation is related to the "sharpness" of the resonance at frequencies close to ω_0. What is the physical meaning of Eq. (10.9)? Why does the Q-factor increase with increasing the inductance but not the capacitance? To answer these questions, consider the fluid mechanics analogy in Fig. 10.2. The high Q implies a massive wheel (note: high inductance is equivalent to high wheel mass). Simultaneously, it implies a large membrane stiffness (the small capacitance, which is inversely proportional to the stiffness). The mechanical resonator so constructed will be less susceptible to losses at resonance but will not resonate at all if the driving force has a frequency far away from the resonance.

A *general definition* of the quality factor also applicable to mechanical engineering and physics is as follows. The quality factor is 2π times *the ratio per cycle of the energy stored in the resonator to the energy supplied by a source*, while keeping the signal amplitudes constant at the resonant frequency. According to Eq. (10.6c), the instantaneous energies stored in the inductor and capacitor are given by

$$\begin{aligned} E_{\mathrm{L}}(t) &= \frac{1}{2}Li^2(t) = \left(\frac{V_{\mathrm{m}}}{R}\right)^2 \frac{L}{2}\cos^2(\omega_0 t), \\ E_{\mathrm{C}}(t) &= \frac{1}{2}Cv_{\mathrm{C}}^2(t) = \left(\frac{V_{\mathrm{m}}}{R}\right)^2 \frac{1}{2\omega_0^2 C}\sin^2(\omega_0 t) \end{aligned} \tag{10.11}$$

Since $1/\left(\omega_0^2 C\right) = L$, the coefficients in front of the cosine squared and sine squared terms are equal. It means that even though both energies vary over time, their sum is a *constant*:

$$E_{\mathrm{L}}(t) + E_{\mathrm{C}}(t) = \left(\frac{V_{\mathrm{m}}}{R}\right)^2 \frac{L}{2} \tag{10.12a}$$

The energy dissipated in the resistance is the integral of the instantaneous absorbed power over the period; this integral is equal to

$$E_{\mathrm{diss}} = \int_0^{2\pi/\omega_0} \frac{V_{\mathrm{m}}^2}{R}\cos^2(\omega_0 t)dt = \frac{V_{\mathrm{m}}^2}{2R}\int_0^{2\pi/\omega_0}(1 + \cos(2\omega_0 t))dt = \frac{\pi V_{\mathrm{m}}^2}{\omega_0 R} \tag{10.12b}$$

The ratio of the two energies times 2π precisely equals Eq. (10.9).

Example 10.3: A series resonant LC circuit is driven by a laboratory AC voltage source with amplitude $V_{\mathrm{m}} = 10$ V and an internal resistance of 50 Ω (a function generator). Which value should the ratio L/C have to obtain the amplitude of the capacitor voltage to be equal to 50 V at resonance?

Solution: We replace the realistic voltage source by its Thévenin equivalent: the ideal voltage source with the amplitude of 10 V and the series resistance of 50 Ω. Here, we again arrive at the standard RLC circuit shown in Fig. 10.4a. According to Eq. (10.8), the Q-factor of the RLC circuit should be equal to 5. From the definition of the Q-factor given by Eq. (10.9), one has

$$\sqrt{L/C} = RQ = 250 \Rightarrow L/C = 62,500 \ \Omega^2 \tag{10.13}$$

This result is valid for *any* resonant frequency. For example, the set $L = 1\,\mathrm{mH},\ C = 16$ nF will give us the desired amplitude value.

Note that large Q-factors usually imply large inductances which increase the series resistance of the inductor coil and thus increase the net circuit resistance (increase circuit loss). Therefore, there is a *trade-off* between the circuit Q and the inductance value.

Example 10.4: A series resonant RLC circuit is needed with the resonant frequency of 100 kHz and a Q-factor of 50. The circuit resistance is 10 Ω. Determine the necessary values of L and C.

Solution: From the definition of the resonant frequency and the Q-factor, we obtain

$$\begin{aligned} &1/\sqrt{LC} = 2\pi \times 10^5, \quad \sqrt{L/C} = RQ = 500 \Rightarrow \\ &\qquad \frac{1}{C} = 2\pi \times 10^5 \times 500 \Rightarrow C = 3.2 \ \text{nF} \end{aligned} \tag{10.14}$$

Consequently, $L = 0.80$ mH.

10.1.5 Bandwidth of the Series Resonant RLC Circuit

The *bandwidth of the series resonant RLC circuit* is obtained by analyzing the behavior of the amplitude of the circuit current as a function of *source frequency*. An alternative definition implies analyzing the behavior of the amplitude of the resistor voltage; however, both quantities are equal to within a constant R. The phasor current at any frequency is obtained from Eq. (10.6a). It will be written here in terms of quality factor Q, frequency f, and resonant frequency f_0 in the form:

$$\mathbf{I}=\frac{V_{\mathrm{m}}}{\mathbf{Z}_{\mathrm{eq}}}=\frac{V_{\mathrm{m}}}{R+j\left(\omega L-\frac{1}{\omega C}\right)}=I_{\mathrm{m}}\frac{1}{1+jQ\left(\frac{f}{f_0}-\frac{f_0}{f}\right)} \tag{10.15a}$$

Here, $I_{\mathrm{m}}=V_{\mathrm{m}}/R$ is the maximum (resonant) current amplitude in the series RLC circuit; $f_0=1/\left(2\pi\sqrt{LC}\right)$ is the resonant frequency in Hz. The real-valued circuit current and the real-valued resistor voltage are both found from the phasor current given by Eq. (10.15a):

$$\left.\begin{array}{r}i(t)=I_{\mathrm{m}}H\cos(\omega t+\varphi)\ \ [\mathrm{A}]\\ \upsilon_{\mathrm{R}}(t)=V_{\mathrm{m}}H\cos(\omega t+\varphi)\ \ [\mathrm{V}]\end{array}\right\},\quad H(f)=\frac{1}{\sqrt{1+Q^2\left(\frac{f}{f_0}-\frac{f_0}{f}\right)^2}} \tag{10.15b}$$

Here, $H(f)$ is the dimensionless function of frequency that peaks at the resonant frequency $f=f_0, H(f_0)=1$. One may treat $H(f)$ as an *amplitude transfer function* of an associated RLC filter with the input voltage being the source voltage and the output voltage being the resistor voltage. In this case, $H(f)$ is equal to the amplitude ratio of the two voltages. Simultaneously, $H(f)$ characterizes how fast the circuit current amplitude decays when the circuit frequency deviates from the resonant frequency f_0. To be specific, we assume $f_0=10$ kHz and $Q=1,\ 2,\ 5$ in Eq. (10.15b). Figure 10.5 plots the function $H(f)$ in decibels, $H(f)_{\mathrm{dB}}=20\log_{10}(H(f))$ [dB], using a log-log scale, i.e., creates its *Bode plot*.

The *bandwidth B of the series resonant RLC circuit* is defined as the interval of frequencies over which the function $H(f)$ is greater than or equal to $1/\sqrt{2}=0.707$. In other words, the signal power at the resistor (which is proportional to the square of $H(f)$) is no less than 50 % of the maximum power at the exact resonance. We call the bandwidth so defined the *half-power bandwidth*. In terms of the transfer function $H(f)_{\mathrm{dB}}$ in decibels, this condition corresponds to the inequality $H(f)_{\mathrm{dB}}\geq -3$ dB.

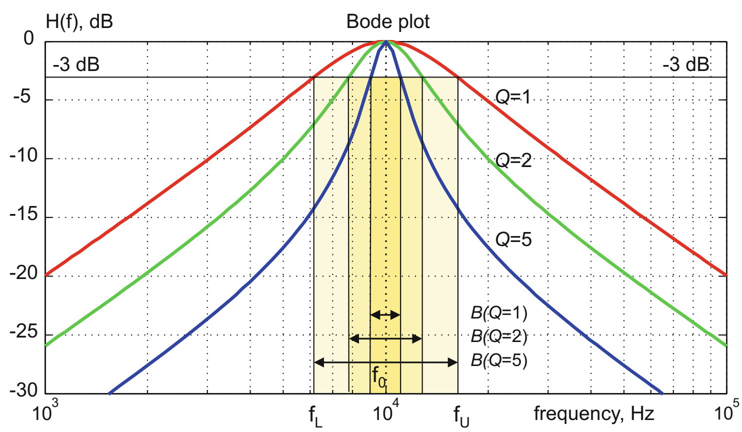


Fig. 10.5. Amplitude of the circuit current (or the amplitude of the resistor voltage) normalized by its peak value at resonance. The resulting graph is the Bode plot.

The first resonant curve in the form of Fig. 10.5 was published by Heinrich Hertz in 1887 although he used a linear, not a logarithmic, frequency scale so that the curve did not look quite symmetric. Figure 10.5 indicates that the bandwidth increases when the quality factor decreases and vice versa. In other words, the low-Q resonant circuit has a large bandwidth; we may say it resonates "equally bad" over a wider band of frequencies. However, the high-Q circuit has a small bandwidth; it resonates well but only over a small band of frequencies. The *lower and upper half-power frequencies*, f_L, f_U, are obtained by setting $H(f) = 1/\sqrt{2}$ in Eq. (10.15b) and solving for f. The resulting quadratic equation has two roots:

$$\begin{aligned} f_L &= f_0\left(\sqrt{1+1/(2Q)^2} - 1/(2Q)\right), \\ f_U &= f_0\left(\sqrt{1+1/(2Q)^2} + 1/(2Q)\right) \quad [\text{Hz}] \end{aligned} \tag{10.16}$$

Despite the complexity of those expressions, the final result for the half-power bandwidth is surprisingly simple and understandable:

$$B \equiv f_U - f_L = \frac{f_0}{Q} = \frac{R}{2\pi L} \quad [\text{Hz}] \tag{10.17}$$

Exercise 10.3: Determine the bandwidth B of the series resonant RLC circuit with the resonant frequency of 1 MHz and a Q-factor of 10.

Answer: $B = 100$ kHz.

Example 10.5: A series resonant RLC circuit is needed with a resonant frequency of 500 kHz and a bandwidth of 20 kHz. Given the circuit resistance of 15 Ω, determine L and C.

Solution: From the bandwidth definition, the required Q-factor is equal to $500/20 = 25$. Further, we may follow the solution developed in Example 10.4. From the definition of the resonant frequency and the Q-factor, we subsequently obtain $1/\sqrt{LC} = 2\pi \times 5 \times 10^5$, $\sqrt{L/C} = RQ = 375 \Rightarrow C \approx 0.85$ nF. Next, we determine $L = C(RQ)^2 \approx 0.12$ mH. Alternatively, one could find the inductance L directly from Eq. (10.16), that is, $L = R/(2\pi B) = 0.12$ mH.

Example 10.6: Create Bode plots as seen in Fig. 10.5 using MATLAB.

Solution: We create only one bandwidth curve, for $Q = 2$. Other curves are obtained similarly, using the command `hold on`.

```
f       = logspace(3, 5, 101); % frequency vector, Hz(from 10^3 to 10^5 Hz)
f0      = 1e4;                 % resonant frequency, Hz
Q       = 2;                   % quality factor, dimensionless
H       = 1./sqrt(1+Q^2*(f/f0-f0./f).^2);
HdB     = 20*log10(H);
semilogx(f, HdB, 'r'); grid on;
title('Bode plot');
xlabel('frequency, Hz'); ylabel('H, dB')
axis([min(f) max(f) -30 0])
```

10.1.6 Parallel Resonant RLC Circuit: Duality

The parallel resonant RLC circuit is shown in Fig. 10.6. It is driven by an alternating current source $i_S(t) = I_m \cos \omega t$. The parallel RLC resonator is a current divider circuit, which is the dual to the series RLC resonator (which is a voltage divider) in Fig. 10.4. While the series RLC resonator is capable of creating large voltages (or "amplifying" the supply voltage), the parallel RLC resonator circuit is capable of producing large currents. The amplitudes of the currents through the inductor and the capacitor may be large, much larger than the supply current itself. The circuit in Fig. 10.6 is also called *the parallel RLC tank circuit*.

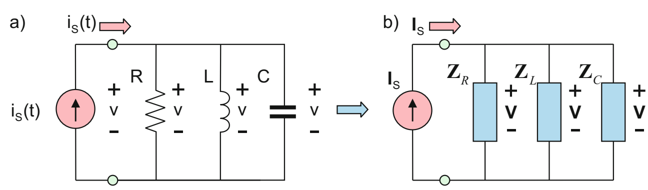


Fig. 10.6. Parallel resonant RLC circuit and its phasor representation.

The circuit in Fig 10.6a is solved by using the phasor method; see Fig. 10.6b. The equivalent impedance is given by

$$\frac{1}{\mathbf{Z}_{\text{eq}}} = \frac{1}{\mathbf{Z}_{\text{R}}} + \frac{1}{\mathbf{Z}_{\text{L}}} + \frac{1}{\mathbf{Z}_{\text{C}}} = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C = \frac{1}{R} - j\frac{1 - LC\omega^2}{\omega L} \quad [\Omega] \tag{10.18}$$

The *resonance condition* for *any* AC circuit states that the impedance $\mathbf{Z}_{\text{eq}}$ must be a purely real number at resonance. If the impedance is real, its reciprocal, the *admittance*, is also real and vice versa. Therefore, from Eq. (10.18), we obtain the *resonant frequency*

$$\omega_0 = \frac{1}{\sqrt{LC}}, \ f_0 = \frac{1}{2\pi\sqrt{LC}} \tag{10.19}$$

which *coincides* with the resonant frequency of the series RLC tank circuit and with the undamped resonant frequency of the LC circuit. Thus, at resonance, $\mathbf{Z}_{eq} = R$ and one has $\mathbf{V} = RI_m$ for the phasor voltage in Fig. 10.6b. Knowing the phasor voltage, we can establish the phasor currents. The corresponding real-valued voltage and currents at resonance take on the forms

$$\begin{aligned} &v(t) = RI_m \cos(\omega_0 t), \quad i_L(t) = \frac{RI_m}{\omega_0 L}\cos(\omega_0 t - 90^\circ), \\ &i_C(t) = RI_m(\omega_0 C)\cos(\omega_0 t + 90^\circ) \end{aligned} \tag{10.20}$$

The amplitude of the circuit voltage, along with the amplitudes of inductor and capacitor currents in Eq. (10.20), reaches a maximum at resonance. Next, we wish to introduce the Q-factor of the circuit, similar to Eq. (10.8) for the series resonator, that is,

$$\begin{aligned} &v(t) = RI_m \cos(\omega_0 t), \quad i_L(t) = I_m Q\cos(\omega_0 t - 90^\circ), \\ &i_C(t) = I_m Q\cos(\omega_0 t + 90^\circ) \end{aligned} \tag{10.21}$$

Comparing Eq. (10.20) with Eq. (10.21), we obtain a different expression:

$$Q = \frac{\sqrt{RC}}{\sqrt{L/R}} = \omega_0 RC = \frac{R}{\omega_0 L} \tag{10.22}$$

which is exactly the reciprocal of the Q-factor of the series RLC circuit. This means that a high-Q parallel resonant circuit will require higher capacitances than inductances.

Fortunately, all the results related to the series resonant RLC circuit can directly be converted to the parallel RLC resonant circuit using the substitutions:

$$v(t) \to Ri(t), \quad i_L(t) \to v_C(t)/R, \quad i_C \to v_L(t)/R \tag{10.23a}$$

Here, the left-hand side corresponds to the parallel RLC circuit, whereas the right-hand side corresponds to the series RLC circuit. Furthermore, we need to replace V_m by RI_m and interchange the role of two partial time constants:

$$\frac{L}{R} \leftrightarrow RC \tag{10.23b}$$

in the original solution for the series RLC circuit; see Eq. (10.22). The solution so constructed will match exactly the solution of the parallel RLC circuit depicted in

Fig. 10.6. This fact is proved by direct substitution. Thus, Eqs. (10.23) reflects the *duality* of the series/parallel RLC AC steady-state electric circuits driven by voltage and current sources, respectively. It means that the results established for one circuit may be applied to the other circuit and vice versa. A similar duality is established for the transient RLC circuits. Consequently, we can concentrate our study on the series RLC circuit.

Exercise 10.4: For the parallel resonant RLC circuit in Fig. 10.6, determine resonant phasor currents $\mathbf{I}_R$, $\mathbf{I}_L$, and $\mathbf{I}_C$ if $I_m = 100$ mA, $L = 30$ μH, $C = 1$ μF, and $R = 100$ Ω.

Answer: $\mathbf{I}_R = \mathbf{I}_S = 0.1\angle 0°$, $\mathbf{I}_L = 1.83\angle -90°$, $\mathbf{I}_C = 1.83\angle +90°$ [A].

Example 10.7: The circuit voltage for the parallel RLC circuit in Fig. 10.6 at any frequency may be written in the form $v(t) = RI_m H(f)\cos(2\pi f t + \varphi)$ where $H(f)$ is a dimensionless amplitude transfer function, which peaks at the resonant frequency, $H(f_0) = 1$. Create the Bode plot of $H(f)$ at $f_0 = 10$ kHz and for $Q = 1, 2, 5$.

Solution: The amplitude transfer function $H(f)$ for the parallel RLC circuit coincides with the expression (10.15b) for the series RLC circuit. The Bode plot also *coincides* with the corresponding result for the series RLC circuit shown in Fig. 10.5. However, the Q-factor is now given by Eq. (10.22). The bandwidth of the parallel resonant circuit is still given by the expression $B = f_0/Q$[Hz] but with the modified Q-factor. This results in $B = 1/(2\pi RC)$ [Hz].

Section 10.2 Construction of Second-Order RLC Filters

An immediate application of the RLC resonant circuits relates to the concept of filter design. We have already studied the first-order low-pass and high-pass filters on the basis of RC and RL steady-state AC circuits. In fact, the RLC circuits could also be used as low-pass and high-pass filters. They perform even better, i.e., more closely linked to the frequency response of the *ideal filter*, which implies passing all the frequency components below (or above) a certain *cutoff frequency* and rejecting all other frequency components. In other words, the frequency responses of the RLC low-pass and high-pass filters are "steeper" than first-order filters. Not only that, resonant RLC circuits can form *band-pass* and *band-reject* (or *band-stop* or *notch*) filters, a task which is impossible with first-order RC or RL circuits.

10.2.1 Second-Order Band-Pass RLC Filter

A series RLC circuit is shown in Fig. 10.7a. We consider the supply voltage as the *input voltage* $v_{\mathrm{in}}(t)$ into the filter and the resistor voltage $v_{\mathrm{R}}(t)$ as the *output voltage* $v_{\mathrm{out}}(t)$. Figure 10.7b depicts the corresponding circuit transformation. This transformation implies that the input voltage is provided by another circuit block and the output voltage is passed to another circuit block. The circuit in Fig. 10.7b is thus a *two-port network*. Qualitatively then, when the frequency of the input voltage is the resonant frequency of the RLC circuit, the LC block in Fig. 10.7 is replaced by a short circuit (a wire). The input voltage *passes* through unchanged. However, if the frequency differs from the resonant one, the LC block exhibits a large finite impedance that is added to the resistance R. As a result, the circuit current decreases in amplitude, as does the output voltage (voltage across the resistor), which is proportional to the current. Those frequencies are thus rejected. The filter so constructed is known as a *second-order band-pass RLC filter.*

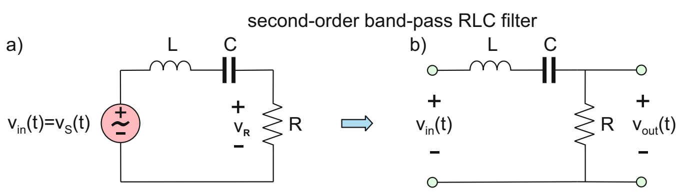


Fig. 10.7. Transformation of the series RLC circuit into the band-pass analog RLC filter.

We assume the source voltage (the input filter voltage) is given in the form $v_{\mathrm{S}}(t) = V_{\mathrm{m}} \cos \omega t$. The phasor current for the corresponding series RLC circuit was found in the previous section (see Eq. (10.15a)):

$$\mathbf{I} = \frac{V_m/R}{1 + jQ\left(\frac{f}{f_0} - \frac{f_0}{f}\right)} \tag{10.24a}$$

Here, $Q = \sqrt{L/C}/R$ is the corresponding quality factor of the series RLC circuit (*quality factor of the filter circuit*), and $f_0 = 1/\sqrt{LC}$ is its resonant frequency. The complex filter transfer function is defined by the ratio of the corresponding phasors:

$$\mathbf{H}(f) \equiv \frac{\mathbf{V}_R}{\mathbf{V}_S} = \frac{R\mathbf{I}}{V_m} \tag{10.24b}$$

Substitution of Eq. (10.24a) into Eq. (10.24b) gives the transfer function in the form:

$$\mathbf{H}(f) = \frac{1}{\sqrt{1 + Q^2\left(\frac{f}{f_0} - \frac{f_0}{f}\right)^2}} \angle - \tan^{-1}\left(Q\left(\frac{f}{f_0} - \frac{f_0}{f}\right)\right) \tag{10.24c}$$

which is equivalent to Eq. (10.15b) of the previous section. Therefore, the results derived for the series resonant RLC circuit are also valid for the band-pass RLC filter in Fig. 10.7. In particular, the *center frequency of the band-pass filter* is the circuit resonant frequency. The *half-power bandwidth of the filter* is given by Eq. (10.17), i.e., $B = f_0/Q = R/(2\pi L)$, and the *lower and upper half-power frequencies* are known from Eq. (10.16) of the previous section. Second-order filter circuits are designed by choosing the values of R, L, C in such a way as to obtain the required values of Q and f_0 (the filter center frequency and the required bandwidth). Thus, we have two equations for three unknowns. The remaining degree of freedom is used to match the filter impedances.

Example 10.8: A band-pass RLC filter is required with the center (resonant) frequency of 1 MHz and a half-power bandwidth B of 100 kHz. Create amplitude and phase Bode plots for the filter in the frequency band from 100 kHz to 10 MHz.

Solution: Clearly, $f_0 = 1$ MHz. The quality factor of the RLC circuit is found to be $Q = f_0/B = 10$. We plot the magnitude of the transfer function, $H(f)$, in decibels and its phase in degrees according to Eq. (10.24c). The result is shown in Fig. 10.8. You should note that far away from the passband, the filter follows a 20-dB-per-decade roll-off.

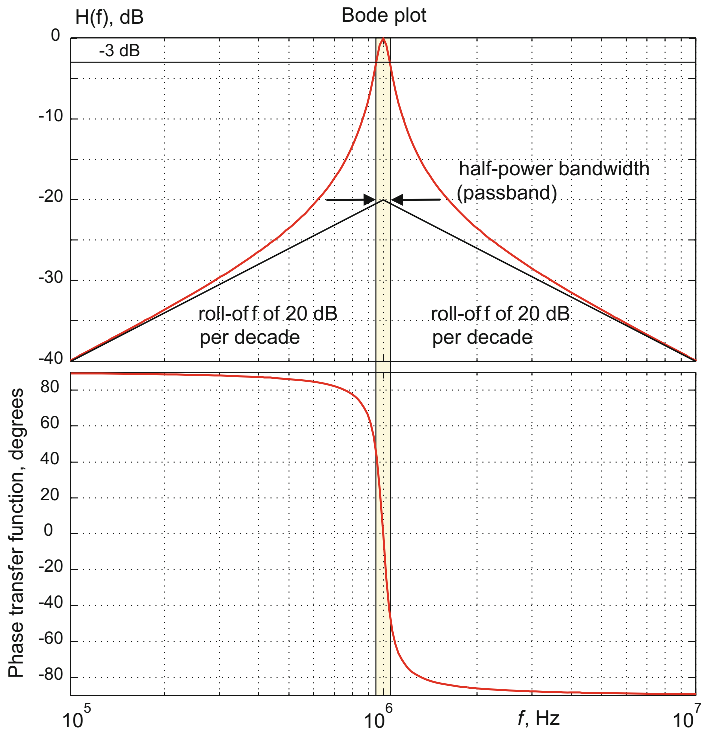


Fig. 10.8. Amplitude and phase Bode plot of a band-pass series RLC filter with $Q = 10$.

Exercise 10.5: In the band-pass filter circuit of Fig.10.7, $L = 100\ \mu\text{H}$, $C = 15.9\ \mu\text{F}$, and $R = 10\ \Omega$. What is the filter bandwidth?

Answer: 15.9 kHz.

Example 10.9: In the previous example, determine the necessary values of L and C given $R = 20\ \Omega$.

Solution: From the definition of the resonant frequency and the Q-factor, we obtain $1/\sqrt{LC} = 2\pi \times 10^6$, $\sqrt{L/C} = RQ = 200 \Rightarrow C \approx 796\ \text{pF}$. Then, we find the required inductance, $L = C(RQ)^2 \approx 31.8\ \mu\text{H}$. Alternatively, one could find the inductance L directly from the definition of the bandwidth for the series RLC resonator, that is, $L = R/(2\pi B) = 31.8\ \mu\text{H}$.

10.2.2 Second-Order Low-Pass RLC Filter

A series RLC circuit is again shown in Fig. 10.9a. We consider the power supply AC voltage as the *input voltage* $v_{in}(t)$ into the filter. We monitor the capacitor voltage $v_C(t)$ as the *output voltage* $v_{out}(t)$ of the filter. Figure 10.9b depicts the corresponding circuit.

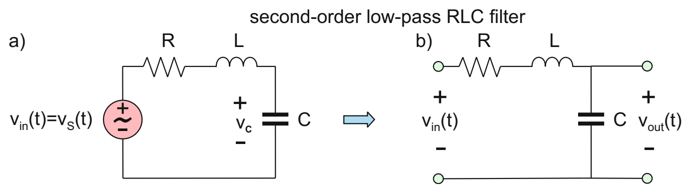


Fig. 10.9. Transformation of the series RLC circuit into the low-pass analog RLC filter.

The filter so constructed is a *second-order low-pass RLC filter*. Qualitatively then, when the frequency of the input voltage is low, the inductor behaves as a short circuit and the capacitor as an open circuit. The input voltage *passes* through unchanged. However, if the frequency is higher than the resonant frequency, both the inductor and the capacitor prevent transmission: the capacitor shorts out the output voltage, whereas the inductor reduces the circuit current. The complex filter transfer function is defined by the ratio of the corresponding phasors:

$$\mathbf{H}(f) \equiv \frac{\mathbf{V}_\mathrm{C}}{\mathbf{V}_\mathrm{S}} = \frac{\mathbf{I}}{j\omega C V_\mathrm{m}} \tag{10.25a}$$

We substitute the expression for the phasor current of the series RLC circuit from Eq. (10.24a) and obtain the transfer function in the form:

$$\mathbf{H}(f) = Q\frac{f_0}{f}\frac{1}{\sqrt{1+Q^2\left(\frac{f}{f_0}-\frac{f_0}{f}\right)^2}}\angle -\frac{\pi}{2} - \tan^{-1}\left(Q\left(\frac{f}{f_0}-\frac{f_0}{f}\right)\right) \tag{10.25b}$$

Example 10.10: A low-pass RLC filter is required with a *passband* from 0 to 1 MHz. In other words, the low-pass *filter bandwidth*, which extends from zero hertz to the *half-power frequency*, should be 1 MHz. Create amplitude and phase Bode plots for the filter in the frequency band from 100 kHz to 10 MHz.

Solution: The critical point for the low-pass RLC filter design is the proper selection of the quality factor. The amplitude transfer function in Eq. (10.25b) can exhibit a sharp peak in the passband with its value higher than one. Such a peak (further investigated in the homework problems) occurs only for $Q = 1/\sqrt{2}$. The value $Q = 1/\sqrt{2}$ corresponds to the maximally flat but still steep transfer function (*maximally flat Butterworth*

Example 10.10 (cont.): response). We will use this value in Eq. (10.25b). Then, the half-power or 3-dB frequency of the filter will be *exactly* the resonant frequency f_0. The resulting Bode plots are shown in Fig. 10.10. The transfer function of the filter has 40-dB-per-decade roll-off.

Figure 10.10 shows three amplitude responses: for an ideal filter with the cutoff frequency of 1 MHz, for a second-order RLC filter with the 3-dB frequency which coincides with $f_0 = 1$ MHz, and for a first-order RC (or RL) filter with the break (half-power) frequency $f_b = f_0 = 1$ MHz. Clearly, the second-order filter better approaches the desired ideal response. This observation encourages us to consider filters of higher order.

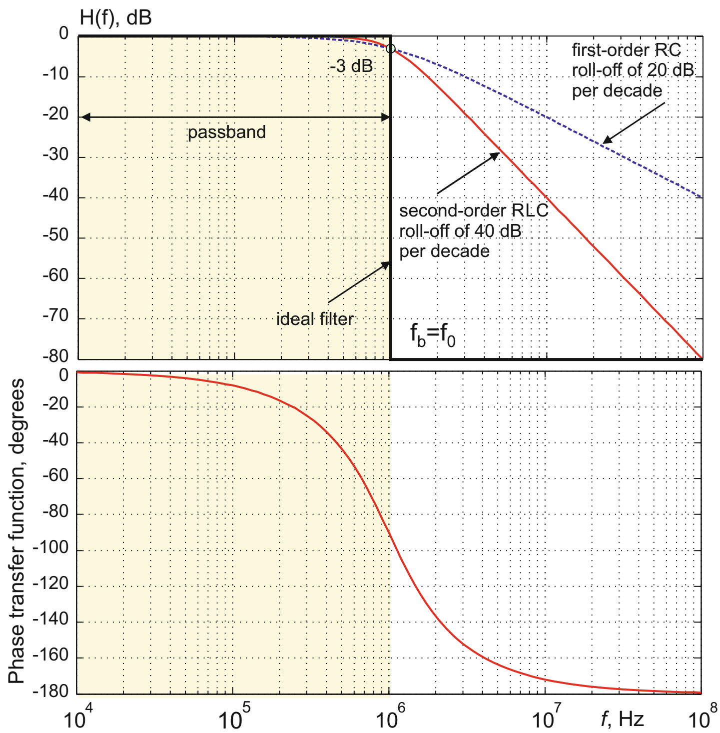


Fig. 10.10. Amplitude and phase Bode plots for the low-pass RLC filter (*solid curves*) compared with a first-order RC filter (*dotted curve*).

10.2.3 Second-Order High-Pass RLC Filter

A series RLC circuit is again shown in Fig. 10.11a. We consider the power supply AC voltage as the *input voltage* $v_{in}(t)$ into the filter. We next consider the inductor voltage $v_L(t)$ as the *output voltage* $v_{out}(t)$ of the filter, see Fig. 10.11b. The constructed circuit is a *second-order high-pass RLC filter*. Qualitatively, when the frequency of the input voltage is low, the capacitor behaves like an open circuit, while the inductor behaves like a short circuit. Both the inductor and the capacitor prevent transmission. However, if the frequency is higher than the resonant frequency, the capacitor becomes a short circuit and the inductor becomes an open circuit. The input voltage is *passed* through the filter nearly unchanged. The complex filter transfer function is defined by the ratio of the corresponding phasors:

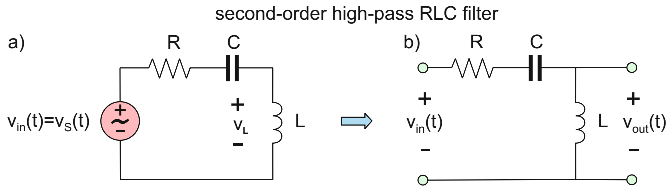


Fig. 10.11. Transformation of the series RLC circuit into the high-pass analog RLC filter.

$$\mathbf{H}(f) \equiv \frac{\mathbf{V}_\mathrm{L}}{\mathbf{V}_\mathrm{S}} = \frac{j\omega L\mathbf{I}}{V_\mathrm{m}} \tag{10.26a}$$

We substitute the expression for the phasor current of the series RLC circuit from Eq. (10.24a) and obtain the transfer function in the form:

$$\mathbf{H}(f) = Q\frac{f}{f_0}\frac{1}{\sqrt{1+Q^2\left(\frac{f}{f_0}-\frac{f_0}{f}\right)^2}}\angle\frac{\pi}{2} - \tan^{-1}\left(Q\left(\frac{f}{f_0}-\frac{f_0}{f}\right)\right) \tag{10.26b}$$

The amplitude transfer function of the high-pass filter in Eq. (10.26b) is the mirror reflection of the amplitude transfer function for the low-pass filter in Eq. (10.25b) if a logarithmic frequency scale is used. This fact is seen by substituting $f \leftrightarrow 1/f, f_0 \leftrightarrow 1/f_0$, which makes both expressions identical.

Example 10.11: A high-pass RLC filter is required with the *passband* from 0 to 1 MHz. The high-pass filter *half-power frequency* should be 1 MHz. Create amplitude and phase Bode plots for the filter in the band from 100 kHz to 10 MHz.

Solution: The important point for the high-pass RLC filter design is again the proper selection of the quality factor. Similar to the low-pass filter, we choose the value $Q = 1/\sqrt{2}$, which corresponds to the maximally flat transfer function (*Butterworth response*). Then, the half-power or 3-dB frequency of the filter will be *exactly* the resonant frequency f_0. The resulting Bode plots are shown in Fig. 10.12 in comparison with the transfer function of the first-order high-pass filter. The amplitude transfer function of the filter again has the 40-dB-per-decade roll-off.

Exercise 10.6: A band-pass filter is as a series combination of the second-order low-pass RLC filter and the second-order high-pass RLC filter, respectively. Both filters have the same half-power frequency. What is the transfer function roll-off far away from the passband per one octave?

Answer: 12 dB.

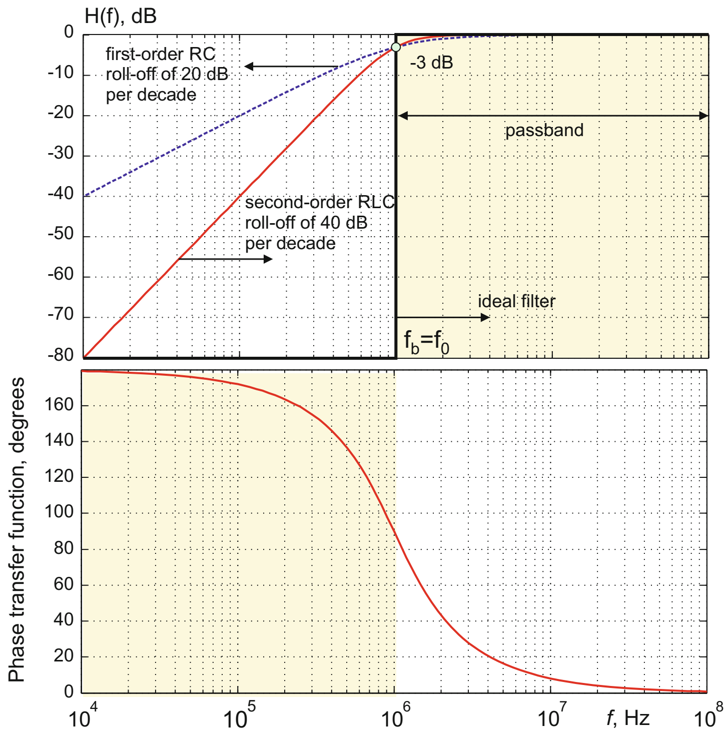


Fig. 10.12. Amplitude and phase Bode plots for the high-pass RLC filter (*solid curves*) and amplitude comparison with a first-order RC filter (*dotted curve*).

10.2.4 Second-Order Band-Reject RLC Filter

A series RLC circuit is again shown in Fig. 10.13a. We consider the power supply AC voltage as the *input voltage* $v_{in}(t)$ into the filter, and the voltage $v_{LC}(t)$ across the LC block is recorded as the *output voltage* $v_{out}(t)$. Figure 10.13b depicts the corresponding filter circuit.

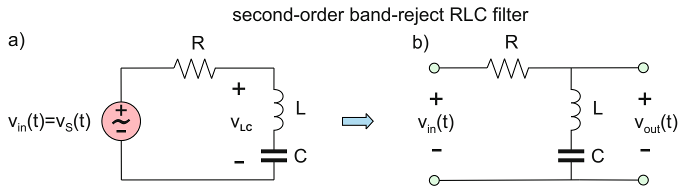


Fig. 10.13. Transformation of the series RLC circuit into the band-reject analog RLC filter.

The filter so constructed is a *second-order band-reject* (known as *band-stop* or *notch*) *RLC filter*. At resonance, the LC block forms a short circuit: the output filter voltage is thus shorted out. All other frequencies pass through. This filter is useful when a single tone (e.g., 60 Hz) needs to be rejected. By KVL, its transfer function is equal to one minus the transfer function of the band-pass filter in Eq. (10.24c), i.e.,

$$\mathbf{H}(f) = 1 - \frac{1}{1 + jQ\left(\frac{f}{f_0} - \frac{f_0}{f}\right)} = Q\left|\frac{f}{f_0} - \frac{f_0}{f}\right| \frac{1}{\sqrt{1 + Q^2\left(\frac{f}{f_0} - \frac{f_0}{f}\right)^2}} \angle \frac{\pi}{2} - \tan^{-1}\left(Q\left(\frac{f}{f_0} - \frac{f_0}{f}\right)\right) \tag{10.27}$$

Therefore, the filter behavior is the opposite of the band-pass filter previously analyzed.

Example 10.12: A band-reject RLC filter is required with the center frequency of 1 MHz and the half-power bandwidth, B, of 100 kHz. Create amplitude and phase Bode plots for the filter in the frequency band from 100 kHz to 10 MHz.

Solution: The quality factor of the RLC circuit is found to be $Q = f_0/B = 10$. We plot the magnitude of the transfer function $H(f)$ in decibels and its phase in degrees according to Eq. (10.27). The result is shown in Fig. 10.14. The filter response is very steep over the specified frequency range. We can lower the Q-factor, which will lead to a wider bandwidth. Note that the amplitude transfer function formally equals zero at the exact resonant frequency. This result is physically unrealizable since real inductors have a small parasitic series resistance.

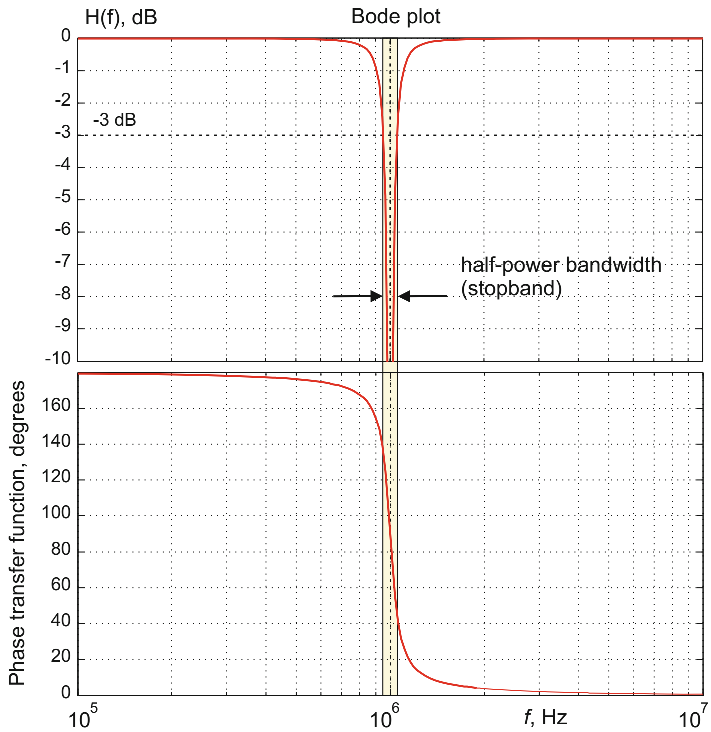


Fig. 10.14. Amplitude and phase Bode plots for the band-reject series RLC filter.

10.2.5 Second-Order RLC Filters Derived from the Parallel RLC Circuit

All second-order filters considered so far are derived from the *series* RLC circuit, with the same quality factor given by $Q = \omega_0 L/R$. The *natural* structure *after shorting out the input voltage source* is shown in Fig. 10.15a. A complementary group of these filter circuits exists; after shorting out the input voltage source, its natural structure is that of the parallel RLC circuit seen in Fig. 10.15b. These circuits operate quite similarly, but all of them have the quality factor of the parallel RLC resonator, that is, $Q = \omega_0 RC$.

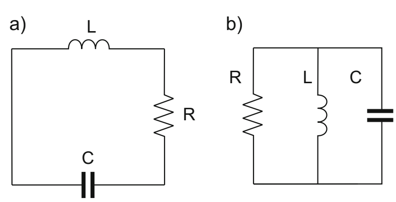


Fig. 10.15. (**a**) Series RLC circuit with no excitation and (**b**) parallel RLC circuit with no excitation.

For the filter circuits derived from the parallel RLC circuit, the resonant frequency still has to satisfy the condition that a *real* circuit impedance is "seen" by the voltage source. The resonant frequency found this way either does not equal the undamped resonant frequency $f_0 = 1/\left(2\pi\sqrt{LC}\right)$ or does not exist at all. However, the structure of the filter equations is *not* affected by this result. Only the undamped resonant frequency f_0 appears to be important for the voltage transfer function, which indeed remains the same for *any* filter circuit containing one inductance and one capacitance.

Section 10.3 RLC Circuits for Near-Field Communications and Proximity Sensors

10.3.1 Near-Field Wireless Link

Near-field wireless communication can transfer data, power, or both of them simultaneously. Common data-related applications include *radio-frequency identification* (RFID) systems of 125/134 kHz and 13.56 MHz, *electronic article surveillance* (EAS) for electronic anti-theft devices in shops, and mobile and other portable device *near-field communication* (NFC). Promising applications in biomedical engineering have also been explored. Figure 10.16 shows the key concept of a *near-field wireless link*. The transmitter and the receiver inductor coils share a common magnetic flux density $\vec{B}$ in the near field. The transmitter/receiver system in Fig. 10.16 is known as an *inductively coupled system*.

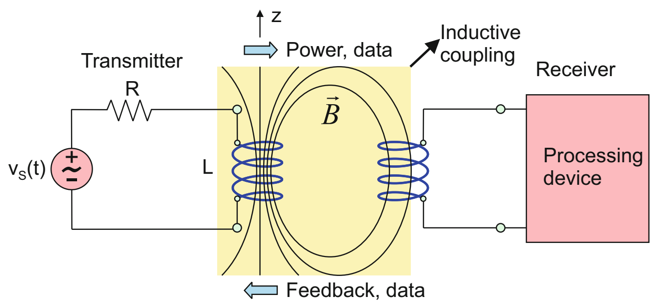


Fig. 10.16. The concept of the near-field wireless link; the magnetic flux density is shared between receiver and transmitter coils.

In contrast to the radio-frequency radiating fields, the near field $\vec{B}$ is very strong in the vicinity of the coil antenna. However, this field very rapidly decays at larger distances from the transmitter. For example, consider a transmitter coil with N loops of area A each. The corresponding near field of the transmitter coil in Fig. 10.16 with current $i(t)$ on the coil axis at the axial distances z much greater than the coil length (and the loop radius) may be found in the form:

$$B = N\left[\frac{\mu_0 A}{2\pi}\frac{i(t)}{z^3}\right] \ [\text{T}], \quad \mu_0 = 4\pi \times 10^{-7}\ \text{H/m} \tag{10.28}$$

where B is recorded in *tesla* T. The expression in square brackets is the contribution of a single loop. Thus, the near-field decay is inversely proportional to the *third* degree of the separation distance. This observation (obtained via an asymptotic analysis of the related magnetostatic expressions) is also valid for any quasi-static *magnetic* (and *electric*) *dipole*. Such a short-range wireless communication is both safe and effective.

10.3.2 Transmitter Circuit

In a transmitter circuit shown in Fig. 10.17a, the function generator is modeled by an ideal voltage source $v_S(t) = V_m \cos \omega t$ in series with resistance R. The function generator is connected to the transmitter coil modeled by the inductance L. The goal is to increase the magnetic flux density $\vec{B}$ of the transmitter. According to Eq. (10.28), the obvious choice is to increase the inductance of the transmitter. However, this operation would decrease the circuit current $i(t)$ due to an increase of the impedance magnitude. We will attempt to solve this problem with the series resonant RLC circuit shown in Fig. 10.17b.

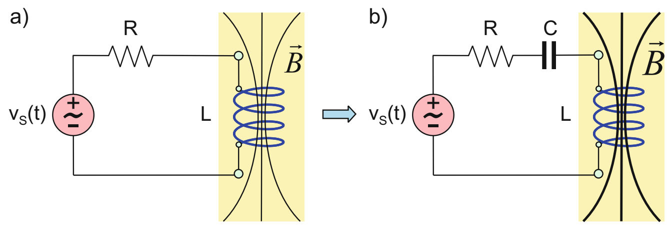


Fig. 10.17. Using a series capacitor in order to increase the circuit current.

The original and the modified circuits in Fig. 10.2 are both solved by using the phasor method. We denote the desired operating frequency by f_0. For the original circuit in Fig. 10.17a, the phasor current may then be written in the form:

$$\mathbf{I} = \frac{V_m}{\mathbf{Z}_{eq}} = \frac{V_m}{R + j\omega L} = \frac{V_m}{R} \frac{1}{\sqrt{1 + Q^2 \left(\frac{f}{f_0}\right)^2}} \angle - \tan^{-1}\left(Q \frac{f}{f_0}\right), \quad Q = \omega_0 (L/R) \tag{10.29}$$

For the series RLC circuit in Fig. 10.17b, the capacitance is chosen in such a way that the resonant frequency of the circuit coincides with the operation frequency f_0. The phasor current for the RLC circuit has been derived in the previous sections. It has the form:

$$\mathbf{I} = \frac{V_m}{\mathbf{Z}_{eq}} = \frac{V_m}{R + j\left(\omega L - \frac{1}{\omega C}\right)} = \frac{V_m}{R} \frac{1}{\sqrt{1 + Q^2 \left(\frac{f}{f_0} - \frac{f_0}{f}\right)^2}} \angle - \tan^{-1}\left(Q\left(\frac{f}{f_0} - \frac{f_0}{f}\right)\right) \tag{10.30}$$

Note the presence of the quality factor, $Q = \omega_0 (L/R)$, for the series RLC circuit in both Eqs. (10.29) and (10.30). At exactly the operation frequency, $f = f_0$, the ratio of current *magnitudes* (both phasors $\angle \cdot$ have the magnitude of one) becomes

$$\frac{I_{m \ \text{circuit with series capacitor}}}{I_{m \ \text{original circuit}}} = \sqrt{1 + Q^2} \approx Q \ \text{ for } Q \gg 1 \tag{10.31}$$

This ratio may significantly exceed one for high Q values. Thus, the series RLC circuit may considerably increase the circuit current and the associated magnetic field.

Exercise 10.7: It is suggested to increase the magnetic field for the circuit without the capacitor in Fig. 10.7a by simply doubling the number of coil turns and increasing the coil length by the factor of two. Given that:

1. $Q = \omega_0(L/R) \gg 1$ at the operation frequency
2. $Q = \omega_0(L/R) \ll 1$ at the operation frequency

how does the magnetic field B change?

Answer: (i) B remains the same. (ii) B doubles.

Example 10.13: Given the operation frequency (center band frequency) of $f_0 = 1$ MHz and $V_m = 10$ V, $L = 50$ μH, $R = 50$ Ω, plot the amplitude of the circuit current as a function of source frequency for the original (RL) and modified (resonant RLC) circuits in in Fig. 10.17 over the frequency band from 0.5 to 1.5 MHz.

Solution: The quality factor is found to be $Q = 6.283$. Next, we plot both current amplitudes based on Eqs. (10.29) and (10.30) using a linear scale. The result is shown in Fig. 10.18. The amplitude of the circuit current increases from 31.4 to 200 mA at the operation frequency f_0 (resonant frequency of the RLC circuit).

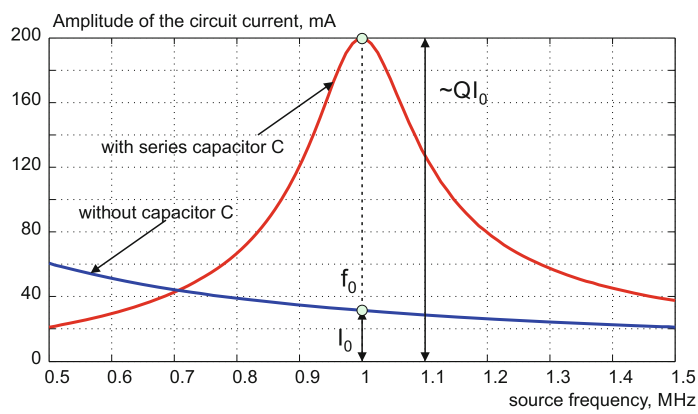


Fig. 10.18. Amplitudes of the circuit current for the original and modified circuits in Fig. 10.17.

10.3.3 Receiver Circuit

Consider the receiver coil in Fig. 10.19a. Its equivalent circuit includes the ideal inductor L in series with R, which is the resistance of the coil winding. It also includes an induced emf (electromotive force) voltage source $v_{emf}(t)$, which follows Faraday's law of induction:

$$v_{\text{emf}}(t) \equiv AN\frac{dB(t)}{dt} = V_{\text{m}}\cos(\omega t), \quad B(t) = B_{\text{m}}\sin(\omega t) \tag{10.32}$$

Here, $B(t)$ is the coaxial component of the external, time-varying magnetic flux density of the transmitter at the receiver location. The source voltage amplitude is given by $V_{\text{m}} = AN\omega B_{\text{m}}$ where A is the area of the receiver coil and N is the number of coil turns. The major parameter of interest is the (small) open-circuit voltage of the receiver coil, $v_{\text{out}}(t)$. It is desired to increase this voltage. For the circuit shown in Fig. 10.19a, $v_{\text{out}}(t)$ is always equal to $v_{\text{emf}}(t)$. However, the situation will change if we create a series RLC circuit as shown in Fig. 10.19b. The output voltage becomes the capacitor voltage. We will attempt to increase $v_{\text{out}}(t)$ by using the resonance condition.

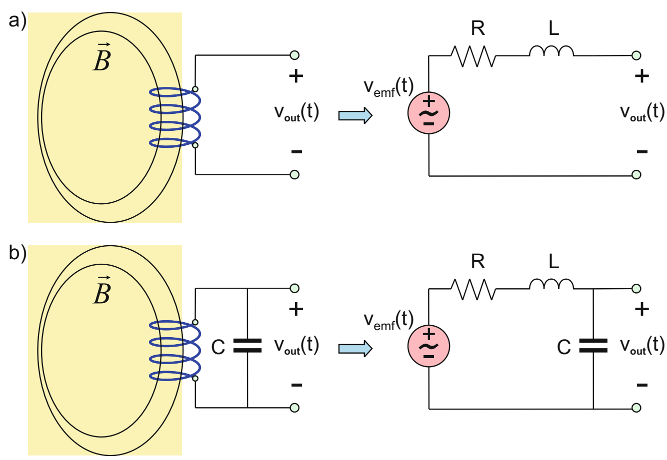


Fig. 10.19. (**a**) Receiver coil and (**b**) receiver coil with capacitance to increase the open-circuit voltage.

The circuit in Fig. 10.19b is solved using the phasor method. The desired resonant frequency (operating frequency) is f_0. The phasor for the output voltage has the form:

$$\mathbf{V}_{\text{out}} = \mathbf{V}_{\text{C}} = \frac{V_{\text{m}}/(j\omega C)}{R + j\omega L + \frac{1}{j\omega C}} = Q\frac{f_0}{f}\frac{V_{\text{m}}}{\sqrt{1 + Q^2\left(\frac{f}{f_0} - \frac{f_0}{f}\right)^2}}\left\{\angle -\frac{\pi}{2} - \tan^{-1}\left(Q\left(\frac{f}{f_0} - \frac{f_0}{f}\right)\right)\right\} \tag{10.33}$$

where $Q = \omega_0(L/R)$ is again the quality factor of the series RLC resonant circuit (and simultaneously the *quality factor of the nonideal inductor* with series resistance R). At the exact resonant frequency, the output voltage amplitude becomes

$$V_{\text{out}} = QV_{\text{m}} \tag{10.34}$$

This value may significantly exceed V_m given a high value of Q. Thus, the series RLC circuit formed with the help of the shunt capacitor C in Fig. 10.19b may considerably increase the received voltage.

Example 10.14: Given the operating frequency (center band frequency) of $f_0 = 1\ \text{MHz}$ and $V_m = 10\ \text{mV}$, $L = 78\ \mu\text{H}$, $R = 10\ \Omega$, plot the amplitude of the output voltage for the original and modified circuits in Fig. 10.19 as a function of source frequency over the band from 0.5 to 1.5 MHz.

Solution: We find the required capacitance value first. Specifically, $C = 1/\left(L(2\pi f_0)^2\right) \approx 325\ \text{pF}$. The quality factor is given by $Q \approx 49.0$. Next, we plot both voltage amplitudes. The first one is simply V_m. The second plot is based on Eq. (10.33). The results are shown in Fig. 10.20. The amplitude of the output voltage increases from 10 to 490 mV at the operating frequency f_0 (resonant frequency of the series RLC circuit).

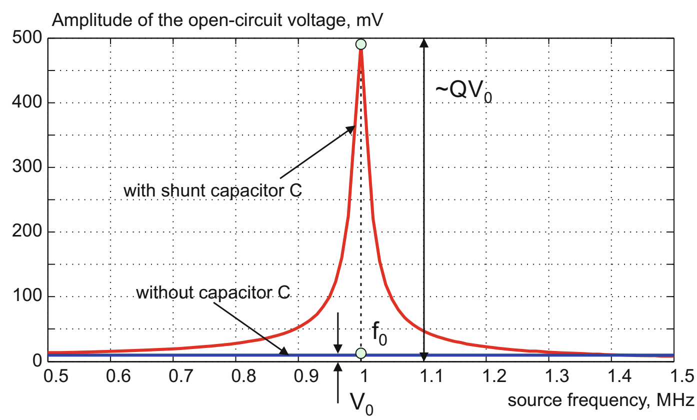


Fig. 10.20. Amplitudes of the output voltage for the original and modified circuits in Fig. 10.19.

The circuit in Fig. 10.19b is the low-pass second-order RLC filter studied in the previous section, right? Why is it boosting the source voltage instead of just passing it through? The key is the Q-factor. The present circuit operates as a filter at relatively small values of the quality factors, i.e., $Q \le 1$. At higher Q values, the circuit generates a voltage spike close to the resonant frequency and operates as a *voltage multiplier*. This operation is similar to the operation of an electric transformer.

10.3.4 Application Example: Near-Field Wireless Link in Laboratory

Figure 10.21 shows a prototype of the near-field link implemented in an undergraduate laboratory. The operating frequency of the transmitter is tunable; it ranges from 400 kHz to 1.2 MHz. Despite this relatively high frequencies, the circuitry can still be implemented on standard protoboards. The key is the tunability of both the transmitter and the receiver, which simultaneously accounts for the parasitic capacitance of the board.

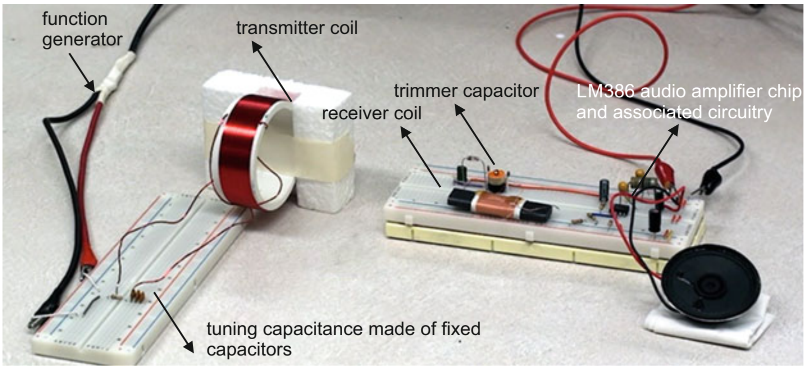


Fig. 10.21. Prototype of the near-field link implemented in laboratory.

The transmitter is driven by a function generator; the tuning is made by a bank of fixed capacitors from a laboratory kit. The receiver uses a single-ended magnetic-core coil (a *loopstick antenna*) with an inductance of approximately 1 mH and a series resistance of about 10 Ω. The RLC circuit at the receiver uses a trimmer capacitor of 10–180 pF range for tuning. When the transmitter is sending an amplitude-modulated signal from the function generator, the receiver operates as an AM radio given the subsequent rectifying circuit with a germanium diode and an audio amplifier IC (LM386). Frequency modulation is also possible; however, the receiver circuit has to be modified accordingly. The system operating range is up to two feet on average. When an external modulation input of the function generator is used, an audio clip may be transmitted.

10.3.5 Application Example: Proximity Sensors

The idea of the resonant RLC proximity sensor is quite simple. Assume that the inductance is a large coil or simply a loop of wire. When a metal object to be detected is brought in proximity to the loop, its (self) inductance changes. This causes a detectable change in the resonant frequency f_0 of an RLC tank circuit. After encoding, information may be extracted about the presence of the object and sometimes of its size. This is the well-known principle of a *metal detector*. A large variety of metal-detecting circuits already exist, and more are still awaiting discovery. Another idea is to change the capacitance by putting a dielectric object (such as a medical pill) inside the capacitor. A similar change in the resonant frequency may be observed and detected. Such a device may be used, for example, as an automatic pill counter.

The detector circuit itself can operate based on three different principles. First, a simple method is to use the series RLC tank circuit with the external AC power supply. The measured parameter is the amplitude of the circuit current (resistor voltage) at the frequency of the AC source. When the resonant frequency is close to the AC frequency, the circuit voltage is large. However, when the resonant frequency deviates from the source frequency, the circuit voltage becomes smaller. The change in the circuit voltage is detected. A second method is to make the tank circuit *self-resonant*, by using an amplifier with a *positive feedback*. A resonant circuit so built does not need an AC power supply. It oscillates *exactly* at f_0 when there is no object to be detected. When the object is present the oscillation frequency changes. The change in the AC frequency is encoded by another electronic circuit. Using self-resonant tank circuits is perhaps the most common method in practice. A third method is based on the effect of the resistance in the tank circuit. When a metal object is placed close to the coil, the coil's series resistance significantly increases, due to the so-called eddy current losses (for all metals) and, possibly, hysteresis losses (for magnetic metals such as iron, nickel, steel alloys, etc.). The increase in the resistance leads to smaller voltage oscillations in the self-resonant circuit. The circuit may be tuned in such a way as to stop oscillating at a given value of the extra resistance. Great sensitivity may be achieved with this method.

Figure 10.22 shows the inductor assembly in a resonant sensor for an *automatic traffic light.* The inductor now is a single-turn (or multi-turn) pavement loop. When a vehicle is located above the loop, its (self) inductance L decreases. This leads to an increase in the resonance frequency. The change in frequency, not the change in the amplitude, is typically detected and encoded. The latter is used to indicate the presence of a vehicle and to adjust the traffic light control. Most vehicle detectors based on loop inductors operate with frequencies from 10 to 100 kHz. A (simplified) equivalent tank circuit for the traffic light control is shown in Fig. 10.23b. We note the series resistance R, which is the parasitic resistance of the loop. The parasitic resistance includes both the effect of the passing vehicle and of the ground.

Fig. 10.22. Multiple vehicle detection loops after installation at an intersection. Courtesy of the US Traffic Corporation, Loop Application Note of 3/10/03.

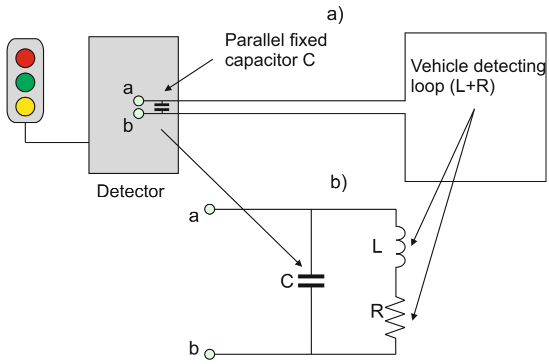


Fig. 10.23. (**a**) Simplified schematic of the vehicle detecting system and (**b**) equivalent resonant circuit.

The circuit in Fig. 10.23b may be analyzed exactly in the same way as the series/parallel RLC tank circuits in Section 10.1. The tricky part for the tank circuit block in Fig. 10.23b is that its resonant angular frequency is no longer $1/\sqrt{LC}$. However, it can still be found from the condition of a purely real equivalent impedance $\mathbf{Z}_{eq}$, see the summary of this chapter.

Single coils of special shapes—the *horseshoe shape*—may be used to detect the level and the presence of molten metals through the walls of (large) casting molds and for other purposes. The equivalent circuit is the parallel RLC tank circuit. When properly tuned, the self-resonating circuit quantitatively detects variations in molten metal level through 4–5″ thick walls, see Fig. 10.24.

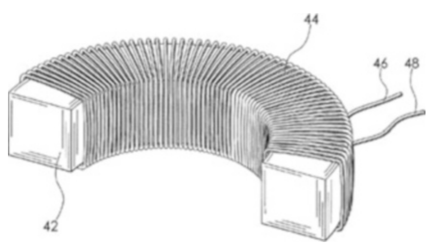
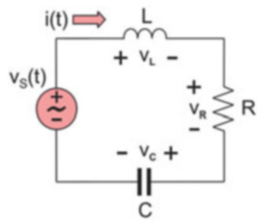
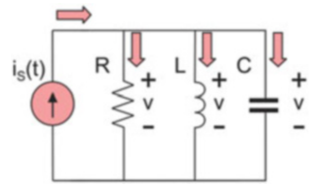
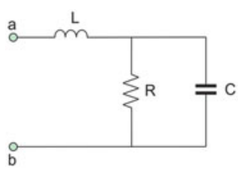
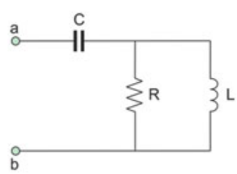
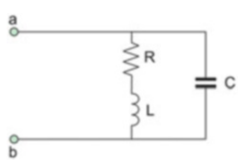
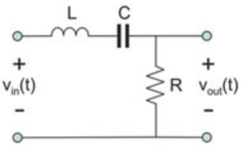
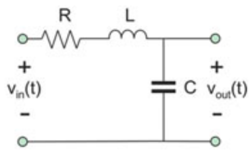
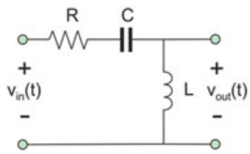
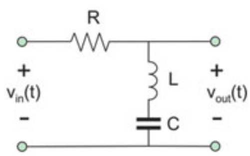
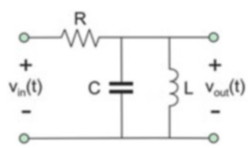
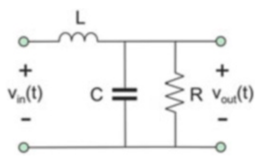
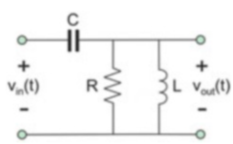
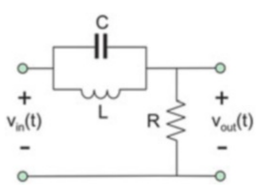


Fig. 10.24. A half-toroidal (*horseshoe*) coil used to concentrate the magnetic field between its tips in a molten metal detector (Foley, Biederman, Ludwig, and Makarov, US Patent 7,828,043 Nov. 9th 2010).

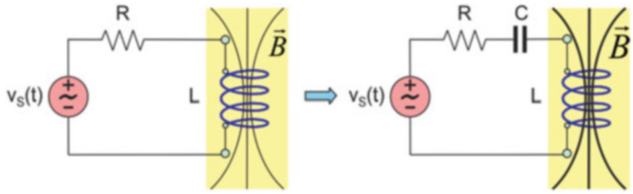
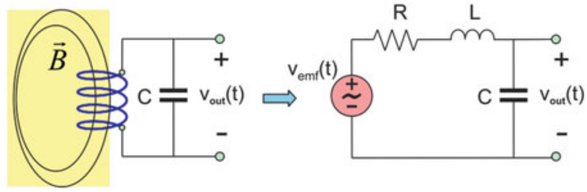
Summary

TERM	Series RLC circuit	Parallel RLC circuit
Series and parallel RLC resonators		
Resonant frequency	$\omega_0 = 1/\sqrt{LC},\ f_0 = 1/\left(2\pi\sqrt{LC}\right)$ Coincides with the undamped resonant frequency	$\omega_0 = 1/\sqrt{LC},\ f_0 = 1/\left(2\pi\sqrt{LC}\right)$ Coincides with the undamped resonant frequency
Quality factor of the resonant circuit	$Q = \dfrac{\sqrt{L/R}}{\sqrt{RC}} = \dfrac{1}{\omega_0 RC} = \omega_0(L/R)$ dimensionless	$Q = \dfrac{\sqrt{RC}}{\sqrt{L/R}} = \omega_0 RC = \dfrac{1}{\omega_0(L/R)}$ dimensionless
Bandwidth of the resonant circuit	$B \equiv f_U - f_L = \dfrac{f_0}{Q} = \dfrac{1}{2\pi(L/R)}$ [Hz]	$B \equiv f_U - f_L = \dfrac{f_0}{Q} = \dfrac{1}{2\pi RC}$ [Hz]
Half-power lower and upper frequencies	$f_{L,U} = f_0\left(\sqrt{1+\dfrac{1}{(2Q)^2}} \mp \dfrac{1}{2Q}\right)$ [Hz]	$f_{L,U} = f_0\left(\sqrt{1+\dfrac{1}{(2Q)^2}} \mp \dfrac{1}{2Q}\right)$ [Hz]
Other RLC resonators	Circuit diagram	Resonant frequency
L+R||C		$\omega_0 = \dfrac{1}{(RC)}\sqrt{\dfrac{RC}{L/R} - 1}$ Different from the undamped resonant frequency
C+L||R		$\omega_0 = \dfrac{1}{(L/R)}\dfrac{1}{\sqrt{\dfrac{RC}{L/R} - 1}}$ Different from the undamped resonant frequency
(R+L)||C		$\omega_0 = \dfrac{1}{(L/R)}\sqrt{\dfrac{L/R}{RC} - 1}$ Different from the undamped resonant frequency

(continued)

RLC filter circuits derived from the *series* RLC circuit: $f_0 = 1/\left(2\pi\sqrt{LC}\right)$, $Q = 1/(\omega_0 RC)$		
Band-pass filter		$\mathbf{H}_0(f) = \dfrac{1}{1 + jQ\left(\dfrac{f}{f_0} - \dfrac{f_0}{f}\right)}$
Low-pass filter		$\mathbf{H}(f) = Q\dfrac{f_0}{f}\mathbf{H}_0(f)$
High-pass filter		$\mathbf{H}(f) = Q\dfrac{f}{f_0}\mathbf{H}_0(f)$
Band-reject filter		$\mathbf{H}(f) = jQ\left(\dfrac{f}{f_0} - \dfrac{f_0}{f}\right)\mathbf{H}_0(f)$
RLC filter circuits derived from the *parallel* RLC circuit: $f_0 = 1/\left(2\pi\sqrt{LC}\right)$, $Q = \omega_0 RC$		
Band-pass filter based on parallel RLC circuit		$\mathbf{H}_0(f) = \dfrac{1}{1 + jQ\left(\dfrac{f}{f_0} - \dfrac{f_0}{f}\right)}$
Low-pass filter based on parallel RLC circuit		$\mathbf{H}(f) = Q\dfrac{f_0}{f}\mathbf{H}_0(f)$
High-pass filter based on parallel RLC circuit		$\mathbf{H}(f) = Q\dfrac{f}{f_0}\mathbf{H}_0(f)$
Band-reject filter based on parallel RLC circuit		$\mathbf{H}(f) = jQ\left(\dfrac{f}{f_0} - \dfrac{f_0}{f}\right)\mathbf{H}_0(f)$

(continued)

Near-field wireless transmitter/receiver	
Resonant circuit at the transmitter (TX)	 The series capacitor forms the series RLC circuit and increases the amplitude of the magnetic flux density anywhere in space by the factor $\sqrt{1+Q^2}$, $Q=\omega_0(L/R)$
Resonant circuit at the receiver (RX)	 The shunt capacitor again forms the series RLC circuit and increases the amplitude of the output voltage by the factor Q, $Q=\omega_0(L/R)$

Problems

10.1. Theory of the Second-Order RLC Resonator

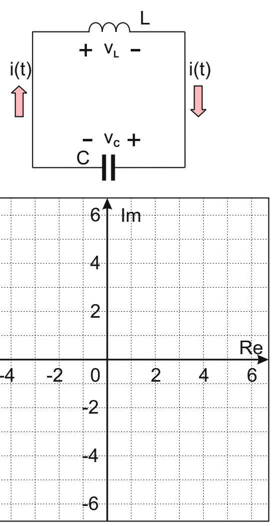
10.1.1 Self-Oscillating Ideal LC Circuit

10.1.2 Series Resonant Ideal LC Circuit

Problem 10.1. Give an example of a self-oscillating (resonant) mechanical system different from that in Fig. 10.2 of this section.

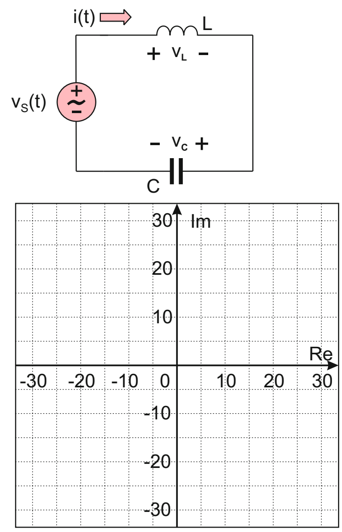
Problem 10.2. For the self-oscillating circuit shown in the figure below, the circuit current is specified by $i(t) = I_m \cos \omega t$. Given $I_m = 200$ mA, $L = 0.63$ mH, $C = 1$ μF:

A. Determine the undamped resonant frequency f_0.
B. Construct the phasor diagram for phasor voltages $\mathbf{V}_L$ and $\mathbf{V}_C$ and phasor current $\mathbf{I}$ on the same plot. Assume that every plot division corresponds to 1 V or to 100 mA.



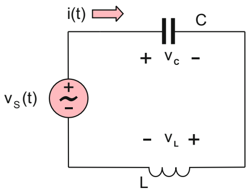
Problem 10.3. For a series ideal LC circuit shown in the figure below, the voltage source has the form $v_S(t) = V_m \cos \omega t$. Given $V_m = 5$ V, $L = 0.5$ mH, $C = 1$ μF:

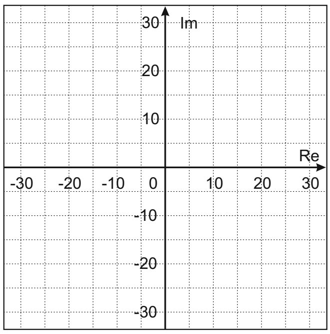
A. Determine the undamped resonant frequency f_0 of the circuit.
B. Construct the phasor diagram for phasor voltages $\mathbf{V}_S$, $\mathbf{V}_L$, and $\mathbf{V}_C$ when the source frequency is 90 % of the resonant frequency.
C. Describe how your phasor diagram would change if the inductance becomes 1 mH instead of 0.5 mH.



Problem 10.4. For a series ideal LC circuit shown in the figure below, the voltage source has the form $v_S(t) = V_m \cos \omega t$. Given $V_m = 5$ V, $L = 1$ mH, $C = 0.5$μF:

A. Determine the undamped resonant frequency f_0 of the circuit.
B. Construct the phasor diagram for phasor voltages $\mathbf{V}_S$, $\mathbf{V}_L$, and $\mathbf{V}_C$ when the source frequency is 111 % of the resonant frequency.
C. Describe how your phasor diagram would change if the capacitance becomes 1 μF instead of 0.5 μF.





10.1.3 Series Resonant RLC Circuit: Resonance Condition
10.1.4 Quality Factor Q of the Series Resonant RLC Circuit
10.1.5 Bandwidth of the Series Resonant RLC Circuit

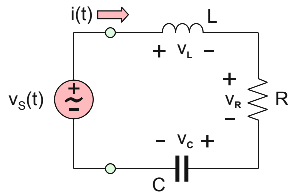
Problem 10.5. For a generic series resonant RLC circuit with the supply voltage $\upsilon_S(t) = V_m \cos \omega t$, resistance R, inductance L, and capacitance C, give the expressions (show units) for:

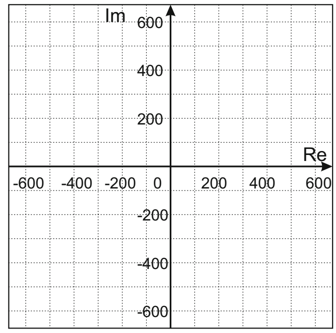
A. Equivalent circuit impedance at the resonance
B. Resonant frequency
C. Quality factor of the resonant circuit

Problem 10.6. Describe the physical meaning of the quality factor of the series RLC resonator circuit in your own words.

Problem 10.7. In the series resonant RLC circuit shown in the figure that follows, given $V_m = 1$ V, $L = 1$ mH, $C = 80$ pF, $R = 10$ Ω:

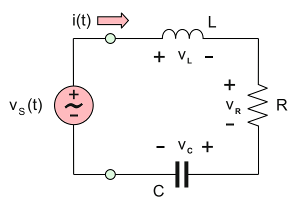
A. Determine resonant frequency and the Q-factor.
B. Determine resonant phasor current and phasor voltages $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$; construct the phasor diagram. Assume voltage scale in volts and current scale in milliamperes.
C. Determine the real-valued circuit current $i(t)$ and the inductor/capacitor voltages $\upsilon_L(t)$, $\upsilon_C(t)$ at resonance.

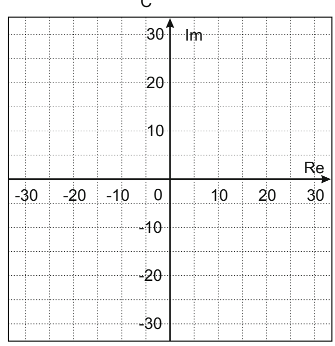




Problem 10.8. In the series resonant RLC circuit shown in the figure below, given $V_m = 5$ V, $L = 30$ μH, $C = 0.48$ nF, $R = 50$ Ω:

A. Determine resonant frequency and the Q-factor.
B. Determine resonant phasor voltages $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$; construct the phasor diagram.
C. Determine the real-valued resistor voltage $\upsilon_R(t)$ and the inductor/capacitor voltages $\upsilon_L(t)$, $\upsilon_C(t)$ at the resonance.



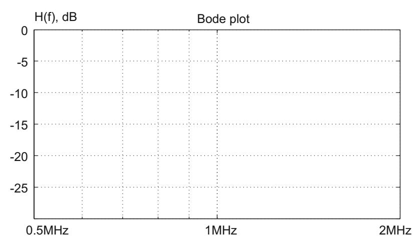


Problem 10.9. A series resonant LC circuit is driven by a laboratory AC voltage source with an amplitude $V_m = 12$ V and an internal resistance of 50 Ω (a function generator). Which value should the ratio L/C have to obtain the amplitude of the capacitor voltage equal to 200 V at the resonance?

Problem 10.10. A series resonant RLC circuit is needed with the resonant frequency of 1 MHz and a Q-factor of 100. The circuit resistance is 10 Ω. Determine the necessary values of L and C.

Problem 10.11. Describe the physical meaning of the resonance bandwidth of the series resonant RLC circuit in your own words.

Problem 10.12. A series resonant RLC circuit has the resonant frequency of 1 MHz and the quality factor of 10. Create the Bode plot for the amplitude of the circuit current normalized by its maximum value at the resonance over frequency band from 0.5 to 2 MHz.

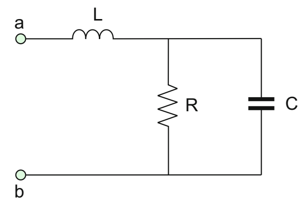


Problem 10.13. Determine the bandwidth, B, of the series resonant RLC circuit with the resonant frequency of 1 MHz and a Q-factor of 100.

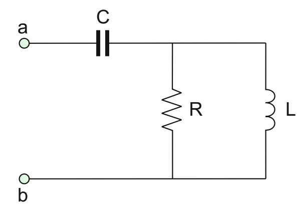
Problem 10.14. A series resonant RLC circuit is needed with the resonant frequency of 1 MHz and the bandwidth of 10 kHz. Given the circuit resistance of 10 Ω, determine L and C.

Problem 10.15. For the RLC circuit block shown in the figure, establish the resonant frequency in terms of component values.

Hint: The resonance is defined by the condition of the purely real equivalent impedance between terminals a and b. In other words, $\mathrm{Im}\left(\mathbf{Z}_{eq}\right) = 0$ at the resonance.



Problem 10.16. Repeat the previous problem for the circuit shown in the figure that follows.



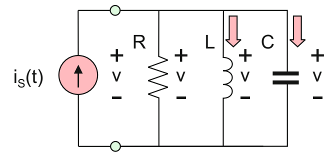
10.1.6 Parallel Resonant RLC Circuit: Duality

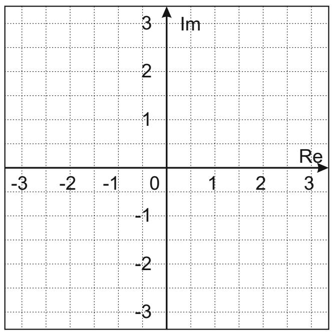
Problem 10.17. For a generic parallel RLC resonant circuit with the supply current $i_S(t) = I_m \cos \omega t$, resistance R, inductance L, and capacitance C, give the expressions (show units) for:

A. Equivalent circuit impedance at the resonance
B. Resonant frequency
C. Quality factor of the resonant circuit

Problem 10.18. In the parallel resonant RLC circuit shown in the figure that follows, given $I_m = 0.5$ A, $L = 30$ μH, $C = 0.43$ μF, $R = 50$ Ω:

A. Determine the resonant frequency and the Q-factor.
B. Determine resonant phasor currents $\mathbf{I}_R$, $\mathbf{I}_L$, and $\mathbf{I}_C$; construct the phasor diagram.
C. Determine the real-valued resistor current $i_R(t)$ and the inductor/capacitor currents $i_L(t)$, $i_C(t)$ at the resonance.





Problem 10.19. Determine the bandwidth, B, of the parallel resonant RLC circuit with the resonant frequency of 0.5 MHz and a Q-factor of 50.

Problem 10.20. A parallel resonant RLC circuit is needed with the resonant frequency of 1 MHz and the bandwidth of 10 kHz. Given the circuit resistance of 100 Ω, determine L and C.

10.2: Construction of Second-Order RLC Filters

10.2.1 Second-Order Band-Pass RLC Filter

10.2.2 Second-Order Low-Pass RLC Filter

10.2.3 Second-Order High-Pass RLC Filter

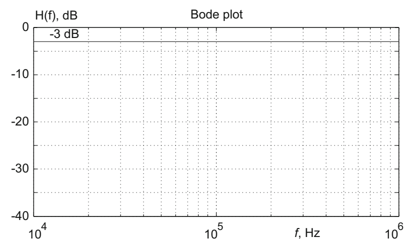
10.2.4 Second-Order Band-Reject RLC Filter

Problem 10.21

A. Draw the circuit diagram of the second-order RLC band-pass filter. Label R, L, and C.
B. Show the input and output ports (input and output voltages)
C. Define the resonant frequency and the Q-factor of the filter circuit.

Problem 10.22. A band-pass RLC filter is required with the center (resonant) frequency of 100 kHz and the half-power bandwidth, B, of 20 kHz.

A. Create its amplitude Bode plot in the frequency band from 10 kHz to 1 MHz.
B. Label the filter passband.
C. Determine the necessary values of L and C given $R = 20\ \Omega$.

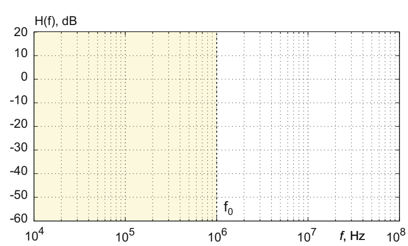


Problem 10.23

A. Draw the circuit diagram of the second-order RLC low-pass filter. Label R, L, and C.
B. Show the input and output ports (input and output voltages)
C. Define the resonant frequency and the Q-factor of the filter circuit.
D. Which Q-factor is required for the maximally flat response?
E. What is the filter's half-power frequency for the maximally flat response?

Problem 10.24. A low-pass RLC filter is required with the *passband* from 0 to 1 MHz. Create amplitude Bode plots for the filter in the frequency band from 100 kHz to 10 MHz given the resonant frequency of the filter circuit of 1 MHz and

A. $Q = 5$
B. $Q = 1/\sqrt{2}$
C. $Q = 0.2$



Problem 10.25*. Generate Fig. 10.4 of this section, the Bode plots for the low-pass filter using MATLAB.

Problem 10.26

A. Draw the circuit diagram of the second-order RLC high-pass filter. Label *R*, *L*, and *C*.
B. Show the input and output ports (input and output voltages)
C. Define the resonant frequency and the *Q*-factor of the filter circuit.
D. Which *Q*-factor is required for the maximally flat response?
E. What is the filter's half-power frequency for the maximally flat response?

Problem 10.27. A high-pass RLC filter is required with the *passband* from 0 to 1 MHz. Create amplitude Bode plots for the filter in the frequency band from 100 kHz to 10 MHz given the resonant frequency of the filter circuit of 1 MHz and

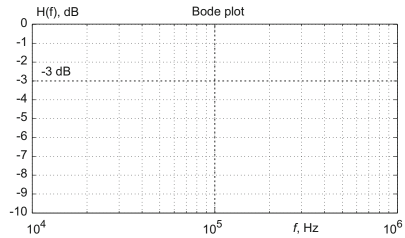
A. $Q = 10$
B. $Q = 1/\sqrt{2}$
C. $Q = 0.1$

Problem 10.28

A. Draw the circuit diagram of the second-order RLC band-reject filter. Label *R*, *L*, and *C*.
B. Show the input and output ports (input and output voltages)
C. Define the resonant frequency and the *Q*-factor of the filter circuit.

Problem 10.29. A band-reject RLC filter is required with the center (resonant) frequency of 100 kHz and the half-power bandwidth, *B*, of 20 kHz.

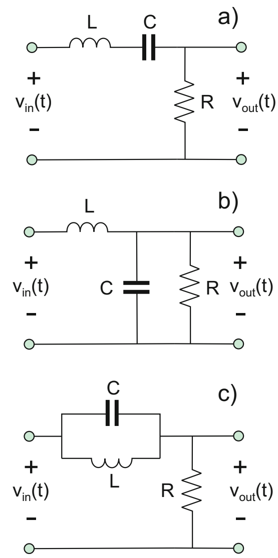
A. Create its amplitude Bode plot in the frequency band from 10 kHz to 1 MHz.
B. Label the filter passband.
C. Determine the necessary values of *L* and *C* given $R = 20\ \Omega$.



10.2.6. Second-Order RLC Filters Derived from the Parallel RLC Circuit

Problem 10.30. Second-order RLC filters may be constructed either on the basis of the series RLC circuit or on the basis of the parallel RLC circuit. The undamped resonant frequency, $f_0 = 1/\left(2\pi\sqrt{LC}\right)$, which is present in the filter equations, remains the same in either case. However, the quality factor does not. Three unknown second-order RLC filter circuits are shown in the figure that follows.

A. Determine the filter function (band-pass, low-pass, high-pass, or band-reject).
B. By analyzing filter's natural structure (after shorting out the input voltage source), determine the expression for the filter quality factor.



10.3. RLC Circuits for Near-Field Communications and Proximity Sensors

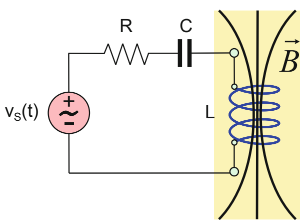
10.3.1 Near-Field Wireless Link
10.3.2 Transmitter Circuit
10.3.3 Receiver Circuit
10.3.4 Application Example: Near-Field Wireless Link in Laboratory
10.3.5 Application Example: Proximity Sensors

Problem 10.31. Describe the concept and purpose of the near-field wireless link in your own words. Think of an example where the link may be used solely for the power transfer.

Problem 10.32. In the circuit shown in the following figure, a capacitor C is introduced in series with an ideal coil having inductance L in order to set up a series resonant RLC circuit and increase the amplitude of the magnetic flux density $\vec{B}$ oscillating at 1 MHz. Determine the ratio of the magnetic flux amplitudes

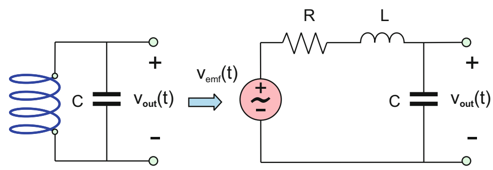
$$\frac{B_{\mathrm{m}\ \text{circuit with series capacitor}}}{B_{\mathrm{m}\ \text{original circuit}}}$$

with and without the capacitor anywhere in space given that $L = 100\ \mu\text{H},\ R = 25\ \Omega$.



Problem 10.33. Given the operation frequency (center band frequency) of $f_0 = 1$ MHz and $V_\mathrm{m} = 1$ V, $L = 100\ \mu\text{H}$, $R = 20\ \Omega$, plot to scale the amplitude of the circuit current as a function of source frequency for the original (RL) and modified (resonant RLC) circuits in the figure to the previous problem over the frequency band from 0.5 to 1.5 MHz. Label the amplitude values at the operation frequency.

Problem 10.34. In the circuit shown in the figure that follows, a receiver coil antenna is subject to an external magnetic field oscillating at 1 MHz. A capacitor C is introduced in parallel with the coil having inductance L and series resistance R in order to set up a *series* resonant RLC circuit and increase the amplitude of the output voltage $v_\mathrm{out}(t)$. Determine the ratio of the output voltage amplitudes with and without the capacitor given that $L = 1000\ \mu\text{H},\ R = 10\ \Omega$.



Problem 10.35. Given the operation frequency (center band frequency) of $f_0 = 1$ MHz and $v_\mathrm{emf}(t) = V_\mathrm{m}\cos(\omega t)$, $V_\mathrm{m} = 1$ mV, $L = 500\ \mu\text{H}$, $R = 50\ \Omega$, plot to scale the amplitude of the output voltage as a function of source frequency for the original (RL) and modified (resonant series RLC) circuits in the figure to the previous problem over the frequency band from 0.5 to 1.5 MHz. Label the amplitude values at the operation frequency.

Problem 10.36. In the circuit shown in the figure below, a receive coil antenna is subject to an external magnetic field oscillating at 1 MHz. A capacitor C is introduced in parallel with the coil having inductance L and series resistance R in order to set up a *series* resonant RLC circuit and increase the amplitude of the output voltage $v_\mathrm{out}(t)$. Determine the output voltage amplitudes with and without the capacitor given that $v_\mathrm{emf}(t) = 1\cos(\omega t)$ [mV] and that $L = 1000\ \mu\text{H},\ R = 10\ \Omega,\ R_f = 100\ \Omega$.