Chapter 7: Transient Circuit Fundamentals

Overview

Prerequisites:

- Knowledge of first-order ordinary differential equations (calculus)
- Knowledge of Thévenin/Norton equivalent circuits (Chapter 4)
- Knowledge of constitutive relations for dynamic circuit elements (Chapter 6)
- Knowledge of basic amplifier theory (Chapter 5)

Objectives of Section 7.1:

- Demonstrate the universal character of the KVL/KCL as applied to any electric circuit including transient circuits
- Establish the general character of the time constant $\tau = RC$ for RC circuits
- Establish the continuity of the capacitor voltage and its role in circuit ODEs
- Solve *any* first-order transient RC circuit configuration and understand the practical meaning of the RC circuit using different application examples

Objectives of Section 7.2:

- Demonstrate the universal character of the KVL/KCL as applicable to any electric circuit including transient circuits
- Establish the general character of the time constant $\tau = L/R$ for RL circuits
- Establish the continuity of the inductor current and its role in circuit ODEs
- Solve *any* first-order transient RL circuit configuration and understand the practical meaning of the RL circuit using an application example

Objectives of Section 7.3:

- Obtain initial exposure to a bistable amplifier circuit with positive feedback
- Understand the principle of operation of a relaxation oscillator—RC timer—on the base of the bistable amplifier circuit
- Establish oscillation frequency and voltage amplitudes from the relaxation oscillator; demonstrate the corresponding laboratory setup
- Briefly discuss the 555 timer IC

S.N. Makarov et al., *Practical Electrical Engineering*, DOI 10.1007/978-3-319-21173-2_7

Objectives of Section 7.4:

- Define the single-time-constant (STC) transient circuit
- Be able to classify any transient circuit with dynamic elements of the same type as either a single-time-constant circuit or a more complicated circuit
- Solve an example of a non-STC circuit
- Convert an arbitrary transient circuit with one capacitance or one inductance to the basic RC/RL first-order circuit
- Solve a first-order transient circuit with a harmonic forcing function

Objectives of Section 7.5:

- Understand topology and classification for the second-order transient circuits
- Convert a transient circuit with a series/parallel LC block to the standard second-order RLC series/parallel transient circuits
- Introduce two major RLC circuit parameters: damping coefficient and undamped resonant frequency
- Introduce the step response of a second-order transient circuit as a solution with a DC source and a switch. Understand the general value of the step response
- Properly select the independent function (capacitor voltage or inductor current) for the standard form of the step response with zero initial conditions

Objectives of Section 7.6:

- Use the method of characteristic equation for second-order transient circuits
- Understand the meaning of overdamped, critically damped, and underdamped circuits
- Use the value of damping ratio ς to distinguish between three different cases of circuit behavior
- Obtain the complete analytical solution for the step response of the RLC circuit
- Apply this solution for modeling a nonideal (realistic) digital waveform

Application Examples:

- Electromagnetic railgun
- Electromagnetic material processing
- Digital memory cell
- Laboratory ignition system
- RC timer or clock circuit in laboratory
- Transient circuit with a bypass capacitor
- Modeling and origin of the nonideal digital waveform

Keywords:

Transient RC circuit, Transient RL circuit, Energy-release RC/RL circuit, Energy-accumulating RC/RL circuit, Time constant of RC circuit, Time constant of RL circuit, Relaxation time, Voltage continuity across the capacitor, Fluid mechanics analogy of transient RC circuit, Lorentz force, Self-induced Lorentz force, Railgun, Electromagnetic material processing, Electromagnetic forming, Current continuity through the inductor, Fluid mechanics analogy of transient RL circuit, Forced response, Electronic ignition system, Piezoelectric effect, Clock frequency, Clock signal, Positive feedback, Linear oscillators, Switching oscillators, Switching RC oscillator, Astable multivibrator, Relaxation oscillator, Bistable amplifier circuit (operation, threshold voltage, mechanical analogy, triggering, trigger signal), Digital memory element, Inverting Schmitt trigger, Non-inverting Schmitt trigger, 555 timer IC, Single-time-constant circuits (definition, classification of, examples of, with general sources), STC circuits, Non-STC circuits (definition, examples of), Series RLC circuit (generic representation, qualitative description, mechanical analogy, step response, duality), Parallel RLC circuit (generic representation, qualitative description, mechanical analogy, step response, duality), Second-order ODE (homogeneous, nonhomogeneous, initial conditions, in terms of current, in terms of voltage, forcing function, general solution, forced response, particular solution, complementary solution natural response, step response), Damping coefficient, Neper, Time constant of the decay envelope, Undamped resonant frequency, Step response, Impulse response, Damping ratio, Natural frequency, Overdamped circuit, Critically damped circuit, Underdamped circuit, Overshoot, Undershoot, Rise time, Fall time, Ringing, Nonideal digital waveform

Section 7.1 RC Circuits

The first-order RC circuits explored in this section involve the process of discharging or charging a capacitor. This is a time-dependent, or transient, circuit behavior, and to understand it, we are required to solve dynamic circuit equations. Mathematically, this implies the solution of first-order ordinary differential equations (ODEs) with time as one independent variable. Fortunately, KVL and KCL remain valid for any static or dynamic circuit. These laws can be employed to derive the circuit equations. After that, it is either solved analytically for simple circuits or numerically for realistic RC circuits.

7.1.1 Energy-Release Capacitor Circuit

The circuit in Fig. 7.1 depicts a capacitor, *C,* that has been charged to a certain voltage

$$V_0 = \upsilon_C(t \le 0) \tag{7.1}$$

prior to use. Through a switch, the capacitor is connected to a load, represented by a resistor $R = 10\ \Omega$. The switch shown in Fig. 7.1 may be a transistor switch. We assume that the switch closes and thereby connects the load to the capacitor at $t = 0$. Our goal is to find all circuit parameters, plus the power delivered to the load as functions of time.

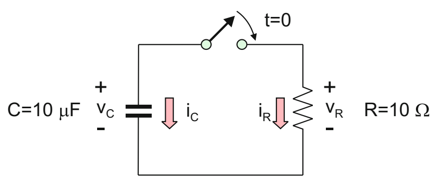


Fig. 7.1. Discharging a capacitor through a load resistor.

The solution to this dynamic circuit is based on applying KVL and KCL, which are valid for all electric circuits. Using KCL gives the result:

$$i_C = -i_R \tag{7.2}$$

at *any* instance of time, *t*. Since both circuit elements in Fig. 7.1 are passive, we can apply the constitutive relations between currents and voltages without changing the sign:

$$\overbrace{C\frac{d\upsilon_C}{dt}}^{i_C} = -\overbrace{\frac{\upsilon_R}{R}}^{i_R} = -\frac{\upsilon_C}{R} \tag{7.3}$$

This is true because KVL states for any *positive* time, $t > 0$,

$$\upsilon_R = \upsilon_C \tag{7.4}$$

Equation (7.3) therefore yields

$$C\frac{d\upsilon_C}{dt} + \frac{\upsilon_C}{R} = 0 \Rightarrow \frac{d\upsilon_C}{dt} + \frac{\upsilon_C}{\tau} = 0, \quad \tau = RC \tag{7.5}$$

This is the famous first-order *transient* circuit equation. Here, τ carries units of seconds since R is recorded in Ω and C is given in F = A × s/V and is called the *time constant* or the *relaxation constant* of the circuit. It is the *only* constant that is present in the first-order differential equation. The solution of an ODE of this type has the generic form

$$\upsilon_C(t) = K\exp\left(-\frac{t}{\tau}\right) \tag{7.6}$$

This fact is proven by direct substitution. The constant K is determined from the initial condition, Eq. (7.1), which yields

$$K = V_0 \tag{7.7}$$

Thus, the circuit voltages have the same form

$$\upsilon_C(t) = \upsilon_R(t) = V_0\exp\left(-\frac{t}{\tau}\right); \quad t \geq 0 \tag{7.8a}$$

for nonnegative values of t. However, although the capacitor voltage is equal to V_0 at $t < 0$, the resistor voltage is exactly zero at $t < 0$, since the switch was open. The current through the load resistor is

$$i_R(t) = \frac{\upsilon_R(t)}{R} = \frac{V_0}{R}\exp\left(-\frac{t}{\tau}\right); \quad t \geq 0 \tag{7.8b}$$

and is zero for negative t. We recall that the capacitor current is the negative of the load current. The instantaneous power delivered to the load resistance is expressed in the form

$$p_R(t) = \upsilon_R(t)i_R(t) = \frac{{V_0}^2}{R}\exp\left(-2\frac{t}{\tau}\right); \quad t \geq 0 \tag{7.8c}$$

Equations (7.8a–c) provide the complete solution of the circuit shown in Fig. 7.1. What is the most remarkable and perhaps most important property of the solution? The answer to this question is linked to the amount of power that can be discharged in a finite amount of time. Let us examine Eq. (7.8c) more closely. When the load resistance, R, becomes small, the delivered power can reach an arbitrarily high value at small positive t. Expressed in another way, when discharged through a small resistance, the (ideal) capacitor delivers an extremely high power pulse during a short period of time!

This conclusion is not affected by the specific capacitance value; the capacitance value only affects the discharge duration. In reality, however, an infinitely small resistance cannot be achieved. How can we use the ability of the charged capacitor to create a large current and, consequently, supply a large power for a short period of time? There are a number of well-known applications such as an electronic photoflash or drivers for the light-emitting diodes (LEDs) or even *electromagnetic material processing*.

7.1.2 Time Constant of the RC Circuit and Its Meaning

To appreciate the value of the time constant as a fundamental property of an *RC* circuit, we consider two examples with explicit component values. Our objective is to find the dynamic voltage and current responses of the circuit as the capacitor discharges. You should note that the time constant τ determines the duration at which the capacitor voltage will have dropped to $1/e$ or 0.368 (36.8 %) of the initial voltage V_0. This number arises from the fact that, at the time instance $t = \tau$, we obtain from Eq. (7.8a)

$$v_C(t) = v_R(t) = V_0 \exp\left(-\frac{\tau}{\tau}\right) = V_0 e^{-1} = V_0/e = 0.368 \ \ V_0 \tag{7.9a}$$

It is sometimes useful to study the dynamic response at $t = 2\tau$, in which case we obtain

$$v_C(t) = v_R(t) = V_0 e^{-2} = 0.135 \ \ V_0 \tag{7.9b}$$

or 13.5 % of its original value, V_0. At 3 τ, we already see the voltage drop less than 5 %.

Example 7.1: In Fig. 7.1, a 10-μF capacitor discharges into a 10-Ω load. The capacitor is initially charged to $V_0 = 10$ V. Plot the capacitor voltage v_C, load current i_R, load voltage v_R, and load power p_R, over the interval from −0.2 ms to 0.5 ms.

Solution: First, we determine the time constant τ. According to Eq. (7.5),

$$\tau = RC = 10^{-5}\text{F} \times 10\,\Omega = 10^{-4}\text{s} = 0.1 \ \ \text{ms} \tag{7.9c}$$

The solution then relies on Eqs. (7.8a) through (7.8c) based on $V_0 = 10$ V. Figure 7.2 shows the behavior of voltage, load current, and load power. The vertical line is the time constant τ. This constant determines how fast the capacitor discharges. At $t = \tau$, the voltage is equal to $1/e$ or 0.368 of the initial capacitor voltage, V_0. Note that a rather low capacitance value of 10 μF is used. We can purchase a 10-μF electrolytic capacitor of 5 mm diameter and 12 mm height and rated at 25 V or 50 V. As seen in Fig. 7.2c, an appreciable load power of 10 W (!) can be created. Unfortunately, it is created for only a very short period of time, on the order of τ.

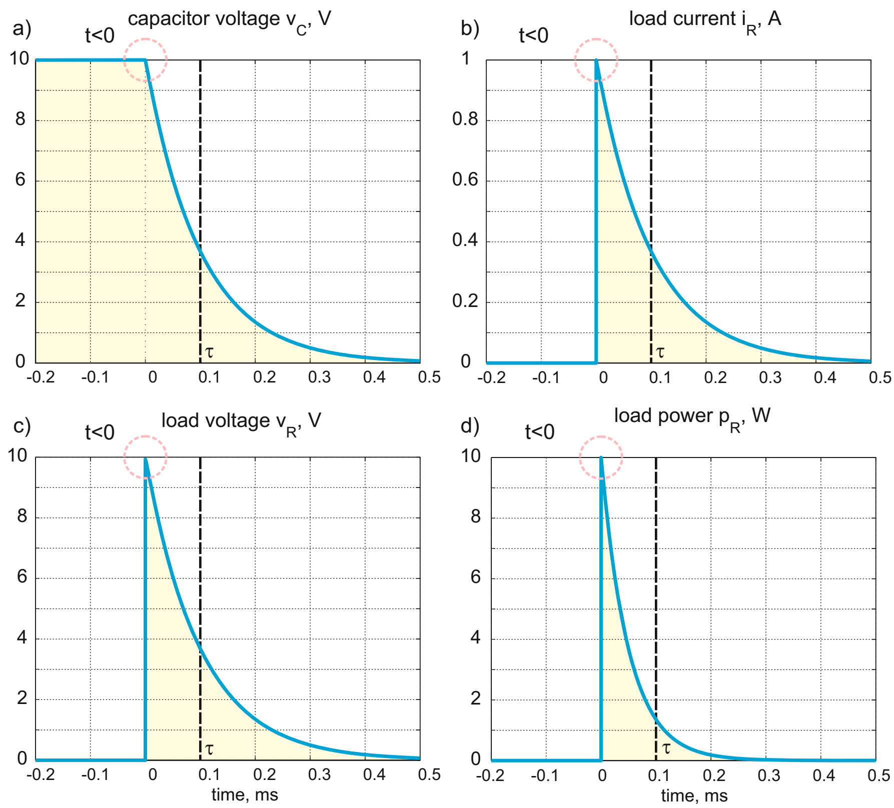


Fig. 7.2. (**a**) Capacitor voltage, (**b**) load current (**c**) load voltage, and (**d**) load power for a 10-μF capacitor discharging into a 10-Ω load resistor.

Exercise 7.1: The capacitor in Fig. 7.1 is initially charged to $V_0 = 20$ V. Determine the capacitor voltage and the instantaneous power delivered to the load resistance at (i) $t = 50$ μs and (ii) $t = 1$ ms.

Answer: (i) –12.13 V and 1472 W; (ii) –0.9 mV and 8.2 μW.

7.1.3 Continuity of the Capacitor Voltage

Energy Consideration

The voltage across the discharging capacitor remains a *continuous* function of time over the breakpoint $t = 0$. On the other hand, *all* other quantities in Fig. 7.2 such as the circuit current, the load voltage, and the load power are subject to a sudden jump when the switch closes. Why is that so? The electric field energy stored in the capacitor is given by

$$E_C = \frac{1}{2} C v_C^2(t) \tag{7.10a}$$

at any time instant. Any energy cannot be released instantaneously. For instance, a vehicle with mass m and speed υ possesses the kinetic energy

$$E_{\mathrm{T}} = \frac{1}{2} m\upsilon^2 \tag{7.10b}$$

and cannot be stopped instantaneously. The kinetic energy must be a continuous function of time, as does the vehicle speed. Similarly, the capacitor energy must be a continuous function of time and so does the capacitor voltage. Such an effect might be called the "capacitor inertia" in reference to mechanical inertia. Thus, the capacitor voltage $\upsilon_{\mathrm{C}}(t)$ is the *only* variable which is always a continuous function of time in an RC circuit. Therefore, it must be used as an *independent* function in the ODEs for RC circuits. Using any other function (circuit current or resistor voltage) is prohibited since we cannot specify the initial conditions for a noncontinuous function.

Fluid Mechanics Analogy

The continuity of the capacitor voltage may be illustrated by a fluid-flow analogy of the discharging capacitor shown in Fig. 7.3. The voltage corresponds to the fluid level in the water-filled tank, which gradually decreases, but cannot jump instantaneously. On the other hand, the fluid acquires a certain velocity (the equivalence to electric current) immediately after the switch in Fig. 7.3 opens. Interestingly, the value of the load resistance in Fig. 7.1 is the reciprocal of the cross section of the pipe in Fig. 7.3. The smaller the cross section of the pipe (i.e., the greater the resistance), the slower the observed fluid flow from the tank (i.e., the smaller load current). At the same time, the leakage time (or the discharge time) increases accordingly. In Fig. 7.3, we actually need to *open* the mechanical valve, whereas in Fig. 7.1 we *close* the electric switch. There is no real contradiction though since both operations really enable the flow of a substance: either the flow of electric current in Fig. 7.1 or the water flow in Fig. 7.3.

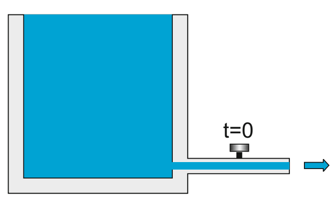


Fig. 7.3. A fluid-flow analogy for the circuit shown in Fig. 7.1.

7.1.4 Application Example: Electromagnetic Railgun

Figure 7.4 shows a generic structure of an electromagnetic accelerator, sometimes called an electromagnetic *railgun*. Apart from potential high-power applications, this setup helps us to visualize the operation of linear motors and generators. The discharging capacitor is connected via a resistor to the rails, as shown on the left in Fig. 7.4. Resistor *R* models ohmic losses in the metal rails and a (typically small) series resistance of the capacitor. The capacitor current flows through two *metal* rails and through a sliding or rolling *metal* rod to be accelerated. Also shown in Fig. 7.4 are two permanent magnets responsible for creating a magnetic flux emanating from the north pole (N) and terminating at the south pole (S). When this perpendicular magnetic flux density B, measured in tesla (T), is applied between the rails, the *Lorentz force* will act on the moving object of length $l = \left|\vec{l}\right|$ and accelerate this object in the direction of the rails. This force is given by

$$\vec{F} = i_C\left(\vec{l} \times \vec{B}\right) \quad [\mathrm{N}] \tag{7.11}$$

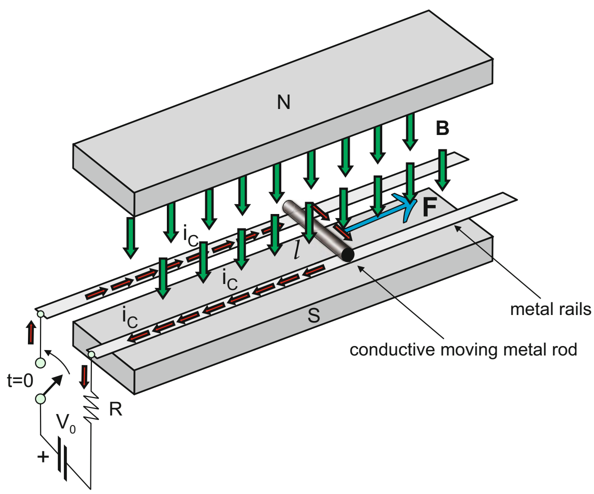


Fig. 7.4. An electromagnetic accelerator based on the Lorentz force effect in the magnetic field created by two permanent magnets and the capacitor discharge current.

The Lorentz force was named after Hendrik Antoon Lorentz (1853–1928), a Dutch physicist and Nobel Prize laureate. Only 24 years of age, Lorentz was appointed to the newly established chair in theoretical physics at the University of Leiden, the oldest university in the Netherlands founded by William, Prince of Orange. Lorentz made significant contributions to field theory ranging from hydrodynamics to general relativity. The Lorentz force is the driving force of any electric motor you are using. The cross product in Eq. (7.11) is consistent with the right-hand rule: the current direction of the moving object represents the fingers of your right hand, and they are turned into the magnetic flux direction so that the thumb points in the direction of the Lorentz force. Another way is to picture a screw whose body points along the force and which is turned in the plane spanned by $\vec{l}$ and $\vec{B}$ such that $\vec{l}$ is rotated into $\vec{B}$. From a circuit point of view, the construction in Fig. 7.4 could be replaced by an RC circuit, with the capacitor connected to the rails through a switch. Resistance R in Fig. 7.4 is formed by the rail resistance combined with the object's resistance and with the contact resistance between the object and the rails. The force is directly proportional to the discharge current i_C.

Example 7.2: The capacitor in Fig. 7.4 is initially charged to a voltage $V_0 = 100$ V and has a capacitance of 1000 μF. The total system resistance R is 1 Ω. For the above example, what force in N is to be expected and for how long?

Solution: According to Eq. (7.8b), the maximum current value, which occurs when the switch has just closed, is $V_0/R = 100$ A. Using Eq. (7.11), we calculate the initial force value, which is 0.6 N.

As time progresses, the current and the force both quickly decrease. Over a time duration of $\tau = RC = 1\,\text{ms}$, both of these values decrease to 36.8 % of their initial values. For simplicity, we assume that an average force acts over the time duration τ. Its value is estimated as approximately 60 % of the initial force value. We then obtain an average of 0.36 N over the time interval $\tau = 1$ ms, which is a rather modest result. Realistic capacitors used for electromagnetic (EM) acceleration are the so-called pulsed capacitors. They have a high charge voltage of $V_0 \geq 10{,}000$ V and capacitances on the order of 100 μF. Therefore, a high-voltage power supply is needed. A number of capacitors are put in parallel to increase the overall capacitance. Large currents, on the order of 10,000 A, into the 1-Ω load may then produce much higher force values.

7.1.5 Application Example: Electromagnetic Material Processing

Electromagnetic Forming

The moving object in Fig. 7.4 may be implemented in various forms. For example, it could be replaced by a liquid metal such as molten aluminum. In principle, an electromagnetic "die casting" machine could be constructed that creates a high-speed liquid metal jet. The key point is the small mass of the object in order to enable a fast acceleration. *Electromagnetic forming* is used to accelerate solid metal sheets at velocities up to a few hundred meters per second, which are 100–1000 times greater than the deformation rates of conventional forming such as sheet metal stamping. The noncontact electromagnetic forming of metals is a process that has been applied since the 1960s but has not seen extensive use. Its common application is to expand, or compress, axisymmetric metal parts as shown in Fig. 7.5a. It has been commercially applied for the joining and assembly of concentric parts and compression crimp seals. Figure 7.5b shows a more recent experiment at the Ohio State University and made with aluminum car door panels.

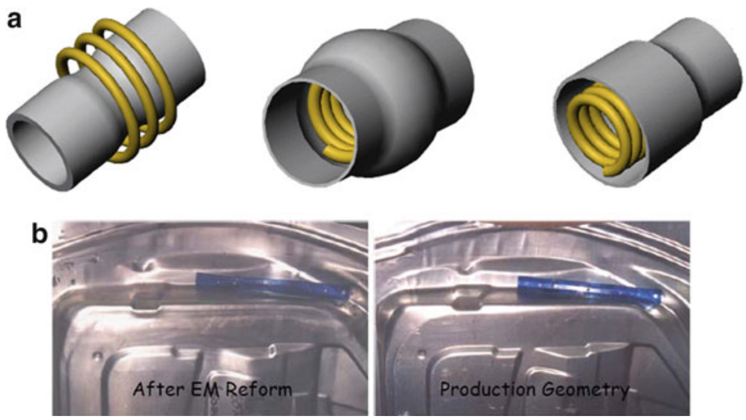


Fig. 7.5. (**a**) Electromagnetic forming of metal joints. The current in the windings generates a magnetic flux that induces eddy currents in the metal. Their product is the Lorentz force. (**b**) Electromagnetically reformed door panel compared with the production geometry.

Self-Induced Lorentz Force

Electromagnetic forming processes shown in Fig. 7.5 do not use permanent magnets. Instead, the so-called *self-induced* Lorentz force is employed. The idea is to generate the magnetic flux B with the same current i_C. Figure 7.5a shows the related concept used in noncontact electromagnetic forming of metal joints. The high discharge current, i_C, creates a strong time-varying magnetic field, both inside and outside of the coils in Fig. 7.5a. In turn, the time-varying magnetic field induces so-called eddy currents in the metal sample. The product of these eddy currents and the magnetic field gives rise to a Lorentz force according to Eq. (7.11). This force is strong enough to deform the joints.

7.1.6 Application Example: Digital Memory Cell

This completely different example investigates a digital circuit that stores binary information. Figure 7.6 shows a schematic of a *dynamic random-access memory* (DRAM) memory cell. The cell stores its bit of information as charge deposited on the cell capacitor C. When the cell is storing a logic 1, the capacitor is charged to a positive voltage V_0; when a logic zero is stored, the capacitor is discharged to a zero voltage. Because of leakage effects, there is always a nonzero resistance R to ground (not shown in the figure). Thus, the cell circuit becomes that of Fig. 7.1. The capacitor will discharge and must be refreshed periodically. During refresh, the capacitor voltage is restored to V_0 if necessary. The refresh operation is in fact performed every 5–10 ms!

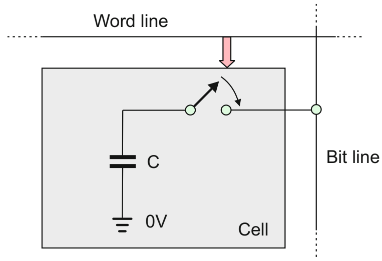


Fig. 7.6. Dynamic RAM memory cell. The bit line carries either logic 1 or 0 information.

7.1.7 Energy-Accumulating Capacitor Circuit

Charging is the inverse process of discharging a capacitor and it involves a power supply. The corresponding circuit is shown in Fig. 7.7. The switch closes at $t = 0$. The resistor R can be either the Thévenin resistance of the practical voltage source or the series parasitic resistance of the capacitor itself or even a combination of both.

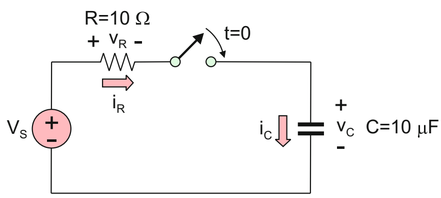


Fig. 7.7. Charging a capacitor with a DC voltage source as an example of another RC circuit.

To support this conclusion, we note that the positions of the switch and the resistor in Fig. 7.7 may be interchanged without affecting the circuit analysis. Similarly, two resistors may be placed on both sides of the switch; the circuit solution will display their series equivalent. The solution to the circuit is once again based on KVL and KCL. By KCL,

$$i_C = i_R \tag{7.12}$$

at *any* time instance t. Since both R and C in Fig. 7.7 are passive elements, we can apply the constitutive relations between currents and voltages without changing the sign:

$$\overbrace{C\frac{dv_C}{dt}}^{i_C} = \overbrace{\frac{v_R}{R}}^{i_R} = \frac{V_S - v_C}{R} \tag{7.13}$$

Next, KVL states

$$v_R = V_S - v_C \tag{7.14}$$

at any *positive* time instance t. Therefore, Eq. (7.13) yields

$$C\frac{dv_C}{dt}+\frac{v_C}{R}=\frac{V_S}{R}\Rightarrow\frac{dv_C}{dt}+\frac{v_C}{\tau}=\frac{V_S}{\tau},\quad \tau=RC \tag{7.15}$$

Equation (7.15) has now an excitation term (the power supply voltage) on its right-hand side. Consequently, the solution to this equation is called a *forced response*. Equation (7.15) is known as an *inhomogeneous* first-order differential equation. This is in contrast to the *homogeneous* differential equation (7.5). Nonetheless, Eq. (7.15) still remains a first-order transient equation with the same time constant τ. The solution to any first-order ordinary differential equation of that type has the generic form

$$v_C(t)=K_1\exp\left(-\frac{t}{\tau}\right)+K_2 \tag{7.16a}$$

This fact can be checked by direct substitution. The two terms containing the exponential factor will cancel out after differentiation. The remaining terms in Eq. (7.15) yield

$$\frac{K_2}{\tau}=\frac{V_S}{\tau}\Rightarrow K_2=V_S \tag{7.16b}$$

The constant parameter K_1 can be determined from the initial condition, $v_C(t=0)=0$. Since $\exp(0)=1$ in Eq. (7.16a), we conclude

$$K_1+K_2=0\Rightarrow K_1=-V_S \tag{7.16c}$$

Thus, the circuit voltages have the form

$$v_C(t)=V_S\left[1-\exp\left(-\frac{t}{\tau}\right)\right],\quad v_R(t)=V_S\exp\left(-\frac{t}{\tau}\right) \tag{7.16d}$$

The resistor voltage has the same form as the resistor voltage in Eq. (7.8a) for the discharging capacitor. However, the capacitor voltage has not. The capacitor current is

$$i_C(t)=C\frac{dv_C(t)}{dt}=\frac{V_S}{R}\exp\left(-\frac{t}{\tau}\right) \tag{7.16e}$$

It is equivalent to Eq. (7.8b), the discharge current for the RC circuit. What is the most remarkable property of the solution given? According to Eq. (7.16d), we always need a certain amount of time to charge the capacitor. It is clear that this time will be on the order of the time constant τ. Moreover, from a formal point of view, the capacitor voltage will never exactly reach the source voltage (the exact equality only occurs at $t\to\infty$), see Fig. 7.8.

Example 7.3: A 10-μF capacitor in Fig. 7.7 is charged by a 10-V voltage source. The switch closes at $t = 0$, and the system resistance is 10 Ω. Plot capacitor voltage v_C and capacitor current i_C to scale over the time interval from −0.2 ms to 0.5 ms.

Solution: First, we find the time constant τ. According to Eq. (7.15) or (7.5), $\tau = RC = 10^{-5}\text{F} \times 10\ \Omega = 0.1\,\text{ms}$. The solutions for this example are given by Eqs. (7.16d) and (7.16e) with $V_S = 10\,\text{V}$. Solutions for the capacitor voltage and capacitor current are plotted in Fig. 7.29. The vertical line denotes the time constant τ so that you can see how fast the capacitor charges. At one time constant, i.e., $\tau = 0.1$ ms, the capacitor is charged to $(1 - 1/e)V_S$ or to 63.2 % of the source voltage V_S.

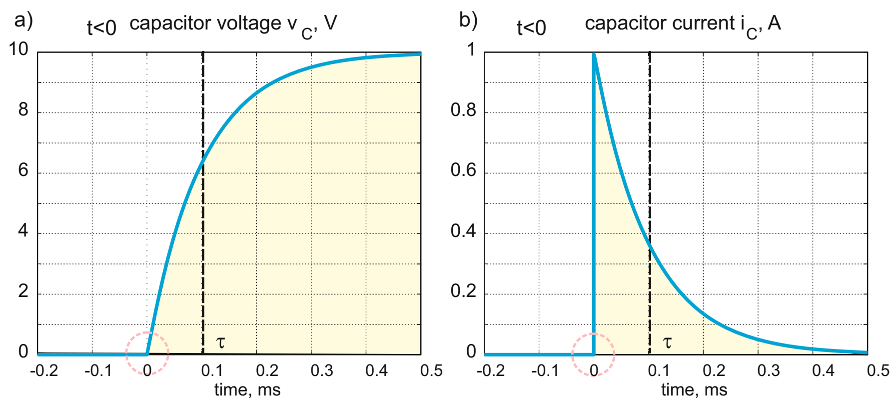


Fig. 7.8. Voltage/current plots for charging a 10-μF capacitor in series with a 10-Ω resistor. We again observe continuity of the capacitor voltage (the "capacitor inertia").

Exercise 7.2: The source voltage in Fig. 7.7 is 20 V. Determine the capacitor voltage and the circuit current at (i) $t = 5$ μs and (ii) $t = 1$ ms.

Answer: (i) –0.98 V and 1.90 A; (ii) –19.999 V and 90 μA.

Section 7.2 RL Circuits

The first-order *RL* circuits studied in this section lay the foundation of understanding the behavior of any transient circuit containing inductances. The transient circuit response is a dynamic process; we are once again required to solve dynamic circuit equations that can be formulated as ordinary differential equations. You can again rely on KVL and KCL since they apply to any static (DC) or dynamic (transient and AC) circuit. Along with the inductor's voltage/current relation, they are used to derive the circuit ODE.

7.2.1 Energy-Release Inductor Circuit

The inductor stores the magnetic-field energy created by an electric current. Thus, in order to be "charged," the inductor must carry some current. A natural choice is therefore a circuit with the *current source* shown in Fig. 7.9a. This is in contrast to the charged capacitor, which does not need a voltage supply to stay charged. If the switch in Fig. 7.9a is open ($t < 0$), the entire current I_S flows through the inductor. The inductor is thus "charged." When the switch closes at $t = 0$, the current source still generates the same current I_S at its terminals. However, the supply is now shorted out, i.e., no current flows into the circuit. In other words, the current supply is effectively disconnected so that the RL circuit becomes a stand-alone circuit in Fig. 7.9b, with the initial current I_S still flowing in the inductor. As time progresses, the inductor releases its energy to the load.

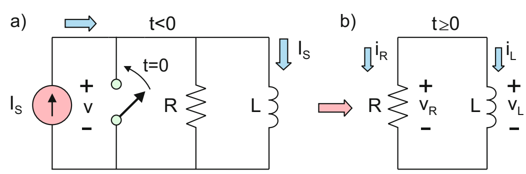


Fig. 7.9. The concept of "discharging" the previously charged inductor through a load resistor *R*.

The solution of the dynamic circuit in Fig. 7.9b is again based on KVL and KCL. With the voltage polarities shown in Fig. 7.9b, the use of KVL yields

$$v_L = v_R \tag{7.17}$$

at *any* time instance, *t*. Since both circuit elements in Fig. 7.9b are passive, we can directly apply the constitutive relations between voltages and currents without changing the sign:

$$\overbrace{L\frac{di_L}{dt}}^{v_L} = \overbrace{Ri_R}^{v_R} = -Ri_L \tag{7.18}$$

According to KCL, $i_R = -i_L$, at any *positive* time instance t, and Eq. (7.18) yields

$$L\frac{di_L}{dt} + Ri_L = 0 \Rightarrow \frac{di_L}{dt} + \frac{i_L}{\tau} = 0, \quad \tau = L/R \tag{7.19}$$

Here again we encounter a first-order transient equation. The constant τ carries the unit of seconds (since R is in ohms and L is in henrys) and is known as the *time constant*, or *relaxation constant,* of the *RL* circuit. It is the only constant that is present in the first-order transient differential equation. We can observe a remarkable similarity between the transient *RL* circuit and the RC circuit of discharging a capacitor. The mathematics is the same, but the capacitor voltage is replaced by the inductor current, and the value of the time constant changes from *RC* to *L/R*. The initial condition $i_L(t=0) = I_S$ includes the past inductor current instead of the past capacitor voltage. The solution of Eq. (7.19) is

$$i_L(t) = K\exp\left(-\frac{t}{\tau}\right), \quad t \geq 0 \tag{7.20}$$

The validity of Eq. (7.20) is seen by direct substitution. The constant K is determined from the initial condition. Setting $t = 0$ yields $K = I_S$. Both currents in Fig. 7.9b are

$$i_L(t) = -i_R(t) = I_S\exp\left(-\frac{t}{\tau}\right) \quad t \geq 0 \tag{7.21a}$$

At $t < 0$, the inductor current maintains its value I_S but the resistor current is zero, as shown in Fig. 7.9a. The resistor (or load) voltage is given by

$$v_R(t) = Ri_R(t) = -RI_S\exp\left(-\frac{t}{\tau}\right) \tag{7.21b}$$

At $t < 0$, the load voltage is zero. The instantaneous power delivered to the load is

$$p_R(t) = v_R(t)i_R(t) = RI_S^2\exp\left(-2\frac{t}{\tau}\right) \tag{7.21c}$$

Equations (7.21a)–(7.21c) provide the complete solutions for the circuit depicted in Fig. 7.9.

Example 7.4: A 1-mH inductor in Fig. 7.9 is connected to a 1-kΩ load. The supply current (disconnected at $t = 0$) is 1 A. Plot inductor current i_L, inductor (or load) voltage v_R, load current, i_R, and load power, p_R, over the interval from –2τ to 5τ.

Example 7.4 (cont.):

Solution: First, we find the time constant τ. According to Eq. (7.19),

$$\tau = L/R = 10^{-3}\text{H}/1000\ \ \Omega = 10^{-6}\text{s} = 1\ \ \mu\text{s} \tag{7.22}$$

which is a rather small value. The solution for this example is given by Eqs. (7.21a) through (7.21c), with $I_S = 1$ A, and is shown in Fig. 7.10. The vertical line in all plots is the time constant τ. One can see that the time constant determines how quickly the inductor current and the load voltage decrease. At 1τ, the load voltage is equal to $1/e$ or 0.368 (36.8 %) of the initial voltage RI_S.

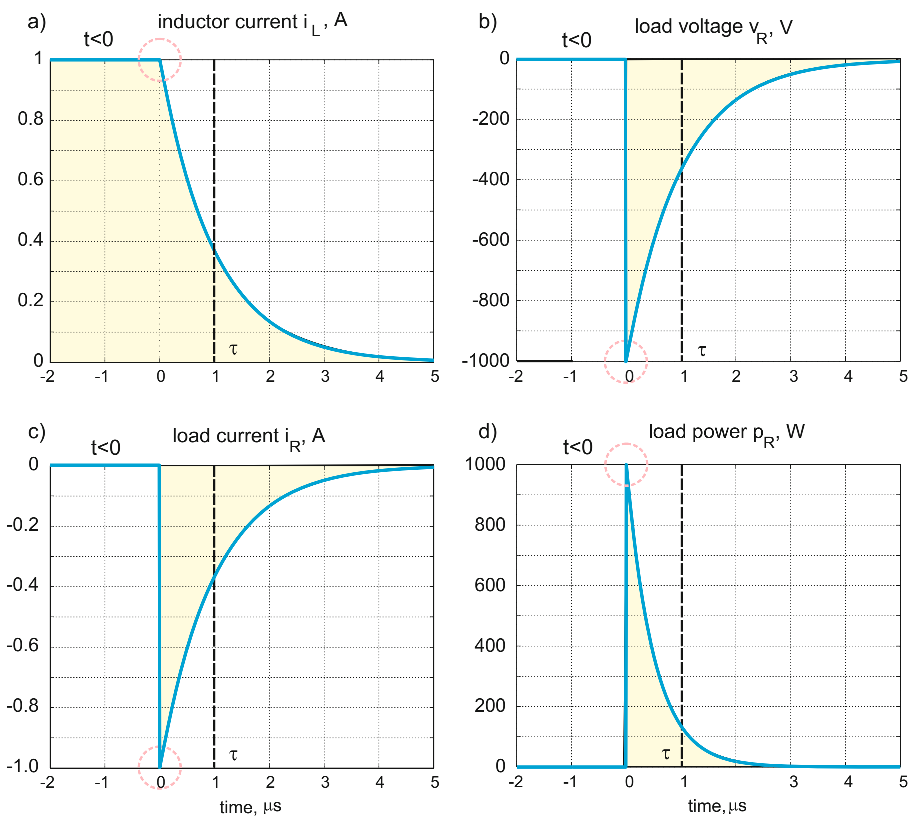


Fig. 7.10. (**a**) Inductor current, (**b**) load voltage, (**c**) load current, and (**d**) load power for a 1-mH inductor connected to a 10-Ω load resistor.

Exercise 7.4: The supply current in Fig. 7.9 is 0.5 A. Given $L = 1$ mH and $R = 1$ kΩ, determine the load voltage and the instantaneous power delivered to the load resistance at (i) $t = 0.2$ μs and (ii) $t = 10$ μs.

Answer: (i) –409.4 V and 167.6 W; (ii) –23 mV and 0.52 μW.

What is the most remarkable property of the solution? According to Fig. 7.10, the high voltage spike across the inductor is created in an RL switching circuit when the load resistance, R, is large. For example, an air gap has a very high resistance. When used as a load, it may possess a very high voltage drop of several kV and more. This is the idea behind any medium-to-high-power *electronic ignition* system, including the 12-V-powered car ignition system, a missile ignition system, etc. Such a circuit must include at least three basic elements: (a) a voltage or current power supply, (b) a switch, and (c) an inductor (coil). The switch can be a transistor switch controlled by a sensor.

7.2.2 Continuity of the Inductor Current

The current through the inductor remains a *continuous* function of time over the breakpoint $t = 0$. However, *all* other quantities in Fig. 7.10, the load voltage, the load current, and the load power are subject to a sudden jump when the switch closes. The reason for such a continuity is the *finite* magnetic-field energy stored in the inductor:

$$E_L = \frac{1}{2} L i_L^2(t), \tag{7.23}$$

at any time instant. This energy cannot be released instantaneously. Such an effect might be called "inductor inertia" in reference to mechanical inertia of a vehicle with mass m and speed v and kinetic energy $E_T = 0.5\,mv^2$, which cannot be stopped instantaneously. The kinetic energy is a continuous function of time, as is the vehicle speed. Similarly, the inductor energy is a continuous function of time, as is the inductor current. The inductor current is the only variable which is always a continuous function. Therefore, it must be used as an *independent* function in the ODEs for RL circuits. Using any other function is prohibited since we cannot state the initial conditions for a noncontinuous function.

Fluid Mechanics Analogy

The continuity of the inductor current may be illustrated by a fluid-flow analogy of the energy-releasing inductor circuit shown in Fig. 7.11. The electric current corresponds to the velocity of the fluid. The inductance is a massive wheel of mass m. When subject to a DC current (constant water flow), it acquires a certain angular velocity. This is the case of Fig. 7.9a at $t < 0$. When the water pump is suddenly turned off, the wheel inertia will still support the same water flow, at least at the initial time moment. After that, the wheel slowly decelerates. This is the case of Fig. 7.9b.

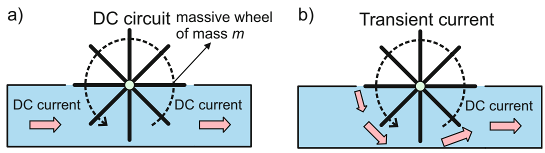


Fig. 7.11. A fluid-flow analogy for the two circuits shown in Fig. 7.9.

7.2.3 Energy-Accumulating Inductor Circuit

The circuit behavior is now exactly the opposite of the circuit shown in Fig. 7.9. For instance, when the switch in Fig. 7.12 is closed, the current supply is shorted out. No current flows into the circuit. However, when the switch opens, the supply current I_S starts to flow into the circuit, and as time approaches infinity, the entire supply current I_S flows through the inductor. Thus, the inductor becomes "charged."

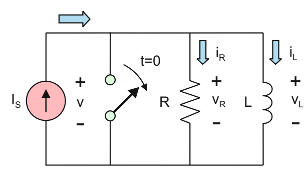


Fig. 7.12. The concept of "charging" an inductor using a current source.

The solution of the dynamic circuit in Fig. 7.12 is based on KVL and KCL. With the voltage polarities in Fig. 7.12, the use of KVL yields

$$v_L = v_R \tag{7.24}$$

at *any* time instance, t. Since both circuit elements in Fig. 7.12 are passive, we can apply the constitutive relations between voltages and currents without changing the sign:

$$\overbrace{L\frac{di_L}{dt}}^{v_L} = \overbrace{Ri_R}^{v_R} = RI_S - Ri_L \tag{7.25}$$

because, according to KCL, $i_R = I_S - i_L$, at any *positive* time instance, t. Equation (7.25) yields

$$\frac{di_L}{dt} + \frac{i_L}{\tau} = \frac{I_S}{\tau}, \quad \tau = L/R \tag{7.26}$$

which is the inhomogeneous first-order transient equation with the forcing function (right-hand side) equal to I_S/τ. Here, τ is the generic time constant of the *RL* circuit. Once again, there is a close similarity between the present *RL* circuit and the series *RC* circuit for charging the capacitor. The mathematics is the same, but the capacitor voltage is replaced by an inductor current and the voltage supply is replaced by the current supply.

The initial condition for Eq. (7.26) now includes the past inductor current of 0 A instead of the past capacitor voltage, which is 0 V. Equation (7.26) has the solution

$$i_L(t) = K_1 \exp\left(-\frac{t}{\tau}\right) + K_2, \quad K_2 = I_S \tag{7.27}$$

This fact is seen by direct substitution. The constant K_1 is found from the initial condition of zero inductor current at $t = 0$, which yields $K_1 = -K_2$. Therefore,

$$i_L(t) = I_S\left[1 - \exp\left(-\frac{t}{\tau}\right)\right], \quad i_R(t) = I_S \exp\left(-\frac{t}{\tau}\right), \quad t \ge 0 \tag{7.28a}$$

Both currents are zero at $t < 0$. However, the inductor current is continuous over the breakpoint while the resistor current is not. The inductor/resistor voltages are given by

$$\upsilon_L(t) = L\frac{di_L(t)}{dt} = RI_S \exp\left(-\frac{t}{\tau}\right), \quad \upsilon_R(t) = \upsilon_L(t) \quad t \ge 0 \tag{7.28b}$$

Both voltages are zero at $t < 0$. This completes our circuit analysis.

Example 7.5: A 1-mH inductor in Fig. 7.12 is connected to a 1-A current power supply. The resistor value is $R = 1\,\mathrm{k\Omega}$. Plot the inductor current, i_L, and the inductor voltage, υ_L, to scale versus time over the interval from -2τ to 5τ.

Solution: First, we find the time constant τ. According to Eq. (7.26), we get $\tau = L/R = 10^{-3}\mathrm{H}/1\,\mathrm{k\Omega} = 10^{-6}\mathrm{s} = 1\,\mathrm{\mu s}$. The solution to the example is given by Eqs. (7.28a, b) with $I_S = 1$ A; see Fig. 7.13. The vertical line in both plots is the time constant τ. One can see that this time constant determines how fast the circuit stabilizes. At 1τ, the inductor current reaches $(1 - 1/e)I_S$, i.e., 63.2 % of the expected DC value. Note that Fig. 7.13 of this section and Fig. 7.8 of the previous section are identical to within interchanging voltage and current terms!

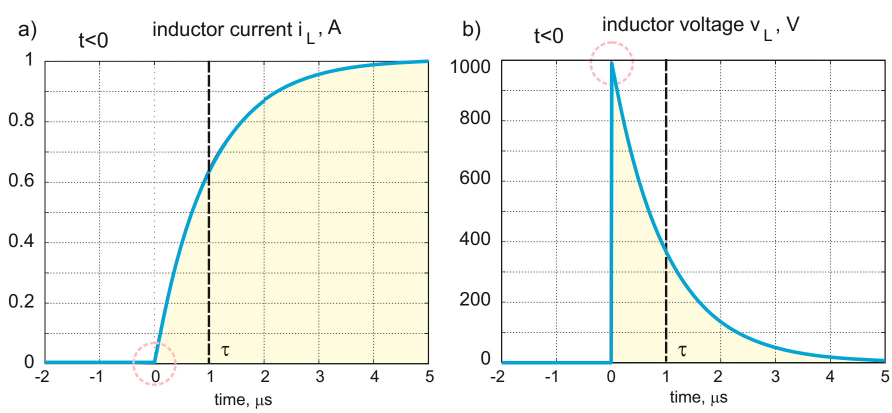


Fig. 7.13. Voltage/current plots for “charging” the 1-mH inductor in parallel with the 1-kΩ resistor.

Exercise 7.5: The supply current in Fig. 7.12 is 0.5 A. Given $L = 1$ mH and $R = 1$ kΩ, determine the inductor voltage and the inductor current at (i) $t = 0.2$ µs and (ii) $t = 10$ µs.

Answer: (i) –409.4 V and 91 mA; (ii) –23 mV and 0.49999 A.

The final question to ask is what is the most remarkable property of the solutions given by Eqs. (7.28a, b)? According to Eq. (7.28a), we always need a certain period of time to create a given current through the inductor. The elapsed time will be on the order of the time constant τ. Moreover, the inductor current will never exactly reach the supply current (the exact equality only occurs as $t \to \infty$). In practice, this effect is masked by noise and by other factors. Interestingly, the resistor carries most of the circuit current when the solution changes rapidly, i.e., close to the initial time $t = 0$. At the same time, when the circuit stabilizes, i.e., when t becomes large compared to τ, the influence of the inductor dominates. This observation leads us to the concept of *impedance* (the "resistance" of dynamic circuit elements) that is considered next. The impedance is similar to a resistance (and has the same unit), but it depends on how fast circuit current and voltage change. When the changes are very fast, the inductor exhibits a much greater "resistance" than the resistor; it becomes virtually an open circuit with no current flow.

7.2.4 Energy-Release RL Circuit with the Voltage Supply

The combination of the current supply I_S and resistor R in Figs. 7.9 and 7.12 is in fact the Norton equivalent circuit of any network of power supplies and resistors. The RL circuit may be modeled a Thévenin equivalent too. The concept is shown in Fig. 7.14.

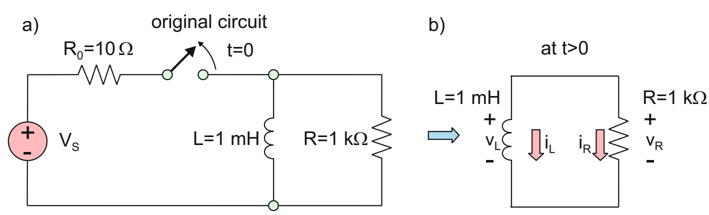


Fig. 7.14. "Discharging" an inductor with a voltage supply, as an example of an RL circuit.

Prior to opening the switch in Fig. 7.14a, the inductor current is found to be V_S/R_0. When the switch opens, the supply is disconnected from the RL circuit; see Fig. 7.14b. As time progresses, the inductor releases its energy into the load. The circuit in Fig. 7.14b is identical to the circuit in Fig. 7.9b. Therefore, all prior results related to the energy-release RL circuit will remain valid if we replace the initial inductor current I_S by V_S/R_0. According to Eq. (7.21b), the load voltage is given by

$$v_R(t) = R i_R(t) = -\frac{R}{R_0} V_S \exp\left(-\frac{t}{\tau}\right) \tag{7.29}$$

If the ratio R/R_0 is large, the initial voltage spike of the inductor is large too. The magnitude of the initial voltage spike for the circuit in Fig. 7.14 is a hundred times the supply voltage V_S! Can we model an *electronic ignition system* in the laboratory? Yes, according to Fig. 7.14 this can be accomplished relatively easily. The key, however, is the construction of a fast switch. A proper choice may be a power transistor switch.

7.2.5 Application Example: Laboratory Ignition Circuit

A circuit rated at 6 V for safety purposes is shown in Fig. 7.15. The laboratory DC voltage source usually delivers up to 3 A of current, if not current limited. The electric step-up transformer with two coils is a 6-V car ignition coil. Instead of a simple coil, a transformer is used to further boost the inductor voltage spike of the *RL* circuit. The small resistance R_0 is the transistor/wire resistance. The very large resistance R is the resistance of the spark plug in series with the large resistance of the spark plug cable R_{CABLE}, which is a carbon core wire. This carbon core wire is used to prevent higher EM radiation and its possible influence (we would hear it as noise) on the car audio receiving equipment. Figure 7.16 shows the operating circuit. The sparking frequency ranges from 2 Hz to 100 Hz.

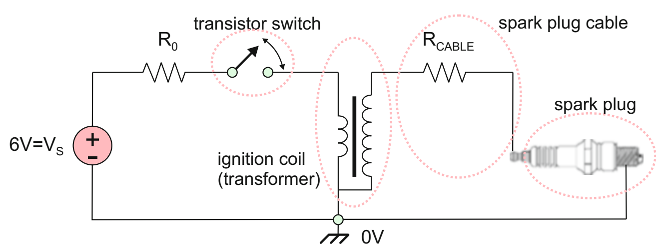


Fig. 7.15. Modeling the ignition system as in a laboratory. The spark plug voltage is about 3–10 kV.

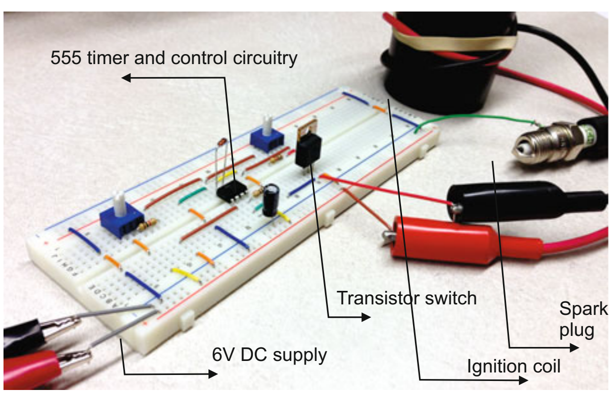


Fig. 7.16. A constructed ignition system.

Spark Gap Radio

The spark gap in the circuit in Fig. 7.15 and in Fig. 7.16 is a powerful and broadband source of electromagnetic radiation. One favorite story in science fiction novels is that, after crashing his or her spaceship on an inhabited planet, the commander can quickly construct a spark generator out of remaining parts of the ship and send an SOS signal out to space. An AM radio can "listen" to the circuit shown in Fig. 7.16 over the entire AM band from 540 to 1610 kHz used in the USA. Even better results are observed for the long-wave AM band from 153 to 279 kHz used in Europe, Africa, and parts of Asia.

Exercise 7.6: The supply voltage in Fig. 7.14 is 10 V. Determine the inductor voltage and the inductor current at (i) $t = 0.2$ μs and (ii) $t = 10$ μs.

Answer: (i) –818.7 V and 0.82 A; (ii) –45 mV and 45 μA.

Section 7.3 Switching RC Oscillator

The time constant τ of an RC circuit provides a natural time scale. It is widely employed for timing purposes. Some of you may have already used microcontroller starter kits. The microcontroller manual discussing the various settings will likely feature a topic entitled "RC oscillator." Here you will discover that the microcontroller *clock frequency* can be controlled by an external resistor R and capacitor C. How is this possible? We have just seen that the RC circuit discharges the capacitor through the resistor, but how can it be used to create a periodic *clock signal* at a given frequency? The present section aims to augment an RC circuit with an amplifier circuit and establish a *clock circuit*.

7.3.1 About Electronic Oscillators

An *electronic oscillator* is a circuit that has an output, but no input in the common sense. It generates a certain periodic waveform at the output. The period, amplitude, and shape of this waveform are determined by the circuit topology. The "heart" of any oscillator circuit is an amplifier block with some sort of a *positive feedback*. The positive feedback is the opposite of the negative feedback. A part of the output amplifier's voltage is fed back into the input with the sign plus. All oscillator circuits may be divided into *linear oscillators*, which create sinusoidal waveforms, and *switching oscillators*, which create square and other periodic nonharmonic waveforms. The subject of this section is a switching oscillator circuit, which is called an *astable multivibrator* or a *relaxation oscillator*. This circuit uses the comparator amplifier but with a positive feedback loop and a transient RC block. It is perhaps the simplest and yet efficient oscillator circuit.

7.3.2 Bistable Amplifier Circuit with the Positive Feedback

Saturation Mechanism

Consider the circuit shown in Fig. 7.17. At first sight, it is similar to the inverting amplifier configuration. However, the amplifier polarity is interchanged, which means that the feedback is now positive. The circuit has no input: both potential inputs are grounded. Since there is no current into the amplifier itself (the first summing-point constraint still applies), two resistors of the feedback loop form a voltage divider between the output voltage υ_{out} and ground. Therefore, the voltage at node (+) becomes

$$\upsilon^+ = \frac{R_1}{R_1 + R_2}\upsilon_{out} \tag{7.30}$$

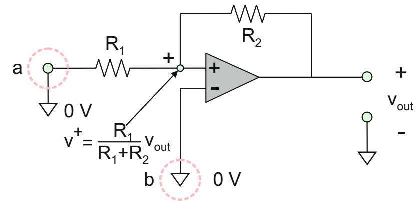


Fig. 7.17. A bistable amplifier circuit.

To analyze the circuit in Fig. 7.17, we again consider the feedback as a fast dynamic process with a very short delay in the feedback loop. We assume that $R_1 = R_2$ for simplicity. It means that 50 % of v_{out} is returned back to the non-inverting input in, say, 1 μs. The open-loop amplifier's gain will be $A = 10^6$; the amplifier hits the power rails $v_{\text{out}} = \pm V_{\text{CC}}$ in saturation. The initial value of v^+ will be 0 V, and the initial value of v_{out} will be 1 μV (at the noise level). Table 7.1 shows the dynamics of the feedback process where the amplifier operates as $v_{\text{out}} = Av^+$, but it takes 1 μs to return 50 % of v_{out}. It follows from Table 7.1 that the amplifier will be very quickly saturated; its output will be the positive rail voltage $v_{\text{out}} = +V_{\text{CC}}$. All other positive initial values of v_{out} will lead to the same result. Simultaneously, all negative initial values of v_{out} will lead to the saturation at the negative rail $v_{\text{out}} = -V_{\text{CC}}$.

Table 7.1. Dynamics of the output voltage for the bistable amplifier circuit.

Time, μs	v^+	v_{out}
0	0 V	10^{-6} V
1	0.5×10^{-6} V	0.5 V
2	0.25 V	$+V_{\text{CC}}$

Two Stable States

The key point is that once the saturation state

$$v_{\text{out}} = +V_{\text{CC}},\ v^+ = +\frac{R_1}{R_1 + R_2} V_{\text{CC}} \tag{7.31}$$

has been reached, the amplifier circuit will exist in this state *indefinitely*, despite all the subsequent electric noise. To prove this fact, we may introduce a small perturbation in v_{out} and/or in v^+; the circuit will quickly return to the solution given by Eq. (7.31). Quite similarly, once the opposite saturation state

$$v_{\text{out}} = -V_{\text{CC}}\quad v^+ = -\frac{R_1}{R_1 + R_2} V_{\text{CC}} \tag{7.32}$$

has been reached, the amplifier circuit will exist in this state *indefinitely*. Thus, the positive feedback always forces the comparator to operate in saturation, i.e., in either of the two *stable* states, $v_{\rm out} = \pm V_{\rm CC}$, where $\pm V_{\rm CC}$ is the supply voltage of the amplifier. This result is valid for any pair of the resistances R_1, R_2. The resistance pair specifies values of v^+ in Eqs. (7.31) and (7.32), respectively, known as *threshold voltages*. Figure 7.18 shows the corresponding *mechanical analogy* of the bistable amplifier circuit. Note that a grounded comparator amplifier *without* the positive feedback loop would also be always saturated due to inherent electric noise. However, there are *no* stable states whatsoever; the switching between the rails is random; it is controlled by random noise.

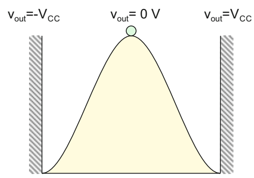


Fig. 7.18. Mechanical analogy of two stable states for the bistable amplifier circuit.

7.3.3 Triggering

The amplifier circuit in Fig. 7.17 can be in either of two stable states defined by the initial conditions. As such, it is useless as long as we do not have a mean to change the state. A *trigger signal* (an input voltage signal) may be applied to switch between the states. After introducing an *external* trigger signal in the form of short pulses, the bistable amplifier circuit becomes a basic *digital memory element* capable of saving and retrieving one bit of data. When the input voltage signal is applied to node (a) in Fig. 7.17 instead of grounding it, the corresponding circuit becomes the *non-inverting Schmitt trigger*. When the input voltage is applied to node (b) in Fig. 7.17 instead of grounding it, the corresponding circuit becomes the *inverting Schmitt trigger*. The Schmitt triggers are used as zero-level detectors in analog electronics and for many other purposes. When triggered, the bistable amplifier circuit operates as a *comparator*.

Exercise 7.7: The bistable amplifier circuit with $R_1 = R_2$ in Fig. 7.17 exists in the positive stable state with $v_{\rm out} = +V_{\rm CC}$. A trigger signal is applied to node (b) in Fig. 7.17. Determine output voltage when the applied trigger signal is (i) $-V_{CC}$, (ii) $+0.4V_{\rm CC}$, and (iii) $+0.6V_{\rm CC}$, where $\pm V_{\rm CC}$ is the supply voltage of the amplifier.

Answer: (i) $+V_{\rm CC}$; (ii) $+V_{\rm CC}$; (iii) $-V_{\rm CC}$.

7.3.4 Switching RC Oscillator

The idea of the *switching RC oscillator* (or the *relaxation oscillator*) is not to use an external trigger, but rather to derive the trigger signal from an RC circuit connected to the output of the amplifier itself. The circuit diagram of the relaxation oscillator is shown in Fig. 7.19. The RC circuit with the source voltage v_{out} forms a negative feedback loop. The capacitor may either charge or discharge depending on the source voltage.

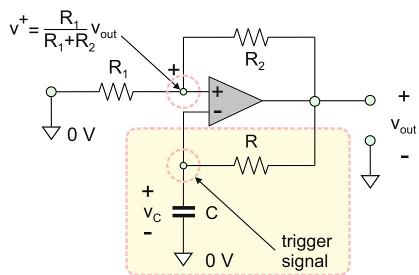


Fig. 7.19. Bistable amplifier circuit with an RC circuit in the negative feedback loop.

To analyze the amplifier circuit in Fig 7.19, we assume an infinitely high open-loop DC gain A and use basic amplifier equations 5.4 with $v^- = v_C$. This yields

$$v_{\text{out}} = +V_{\text{CC}} \quad \text{if } \frac{R_1}{R_1 + R_2} v_{\text{out}} > v_{\text{C}} \tag{7.33a}$$

$$v_{\text{out}} = -V_{\text{CC}} \quad \text{if } \frac{R_1}{R_1 + R_2} v_{\text{out}} < v_{\text{C}} \tag{7.33b}$$

Thus, the circuit in Fig. 7.19 becomes equivalent to a simple RC circuit given in Fig. 7.20 where the dependent (or rather switching) voltage source is defined by Eqs. (7.33a, b). The corresponding transient analysis is performed starting with some initial conditions.

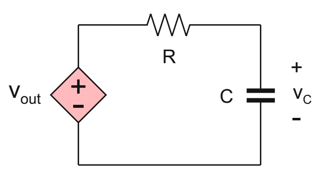


Fig. 7.20. Equivalent circuit for finding oscillation behavior. The dependent voltage source is controlled by the capacitor voltage.

Example 7.6: Given initial conditions $v_{out} = 10$ V and $v_C = 0$ V at $t = 0$, solve the circuit in Fig. 7.19 and Fig. 7.20. Assume that $V_{CC} = 10$ V and $R_1 = R_2$.

Solution: Step 1. The source voltage in Fig. 7.20 is $v_{out} = 10$ V and the capacitor voltage is zero, i.e., $v_C = 0$ V at the initial time moment $t = 0$.

Step 2. The capacitor starts to charge. When its voltage reaches $v_C = 5$ V, the differential voltage at the amplifier's input becomes negative so that Eq. (7.33a) is no longer valid. The source voltage in Fig. 7.20 switches to
$v_{out} = -10$ V according to Eq. (7.33b).

Step 3. The capacitor starts to discharge. When it reaches $v_C = -5$ V, the differential voltage at the amplifier's input becomes positive, and the output voltage therefore switches back to $v_{out} = 10$ V according to Eq.(7.33a).
After Step 3, the circuit returns to Step 1, and the process continues periodically. This results in the output voltage shown in Fig. 7.21.

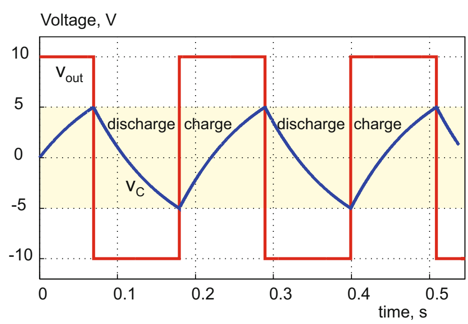


Fig. 7.21. Relaxation oscillator operation for the idealized amplifier model (amplifier hits the rails in saturation); the amplifier supply voltage is 10 V.

Exercise 7.8: The relaxation oscillator circuit in Fig. 7.19 uses $R_1 = 1$ kΩ and $R_2 = 4$ kΩ. The supply voltage of the amplifier is ±15 V. Determine the amplitude (peak value) of the oscillating capacitor voltage and the oscillating output voltage.

Answer: 15 V and 3 V, respectively.

7.3.5 Oscillation Frequency

Consider the positive half cycle in Fig. 7.21. The solution for the RC circuit in Fig. 7.20 is given by Eqs. (7.16a, b) with $V_S = V_{CC}$. The constant K_1 is found from the initial condition of $v_C(t = 0) = -\beta V_{CC}$ where $\beta = R_1/(R_1 + R_2)$ is the amount of the positive feedback. We assume that the initial time instance has been switched to the start of the half cycle. Therefore, during the entire positive half cycle

$$\upsilon_C(t) = V_{CC} - (1+\beta)V_{CC}\exp\left(-\frac{t}{\tau}\right), \quad \tau = RC \tag{7.34}$$

At the end of the positive half cycle, the capacitor voltage becomes $\beta\, V_{CC}$. This allows us to find the half cycle duration $T/2$ and the oscillation period T. Solving Eq. (7.34) with $\upsilon_C = \beta\, V_{CC}$ and $t = T/2$, one has (f is the oscillation frequency in hertz)

$$T = 2\tau \ln\frac{1+\beta}{1-\beta}, \quad f = \frac{1}{T} = \frac{1}{2\tau}\left(\ln\frac{1+\beta}{1-\beta}\right)^{-1} \tag{7.35}$$

Exercise 7.9: For the relaxation oscillator with $R_1 = R_2$, express the oscillation frequency in terms of its time constant τ.

Answer: $f = \frac{0.455}{\tau}$ [Hz].

7.3.6 Circuit Implementation: 555 Timer

Figure 7.22 shows the relaxation oscillator circuit implemented in a laboratory and its output voltages when the supply voltages are ± 5 V. Since the realistic amplifier never reaches the supply rails, the peak-to-peak (Pk-Pk) value of the output voltage is now less than 10 V. The output current limitations of the amplifier IC may severely affect circuit performance. Also, the oscillation frequency only *approximately* follows Eq. (7.35).

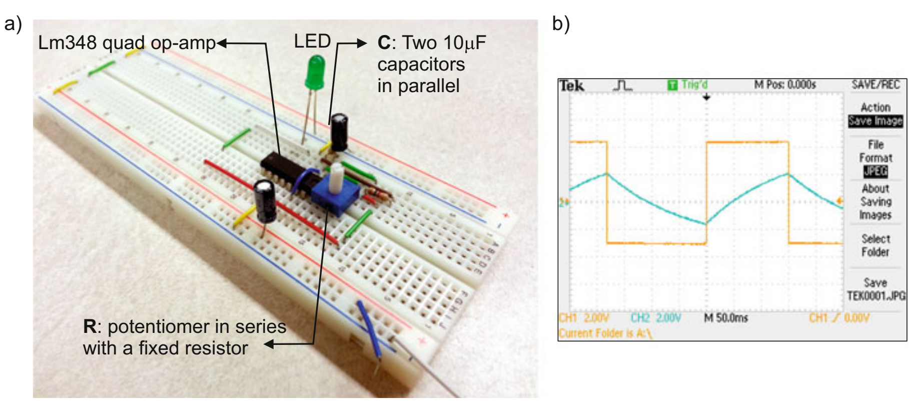


Fig. 7.22. (a) Timer circuit operation in laboratory. The square waveform is the output voltage $\upsilon_{out}(t)$. The curved waveform is the capacitor voltage $\upsilon_C(t)$. A variable resistance R makes it possible to visually control the oscillation frequency using an LED connected to the output through a *buffer amplifier* . The oscillation frequency changes from 0.5 to about 100 Hz.

555 Timer Integrated Circuit

Rectangular pulse forms at lower frequencies are routinely created by the well-known *555 timer IC* (integrated circuit). The 555 timer operates conceptually similarly to the relaxation oscillator circuit described above; it is more versatile though. The 555 timer creates a waveform of relatively sharp and clean rectangular voltage pulses whose frequency is controlled by an external capacitor and resistor. The *duty cycle* (ratio of the positive phase duration to the signal period) can also be controlled. The 555 timer is perhaps one of the most popular integrated circuits ever built.

Section 7.4 Single-Time-Constant (STC) Transient Circuits

7.4.1 Circuits with Resistances and Capacitances

Consider a transient circuit with an arbitrary number of capacitances and resistances. The circuit has an independent voltage source (or sources) and a switch. Instead of the voltage source, some capacitors may be charged prior to closing the switch. A *single-time-constant transient circuit* (*STC circuit*) is that which solution has the form

$$v(t) = K_1 \exp\left(-\frac{t}{\tau}\right) + K_2 \tag{7.36}$$

for *any* branch voltage in the circuit. In other words, only *one* exponential function is involved, similar to the basic RC circuits studied previously. Here, τ is the only *time constant* of the circuit. The STC transient circuits are frequently encountered in practice, in particular, in the study of transistor amplifiers. The STC transient circuits include:

1. Transient circuits with only one capacitance C. According to Thévenin's theorem, the network of resistances and source(s) seen by the capacitor is reduced to its Thévenin equivalent. As a result, we obtain the circuit shown in Fig. 7.7. Its time constant is given by

   $$\tau = R_{\mathrm{T}} C \tag{7.37}$$

   where R_{T} is Thévenin resistance – the equivalent resistance of the network with the independent voltage source(s) shorted out.
2. Transient circuits with only one resistance R. Thévenin's theorem may be applied again, this time to the network of capacitances and source(s) seen by the resistance. As a result, we again obtain the circuit from Fig. 7.7. Its time constant is given by

   $$\tau = R C_{\mathrm{T}} \tag{7.38}$$

   where C_{T} is the equivalent capacitance of the network with the independent voltage source(s) shorted out.
3. Transient circuits with an arbitrary number of capacitances and resistances given that the solutions for different capacitor voltages obtained by the simultaneous use of KVL and KCL are all *linear functions* of each other. Consider a circuit with multiple capacitances. Assume that N is the final number of capacitances after all possible series/parallel combinations. For the STC condition to hold, there should be $N-1$ independent closed loops that include *only* capacitances (and possibly independent voltage source(s)) but do not include resistances. This useful result has been confirmed by the authors based on an extensive circuit analysis.

As an example, we consider here simple transient circuits shown in Fig. 7.23.

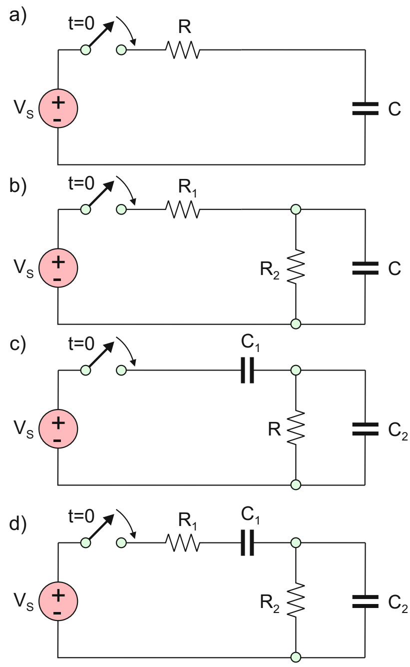


Fig. 7.23. Transient circuits with multiple resistances and capacitances.

Exercise 7.10: For the circuits in Fig. 7.23, establish the number of independent closed loops that include only capacitance(s) and independent voltage source but do not include resistances.

Answer: (a) Zero; (b) zero; (c) one; (d) zero.

The first case in Fig. 7.23 is the transient circuit of Fig. 7.7. The positions of the switch and the resistor may be interchanged without affecting the circuit solution, which has the form of Eq. (7.36)—see also Eqs. (7.16). The second case is again the STC circuit; the solution for the capacitor voltage (and any other voltage in the circuit) is given by Eq. (7.36) with $\tau = \left(R_1 \middle|\middle| R_2\right)C$. The constants K_1 and K_2 in this equation will be different for voltages across different circuit elements. The third case is also the STC circuit; the solution for the capacitor voltage (and any other voltage in the circuit) is still given by Eq. (7.36) with $\tau = R(C_1 + C_2)$. This case requires extra care since two capacitances and the source form a closed loop. Therefore, according to KVL, capacitor voltages cannot be both equal to zero at the initial time moment (and at any other time moment). Finally, consider the last case in Fig. 7.23. For this circuit with two capacitances and two resistances, there is no closed loop that includes only the capacitances but does not include resistances. Therefore, this circuit is *not* a STC circuit. This result will be confirmed shortly.

Exercise 7.11: Given initially uncharged capacitor(s), write solutions for the capacitor voltage for the circuits shown in Fig. 7.23a–c. For the circuit in Fig. 7.23c, assume that that C_1 was initially uncharged but C_2 was initially charged to V_S and connected to the rest of the circuit at $t = 0$ via a second switch.

Answer:

$$\upsilon_C(t) = V_S\left[1 - \exp\left(-\frac{t}{\tau}\right)\right], \quad \tau = RC \quad \text{in Fig. 7.23a} \tag{7.39}$$

$$\upsilon_C(t) = V_S\frac{R_2}{R_1 + R_2}\left[1 - \exp\left(-\frac{t}{\tau}\right)\right], \quad \tau = (R_1||R_2)C \quad \text{in Fig. 7.23b} \tag{7.40}$$

$$\upsilon_{C1}(t) = V_S\left[1 - \exp\left(-\frac{t}{\tau}\right)\right], \quad \upsilon_{C2}(t) = V_S\exp\left(-\frac{t}{\tau}\right), \quad \tau = R(C_1 + C_2) \quad \text{in Fig. 7.23c} \tag{7.41}$$

7.4.2 Circuits with Resistances and Inductances

The forthcoming analysis is quite similar to the analysis performed previously for the capacitances and resistances. The *single-time-constant transient circuit* (*STC circuit*) with resistances and inductances is that which solution has the form

$$i(t) = K_1\exp\left(-\frac{t}{\tau}\right) + K_2 \tag{7.42}$$

for *any* branch current in the circuit. The list of the corresponding STC circuits also becomes identical to the previous case with a few modifications:

1. Capacitances are replaced by inductances.
2. Independent current source(s) will be used. They may be converted to voltage sources according to the source transformation theorem.
3. In condition #3, instead of loops, we use circuit *node*s. This condition is now formulated as follows. Consider a circuit with multiple inductances. Assume that N is the final number of inductances after all possible series/parallel combinations. For the STC condition to hold, there should be $N - 1$ independent (single) nodes, every branch of which is either an inductance or an independent current source.

As an example, we consider here simple transient circuits shown in Fig. 7.24.

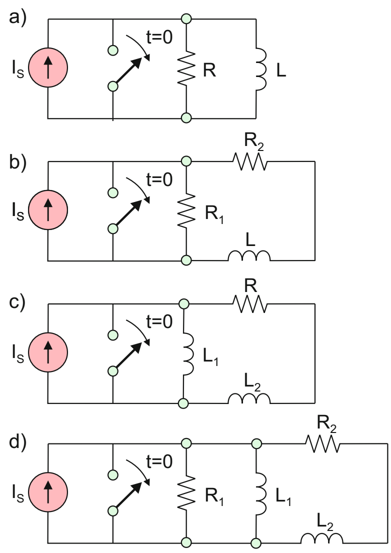


Fig. 7.24. Transient circuits with multiple resistances and inductances.

Exercise 7.12: For the circuits in Fig. 7.24, establish the number of independent nodes, every branch of which is either an inductance or an independent current source.

Answer: (a) Zero; (b) zero; (c) one; (d) zero.

The first case in Fig. 7.24 is the transient circuit of Fig. 7.12. The corresponding solution for the inductor current is given by Eq. (7.42); see also Eqs. (7.28). The second case is again the STC circuit; the solution for the inductor current (and any other current in the circuit) is given by Eq. (7.42) with $\tau = L/(R_1 + R_2)$. The constants K_1 and K_2 in this equation will be different for voltages across different circuit elements. The third case is also the STC circuit; the solution for the inductor current (and any other current in the circuit) is still given by Eq. (7.42) with $\tau = (L_1 + L_2)/R$. This case requires extra care since two inductances and the current source are three branches of the same node. Therefore, according to KCL, inductance currents cannot be both equal to zero at the initial time moment (and at any other time moment). We may assume, for example, that inductance L_2 carried initial current I_S and employs a second switch. Finally, consider the last case in Fig. 7.24. For this circuit with two inductances and two resistances, there is no nontrivial *single* node that includes only the inductances but does not include resistances. Therefore, this circuit is not the STC circuit.

Exercise 7.13: Given zero inductor current at the initial time moment, write solutions for the inductor current for the circuits shown in Fig. 7.24a–c. For the circuit in Fig. 7.24c, additionally assume that that inductor L_1 was carrying zero initial current but inductor L_2 was initially carrying current I_S and connected to the rest of the circuit at $t = 0$ via a second switch.

Answer:

$$i_L(t) = I_S\left[1 - \exp\left(-\frac{t}{\tau}\right)\right], \quad \tau = L/R \quad \text{in Fig. 7.24a} \tag{7.43}$$

$$i_L(t) = I_S\frac{R_1}{R_1 + R_2}\left[1 - \exp\left(-\frac{t}{\tau}\right)\right], \quad \tau = L/(R_1 + R_2) \text{ in Fig. 7.24b} \tag{7.44}$$

$$\begin{aligned} &i_{L1}(t) = I_S\left[1 - \exp\left(-\frac{t}{\tau}\right)\right], \quad i_{L2}(t) = I_S\exp\left(-\frac{t}{\tau}\right), \\ &\tau = (L_1 + L_2)/R \quad \text{in Fig. 7.24c} \end{aligned} \tag{7.45}$$

7.4.3 Example of a Non-STC Transient Circuit

Figure 7.25a shows the last circuit from Fig. 7.23 to be analyzed here in detail. With reference to Fig. 7.25a, KCL and KVL give

$$\begin{aligned} &i = i_1 + i_2 \Rightarrow C_1\frac{dv_1}{dt} = C_2\frac{dv_2}{dt} + \frac{v_2}{R_2}; \\ &-V_S + v_1 + v_2 = 0 \Rightarrow R_1C_1\frac{dv_1}{dt} = V_S - v_1 - v_2 \end{aligned} \tag{7.46}$$

Expressing either v_1 in terms of v_2 or vice versa, we obtain from Eq. (7.46) a *second-order ODE* for either of the capacitor voltages. For simplicity, we will assume that $C_1 = C_2 = C$, $R_1 = R_2 = R$, and $\tau_0 = RC$. Then, the corresponding ODE for v_1 has the form

$$\frac{d^2v_1}{dt^2} + \frac{3}{\tau_0}\frac{dv_1}{dt} + \frac{v_1}{\tau_0^2} = \frac{V_S}{\tau_0^2} \tag{7.47}$$

The analysis of the second-order ODEs like Eq. (7.47) is thoroughly explained in the last section of this chapter. Here, we present its succinct version suitable for our immediate purposes. The solution of the *homogeneous second-order ODE* is sought in the form $\exp(-\alpha t/\tau_0)$ with α being a dimensionless constant. Substitution of this expression into the homogeneous ODE gives a quadratic equation for α, $\alpha^2 + 3\alpha + 1 = 0$, with the two positive roots $\alpha_1 = 2.62$ and $\alpha_2 = 0.38$. Therefore, the solution for Eq. (7.47) should have the form

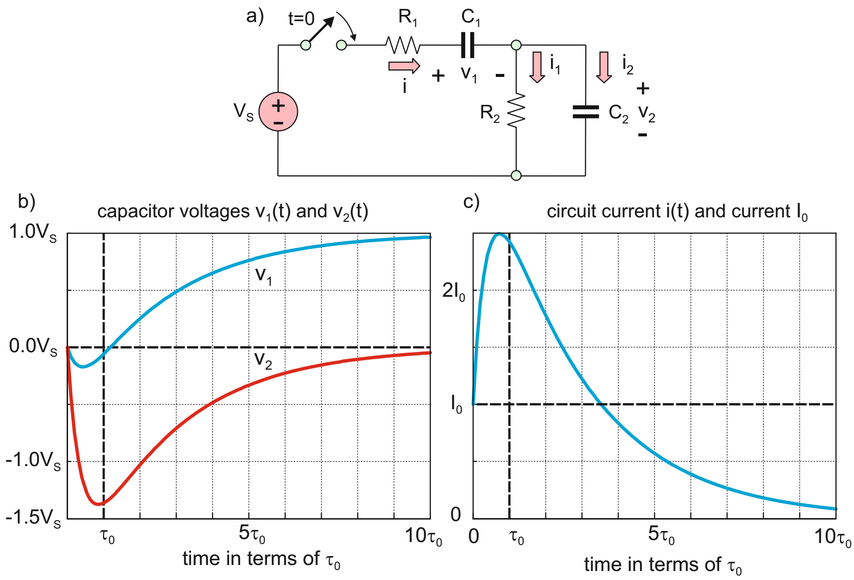


Fig. 7.25. A non-STC circuit with two resistances and two capacitances and its solution behavior.

$$v_1(t) = V_S + K_1 \exp\left(-\frac{\alpha_1 t}{\tau_0}\right) + K_2 \exp\left(-\frac{\alpha_2 t}{\tau_0}\right) \tag{7.48}$$

where K_1, K_2 are two constants determined by the initial conditions The solution for voltage $v_2(t)$ is found using the second of equations (7.46):

$$v_2(t) = (1+\alpha_1)K_1 \exp\left(-\frac{\alpha_1 t}{\tau_0}\right) + (1+\alpha_2)K_2 \exp\left(-\frac{\alpha_2 t}{\tau_0}\right) \tag{7.49}$$

The initial conditions imply that both capacitors are uncharged prior to closing the switch. This gives $K_1 = (1+\alpha_2)/(\alpha_1 - \alpha_2)V_S$ and $K_2 = (1+\alpha_1)/(\alpha_2 - \alpha_1)V_S$. With this in mind, the solution is complete. Figure 7.25b, c shows the behavior of the two capacitor voltages. A truly remarkable point is that instantaneous voltage across the second capacitor in Fig. 7.25b *exceeds* the (absolute) source voltage. Moreover, the instantaneous circuit current in Fig. 7.25c *exceeds* the initial circuit current $I_0 = V_S/R_1$ by 2.5 times. Those distinct features are observed for other *second-order* transient circuits studied further.

7.4.4 Example of an STC Transient Circuit

In Fig. 7.26, we present an example of a rather complicated circuit, which still follows the STC circuit model. The proof is based on the observation that node (*) in Fig. 7.26 connects three branches: two of which are exactly the inductances and the remaining one is the current source. According to KCL, this circuit again implies that both inductances cannot have zero current simultaneously prior to closing the switch.

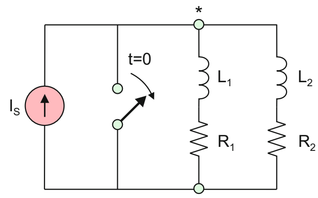


Fig. 7.26. An STC circuit with two inductances and two resistances.

7.4.5 Method of Thévenin Equivalent and Application Example: Circuit with a Bypass Capacitor

Let us consider a transient circuit with a bypass capacitor shown in Fig. 7.27a. It includes a voltage source represented by its Thévenin equivalent $v_S(t)$, R_S and a load represented by its equivalent resistance R_L. The source is turned on at $t = 0$ and it generates a voltage in the form of a (large) DC component V_S and a superimposed (small) AC signal:

$$v_S(t) = V_S + V_m \cos \omega t, \quad \omega = 2\pi f \tag{7.50}$$

where f is frequency in Hz. Generally, $V_S >> V_m$.

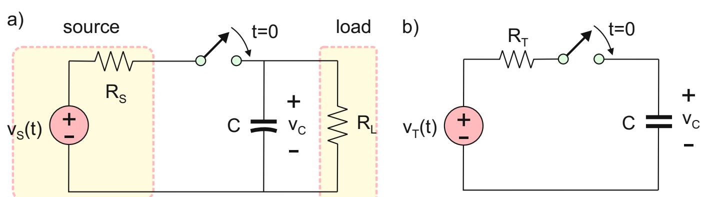


Fig. 7.27. Model of a voltage source connected to a load with a bypass capacitor.

We first interchange positions of capacitance C and resistance R_L in Fig. 7.27. With the switch closed, the voltage source $v_S(t)$ with two resistances R_S, R_L is then converted to its Thévenin equivalent circuit with

$$v_T(t) = \frac{R_L}{R_S + R_L} v_S(t), \quad R_T = R_S \| R_L \tag{7.51}$$

The resulting circuit is the simple RC circuit shown in Fig. 7.27b. The method of a Thévenin (or Norton) equivalent circuit is common for all transient circuits with one capacitor or inductor as explained in the previous section. According to this method, a circuit with *one* capacitance or *one* inductance is *always* converted to the basic RC or RL transient circuit. Therefore, it is always *a first-order* transient circuit which is described by a *first-order* ODE. Once the solution for the capacitor voltage $v_C(t)$ in Fig. 7.27a is found, the voltage across the load resistor is then simply expressed as $v_L(t) = v_C(t)$.

Forced Response and Natural Response

Using KVL, KCL, and the capacitor equation, we obtain the first-order ODE for the circuit in Fig. 7.27b in the form

$$\frac{dv_C(t)}{dt} + \frac{v_C(t)}{\tau} = f(t), \quad f(t) = \frac{1}{\tau}\frac{R_L}{R_S + R_L}(V_S + V_m \cos\omega t), \quad \tau = R_T C \tag{7.52}$$

at any *positive* time instance, $t > 0$. The initial condition is $v_C(t = 0) = 0$. The time-dependent source term $f(t)$ is called a *forcing function*, whether or not the switch is present. The action of the switch may always be included into $f(t)$ by setting $f(t) = 0$ at $t < 0$. A *general solution* of the *homogeneous* Eq. (7.52), let us call it $x_c(t)$, is known as the *complementary solution* (or *natural response*). The natural response is given by $x_c(t) = K\exp(-t/\tau)$ where K is an arbitrary constant. The natural response carries information *about the circuit*. A *particular solution* of the *inhomogeneous* Eq. (7.52), let us call it $x_p(t)$, is known as the *forced response*. The forced response carries information *about the excitation* $f(t)$. It does not need to contain an arbitrary integration constant. By linearity, the complete solution for Eq. (7.53) is the *sum* of both responses, i.e.,

$$v_C(t) = x_p(t) + x_c(t) \tag{7.53}$$

We already know the natural response. The forced response is sought in the form

$$x_p(t) = a\cos\omega t + b\sin\omega t + c \tag{7.54}$$

where constants a, b, c are to be uniquely determined. If the function $f(t)$ is sinusoidal, a combination of sine and cosine would suffice under certain conditions for *any* linear differential equation, not necessarily of the first order. The particular solution in the form of Eq. (7.54) is even useable for second-order circuits considered next.

Example 7.7: Find the forced response for Eq. (7.52).

Solution: Equation (7.54) is substituted into Eq. (7.52); then all terms are pulled to the left-hand side, and all terms with $\cos\omega t$ and $\cos\omega t$ are combined. We obtain:

$$\cos\omega t\left[\frac{a}{\tau} + b\omega - \frac{V_m}{\tau}\frac{R_L}{R_S + R_L}\right] + \sin\omega t\left[\frac{b}{\tau} - a\omega\right] + \left[\frac{c}{\tau} - \frac{V_S}{\tau}\frac{R_L}{R_S + R_L}\right] = 0 \tag{7.55}$$

In order to satisfy Eq. (7.55), we require all three expressions in the square brackets to be zero. This operation yields a system of two equations for a and b, while the constant c is found directly. Working out the algebra produces the results

$$a = \frac{R_L V_m}{R_S + R_L}\frac{1}{1 + (\omega\tau)^2}, \quad b = \frac{R_L V_m}{R_S + R_L}\frac{\omega\tau}{1 + (\omega\tau)^2}, \quad c = \frac{R_L V_S}{R_S + R_L} \tag{7.56}$$

Example 7.8: Obtain the complete solution for Eq. (7.52) and plot it to scale.

Solution: The final solution is based on Eqs. (7.54)–(7.56). The initial condition $\upsilon_C(t=0) = 0$ is satisfied if $K = -a - c$. The required load voltage has the form

$$\begin{aligned} \upsilon_L(t) = \upsilon_C(t) = {} & \frac{R_L V_S}{R_S + R_L}(1 - \exp(-t/\tau)) \\ & + \frac{R_L V_m}{R_S + R_L}\frac{1}{1 + (\omega\tau)^2}[\cos\omega t + \omega\tau\sin\omega t - \exp(-t/\tau)] \end{aligned} \tag{7.57}$$

This solution is compared with the load voltage without bypass capacitor:

$$\upsilon_L(t) = \frac{R_L V_S}{R_S + R_L} + \frac{R_L V_m}{R_S + R_L}\cos\omega t \tag{7.58}$$

Given that $\omega\tau$ and t/τ are both large, the dominant AC term in Eq. (7.57) is the sine function. Comparing Eqs. (7.57) and (7.58) with each other, we can therefore state that the bypass capacitor reduces the amplitude of the unwanted AC component at the load by a factor $1/(\omega\tau)$ while keeping the DC component unchanged! Figure 7.28 plots the load voltage with and without the bypass capacitor. The circuit parameters are $R_S = 5\ \Omega$, $R_L = 1\ \text{k}\Omega$, $C = 1000\ \mu\text{F}$. The source parameters are $V_S = 10$ V, $V_m = 1$ V, $f = 500$ Hz. One hidden yet critical solution parameter is the source resistance R_S. When this parameter is very small, the bypass capacitor has little if any effect on the solution.

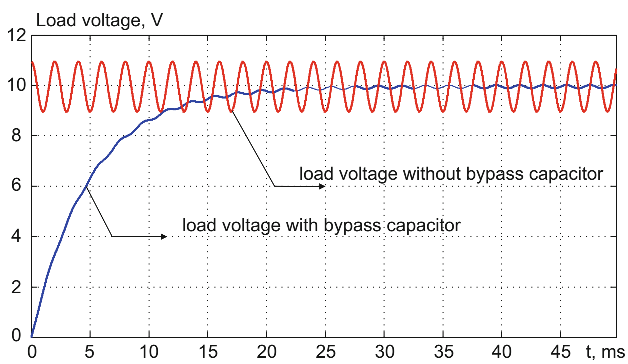


Fig. 7.28. Load voltage with and without the bypass capacitor predicted by Eqs. (7.57) and (7.58).

Section 7.5 Description of the Second-Order Transient Circuits

7.5.1 Types of Second-Order Transient Circuits

A *first-order transient circuit* is described by a *first-order* differential equation. All single-time-constant (STC) circuits considered thus far are the first-order transient circuits. A *second-order transient circuit* is described by a *second-order* differential equation. Figure 7.29 outlines two major types of the second-order order transient circuits.

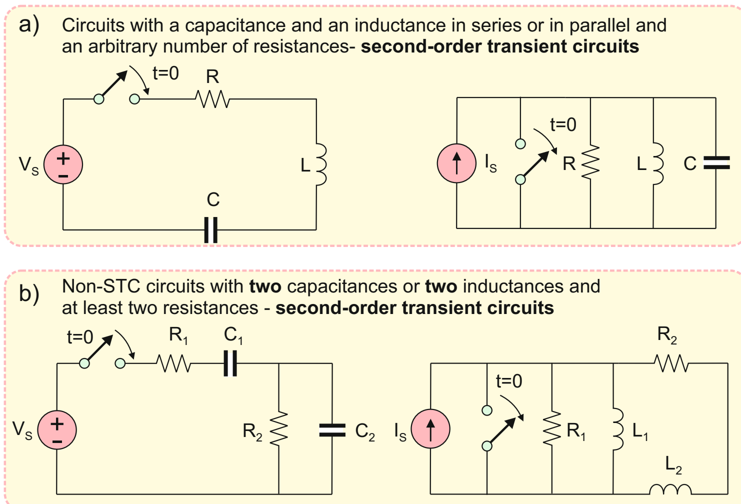


Fig. 7.29. Classification of second-order transient circuits.

The first type involves *two* nonidentical dynamic elements (capacitance and inductance) assembled as a series or parallel two-terminal *LC network*—see Fig. 7.29a. This figure shows two such connections. The number of resistances may be arbitrary. As long as the LC network sees only the combination of source(s) and resistances, the equivalent circuit (obtained with the help of Thévenin or Norton equivalents) will have the form shown in Fig. 7.29a. The second type includes non-STC circuits with two capacitances or two inductances—see Fig. 7.29b. In this section, we will study the second-order circuits on the base of the LC networks.

7.5.2 Series-Connected Second-Order RLC Circuit

Generic Representation of a Series RLC Circuit and Qualitative Operation

Consider a series LC network shown in the shaded box of Fig. 7.30. Using Thévenin's theorem, any network of resistances/independent DC sources connected to this block may be represented as the series combination of the voltage source and the resistance *R*. This is why we arrive at the very generic series RLC circuit shown in Fig. 7.30. The switch implies that the ideal voltage source is to be connected to the circuit at time $t = 0$. Prior to $t = 0$, this source has no effect, it is disconnected. Qualitatively, just after closing the switch the entire voltage drop will be acquired by the inductance. Then, the capacitor voltage will increase. Finally, the capacitor voltage will assume the supply voltage V_S, and no current will flow in the circuit under DC steady-state condition.

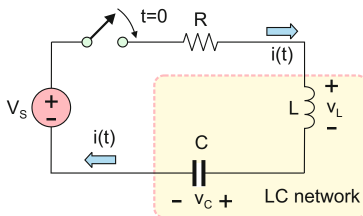


Fig. 7.30. Series RLC second-order transient circuit; the switch closes at $t = 0$.

Mechanical Analogy

Figure 7.31 shows an intuitive analogy between a mechanical mass-spring-damping system and an electric (or electronic) RLC transient circuit depicted in Fig. 7.30.

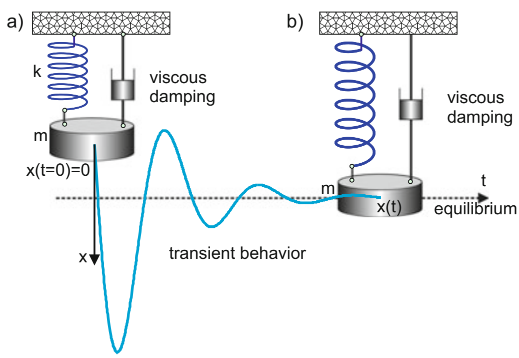


Fig. 7.31. Mechanical counterpart of an RLC circuit; at $t = 0$ the gravity force is applied.

The inductance L corresponds to the mechanical mass m. The capacitance C is the inverse stiffness, $1/k$, of the spring. The resistance R corresponds to viscous damping. At $t = 0$, the gravity force is applied to mass m in Fig. 7.31 initially located at $x = 0$. As a result, the mass reaches a new equilibrium position $x(t \to \infty)$ with or without intermediate oscillations depending on damping. The following correspondences may be established between mechanical and electrical quantities ($q(t)$ is the capacitor charge):

$$i(i) \quad \to \quad dx(t)/dt, \quad q(t) \quad \to \quad x(t), \quad mg \quad \to \quad V_S \tag{7.59}$$

Solution in Terms of Circuit Current

A solution to the dynamic circuit in Fig. 7.30 is based on KVL and KCL. KCL prescribes the same current $i(t)$ through every circuit element at *any* time instant t. KVL states

$$\upsilon_L + \upsilon_R + \upsilon_C = V_S \tag{7.60}$$

at any *positive* time $t > 0$. Indeed, $\upsilon_L, \upsilon_R, \upsilon_C$ are all functions of time. Using constitutive relations for the inductance, resistance, and capacitance (integral form), we obtain

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(t')dt' = V_S \tag{7.61}$$

where we assumed that the initial capacitor voltage is zero. Differentiation over time yields the expected *homogeneous* second-order ODE:

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 0 \tag{7.62}$$

which, after division by L, can be cast in the *standard form*:

$$\frac{d^2i(t)}{dt^2} + 2\alpha\frac{di(t)}{dt} + \omega_0^2 i(t) = 0 \tag{7.63}$$

The two constants present in this equation are given by

$$\alpha = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \tag{7.64}$$

Both constants in Eq. (7.64) have a general mathematical meaning that should be remembered. The first constant α with the units of neper/sec is the *damping coefficient*. It generally characterizes how fast oscillations in Fig. 7.31 decay and reach a steady state. The neper (Np) is a *dimensionless* unit named after John Napier (1550–1617), a Scottish mathematician. The constant α is similar to the inverse time constant $1/\tau$ for the first-order transient circuits. The second constant ω_0 with the units of rad/s is the *undamped resonant frequency* of the RLC circuit. This constant characterizes frequency of oscillations—see again the mechanical analogy in Fig. 7.31. The meaning of the undamped resonant frequency remains the *same* for any LC circuit block, either in transient analysis or in the AC circuit analysis, either for series or parallel configurations.

Exercise 7.14: When is $i(t) = K\cos\omega_0 t$ a solution to Eq. (7.63)?

Answer: At $R = 0$.

7.5.3 Initial Conditions in Terms of Circuit Current and Capacitor Voltage

In contrast to the first-order transient circuits, there are two equally possible choices of the independent function for the RLC circuit in Fig. 7.30:

- Circuit (or inductor) current $i(t)$
- Capacitor voltage, $\upsilon_C(t)$

The circuit current remains continuous over time (inductor "inertia") and so does the capacitor voltage (capacitor "inertia"). We have chosen the circuit current and obtained the second-order ODE Eq. (7.53). You may wonder if this is really the best choice. The answer is nontrivial and is hidden in the initial conditions. Any first-order ODE needs *one initial condition*. Any second-order ODE needs *two initial conditions*. Let us establish these initial conditions for the circuit current first. Following the current continuity through the inductor, the circuit current must be zero at $t = 0$, that is,

$$i(t = 0) = 0 \tag{7.65a}$$

Hence, the first initial condition is established. The second one is that of the initial capacitor voltage equal to zero. According to Eq. (7.60) and Eq. (7.65a),

$$\upsilon_L(t = 0) + \underbrace{\upsilon_R(t = 0)}_{Ri(t=0)=0} + \underbrace{\upsilon_C(t = 0)}_{0} = V_S \Rightarrow L\frac{di}{dt}(t = 0) = V_S \Rightarrow \frac{di}{dt}(t = 0) = \frac{V_S}{L} \tag{7.65b}$$

Thus, the initial circuit current is zero, whereas its first derivative is *not*. This is a drawback of the electric current formulation given by Eq. (7.63) for the series RLC circuit. On the other hand, the capacitor voltage and its derivative (which is proportional to the capacitor/inductor/circuit current) are both zero at $t = 0$, which leads to a simpler "universal" homogeneous formulation of the initial conditions, i.e.,

$$\upsilon_C(t = 0) = 0, \quad \frac{d\upsilon_C(t = 0)}{dt} = 0 \tag{7.66}$$

At the same time, the second-order ODE for the capacitor voltage becomes *inhomogeneous*, i.e., at $t \geq 0$

$$\frac{d^2\upsilon_C(t)}{dt^2} + 2\alpha\frac{d\upsilon_C(t)}{dt} + \omega_0^2\upsilon_C(t) = \omega_0^2 V_S \tag{7.67}$$

It is worth noting that this equation has exactly the same form as Eq. (7.63), but with the nonzero right-hand side. The derivation of Eq. (7.67) is similar to the derivation of Eq. (7.63); it is suggested as a homework problem.

7.5.4 Step Response and Choice of the Independent Function

The selection between current and voltage formulations reflects our desire to convert the circuit differential equation into a standard form, which will allow us to use powerful tools of signals and systems theory. What should be preferred: the homogeneous second-order ODE Eq. (7.63) for the circuit current augmented with inhomogeneous initial conditions Eqs. (7.65a, b), or the inhomogeneous second-order ODE Eq. (7.67) for the capacitor voltage augmented with the homogeneous initial conditions Eq. (7.66)? The answer is as follows. If the initial conditions are all zero, the only remaining excitation is *the forcing function*: the right-hand side of Eq. (7.67). We consider a unit *step function*, $u(t)$, defined by (see Fig. 7.32a)

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{7.68}$$

Equation (7.67) may be conveniently written in the form

$$\frac{d^2 v_C(t)}{dt^2} + 2\alpha \frac{dv_C(t)}{dt} + \omega_0^2 v_C(t) = Au(t) \tag{7.69}$$

at *any* time instant where $A = \omega_0^2 V_S$. The forcing function is thus the product of the constant A and the unit step function $u(t)$, as seen in Fig. 7.32b. The solution to Eq. (7.67) or Eq. (7.69), after division by A, is the *normalized step response* of a second-order system. We call it the response to a unit step voltage excitation. It is generally accepted in signals and systems theory that for the unit step response the initial conditions should be homogeneous or *zero*. Therefore, Eq. (7.67) or Eq. (7.69) with zero initial conditions Eq. (7.66) is preferred when dealing with future applications of the unit step response.

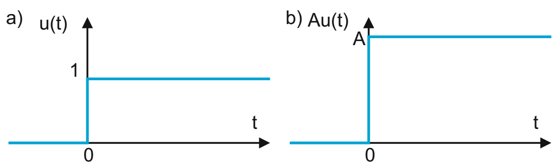


Fig. 7.32. (**a**) Unit step function $u(t)$ and (**b**) the scaled right-hand side of Eq. (7.59).

The step response is the "business card" of the circuit, which actually contains the complete information about its behavior. If the circuit in Fig. 7.30 with an arbitrary time-varying voltage source $V_S \rightarrow v_S(t)$ is considered, this source may be represented as a number of "steps" in time. Hence, the complete solution may be constructed as a sum (or integral) of the elementary unit step response solutions, properly scaled and shifted in time. As an example, we consider a voltage source $v_S(t)$ in Fig. 7.30, which generates a pulse (think of one bit) with the duration T and a 5-V peak value, as depicted in Fig. 7.33.

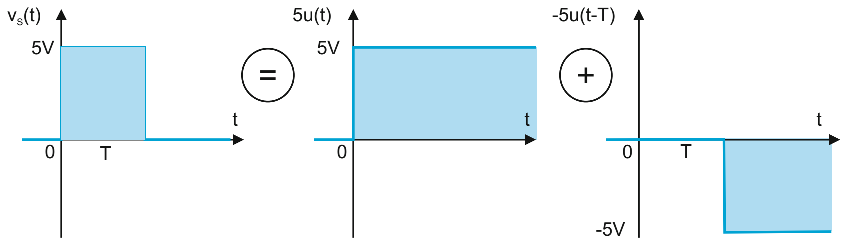


Fig. 7.33. A one-bit excitation voltage constructed as the sum of two step responses.

The switch in Fig. 7.30 is replaced by a short circuit. According to Fig. 7.33, $v_S(t) = 5u(t) - 5u(t-T)$. By linearity, the solution for the bit excitation is simply the sum $5v_C(t)$ $-5v_C(t-T)$ in Fig. 7.33 where $v_C(t)$ is the solution of Eq. (7.69) with $V_S = 1$ V.

7.5.5 Parallel Connected Second-Order RLC Circuit

Generic Representation of the Parallel RLC Circuit and Qualitative Operation

Consider a parallel connected LC block in the shaded box in Fig. 7.34. Using Norton's theorem, any network of resistors and power supplies connected to this block is represented as the parallel combination of a current source and a resistor *R*, as seen in Fig. 7.34. This is why we arrive at the very generic parallel RLC circuit shown in Fig. 7.34. The switch implies that the ideal current power supply is to be connected to the circuit at time instant $t = 0$. Qualitatively, just after closing the switch, the entire circuit current will flow through the capacitor. As time progresses, the capacitor current will decrease. Finally, the entire current will flow through the inductor, which becomes the short circuit under DC conditions.

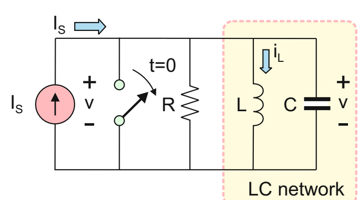


Fig. 7.34. Generic representation of any network of sources and resistances connected to the parallel LC circuit: the standard parallel LCR circuit.

Circuit Equation in Terms of Voltage

The same reasoning as mentioned for the series RLC circuit applies. There are two possible choices of the independent function for the RLC circuit in Fig. 7.34:

- Circuit (or capacitor) voltage, $v(t)$
- Inductor current $i_L(t)$

The capacitor voltage remains continuous over time (inductor “inertia”) and so does the inductor current (inductor “inertia”). We use for derivation $v(t)$ first. KCL yields

$$C\frac{dv(t)}{dt}+\frac{v(t)}{R}+\frac{1}{L}\int_0^t v(t')dt'=I_S \tag{7.70}$$

at any *positive* time $t>0$. Differentiation over time and division by C yield the expected *homogeneous* second-order ODE:

$$\frac{d^2v(t)}{dt^2}+\frac{1}{RC}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=0 \tag{7.71}$$

Eq. (7.71) is written in the form of Eq. (7.63):

$$\frac{d^2v(t)}{dt^2}+2\alpha\frac{dv(t)}{dt}+\omega_0^2v(t)=0 \tag{7.72}$$

if we define the damping coefficient α and the undamped resonant frequency ω_0 as

$$\alpha=\frac{1}{2RC},\quad \omega_0=\frac{1}{\sqrt{LC}} \tag{7.73}$$

Initial Conditions and Choice of Independent Function
The initial conditions for Eq. (7.72) are that the voltage across the capacitor and the inductor current must be continuous. Therefore, they must have the form

$$v(t=0)=0,\quad \frac{dv}{dt}(t=0)=\frac{I_S}{C} \tag{7.74}$$

The voltage derivative is not zero at the initial time moment. Eq. (7.74) is similar to Eqs. (7.65a, b). It has been stated that the step response of the second-order system is generally calculated with the homogeneous (zero) initial conditions. The second-order circuit ODE written in terms of the inductor current

$$\frac{d^2i_L(t)}{dt^2}+2\alpha\frac{di_L(t)}{dt}+\omega_0^2v_L(t)=\omega_0^2I_S \tag{7.75}$$

possesses zero initial conditions, i.e.,

$$i_L(t=0)=0,\quad \frac{di_L(t=0)}{dt}=0 \tag{7.76}$$

Those are the preferred conditions for the step response calculations. The derivation of Eq. (7.75) is similar to the derivation of Eq. (7.72); it is left as a homework problem.

Duality

Comparing Eqs. (7.66) and (7.67) for the series LCR circuit with Eqs. (7.76) and (7.75) for the parallel LCR circuit, we can establish the following substitutions:

$$V_S \leftrightarrow RI_S, \quad \upsilon_C \leftrightarrow Ri_L, \quad \frac{L}{R} \leftrightarrow RC \tag{7.77}$$

These substitutions make both sets of equations including all the constants mathematically *identical*. This fact reflects the *duality* of series/parallel RLC electric circuits. A similar duality is established for the steady-state RLC resonator circuits. Since the initial conditions are also the same, we conclude that the step response of the parallel RLC circuit is equivalent to the step response of the series RLC circuit.

Exercise 7.15: The damping coefficient of a second-order RLC circuit (i) does not depend on capacitance, (ii) decreases when resistance increases, and (iii) equals zero. Determine the circuit topology in every case.

Answer: (i) Series RLC circuit, (ii) parallel RLC circuit, (iii) series LC circuit.

Section 7.6 Step Response of the Series RLC Circuit

7.6.1 General Solution of the Second-order ODE

Solution for Step Response

The starting point is Eq. (7.69) of the previous section for the capacitor voltage of the series RLC circuit augmented with the homogeneous initial conditions, i.e.,

$$\frac{d^2 v_C(t)}{dt^2} + 2\alpha \frac{dv_C(t)}{dt} + \omega_0^2 v_C(t) = Au(t), \quad A = \omega_0^2 V_S \tag{7.78a}$$

$$v_C(t=0) = 0, \quad \frac{dv_C(t=0)}{dt} = 0 \tag{7.78b}$$

Similar to the first-order transient circuits with arbitrary sources, the general solution is also given by the sum of two parts: a *particular solution* of the inhomogeneous equation (7.78a), let us call it $x_p(t)$, and a *complementary solution*, let us call it $x_c(t)$, of the homogeneous equation (7.78a). Homogeneous implies that the right-hand side of the ODE equals zero. The particular solution is known as the *forced response* and the complementary solution is known as the *natural response*. As a result, the total solution is

$$v_C(t) = x_p(t) + x_c(t) \tag{7.79}$$

For the circuit with the DC voltage source and the switch acting as a step excitation, the particular solution is trivial. It is proved by direct substitution:

$$x_p(t) = V_S \Rightarrow v_C(t) = V_S + x_c(t) \tag{7.80}$$

The complementary solution carries information about the entire circuit and requires care.

Solution in Arbitrary Case

What if the right-hand side of Eq. (7.78a) is an arbitrary function of time? How is the solution obtained? We have already established that any such solution can be obtained on the basis of the step response. In general, the solution is expressed in terms of a *convolution integral*, which involves an arbitrary right-hand side of the second-order ODE and the time derivative of the step response. This interesting and fundamental question is studied further in signals and systems theory.

7.6.2 Derivation of the Complementary Solution: Method of Characteristic Equation

Similar to the first-order transient circuits, we seek a complementary solution (natural response) of the homogeneous Eq. (7.78a) in the most general exponential form

$$x_p(t) = K\exp(st) \tag{7.81}$$

where K and s are two arbitrary constants. The substitution yields

$$\left(s^2 + 2\alpha s + \omega_0^2\right)K\exp(st) = 0 \tag{7.82}$$

For a nontrivial solution, the *characteristic equation* $s^2 + 2\alpha s + \omega_0^2 = 0$ must be satisfied, that is,

$$s^2 + 2\alpha s + \omega_0^2 = 0 \Rightarrow s_{1,2} = \begin{cases} -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ -\alpha - \sqrt{\alpha^2 - \omega_0^2} \end{cases} \text{ or } s_{1,2} = \begin{cases} -\alpha\left(1 - \sqrt{\dfrac{\varsigma^2 - 1}{\varsigma^2}}\right) \\ -\alpha\left(1 + \sqrt{\dfrac{\varsigma^2 - 1}{\varsigma^2}}\right) \end{cases} \tag{7.83}$$

where the new constant $\zeta = \alpha/\omega_0$ is the *damping ratio* of the RLC circuit. Formally, this constant has units of 1/rad; it is often considered dimensionless. We must distinguish between three separate cases depending on the value of the damping ratio:

Case A This situation (*overdamping*) corresponds to $\varsigma > 1$. In this case, $s_{1,2}$ are both *real* and *negative*. Since the original ODE is linear, the general solution is simply the combination of two independent decaying exponential functions:

$$x_c(t) = K_1\exp(s_1 t) + K_2\exp(s_2 t) \tag{7.84a}$$

Case B This case (*critical damping*) corresponds to $\varsigma = 1$. Both roots $s_{1,2}$ become identical. Therefore, a solution in the form of Eq. (7.84a) with *two* independent constants can no longer be formed. Only one independent constant may be available. Fortunately, another solution in the form $t\exp(s_1 t)$ exists in this special case. This fact is proved by direct substitution. Thus, the general solution becomes

$$x_c(t) = K_1\exp(s_1 t) + K_2 t\exp(s_1 t) \tag{7.84b}$$

Case C This case (*underdamping*) corresponds to $\varsigma < 1$. Both roots $s_{1,2}$ become complex. This means that our initial simple guess Eq. (7.81) is no longer correct. One can prove by direct substitution that the general solution now has the oscillating form

$$x_c(t) = K_1\exp(-\alpha t)\cos\omega_n t + K_2\exp(-\alpha t)\sin\omega_n t \tag{7.84c}$$

where $\omega_n = \sqrt{\omega_0^2 - \alpha^2}$ is the (radian) *natural frequency* of the circuit. $1/\alpha$ is also called the *time constant* or *the time constant of the decay envelope*. The complementary solution in the form of Eqs. (7.84) always contains two independent integration constants. They should be used to satisfy the initial conditions, which complete the solution.

7.6.3 Finding Integration Constants

According to Eqs. (7.79) and (7.80), the capacitor voltage is $\upsilon_C(t) = x_c(t) + V_S$ where $x_c(t)$ is given by Eqs. (7.84). The integration constants may be found using Eq. (7.78b), which dictates that both the capacitor voltage and its derivative must vanish at the initial time $t = 0$. We then have from Eqs. (7.84)

Case A. $K_1 + K_2 + V_S = 0, \quad s_1K_1 + s_2K_2 = 0$ (7.85a)

Case B. $K_1 + V_S = 0, \quad s_1K_1 + K_2 = 0$ (7.85b)

Case C. $K_1 + V_S = 0, \quad -\alpha K_1 + \omega_n K_2 = 0$ (7.85c)

The solution of Eqs. (7.85) has the form

Case A. $K_1 = \frac{s_2 V_S}{s_1 - s_2}, \quad K_2 = \frac{s_1 V_S}{s_2 - s_1}$ (7.86a)

Case B. $K_1 = -V_S, \quad K_2 = s_1 V_S$ (7.86b)

Case C. $K_1 = -V_S, \quad K_2 = -\frac{\alpha}{\omega_n} V_S$ (7.86c)

Equations (7.83) through (7.86) complete the step response solution for the series RLC circuit. The circuit may behave quite differently depending on the value of the damping ratio ς.

7.6.4 Solution Behavior for Different Damping Ratios

We consider the series RLC circuit shown in Fig. 7.35. We will choose round numbers $L = 1\,\text{mH}$, $C = 1\,\text{nF}$. These values approximately correspond to an RLC transient circuit operating in the 100 kHz–1 MHz frequency band.

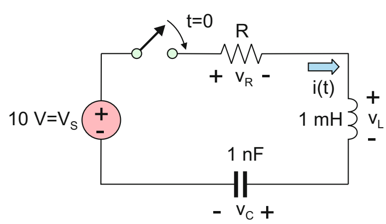


Fig. 7.35. RLC series circuit; the resistance value R may be varied.

Example 7.9: Determine the solution for the capacitor voltage for the circuit shown in Fig. 7.35 over a time interval from 0 to 25 μs for $R = 200\ \Omega,\ 1\ \text{k}\Omega,\ 2\ \text{k}\Omega$, and $20\ \text{k}\Omega$.

Solution: Equations (7.83) through (7.86) give
$R = 200\ \Omega \Rightarrow \alpha = 10^5, \omega_0 = 10^6 \Rightarrow \zeta = \mathbf{0.1}$ **(Case C—underdamped circuit)**

$$\Rightarrow v_C(t) = 10 - 10\exp\left(-10^5 t\right)\cos 9.95 \times 10^5 t - 1.005\exp\left(-10^5 t\right)\sin 9.95 \times 10^5 t \tag{7.87a}$$

$R = 1\ \text{k}\Omega \Rightarrow \alpha = 5 \times 10^5, \omega_0 = 10^6 \Rightarrow \zeta = \mathbf{0.5}$ **(Case C—underdamped circuit)**

$$\Rightarrow v_C(t) = 10 - 10\exp\left(-5 \times 10^5 t\right)\cos 8.66 \times 10^5 t - 5.77\exp\left(-5 \times 10^5 t\right)\sin 8.66 \times 10^5 t \tag{7.87b}$$

$R = 2\ \text{k}\Omega \Rightarrow \alpha = 10^6, \omega_0 = 10^6 \Rightarrow \zeta = \mathbf{1}$ **(Case B—critically damped circuit)**

$$\Rightarrow v_C(t) = 10 - 10\exp\left(-10^6 t\right) - 10^7 t\exp\left(-10^6 t\right) \tag{7.87c}$$

$R = 20\ \text{k}\Omega \Rightarrow \alpha = 10^7, \omega_0 = 10^6 \Rightarrow \zeta = \mathbf{10}$ **(Case A—overdamped circuit)**

$$\Rightarrow v_C(t) = 10 - 10.0252\exp\left(-5.013 \times 10^4 t\right) + 0.0252\exp\left(-1.995 \times 10^7 t\right) \tag{7.87d}$$

Equations (7.87) satisfy both the initial conditions to within numerical rounding error. Figure 7.36 shows the solution behavior for four distinct cases.

The first person who discovered and documented the oscillatory transient response of an electric circuit similar to that depicted in Fig. 7.36 was probably Félix Savary (1797–1841). A renowned astronomer and French academician, he worked with Ampère and discovered an oscillatory discharge of a Leyden jar (an early prototype of the battery) in 1823–1826. Some fifteen years later, the similar observation has been made by Joseph Henry.

7.6.5 Overshoot and Rise Time

One important result seen in Fig. 7.36a, b is the so-called dynamic overshoot caused by a sudden application of a voltage pulse and the associated voltage *ringing*. The dimensionless overshoot (overshoot percentage after multiplying by 100) M_p is the maximum voltage value minus the supply voltage divided by the supply voltage. For a slightly damped circuit, the overshoot may be quite large—see Fig. 7.36a, b. The *rise time* t_r (which is sometimes called the "transition time") in digital circuits is the time taken for the voltage to rise from $0.1V_S$ to $0.9V_S$ (T. L. Floyd, *Digital Fundamentals*, 9th, p. 8); see Figs. 8.14a–c. The circuit designer typically attempts to minimize both the rise time and the overshoot. An important example considered later in our text is a pulse train to be transmitted at a maximum speed (which requires minimum rise time) and with minimum distortion (which requires minimum overshoot). Figure 7.36 indicates that those goals are in fact conflicting. Decreasing the rise time increases the overshoot and vice versa. Designing the damping ratio close to unity, or *slightly below it*, is a reasonable compromise to quickly achieve the desired voltage level without a significant overshoot and ringing. The overshoot and rise time may be estimated analytically. We present here the estimates found in common control theory textbooks:

$$M_p = \frac{\exp(-\pi\zeta)}{\sqrt{1-\zeta^2}} \quad \text{for} \quad \zeta < 1, \quad M_p = 0 \quad \text{for} \quad \zeta \geq 1 \tag{7.88a}$$

$$t_r = \left(1 - 0.4167\zeta + 2.917\zeta^2\right)/\omega_n \quad \text{for} \quad \zeta < 1 \tag{7.88b}$$

Exercise 7.16: The damping coefficient of 15,000 neper/s and the natural frequency of 10 kHz are measured for an unknown series RLC circuit in laboratory via its step response. Given $R = 10\ \Omega$, determine L and C.

Answer: $L = 0.333$ mH, $C = 0.719$ μF.

As to mechanical engineering, it is interesting to note that an automotive suspension system is described by the same step response model and behaves quite similarly to the RLC circuit in Fig. 7.36. Well, driving a car everyday should certainly be a motivation for studying this topic.

7.6.6 Application Example: Nonideal Digital Waveform

Modeling Circuit

The series RLC block is an appropriate model to study the voltage pulse as it realistically occurs in digital circuits and in power electronic circuits involving pulse width modulation (PWM). The ideal square voltage pulse, shown in Fig. 7.33 of the previous section, is a crude approximation of reality. Parasitic capacitance, resistance, and inductance are always present in the circuit. As a result, the pulse form is distorted. To model the pulse form distortion, we consider the circuit shown in Fig. 7.37.

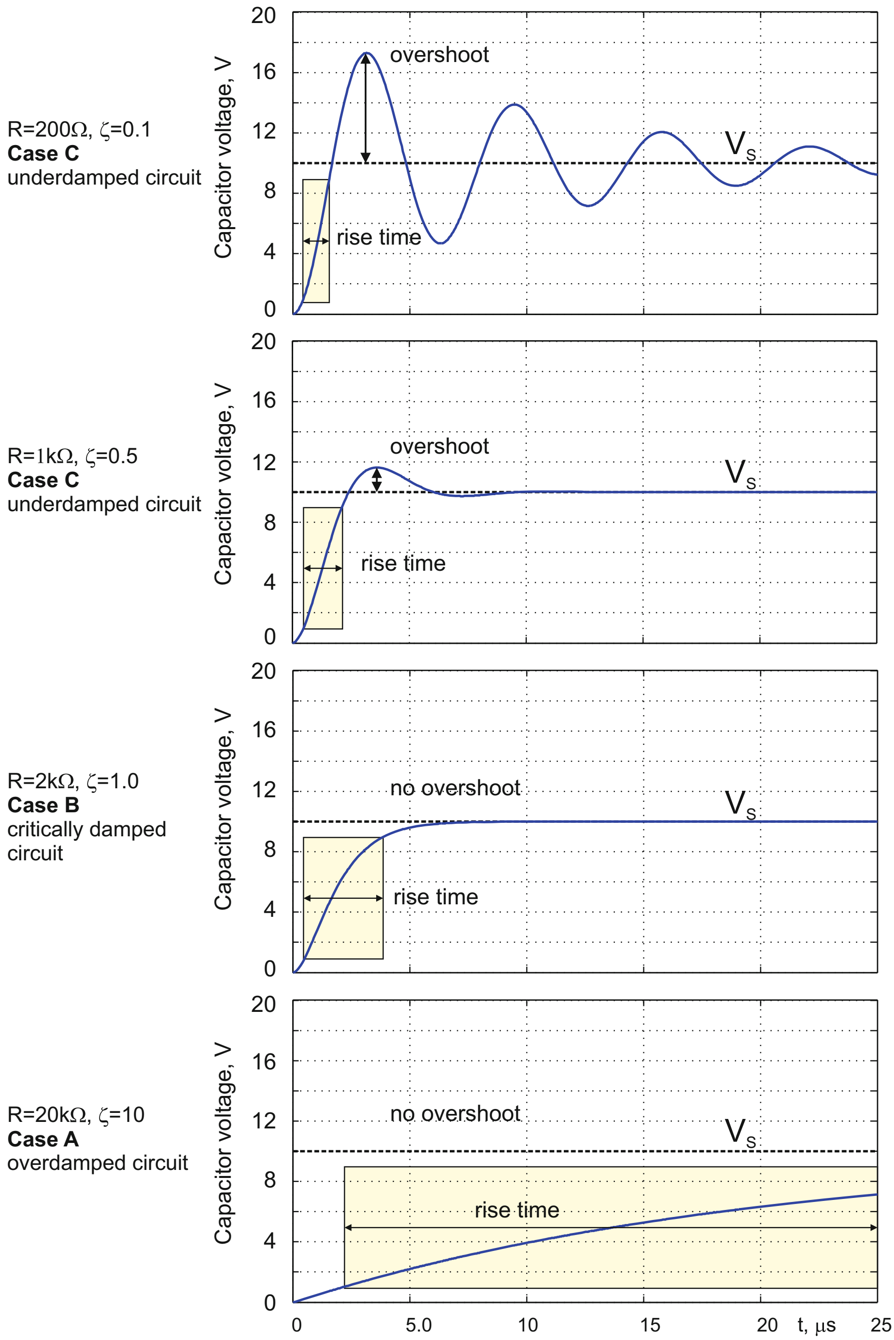


Fig. 7.36. Circuit responses in terms of capacitor voltages for different damping factors.

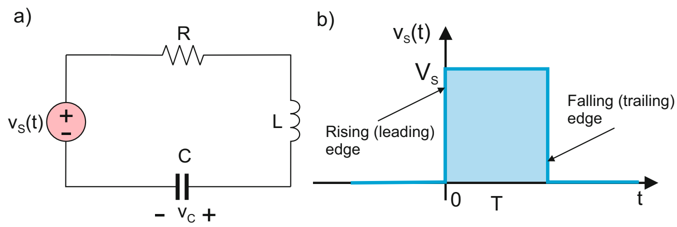


Fig. 7.37. The RLC circuit for studying the nonideal digital (pulse) waveform.

The switch is now removed and a time-varying voltage source $\upsilon_S(t)$ is introduced; it generates the voltage pulse (one bit) of duration T with amplitude V_S as seen in Fig. 7.37b. One may think of the source voltage as an "ideal" digital waveform and, for example, of the capacitor voltage as a nonideal (realistic) waveform influenced by parasitic capacitance, resistance, and inductance.

Solution

The solution to the pulse problem is derived as described at the end of the previous section. We know that Eqs. (7.83) through (7.86) determine the step response—the capacitor voltage $\upsilon_C(t)$ for the circuit with the DC voltage source shown in Fig. 7.35 after closing the switch. To obtain the solution $\upsilon_C^{\text{pulse}}(t)$ for the voltage pulse shown in Fig. 7.37, we simply combine two such step responses, i.e.,

$$\upsilon_C^{\text{pulse}}(t) = \upsilon_C(t) - \upsilon_C(t - T) \tag{7.89}$$

This operation again underscores the importance of the fundamental step response solution. Close inspection of Eq. (7.89) shows that the pulse will possess the dynamic overshoot and the nonzero rise time similar to the step response solution. This happens at the *rising (or leading) edge* of the pulse. At the same time, a *dynamic undershoot* and a nonzero *settling time* will happen at the *falling* or *trailing edge* as depicted in Fig. 7.38.

Example 7.10: Determine the solution for the capacitor voltage, $\upsilon_C^{\text{pulse}}(t)$, for the circuit shown in Fig. 7.37 with $L = 1\ \mu\text{H}$, $C = 1\ \text{nF}$, $V_S = 10\ \text{V}$, $T = 0.5\ \mu\text{s}$ over the time interval from 0 to 1 μs for $R = 15\ \Omega$, $30\ \Omega$, and $60\ \Omega$.

Solution: We find the step response $\upsilon_C(t)$ following Eqs. (7.83) through (7.86) first and then obtain the final solution using Eq. (7.89). For the step response, we obtain
$R = 15\ \Omega \Rightarrow \zeta = $ **0.24 (Case C—underdamped circuit)** $\Rightarrow$

$$\begin{aligned}\upsilon_C(t) = {} & 10 - 10\exp\left(-7.5\times10^6 t\right)\cos 3.07\times10^7 t \\ & - 2.44\exp\left(-7.5\times10^6 t\right)\sin 3.07\times10^7 t\end{aligned} \tag{7.90a}$$

Example 7.10 (cont.):

$R = 30\ \ \Omega \Rightarrow \zeta =$ **0.47 (Case C—underdamped circuit)** $\Rightarrow$

$$v_C(t) = 10 - 10\exp\left(-1.5 \times 10^7 t\right) \cos 2.78 \times 10^7 t$$
$$- 5.39\exp\left(-1.5 \times 10^7 t\right) \sin 2.78 \times 10^7 t \qquad (7.90b)$$

$R = 60\ \ \Omega \Rightarrow \zeta =$ **0.95 (Case C—underdamped circuit)** $\Rightarrow$

$$v_C(t) = 10 - 10\exp\left(-3.0 \times 10^7 t\right) \cos 1.0 \times 10^7 t$$
$$- 30\exp\left(-3.0 \times 10^7 t\right) \sin 1.0 \times 10^7 t \qquad (7.90c)$$

Figure 7.38a–c shows the distorted pulse forms for three particular cases. Figure 7.38a outlines the major pulse parameters: rise time, fall time, overshoot, undershoot, and pulse width. One can see that there is again a conflict between the desire to simultaneously decrease the rise time and the overshoot.

The overshoot and undershoot in Fig. 7.38 approximately coincide, and so do the rise time and the fall or settling time. Note that this is not always the case. The voltage pulse may be very significantly and *unsymmetrically* distorted when the initial pulse width, T, is comparable with the rise time. A good illustration is the previous example solved for $R = 15\,\Omega$ when $T = 0.25\,\mu\text{s}$ or less.

Exercise 7.17: Using a theoretical approximation, find the overshoot for the case of Fig. 7.38a and compare this value with value observed on the figure.

Answer: 48 % (theory) versus 50 % (observation).

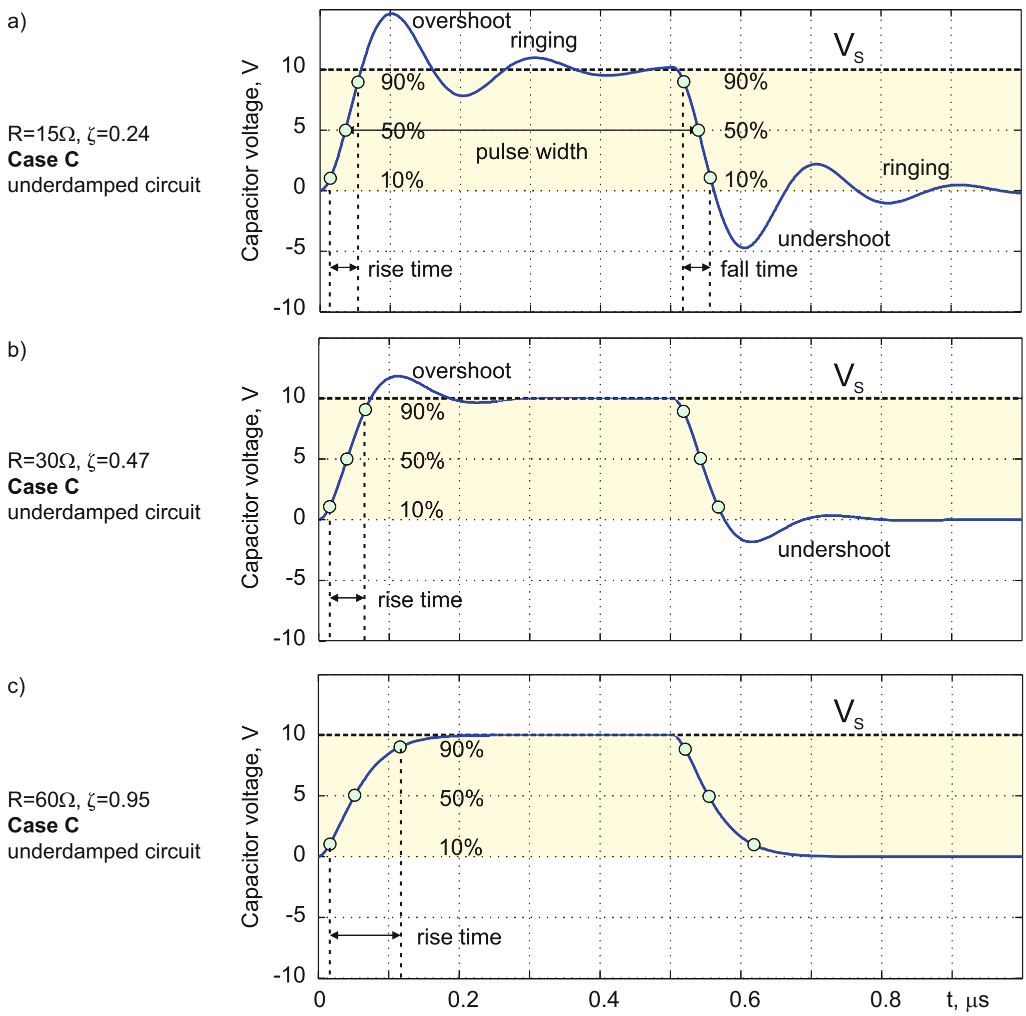
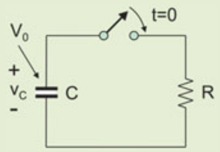
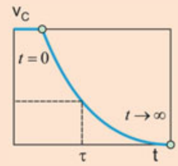
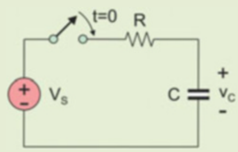
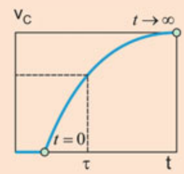
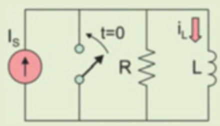
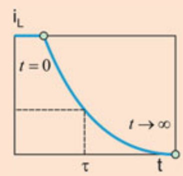
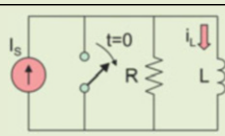
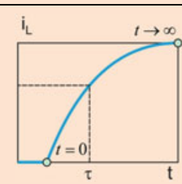
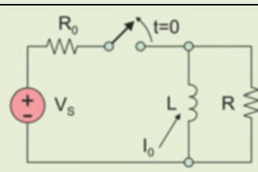
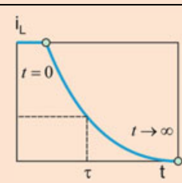


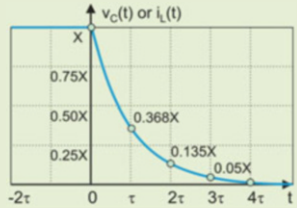
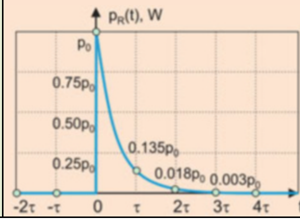
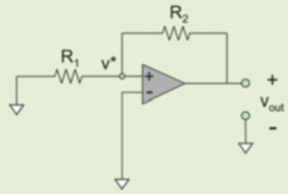
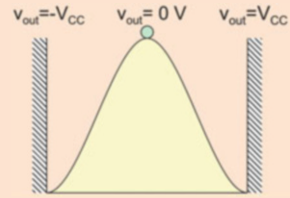
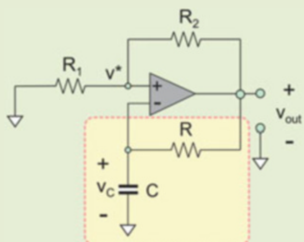
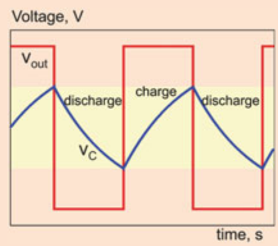
Fig. 7.38. Distorted pulse forms for three different values of the damping ratio. When the damping ratio increases, the overshoot decreases but the rise time increases.

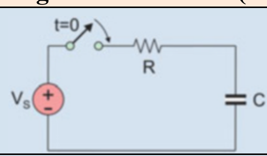
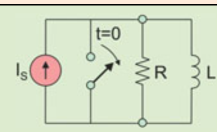
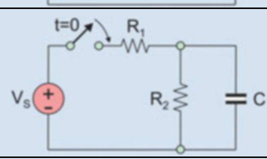
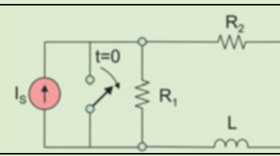
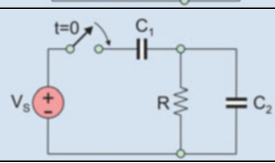
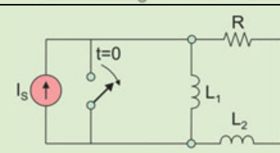
The solution for the second-order circuits with arbitrary sources and arbitrary initial conditions can quite simply be obtained numerically. A straightforward finite-difference second-order method may be implemented in MATLAB or in other software packages with a few lines of the code. This method is the extension of the Euler method used for first-order transient circuits. Interestingly, the same method may be applied to radio-frequency pulse propagation in transmission lines and in free space, including problems such as signal penetration through walls.

Summary

General facts about transient circuits		
Voltage across the capacitor(s) remains continuous for all times—use this voltage as an unknown function		
Current through the inductor(s) remains continuous for all times—use this current as an unknown function		
When discharged through a small resistance, the (ideal) capacitor is able to deliver an extremely high power (and current) during a very short period of time		
When disconnected from the source, the (ideal) inductor in series with a large resistance is able to deliver an extremely high power (and voltage) during a very short period of time		
Transient circuit	**Generic circuit diagram**	**Solution plot**
Energy-release RC circuit $v_C(t) = V_0\exp\left(-\frac{t}{\tau}\right)$ $\tau = RC$ ODE: $\frac{dv_C}{dt} + \frac{v_C}{\tau} = 0$		
Energy-accumulating RC circuit $v_C(t) = V_S\left[1 - \exp\left(-\frac{t}{\tau}\right)\right]$ $\tau = RC$ ODE: $\frac{dv_C}{dt} + \frac{v_C}{\tau} = \frac{V_S}{\tau}$		
Energy-release RL circuit $i_L(t) = I_S\exp\left(-\frac{t}{\tau}\right)$ $\tau = L/R$ ODE: $\frac{di_L}{dt} + \frac{i_L}{\tau} = 0$		
Energy-accumulating RL circuit $i_L(t) = I_S\left[1 - \exp\left(-\frac{t}{\tau}\right)\right]$ $\tau = L/R$ ODE: $\frac{di_L}{dt} + \frac{i_L}{\tau} = \frac{I_S}{\tau}$		
Energy-release RL circuit $i_L(t) = I_0\exp\left(-\frac{t}{\tau}\right)$ $\tau = L/R,\ \ I_0 = V_S/R_0$ ODE: $\frac{di_L}{dt} + \frac{i_L}{\tau} = \frac{V_S}{R_0\tau}$		

(continued)

	Voltage or current	Power
Generic energy-release curves for either dynamic element		
Switching RC oscillator (relaxation oscillator)—RC timer		
Bistable amplifier circuit with positive feedback. Two stable states: $v_{out} = +V_{CC}$ $v_{out} = -V_{CC}$ Threshold voltage(s): $v^* = \frac{R_1}{R_1+R_2} v_{out}$		
Relaxation oscillator with positive feedback: $v_{out} = +V_{CC}$ if $v^* > v_C$ $v_{out} = -V_{CC}$ if $v^* < v_C$ Threshold voltage(s): $v^* = \frac{R_1}{R_1+R_2} v_{out}$		
Period and frequency of the relaxation oscillator ($\beta = R_1/(R_1+R_2)$)	$T = 2\tau \ln \frac{1+\beta}{1-\beta}$	$f = \frac{1}{T} = \frac{1}{2\tau}\left(\ln \frac{1+\beta}{1-\beta}\right)^{-1}$

Single-time-constant (STC) transient circuits		
$\tau = RC$ or $\tau = \frac{L}{R}$		
$\tau = (R_1 \| R_2)C$ or $\tau = \frac{L}{R_1+R_2}$		
$\tau = R(C_1+C_2)$ or $\tau = \frac{L_1+L_2}{R}$		

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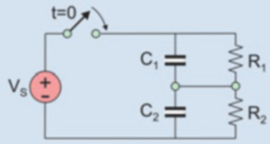
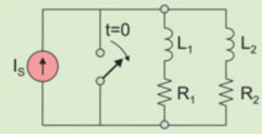
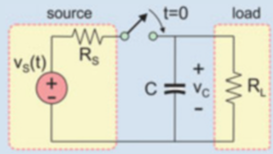
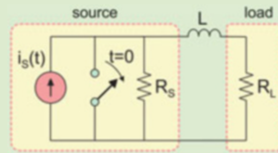
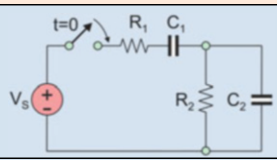
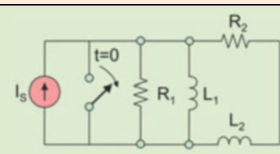
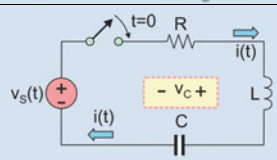
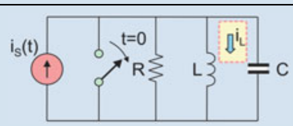
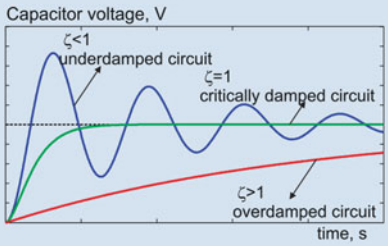
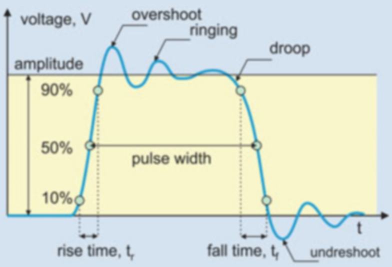
$\tau = \frac{L_1 + L_2}{R_1 + R_2}$ $\tau = R_1 \vert\vert R_2 \times (C_1 + C_2)$	 Arbtr. initial conditions are not allowed	 Arbtr. initial conditions are not allowed
	STC circuits with general sources	
Bypass capacitor and decoupling inductor	 $v_S(t) = V_S + V_m \cos\omega t,\ \tau = (R_S \vert\vert R_L)C$	 $i_S(t) = I_S + I_m \cos\omega t,\ \tau = L/(R_S + R_L)$
Solution for load voltage or load current	$v_L = \frac{R_L V_S}{R_S + R_L}\left(1 - \exp\left(-\frac{t}{\tau}\right)\right)$ $+ \frac{R_L V_m}{R_S + R_L} \times \frac{1}{1 + (\omega\tau)^2}$ $\left[\cos\omega t + \omega\tau \sin\omega t - \exp\left(-\frac{t}{\tau}\right)\right]$	$i_L = \frac{R_S I_S}{R_S + R_L}\left(1 - \exp\left(-\frac{t}{\tau}\right)\right)$ $+ \frac{R_S I_m}{R_S + R_L} \times \frac{1}{1 + (\omega\tau)^2}$ $\left[\cos\omega t + \omega\tau \sin\omega t - \exp\left(-\frac{t}{\tau}\right)\right]$
	Second-order transient circuits	
With two identical dynamic elements		
With series LC network	 • Step response with zero initial conditions: use capacitor voltage $v_C(t)$	• Damping coefficient: $\alpha = R/(2L)$ • Undamped res. freq.: $\omega_0 = 1/\sqrt{LC}$ • Damping ratio: $\zeta = \alpha/\omega_0$ • Natural freq.: $\omega_n = \sqrt{\omega_0^2 - \alpha^2}$ Characteristic Eq.: $s^2 + 2\alpha s + \omega_0^2 = 0$
With parallel LC network	 • Step response with zero initial conditions: use inductor current $i_L(t)$	• Damping coefficient: $\alpha = 1/(2RC)$ • Undamped res. freq.: $\omega_0 = 1/\sqrt{LC}$ • Damping ratio: $\zeta = \alpha/\omega_0$ • Natural freq.: $\omega_n = \sqrt{\omega_0^2 - \alpha^2}$ Characteristic Eq.: $s^2 + 2\alpha s + \omega_0^2 = 0$

(continued)

<table>
<tr>
<td>Overdamped, critically-damped, and underdamped RLC circuits</td>
<td>
</td>
<td>• *Overdamped* ($\zeta > 1$):
$x_c(t) = K_1 \exp(s_1 t) + K_2 \exp(s_2 t)$
$K_1 = \frac{s_2 V_S}{s_1 - s_2}, \quad K_2 = \frac{s_1 V_S}{s_2 - s_1}$
• *Critically damped* ($\zeta = 1$):
$x_c(t) = K_1 \exp(s_1 t) + K_2 t \exp(s_1 t)$
$K_1 = -V_S, \quad K_2 = s_1 V_S$
• *Underdamped* ($\zeta < 1$):
$x_c(t) = K_1 \exp(-\alpha t) \cos \omega_n t +$
$K_2 \exp(-\alpha t) \sin \omega_n t$
$K_1 = -V_S, \quad K_2 = -\frac{\alpha}{\omega_n} V_S$</td>
</tr>
<tr>
<td>Non-ideal digital waveform: second-order circuit</td>
<td>
</td>
<td>• *Overshoot*
$M_p = \frac{\exp(-\pi\zeta)}{\sqrt{1-\zeta^2}} \quad \text{for} \quad \zeta < 1$
• *Undershoot* is approximately overshoot for rise times small compared to pulse width
• *Rise time*
$t_r = (1 - 0.4167\zeta + 2.917\zeta^2)/\omega_n$ for $\zeta < 1$
Fall time is approximately rise time for rise times small compared to pulse width</td>
</tr>
</table>

Problems

7.1 RC Circuits

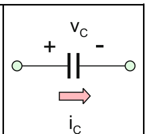
7.1.1 Energy-Release Capacitor Circuit

7.1.2 Time Constant of an RC Circuit and Its Meaning

7.1.3 Continuity of the Capacitor Voltage

Problem 7.1. For the capacitor as a dynamic circuit element, develop:

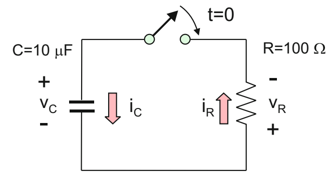
1. Equivalent circuit at DC
2. Relation between voltage and current
3. Expression for the time constant of a transient circuit that includes the dynamic element and a resistor R

Dynamic circuit element	
Equivalent circuit at DC (short or open)	
Relation between voltage and current (passive reference configuration)	
Expression for the time constant of a transient circuit that includes the dynamic element (C) and a resistor R.	$\tau =$

Problem 7.2. Using KCL and KVL, derive the differential equation for the circuit shown in the following figure, keeping the same labeling for the voltages and the currents.

A. Is the final result different from Eq. (7.5) of Section 7.1?

B. Could you give an example of a certain voltage and/or current labeling (by arbitrarily changing polarities and directions in the figure) that causes the differential equations to change?



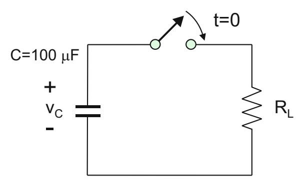
Problem 7.3. Prove that Eq. (7.6) is the solution to Eq. (7.5) (both from Section 7.1) using direct substitution and the differentiation that follows.

Problem 7.4.

A. Show that the time constant, τ, of an RC circuit has the units of seconds.

B. To obtain the slow discharge rate of lesser instantaneous power into the load, should the load resistance be small or large?

Problem 7.5. A 100-μF capacitor discharges into a load as shown in the following figure. The load resistance may have values of 100 Ω, 10 Ω, and 1 Ω. The capacitor is charged to 20 V prior to $t = 0$.



A. Find time constant τ and the maximum instantaneous power delivered to the load resistor in the very first moment for every resistor value—fill out the Table that follows.

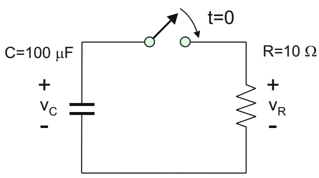
Instantaneous load power right after the switch closes

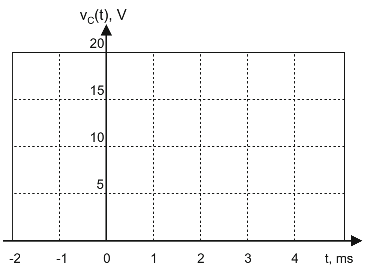
R_L	τ, s	$p_L(t = +0)$, W
100 Ω		
10 Ω		
1 Ω		

B. Do the instantaneous power values from the Table depend on the capacitance value?

Problem 7.6. A 100-μF capacitor, shown in the following figure, discharges into a 10-Ω load resistor. The capacitor is charged to 15 V prior to $t = 0$.

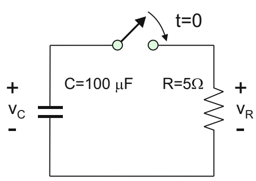
A. Find the time constant of the circuit (show units).
B. Express the voltage across the capacitor as a function of time and sketch it to scale versus time over time interval from -2 ms to 5 ms.

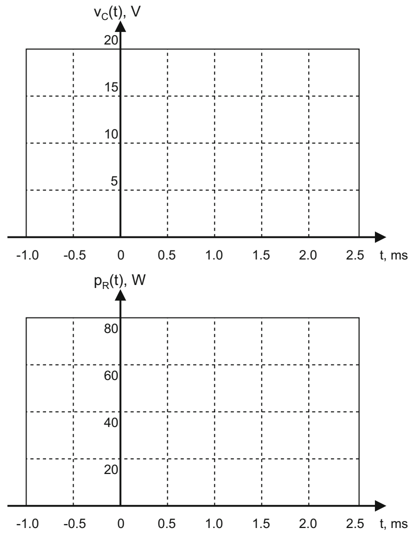




Problem 7.7. A 100-μF capacitor, shown in the following figure, discharges into a 5-Ω load resistor. The capacitor is charged to 20 V prior to $t = 0$.

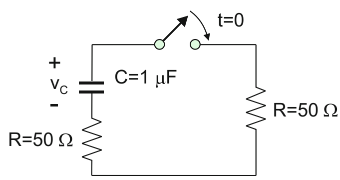
A. Find an expression for the voltage across the capacitor as a function of time and sketch it to scale versus time over the interval from -2τ to 5τ.
B. Repeat the exercise for instantaneous power delivered to the resistor.





Problem 7.8. In the circuit shown in the following figure, the capacitor is charged to 20 V prior to $t = 0$.

A. Find an expression for the voltage across the capacitor as a function of time and sketch it to scale versus time over the interval from -2τ to 5τ.
B. Repeat for instantaneous power delivered to the rightmost resistor.



Problem 7.9. Present the text of a MATLAB script (or of any software of your choice) in order to generate Fig. 7.2d of Section 7.1. Attach the figure so generated to the homework report.

Problem 7.10. Prove that the integral of the load power in Fig. 7.2d given by Eq.(7.8c) is exactly equal to the energy stored in the charged capacitor, $E_C = \frac{1}{2}CV_0^2$ prior to $t = 0$.

Problem 7.11.

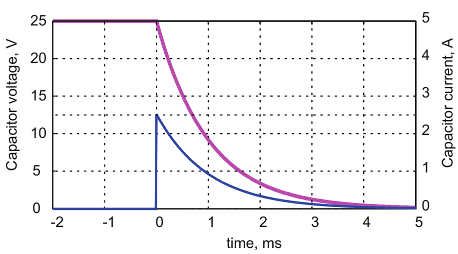
A. Create the generic capacitor voltage discharge curve similar to Fig. 7.2a but for an *arbitrary* capacitor powering an *arbitrary* load resistor over the time interval from -2τ to 5τ. The capacitor is charged to V_0 prior to $t = 0$. To do so, find the capacitor voltage as a fraction of V_0 for every unit of τ and fill out the Table that follows.

Capacitor voltage in terms of V_0

t	-2τ	$-\tau$	0	τ	2τ	3τ	4τ	5τ
$v_C(t)$								

B. Repeat the same task for Fig. 7.2d related to load power. Find the load power in terms of the maximum power just after closing the switch.

Problem 7.12. For an unknown energy-release RC circuit, capacitor voltage and capacitor current were measured in laboratory before and after closing the switch at $t = 0$ as shown in the figure that follows. Approximate R and C.



7.1.4 Application Example: Electromagnetic Railgun

7.1.5 Application Example: Electromagnetic Material Processing

Problem 7.13. An electromagnetic capacitor accelerator with permanent magnets has $B = 0.3$ T. The accelerating object has a length of 2 cm. Plot to scale the Lorentz force as a function of discharge current over the interval $0 < i_C < 1000$ A.

Problem 7.14. An electromagnetic capacitor accelerator needs to create an average force of 5 N over 2 ms on a moving object with length of 1 cm. The load (armature plus object) resistance is 1 Ω, and the external magnetic field is $B = 0.25$ T. Determine:

A. The required capacitor voltage prior to discharge
B. The required capacitance of the capacitor (bank of capacitors)

Hint: Assume that the average force acts over the time interval τ. Its value is approximately equal to 60 % of the initial force value.

Problem 7.15. Solve the previous problem when:

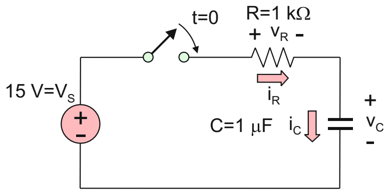
A. The average force increases to 50 N
B. The average force increases to 500 N

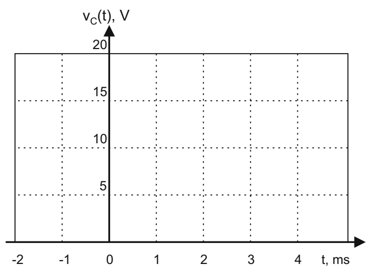
Problem 7.16. The world's largest capacitor bank is located in Dresden, Germany. The pulsed, capacitive power supply system was designed and installed for studying high magnetic fields by experts from Rheinmetall Waffe Munition. The bank delivers 200 kA of discharge current in the initial time moment (just after the switch closes). The time constant is 100 ms. Estimate the bank capacitance if the charging voltage is 200 kV.

7.1.7 Energy-Accumulating Capacitor Circuit

Problem 7.17. A 1-μF capacitor shown in the following figure is charged through the 1-kΩ load resistor. The initial capacitor voltage is zero.

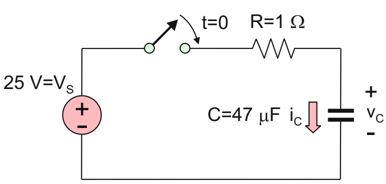
A. Find the time constant of the circuit (show units).
B. Express the voltage across the capacitor as a function of time and sketch it to scale versus time over the interval from −2 ms to 5 ms.





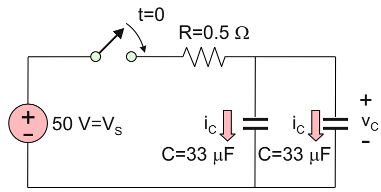
Problem 7.18. For the circuit shown in the following figure:

A. Find an expression for the capacitor voltage, v_C, and the capacitor current, i_C, including the value of time constant.
B. Sketch the capacitor voltage, v_C, and the capacitor current, i_C, to scale versus time over the interval from -2τ to 5τ.

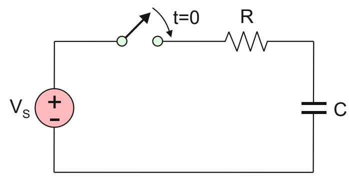


Problem 7.19. For the circuit shown in the following figure:

A. Find an expression for the capacitor voltage, v_C, and the capacitor current, i_C, including the value of time constant.
B. Sketch the capacitor voltage, v_C, and the capacitor current, i_C, to scale versus time over the time interval from -2τ to 5τ.

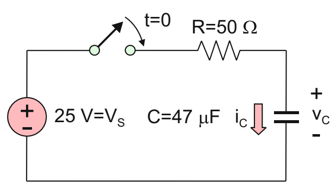


Problem 7.20. Sketch your own fluid-flow counterpart of the charging circuit shown in the figure and establish as many analogies between electrical (R, C, V_S) and mechanical parameters of your drawing as possible.



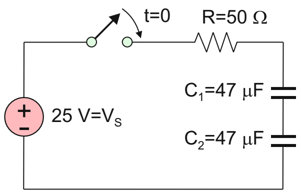
Problem 7.21. For the circuit shown in the following figure:

A. How much time does it take to charge the capacitor to 10 V?
B. To 25 V?

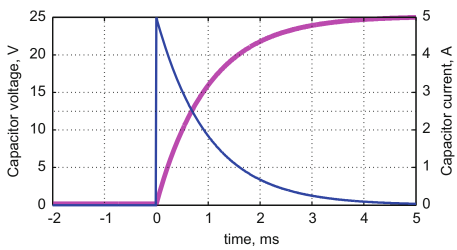


Problem 7.22. For the circuit shown in the figure, how much time does it take to charge

the capacitor, C_1, to 10 V? Assume that the initial voltages of both capacitors are zero.



Problem 7.23. For an unknown energy-accumulating RC circuit, capacitor voltage and capacitor current were measured in laboratory before and after closing the switch at $t = 0$ as shown in the figure that follows. Approximate R and C.

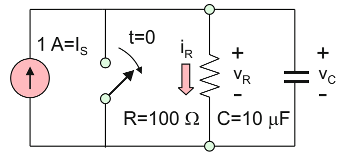


Problem 7.24.

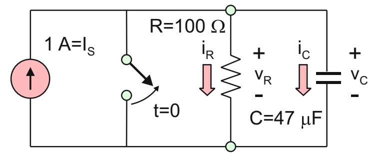
A. Obtain an analytical solution for the capacitor voltage in the circuit shown in the following figure. When the switch is closed, the current source still generates current I_S at its terminals. However, the supply is shorted out – no current flows into the circuit. When the switch is open, the current flows into the circuit.

B. Could you convert this circuit to an equivalent RC transient circuit with the voltage source?

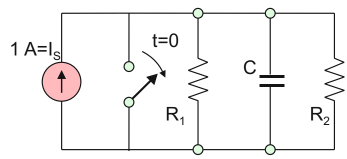
C. Plot the voltage across the resistor versus time over the time interval from -2τ to 5τ.



Problem 7.25. Repeat the previous problem for the circuit shown in the following figure.



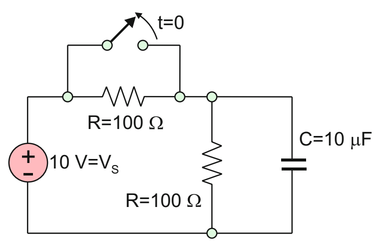
Problem 7.26. Obtain an analytical solution for the capacitor voltage in the circuit shown in the following figure at any time and express it in terms of I_s, R_1, R_2, C. Find the time constant of the circuit when $R_1 = R_2 = 100\ \Omega$, $C = 47\ \mu F$.



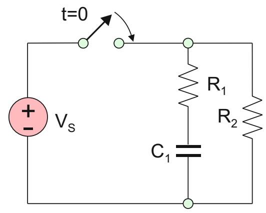
Problem 7.27. For the circuit shown in the figure:

A. Derive and solve the dynamic circuit equation after the switch opens. Assume the initial capacitor voltage equal to zero.

B. Plot the capacitor voltage to scale versus time over the time interval from –2τ to 5τ.



Problem 7.28. In the circuit that follows, the capacitor, C_1, is initially uncharged. The switch is closed at $t = 0$.



Give answers to the following questions based on known circuit parameters C_1, R_1, R_2, V_S:

A. What is the current through resistor R_2 as a function of time?
B. What is the maximum current through resistor R_1?
C. What is the current through resistor R_1 at a time long after the switch closes?
D. What is the charge, $Q^+(t)$, of the capacitor, C_1, as a function of time?

The switch is then opened a very long time after it has been closed – reset the time to $t = 0$.

A. What is the charge $Q^+(t)$ of the capacitor, C_1, as a function of time?
B. What is the current through resistor R_2 as a function of time? Specify the current direction in the figure.

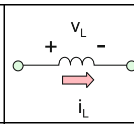
7.2 RL Circuits

7.2.1 Energy-Release Inductor Circuit

7.2.2 Continuity of the Inductor Current

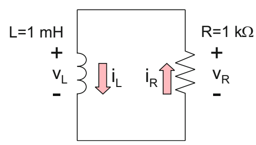
Problem 7.29. For the inductor as a dynamic circuit element, present:

1. Equivalent circuit at DC
2. Relation between voltage and current
3. Expression for the time constant of a transient circuit that includes the dynamic element and a resistor, R

Dynamic circuit element	
Equivalent circuit at DC (short or open)	
Relation between voltage and current (passive reference configuration)	
Expression for the time constant of a transient circuit that includes the dynamic element (L) and a resistor R	$\tau =$

Problem 7.30.

A. Using KCL and KVL, derive the differential equation for the inductor current in the circuit shown in the figure that follows, keeping the same labeling for the voltages and the currents.
B. Is the final result different from Eq. (7.19) of Section 7.2?

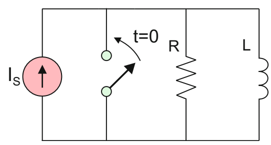


Problem 7.31. Prove that Eq. (7.20) is the solution to Eq. (7.19) using direct substitution and the corresponding differentiation.

Problem 7.32.

A. Show that the time constant τ has units of seconds for the RL circuit.
B. To ensure a slower energy release rate of the inductor, should the load resistance be small or large?
C. To ensure a faster energy release rate of the inductor, should the load resistance be small or large?

Problem 7.33. A 6.8-μH inductor releases its energy into a load resistor as shown in the following figure. The load resistance may have values of 10 Ω, 100 Ω, and 1 kΩ. The inductor current is 1 A prior to $t = 0$.



A. Find time constant τ and the maximum instantaneous power delivered to the load resistor in the very first moment for every resistor value—fill out the Table that follows.

Instantaneous load power just after the switch closes

R	τ, s	$p_R(t=+0)$, W
10 Ω		
100 Ω		
1 kΩ		

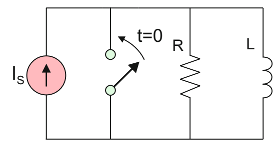
B. Do those instantaneous power values from the Table depend on the inductance value?

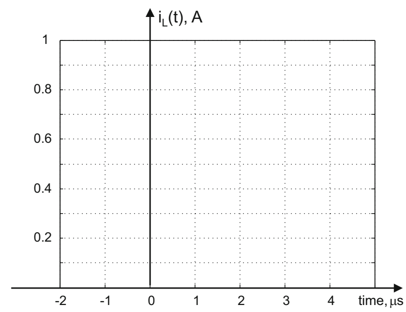
Problem 7.34. Prove that the integral from 0 to ∞ of the load power in Fig. 7.10d is exactly equal to the energy stored in the inductor $E_L = \frac{1}{2}LI_S^2$ prior to $t = 0$. *Hint:* The proof should include analytical integration of the instantaneous power in Eq. (7.21c).

Problem 7.35. A 2-mH inductor, shown in the following figure, releases its energy into the 2-kΩ load resistor. The supply current is 0.8 A.

A. Find the time constant of the RL circuit (show units).

B. Express the current through the inductor as a function of time and sketch it to scale versus time over time interval from −2 μs to 5 μs.



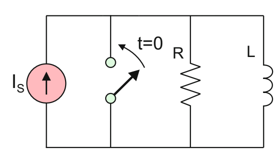


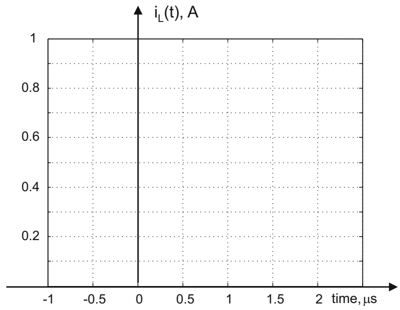
Problem 7.36. A 270-μH inductor, shown in the following figure, releases its energy into the 510-Ω load resistor. The supply current is 0.8 A.

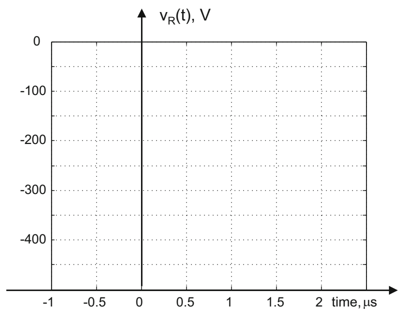
A. Find the time constant of the RL circuit (show units).

B. Express the current through the inductor as a function of time and sketch it to scale versus time over time interval from −1 μs to 2.5 μs.

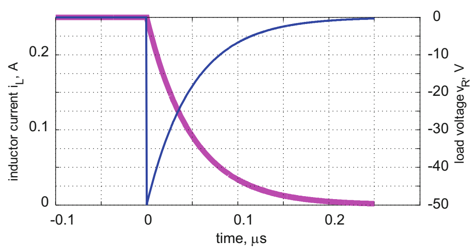
C. Express the resistor voltage as a function of time and sketch it to scale versus time over time interval from −1 μs to 2.5 μs.







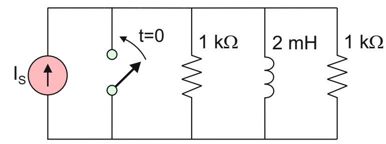
Problem 7.37. For an unknown energy-release RL circuit, inductor current and resistor voltage were measured before and after closing the switch at $t = 0$ as shown in the figure that follows. Approximate R and L.

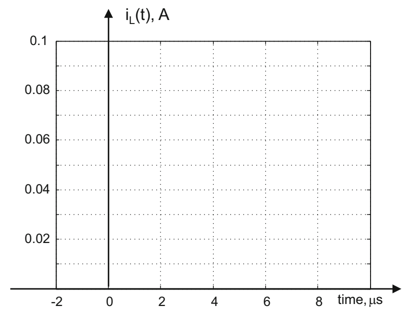


Problem 7.38. A 2-mH inductor, shown in the following figure, releases its energy into two 1-kΩ load resistors. The supply current is 100 mA.

A. Find the time constant of the RL circuit (show units).

B. Express the current through the inductor as a function of time and sketch it to scale versus time over time interval from −2 µs to 10 µs.

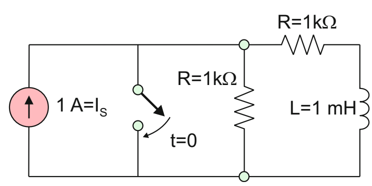




Problem 7.39.

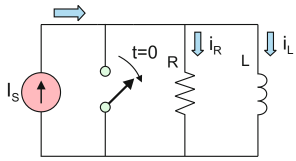
A. Obtain the solution for the inductor current in the circuit shown in the figure at any time.

B. Plot to scale the current through the inductor versus time over the interval from -2τ to 10τ.

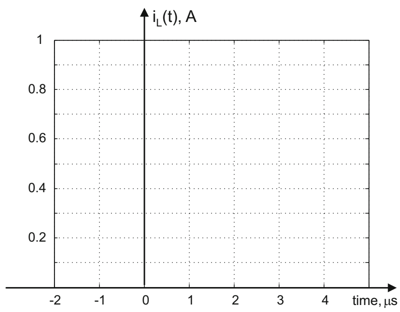


7.2.3 Energy-Accumulating Inductor Circuit

Problem 7.40. In the energy-accumulating RL circuit shown in the following figure, $R = 2$ kΩ and $L = 2$ mH. The supply current is 1 A.

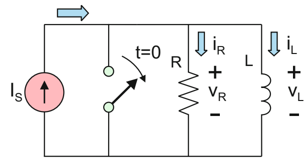


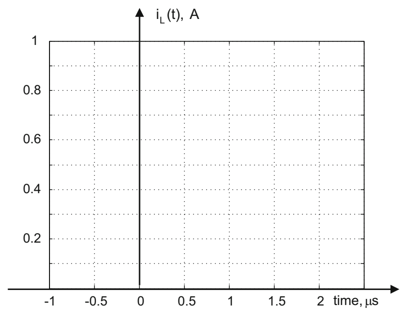
A. Find the time constant of the RL circuit (show units).
B. Express the current through the inductor as a function of time and sketch it to scale versus time over time interval from −2 µs to 5 µs.

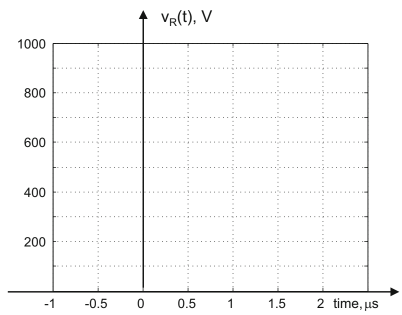


Problem 7.41. In the energy-accumulating RL circuit shown in the following figure, $R = 510$ Ω and $L = 270$ µH. The supply current is 1 A.

A. Find the time constant of the RL circuit (show units).
B. Express the current through the inductor as a function of time and sketch it to scale versus time over time interval from −1 µs to 2.5 µs.
C. Express the resistor voltage as a function of time and sketch it to scale versus time over time interval from −1 µs to 2.5 µs.

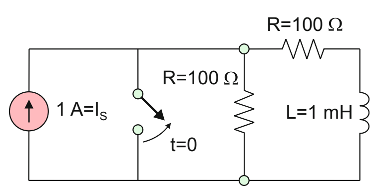






Problem 7.42.

A. Obtain the solution for the inductor current in the circuit shown in the figure at any time.
B. Plot the voltage across the rightmost resistor versus time over the interval from -2τ to 5τ.

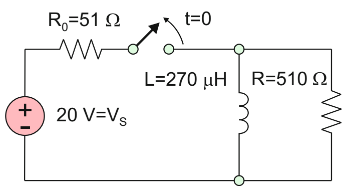


7.2.4 Energy-Release RL Circuit with the Voltage Supply

7.2.5 Application Example: Laboratory Ignition Circuit

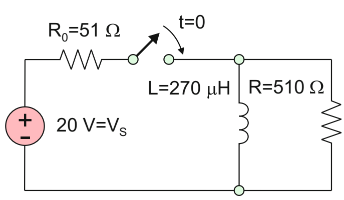
Problem 7.43. A 270-μH inductor shown in the following figure releases its energy into the 510-Ω load resistor. The power supply voltage is 20 V. The switch opens at $t = 0$.

A. Present an expression for the inductor current as a function of time and sketch it to scale versus time over the interval from -1 μs to 2.5 μs.
B. Repeat the same task for the resistor voltage.



Problem 7.44. The circuit for the previous problem is converted to the energy-accumulating RL circuit by inversing the switch operation. Assume that the switch was open prior to $t = 0$. The switch closes at $t = 0$.

A. Derive an expression for the inductor current as a function of time.
B. Repeat the same task for the voltage across resistor R.
C. Could this circuit generate large voltage spikes, similar to the circuit from the previous problem?



7.3 Switching RC Oscillator

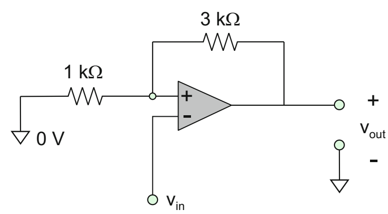
7.3.2 Bistable Amplifier Circuit with the Positive Feedback

7.3.3 Triggering

Problem 7.45. The bistable amplifier circuit shown in the following figure (inverting Schmitt trigger) exists in the positive stable state. Amplifier's power supply rails are ±12 V. Determine output voltage when the applied trigger signal is

1. $v_{in} = 6$ V
2. $v_{in} = 2$ V
3. $v_{in} = -4$ V

Assume that the amplifier hits the power rails in saturation.

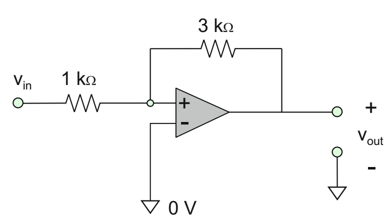


Problem 7.46. Repeat the previous problem when the initial stable state of the amplifier circuit is negative.

Problem 7.47. The bistable amplifier circuit shown in the following figure (non-inverting Schmitt trigger) exists in the positive stable state. Amplifier's power supply rails are ±15 V. Determine output voltage when the applied trigger signal is

A. $v_{in} = -1$ V
B. $v_{in} = -2$ V
C. $v_{in} = -4$ V

Assume that the amplifier hits the power rails in saturation.

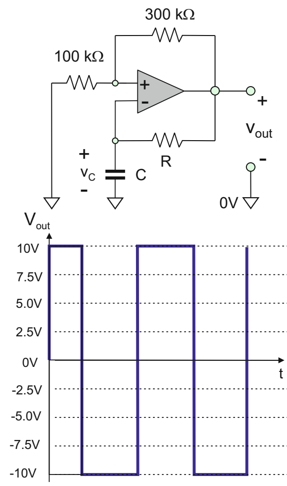


Problem 7.48. Repeat the previous problem when the initial stable state of the amplifier circuit is negative.

7.3.4 Switching RC Oscillator

7.3.5 Oscillation Frequency

Problem 7.49. A clock circuit (relaxation oscillator circuit) shown in the following figure is powered by a ±10-V power supply.



Sketch to scale the capacitor voltage, v_C, as a function of time. Assume that $v_C(t = 0) = 0$. Assume the ideal amplifier model. The specific values of R and C do not matter; they are already included in the time scale.

Problem 7.50. An RC clock circuit is needed with the oscillation frequency of 1 kHz and amplitude of the capacitor voltage of 4 V. Determine one possible set of circuit parameters R_1, R_2, R given that the capacitance of 100 nF is used. The power supply voltage of the amplifier is ±12 V. Assume that the amplifier hits the power rails in saturation.

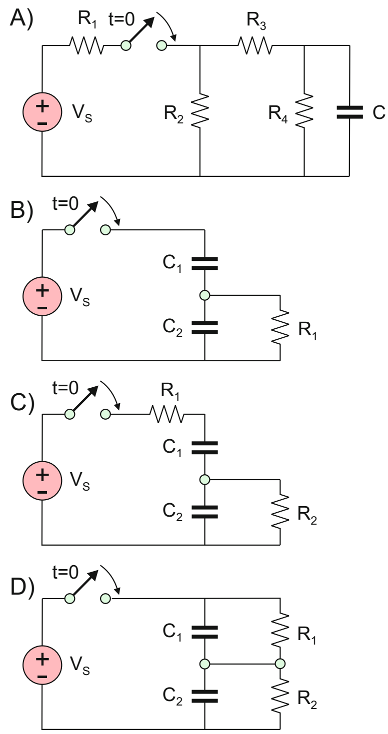
Problem 7.51. A relaxation oscillator circuit may generate nearly triangular waveforms at the capacitor. Which values should the feedback factor $\beta = \frac{R_1}{R_1+R_2}$ attain to make it possible?

7.4 Single-Time-Constant (STC) Transient Circuits

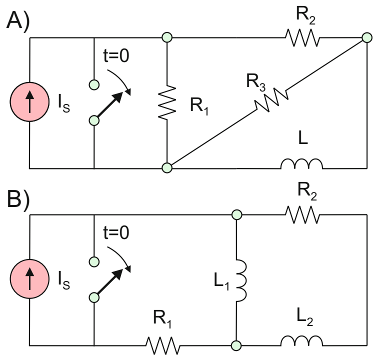
7.4.1 Circuits with Resistances and Capacitances

7.4.2 Circuits with Resistances and Inductances

Problem 7.52. Determine whether or not the transient circuits shown in the following figure are the STC circuits. If this is the case, express the corresponding time constant in terms of the circuit parameters.



Problem 7.53. Repeat the previous problem for the circuits shown in the following figure.

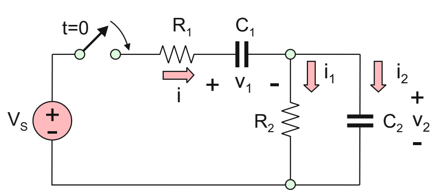


7.4.3 Example of a Non-STC Transient Circuit

Problem 7.54. Using software of your choice, generate the solution for the non-STC circuit in Fig. 7.25a over the time interval from 0 to $15\tau_0$ where $\tau_0 = RC$ with $C_1 = C_2 = C$, $R_1 = R_2 = R$. Use $R = 1$ kΩ, $C = 1$ μF, and $V_S = 10$ V. Plot two capacitor voltages and the circuit current as functions of time to scale.

Problem 7.55.

A. Derive the general ODE for the non-STC circuit in Fig. 7.25a in terms of υ_1 for arbitrary circuit parameters.
B. Present its particular form when $C_1 = C_2 = C$ and $R_1 = 2R_2 = R$. Express all coefficients in terms of $\tau_0 = RC$.
C. Given that the solution for the homogeneous ODE has the form $\exp(-\alpha t/\tau_0)$, determine two possible solutions for the dimensionless coefficient α.
D. Using software of your choice, generate the circuit solution over the time interval from 0 to $15\tau_0$. Use $R = 1$ kΩ, $C = 1$ μF, and $V_S = 10$ V. Plot two capacitor voltages and the circuit current as functions of time to scale.



7.4.4 Example of a STC Circuit

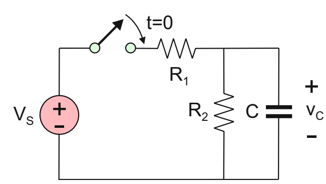
Problem 7.56. For the circuit shown in Fig. 7.26, derive ODEs for inductor currents i_1, i_2.

7.4.5 Method of Thévenin Equivalent. Application Example: Circuit with a Bypass Capacitor

Problem 7.57. In the circuit from Fig. 7.27a, another resistance R_0 is present *in series* with the capacitance C. Determine the natural response of the circuit and find the corresponding time constant, τ.

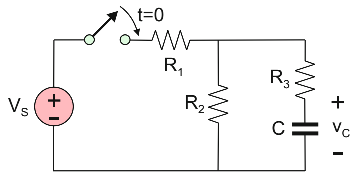
Problem 7.58. A transient circuit with the DC voltage source V_S is shown in the following figure. Given that $V_S = 10$ V and $R_1 = 1$ kΩ, $R_2 = 1$ kΩ, and $C = 1$ μF:

A. Present the ODE for the capacitor voltage $\upsilon_C(t)$.
B. Determine the value of the time constant τ and the ODE right-hand side (the forcing function).
C. Present the solution for the capacitor voltage as a function of time assuming an initially uncharged capacitor



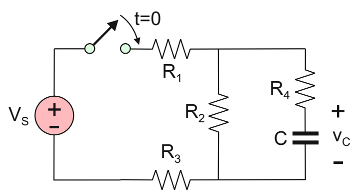
Problem 7.59. A transient circuit with the DC voltage source V_S is shown in the following figure. Given that $V_S = 10$ V and $R_1 = R_2 = 1$ kΩ, $R_3 = 2$ kΩ, and $C = 1$ μF:

A. Present the ODE for the capacitor voltage $\upsilon_C(t)$.
B. Determine the value of the time constant τ and the ODE right-hand side (the forcing function).
C. Present the solution for the capacitor voltage as a function of time assuming an initially uncharged capacitor.



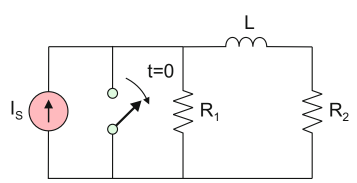
Problem 7.60. A transient circuit with the DC voltage source V_S is shown in the following figure. Given that $V_S = 10\,\text{V}$ and $R_1 = R_2 = 1\,\text{k}\Omega$, $R_3 = R_4 = 2\ \text{k}\Omega$, and $C = 1\ \mu\text{F}$:

A. Present the ODE for the capacitor voltage $v_C(t)$.
B. Determine the value of the time constant τ and the ODE right-hand side (the forcing function).
C. Present the solution for the capacitor voltage as a function of time assuming an initially uncharged capacitor.



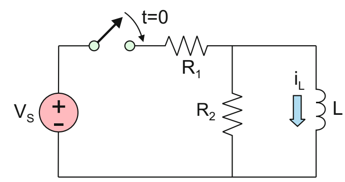
Problem 7.61. A transient circuit with the current source I_S is shown in the following figure. Given that $I_S = 10\ \text{mA}$ and $R_1 = R_2 = 1\ \text{k}\Omega$ and $L = 1\ \text{mH}$:

A. Present the ODE for the inductor current $i_L(t)$.
B. Determine the value of the time constant τ and the ODE right-hand side (the forcing function).
C. Present the solution for the inductor current as a function of time assuming the initial current equal to zero.



Problem 7.62. A transient circuit with the DC voltage source V_S is shown in the figure below. Given that $V_S = 10\ \text{V}$ and $R_1 = R_2 = 1\ \text{k}\Omega$, $L = 1\ \text{mH}$:

A. Present the ODE for the inductor current $i_L(t)$.
B. Determine the value of the time constant τ and the ODE right-hand side (the forcing function).
C. Present the solution for the inductor current as a function of time assuming the initial current equal to zero.



Problem 7.63. Consider the circuits in two previous problems at arbitrary values of R_1, R_2, L, I_S, V_S. What should be the relation between these parameters to guarantee the *same* solution for the inductor current in every case?

Problem 7.64. Describe the mathematical meaning of

A. Natural response
B. Forced response

for a first-order transient circuit in your own words. Do you think that this concept can be applied to any transient circuit?

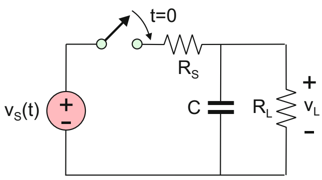
Problem 7.65. If a transient circuit uses a DC supply (either voltage or current) and a switch, what is a general form of the forcing function?

Problem 7.66. If the forcing function of a first-order transient circuit is a combination of sine/cosine function and a constant, what is the general form of the forced response?

Problem 7.67. In the transient circuit shown in the figure below, $v_S(t) = V_S + V_m \sin \omega t$.

A. Write the solution for the voltage across the load resistor R_L in terms of the circuit parameters assuming an initially uncharged capacitor.

B. Write the solution for the voltage across the load resistor R_L when the bypass capacitor is absent.



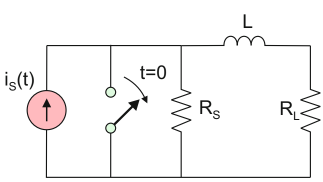
Problem 7.68. Plot to scale the load voltage for the circuit shown in Fig. 7.27a with and without the bypass capacitor over time interval from 0 to 50 ms. The circuit parameters are $R_S = 5\ \Omega$, $R_L = 1\ \text{k}\Omega$, $C = 500\ \mu\text{F}$. The source given by Eq. (7.50) is the superposition of the DC and AC components. The source parameters are $V_S = 10\ \text{V}$, $V_m = 1\text{V}$, $f = 250\ \text{Hz}$.

Problem 7.69. What is the asymptotic form of the solution given by Eqs. (7.57)–(7.58) when the source resistance, R_S, tends to zero?

Problem 7.70. In the transient circuit shown in the figure below, assume $i_S(t) = I_S + I_m \sin \omega t$.

A. Write the solution for the current through the load resistor R_L in terms of the circuit parameters assuming that the initial inductor current is equal to zero.

B. Write the solution for the current through the load resistor R_L when the decoupling inductor is absent

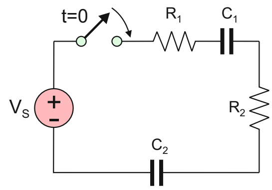


Problem 7.71. In the circuit for the previous problem, another resistor R_0 is present *in parallel* with the inductance L. Determine the natural response of the circuit and find the time constant, τ.

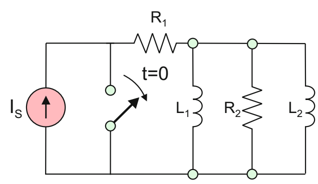
7.5 Description of the Second-Order Transient Circuits

7.5.1 First-order Transient Circuits Versus Second-order Transient Circuits

Problem 7.72. A transient circuit is shown in the following figure. Is it a first- or second-order transient circuit? Justify your answer.



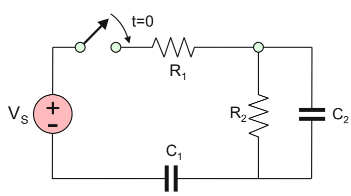
Problem 7.73. A transient circuit is shown in the figure below. Is it a first- or second-order transient circuit? Justify your answer.



Problem 7.74. Establish the ODE for the transient circuit shown in the figure below. Both capacitors have zero voltage prior to closing the switch.

A. Assume $R_1 = R_2 = R$, $C_1 = C_2 = C$.

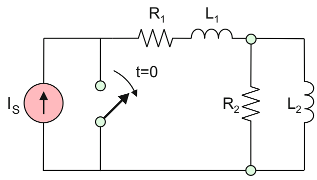
B. Assume arbitrary values of $R_{1,2}, C_{1,2}$.



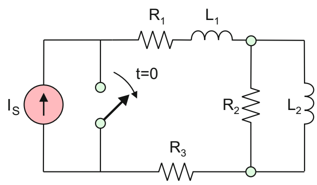
Problem 7.75. Establish the ODE for the transient circuit shown in the figure below. Inductor currents are zero prior opening the switch.

A. Assume $R_1 = R_2 = R$, $L_1 = L_2 = L$.

B. Assume arbitrary values of $R_{1,2}, L_{1,2}$.



Problem 7.76. In the previous problem, add resistance R_3 as shown in the figure that follows and solve task B.



7.5.2 Series Connected Second-order RLC Circuit

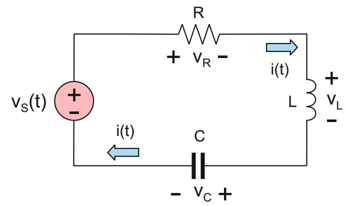
Problem 7.77. Describe in your own words the mechanical counterpart of the series RLC circuit.

Problem 7.78. An RLC circuit in Fig. 7.30 has $R = 1\ \text{k}\Omega$, $C = 1\ \mu\text{F}$, $L = 1\ \text{mH}$.

A. Find the value of the damping coefficient, α (show units).
B. Find the value of undamped resonant frequency, ω_0.

Problem 7.79. How does the second-order ODE Eq. (7.63) for the circuit current in the series RLC circuit from Fig. 7.30 change if the capacitor was charged to $V_S/2$ prior to $t = 0$?

Problem 7.80. In the circuit shown in Fig. 7.30, the switch is replaced by a short circuit. The constant voltage source is replaced by an arbitrary time-varying voltage source $V_S \to v_S(t)$ as shown in the figure that follows. Derive the dynamic circuit equation for the circuit current similar to Eq. (7.63) of this section. Present your result in terms of damping coefficient α and undamped resonant frequency ω_0.



7.5.3 Choice of Independent Function: Initial Conditions

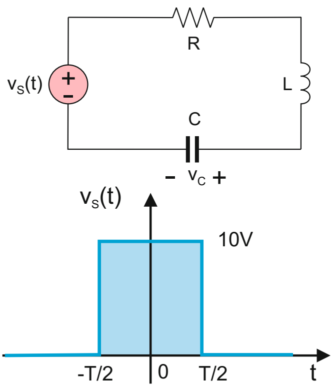
7.5.4 Step Response

Problem 7.81. The RLC circuit shown in Fig. 7.30 is described by dynamic equation (7.63) written in terms of the electric current. How do the initial conditions to this equation change if the capacitor was charged to $V_S/2$ prior to $t = 0$?

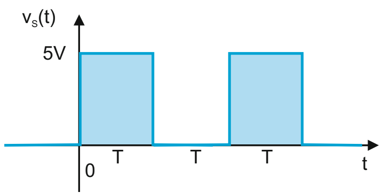
Problem 7.82. For the circuit shown in Fig. 7.30:

A. Derive the dynamic circuit equation (7.67) in terms of the capacitor voltage $v_C(t)$.
B. How does this equation change if the capacitor was charged to $V_S/2$ prior to $t = 0$?

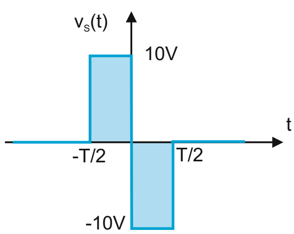
Problem 7.83. For the series RLC circuit with the switch and the DC supply shown in Fig. 7.30, we know the solution, $v_C(t)$, of Eq. (7.67) or Eq. (7.69) for $V_S = 1$ V. The circuit shown in the following figure is now considered, with the voltage source in the form of a voltage pulse (one bit) centered about $t = 0$. Express the solution to the present problem in terms of $v_C(t)$.



Problem 7.84. Repeat the previous problem for the voltage source shown in the figure that follows (a voltage pulse train of two bits).



Problem 7.85. Repeat Problem 7.83 for the voltage source shown in the figure that follows (a bipolar voltage pulse).



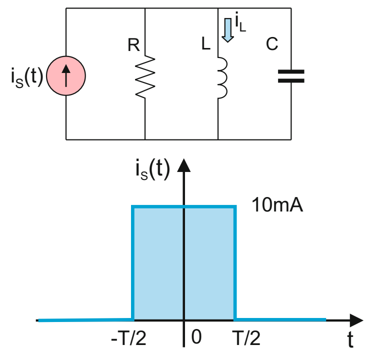
7.5.5 Parallel Connected Second-Order RLC Circuit

Problem 7.86. For the circuit shown in Fig. 7.34:

A. Derive the dynamic circuit equation (7.75) written in terms of the inductor current $i_L(t)$.
B. How does this equation change if the switch in Fig. 7.34 was open prior to $t = 0$ and closes at $t = 0$?
C. How do the initial conditions change in this case?

Problem 7.87. Describe the duality between series and parallel RLC circuits in your own words.

Problem 7.88. For the parallel RLC circuit with the switch and the DC current source shown in Fig. 7.34, we know the solution, $i_L(t)$, of Eq. (7.75) for $I_S = 1$ mA. The circuit shown in the following figure is now considered, with the current source in the form of a pulse (one bit) centered about $t = 0$. Express the solution to in terms of $i_L(t)$.



7.6. Step Response of the Series RLC Circuit

7.6.1 General Solution of the Second-order ODE

7.6.2 Derivation of Complementary Solution

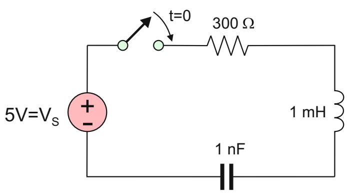
Problem 7.89.

A. The complete solution to a second-order homogeneous ODE is a sum of two distinct components. Describe each of them.
B. Write three forms of the complementary solution (natural response) for the second-order homogeneous ODE.
C. What is a new parameter to be introduced for the underdamped circuit?

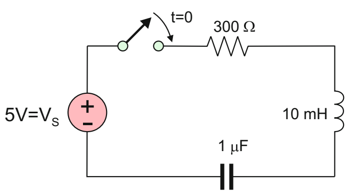
Problem 7.90. For the series RLC circuit shown in the following figure:

A. Find the value of the damping coefficient, α (show units).

B. Find the value of the undamped resonant frequency, ω_0 (show units).
C. Find the value of the damping ratio, ζ (show units).
D. Find the particular solution (forced response).
E. Outline the form of the complementary solution (natural response).
F. Which value should the circuit resistance have for a critically damped circuit?



Problem 7.91. Repeat the previous problem for the series RLC circuit shown in the figure below.



Problem 7.92. For the series RLC circuit shown in the following figure, fill out the table of circuit parameters.

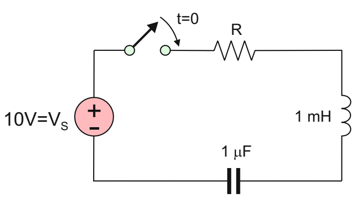


Table of circuit parameters

R, Ω	ζ	Circuit type (overdamped, critically damped, underdamped)
25		
50		
75		
100		

Given fixed *L* and *C*, which values of resistance (large or small) lead to the overdamped circuit?

Problem 7.93. For the series RLC circuit shown in the figure below, fill out the table of circuit parameters.

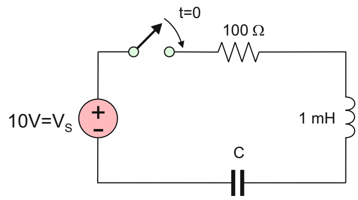


Table of circuit parameters

C, μF	ζ	Circuit type (overdamped, critically damped, underdamped)
0.01		
0.1		
0.4		
1.0		

Given fixed *L* and *R*, which values of capacitance (large or small) lead to the overdamped circuit?

Problem 7.94. For the series RLC circuit shown in the following figure, fill out the table of circuit parameters.

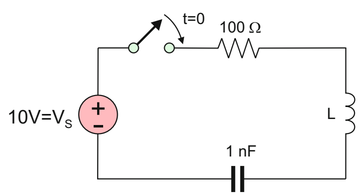


Table of circuit parameters

L, μH	ζ	Circuit type (overdamped, critically damped, underdamped)
0.1		
1		
2.5		
10		

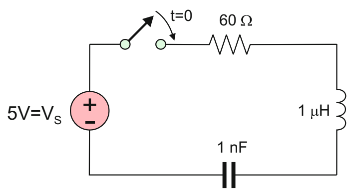
Given fixed R and C, which values of inductance (large or small) lead to the overdamped circuit?

Problem 7.95. Show that underdamped solution and critically damped solutions coincide with each other when $\varsigma \to 1$.

7.6.3 Finding Integration Constants

7.6.4 Solution Behavior for Different Damping Ratios

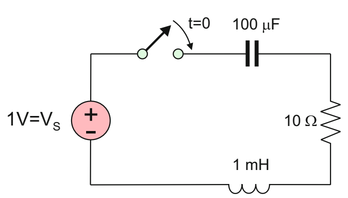
Problem 7.96. For the circuit shown in the figure below:



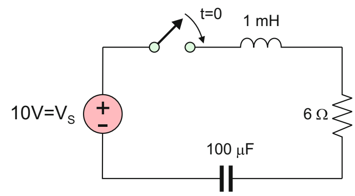
A. Determine damping coefficient α, undamped resonant frequency ω_0, and damping ratio ζ.
B. Determine constants K_1, K_2.
C. Write solution for the capacitor voltage with all constants defined.
D. Calculate and plot to scale capacitor voltage at 0, 0.05, 0.1, 0.2, and 0.3 μs.

Problem 7.97. For the circuit shown in the following figure:

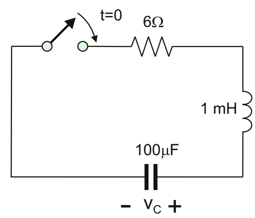
A. Determine damping coefficient α, undamped resonant frequency ω_0, and damping ratio ζ.
B. Determine constants K_1, K_2.
C. Write the solution for the capacitor voltage with all constants defined.
D. Calculate capacitor voltage at 0, 1, 2, 3, 4, and 5 ms and plot it to scale versus time.



Problem 7.98. Repeat the previous problem for the circuit shown in the following figure.



Problem 7.99. In the circuit shown in the figure below, the capacitor was charged to 10 V prior to closing the switch.



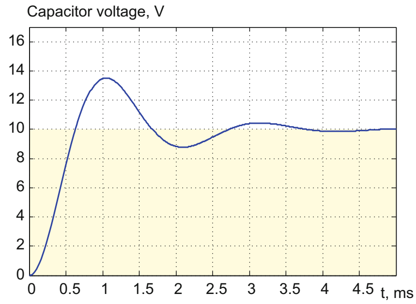
A. How are the circuit equation and initial conditions different from Eqs. (7.78a, b)?
B. Determine damping coefficient α, undamped resonant frequency ω_0, and damping ratio ζ.
C. Determine constants K_1, K_2.
D. Write the solution for capacitor voltage with all constants defined.
E. Calculate the capacitor voltage at 0, 1, 2, 3, 4, and 5 ms and plot it to scale versus time.

7.6.5 Overshoot and Rise Time

7.6.6 Application: Non-ideal Digital Waveform

Problem 7.100. The following figure shows the underdamped step response for a series RLC circuit. The DC source has the voltage of 10 V. Using the figure:

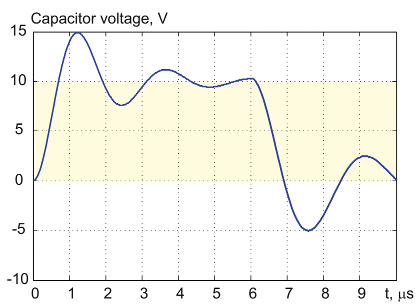
A. Estimate the overshoot percentage.
B. Estimate the rise time.
C. Do these estimates (approximately) agree with Eqs. (7.88a, b)?



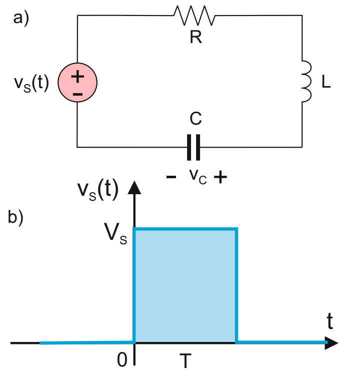
Problem 7.101. Capacitor voltage is measured in a series RLC circuit as shown in the figure to the previous problem. Given $R = 2\,\Omega$, estimate circuit inductance L and circuit capacitance C.

Problem 7.102. The figure that follows shows the distorted rectangular waveform (capacitor voltage) for the circuit shown in Fig. 7.33. The DC source has the voltage of 10 V.

A. Using the figure, estimate the overshoot and undershoot percentages.
B. Using the figure, estimate the rise time and the fall time.
C. Do these estimates (approximately) agree with Eqs. (7.88a, 7.88b)?



Problem 7.103. For the circuit shown in the following figure:



A. Determine the step response $v_C(t)$ for the circuit shown in figure (a) given that $L = 1$ μH, $C = 1$ nF, $V_S = 10$ V, and $R = 75\ \Omega$.
B. Express the solution $v_C^{\text{pulse}}(t)$ for the voltage pulse shown in figure (b) in terms of the step response.
C. Given $T = 0.5$ μs, calculate the solution for the voltage pulse over the time interval from 0 to 0.7 μs in steps of 0.1 μs and plot it to scale.

Problem 7.104. Repeat the previous problem assuming $T = 0.2$ μs. Calculate the solution for the voltage pulse over the time interval from 0 to 0.5 μs in steps of 0.05 μs and plot it to scale versus time.