12 Stationary Time Series Models: Vector Autoregressive Moving-Average Processes (VARMA Processes)

The most important class of models is obtained by requiring $\{X_t\}$ to be the solution of a linear stochastic difference equation with constant coefficients. In analogy to the univariate case, this leads to the theory of vector autoregressive moving-average processes (VARMA processes or just ARMA processes).

Definition 12.1 (VARMA process)**.** A multivariate stochastic process $\{X_t\}$ is a *vector autoregressive moving-average process* of order (p, q), denoted as VARMA(p, q) process, if it is stationary and fulfills the stochastic difference equation

$$X_t - \Phi_1 X_{t-1} - \ldots - \Phi_p X_{t-p} = Z_t + \Theta_1 Z_{t-1} + \ldots + \Theta_q Z_{t-q} \tag{12.1}$$

where $\Phi_p \neq 0$, $\Theta_q \neq 0$ *and* $Z_t \sim \mathrm{WN}(0, \Sigma)$. $\{X_t\}$ *is called a VARMA*(p, q) *process with mean* μ *if* $\{X_t - \mu\}$ *is a VARMA*(p, q) *process.*

With the aid of the lag operator we can write the difference equation more compactly as

$$\Phi(\mathrm{L})X_t = \Theta(\mathrm{L})Z_t$$

where $\Phi(\mathrm{L}) = \mathrm{I}_n - \Phi_1\mathrm{L} - \ldots - \Phi_p\mathrm{L}^p$ and $\Theta(\mathrm{L}) = \mathrm{I}_n + \Theta_1\mathrm{L} + \ldots + \Theta_q\mathrm{L}^q$. $\Phi(\mathrm{L})$ and $\Theta(\mathrm{L})$ are $n \times n$ matrices whose elements are lag polynomials of order smaller or equal to p, respectively q. If $q = 0$, $\Theta(\mathrm{L}) = \mathrm{I}_n$ so that there is no moving-average part. The process is then a purely autoregressive one which is simply called a VAR(p) process. Similarly if $p = 0$, $\Phi(\mathrm{L}) = \mathrm{I}_n$ and there is no autoregressive part. The process is then a purely moving-average one and simply called a VMA(q) process. The importance of VARMA processes stems from the fact that every stationary process can be arbitrarily well approximated by a VARMA process, VAR process, or VMA process.

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12.1 The VAR(1) Process

We start our discussion by analyzing the properties of the VAR(1) process which is defined as the solution the following stochastic difference equation:

$$X_t = \Phi X_{t-1} + Z_t \quad \text{with } Z_t \sim \text{WN}(0, \Sigma).$$

We assume that all eigenvalues of Φ are absolutely strictly smaller than one. As the eigenvalues correspond to the inverses of the roots of the matrix polynomial $\det(\Phi(z)) = \det(I_n - \Phi z)$, this assumption implies that all roots must lie outside the unit circle:

$$\det(\mathrm{I}_n - \Phi z) \neq 0 \text{ for all } z \in \mathbb{C} \text{ with } |z| \leq 1.$$

For the sake of exposition, we will further assume that Φ is diagonalizable, i.e. there exists an invertible matrix P such that $J = P^{-1}\Phi P$ is a diagonal matrix with the eigenvalues of Φ on the diagonal.[1]

Consider now the stochastic process

$$X_t = Z_t + \Phi Z_{t-1} + \Phi^2 Z_{t-2} + \ldots = \sum_{j=0}^{\infty} \Phi^j Z_{t-j}.$$

We will show that this process is stationary and fulfills the first order difference equation above. For $\{X_t\}$ to be well-defined, we must show that $\sum_{j=0}^{\infty} \|\Phi^j\| < \infty$. Using the properties of the matrix norm we get:

$$\begin{aligned}\sum_{j=0}^{\infty} \|\Phi^j\| &= \sum_{j=0}^{\infty} \|\mathrm{P} J^j \mathrm{P}^{-1}\| \leq \sum_{j=0}^{\infty} \|\mathrm{P}\| \|J^j\| \|\mathrm{P}^{-1}\| \\ &\leq \sum_{j=0}^{\infty} \|\mathrm{P}\| \; \|\mathrm{P}^{-1}\| \sqrt{\sum_{i=1}^{n} |\lambda_i|^{2j}} \\ &\leq \|\mathrm{P}\| \; \|\mathrm{P}^{-1}\| \sqrt{n} \sum_{j=0}^{\infty} |\lambda_{\max}|^{2j} < \infty,\end{aligned}$$

where $\lambda_{\max}$ denotes the maximal eigenvalue of Φ in absolute terms. As all eigenvalues are required to be strictly smaller than one, this clearly also holds for $\lambda_{\max}$ so that infinite matrix sum converges. This implies that the process $\{X_t\}$ is stationary. In addition, we have that

[1]The following exposition remains valid even if Φ is not diagonalizable. In this case one has to rely on the Jordan form which complicates the computations (Meyer 2000).

$$X_t = \sum_{j=0}^{\infty} \Phi^j Z_{t-j} = Z_t + \Phi \sum_{j=0}^{\infty} \Phi^j Z_{t-1-j} = \Phi X_{t-1} + Z_t.$$

Thus, the process $\{X_t\}$ also fulfills the difference equation.

Next we demonstrate that this process is also the unique stationary solution to the difference equation. Suppose that there exists another stationary process $\{Y_t\}$ which also fulfills the difference equation. By successively iterating the difference equation one obtains:

$$\begin{aligned} Y_t &= Z_t + \Phi Z_{t-1} + \Phi^2 Y_{t-2} \\ &\dots \\ &= Z_t + \Phi Z_{t-1} + \Phi^2 Z_{t-2} + \dots + \Phi^k Z_{t-k} + \Phi^{k+1} Y_{t-k-1}. \end{aligned}$$

Because $\{Y_t\}$ is assumed to be stationary, $\mathbb{V}Y_t = \mathbb{V}Y_{t-k-1} = \Gamma(0)$ so that

$$\mathbb{V}\left(Y_t - \sum_{j=0}^{k} \Phi^j Z_{t-j} \right) = \Phi^{k+1}\mathbb{V}(Y_{t-k-1})\Phi'^{k+1} = \Phi^{k+1}\Gamma(0)\Phi'^{k+1}.$$

The submultiplicativity of the norm then implies:

$$\left\| \Phi^{k+1}\Gamma(0)\Phi'^{k+1} \right\| \leq \left\| \Phi^{k+1} \right\|^2 \|\Gamma(0)\| = \|P\|^2 \|P^{-1}\|^2 \|\Gamma(0)\| \left(\sum_{i=1}^{n} |\lambda_i|^{2(k+1)} \right).$$

As all eigenvalues of Φ are absolutely strictly smaller than one, the right hand side of the above expression converges to zero for k going to infinity. This implies that Y_t and $X_t = \sum_{j=0}^{\infty} \Phi^j Z_{t-j}$ are equal in the mean square sense and thus also in probability.

Based on Theorem 10.2, the mean and the covariance function of the VAR(1) process is:

$$\begin{aligned} \mathbb{E}X_t &= \sum_{j=0}^{\infty} \Phi^j \mathbb{E}Z_{t-j} = 0, \\ \Gamma(h) &= \sum_{j=0}^{\infty} \Phi^{j+h}\Sigma\Phi'^j = \Phi^h \sum_{j=0}^{\infty} \Phi^j \Sigma \Phi'^j = \Phi^h \Gamma(0). \end{aligned}$$

Analogously to the univariate case, it can be shown that there still exists a unique stationary solution if all eigenvalues are absolutely strictly greater than one. This solution is, however, no longer causal with respect to $\{Z_t\}$. If some of the eigenvalues of Φ are on the unit circle, there exists no stationary solution.

12.2 Representation in Companion Form

A VAR(p) process of dimension n can be represented as a VAR(1) process of dimension $p \times n$. For this purpose we define the pn vector $Y_t = (X_t', X_{t-1}', \ldots, X_{t-p+1}')'$. This new process $\{Y_t\}$ is characterized by the following first order stochastic difference equation:

$$Y_t = \begin{pmatrix} X_t \\ X_{t-1} \\ X_{t-2} \\ \vdots \\ X_{t-p+1} \end{pmatrix} = \begin{pmatrix} \Phi_1 & \Phi_2 & \ldots & \Phi_{p-1} & \Phi_p \\ I_n & 0 & \ldots & 0 & 0 \\ 0 & I_n & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I_n & 0 \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_{t-2} \\ X_{t-3} \\ \vdots \\ X_{t-p} \end{pmatrix} + \begin{pmatrix} Z_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \Phi Y_{t-1} + U_t$$

where $U_t = (Z_t, 0, 0, \ldots, 0)'$ with $U_t \sim \text{WN}\left(0, \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}\right)$. This representation is also known as the *companion form* or *state space representation* (see also Chap. 17). In this representation the last $p(n-1)$ equations are simply identities so that there is no error term attached. The latter name stems from the fact that Y_t encompasses all the information necessary to describe the state of the system. The matrix Φ is called the companion matrix of the VAR(p) process.[2]

The main advantage of the companion form is that by studying the properties of the VAR(1) model, one implicitly encompasses VAR models of higher order and also univariate AR(p) models which can be considered as special cases. The relation between the eigenvalues of the companion matrix and the roots of the polynomial matrix $\Phi(z)$ is given by the formula (Gohberg et al. 1982):

$$\det\left(I_{np} - \Phi z\right) = \det\left(I_n - \Phi_1 z - \ldots - \Phi_p z^p\right). \tag{12.2}$$

In the case of the AR(p) process the eigenvalues of Φ are just the inverses of the roots of the polynomial $\Phi(z)$. Further elaboration of state space models is given in Chap. 17.

12.3 Causal Representation

As will become clear in Chap. 15 and particularly in Sect. 15.2, the issue of the existence of a *causal representation* is even more important than in the univariate

[2]The representation of a VAR(p) process in companion form is not uniquely defined. Permutations of the elements in Y_t will lead to changes in the companion matrix.

case. Before stating the main theorem let us generalize the definition of a causal representation from the univariate case (see Definition 2.2 in Sect. 2.3) to the multivariate one.

Definition 12.2. A VARMA((p,q) process $\{X_t\}$ with $\Phi(\mathrm{L})X_t = \Theta(\mathrm{L})Z_t$ is called *causal* with respect to $\{Z_t\}$ if and only if there exists a sequence of absolutely summable matrices $\{\Psi_j\}, j = 0, 1, 2, \ldots$, i.e. $\sum_{j=0}^{\infty} \|\Psi_j\| < \infty$, such that

$$X_t = \sum_{j=0}^{\infty} \Psi_j Z_{t-j}.$$

Theorem 12.1. *Let $\{X_t\}$ be a VARMA(p,q) process with $\Phi(\mathrm{L})X_t = \Theta(\mathrm{L})Z_t$ and assume that*

$$\det \Phi(z) \neq 0 \quad \textit{for all } z \in \mathbb{C} \textit{ with } |z| \leq 1,$$

then the stochastic difference equation $\Phi(\mathrm{L})X_t = \Theta(\mathrm{L})Z_t$ has exactly one stationary solution with causal representation

$$X_t = \sum_{j=0}^{\infty} \Psi_j Z_{t-j},$$

whereby the sequence of matrices $\{\Psi_j\}$ is absolutely summable and where the matrices are uniquely determined by the identity

$$\Phi(z)\Psi(z) = \Theta(z).$$

Proof. The proof is a straightforward extension of the univariate case. □

As in the univariate case, the coefficient matrices which make up the causal representation can be found by the method of undetermined coefficients, i.e. by equating $\Phi(z)\Psi(z) = \Theta(z)$. In the case of the VAR(1) process, the $\{\Psi_j\}$ have to obey the following recursion:

$$\begin{aligned}
0 &: \quad \Psi_0 = I_n \\
z &: \quad \Psi_1 = \Phi\Psi_0 = \Phi \\
z^2 &: \quad \Psi_2 = \Phi\Psi_1 = \Phi^2 \\
&\quad \ldots \\
z^j &: \quad \Psi_j = \Phi\Psi_{j-1} = \Phi^j
\end{aligned}$$

The recursion in the VAR(2) case is:

$$\begin{aligned}
0 &: \quad \Psi_0 = I_n \\
z &: \quad -\Phi_1 + \Psi_1 = 0 && \Rightarrow \quad \Psi_1 = \Phi_1 \\
z^2 &: \quad -\Phi_2 - \Phi_1\Psi_1 + \Psi_2 = 0 && \Rightarrow \quad \Psi_2 = \Phi_2 + \Phi_1^2 \\
z^3 &: \quad -\Phi_1\Psi_2 - \Phi_2\Psi_1 + \Psi_3 = 0 && \Rightarrow \quad \Psi_3 = \Phi_1^3 + \Phi_1\Phi_2 + \Phi_2\Phi_1 \\
&\quad \ldots
\end{aligned}$$

Remark 12.1. Consider a VAR(1) process with $\Phi = \begin{pmatrix} 0 & \phi \\ 0 & 0 \end{pmatrix}$ with $\phi \neq 0$ then the matrices in the causal representation are $\Psi_j = \Phi^j = 0$ for $j > 1$. This means that $\{X_t\}$ has an alternative representation as a VMA(1) process because $X_t = Z_t + \Phi Z_{t-1}$. This simple example demonstrates that the representation of $\{X_t\}$ as a VARMA process is not unique. It is therefore impossible to always distinguish between VAR and VMA process of higher orders without imposing additional assumptions. These additional assumptions are much more complex in the multivariate case and are known as identifying assumptions. Thus, a general treatment of this identification problem is outside the scope of this book. See Hannan and Deistler (1988) for a general treatment of this issue. For this reason we will concentrate exclusively on VAR processes where these identification issues do not arise.

Example

We illustrate the above concept by the following VAR(2) model:

$$X_t = \begin{pmatrix} 0.8 & -0.5 \\ 0.1 & -0.5 \end{pmatrix} X_{t-1} + \begin{pmatrix} -0.3 & -0.3 \\ -0.2 & 0.3 \end{pmatrix} X_{t-2} + Z_t$$

$$\text{with} \quad Z_t \sim \text{WN}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1.0 & 0.4 \\ 0.4 & 2.0 \end{pmatrix} \right).$$

In a first step, we check whether the VAR model admits a causal representation with respect to $\{Z_t\}$. For this purpose we have to compute the roots of the equation $\det(I_2 - \Phi_1 z - \Phi_2 z^2) = 0$:

$$\det \begin{pmatrix} 1 - 0.8z + 0.3z^2 & 0.5z + 0.3z^2 \\ -0.1z + 0.2z^2 & 1 + 0.5z - 0.3z^2 \end{pmatrix}$$

$$= 1 - 0.3z - 0.35z^2 + 0.32z^3 - 0.15z^4 = 0.$$

The four roots are: $-1.1973, 0.8828 \pm 1.6669\iota, 1.5650$. As they are all outside the unit circle, there exists a causal representation which can be found from the equation $\Phi(z)\Psi(z) = I_2$ by the method of undetermined coefficients. Multiplying the equation system out, we get:

$$\begin{aligned} I_2 - \Phi_1 z \quad & -\Phi_2 z^2 \\ + \Psi_1 z & - \Phi_1 \Psi_1 z^2 - \Phi_2 \Psi_1 z^3 \\ & + \Psi_2 z^2 - \Phi_1 \Psi_2 z^3 - \Phi_2 \Psi_2 z^4 \\ & \qquad \ldots \qquad\qquad\qquad = I_2. \end{aligned}$$

Equating the coefficients corresponding to $z^j, j = 1, 2, \ldots$:

$$\begin{aligned} z: &\quad \Psi_1 = \Phi_1 \\ z^2: &\quad \Psi_2 = \Phi_1 \Psi_1 + \Phi_2 \\ z^3: &\quad \Psi_3 = \Phi_1 \Psi_2 + \Phi_2 \Psi_1 \\ \ldots &\quad \ldots \\ z^j: &\quad \Psi_j = \Phi_1 \Psi_{j-1} + \Phi_2 \Psi_{j-2}. \end{aligned}$$

The last equation shows how to compute the sequence $\{\Psi_j\}$ recursively:

$$\Psi_1 = \begin{pmatrix} 0.8 & -0.5 \\ 0.1 & -0.5 \end{pmatrix} \quad \Psi_2 = \begin{pmatrix} 0.29 & -0.45 \\ -0.17 & 0.50 \end{pmatrix}$$

$$\Psi_3 = \begin{pmatrix} 0.047 & -0.310 \\ -0.016 & -0.345 \end{pmatrix} \qquad \ldots$$

12.4 Computation of the Covariance Function of a Causal VAR Process

As in the univariate case, it is important to be able to compute the covariance and the correlation function of VARMA process (see Sect. 2.4). As explained in Remark 12.1 we will concentrate on VAR processes. Consider first the case of a causal VAR(1) process:

$$X_t = \Phi X_{t-1} + Z_t \qquad Z_t \sim \mathrm{WN}(0, \Sigma).$$

Multiplying the above equation first by X'_t and then successively by X'_{t-h} from the left, $h = 1, 2, \ldots$, and taking expectations, we obtain the Yule-Walker equations:

$$\mathbb{E}\left(X_t X'_t\right) = \Gamma(0) = \Phi\, \mathbb{E}\left(X_{t-1} X'_t\right) + \mathbb{E}\left(Z_t X'_t\right) = \Phi\Gamma(-1) + \Sigma,$$

$$\mathbb{E}\left(X_t X'_{t-h}\right) = \Gamma(h) = \Phi\, \mathbb{E}\left(X_{t-1} X'_{t-h}\right) + \mathbb{E}\left(Z_t X'_{t-h}\right) = \Phi\Gamma(h-1).$$

Knowing $\Gamma(0)$ and Φ, $\Gamma(h)$, $h > 0$, can be computed recursively from the second equation as

$$\Gamma(h) = \Phi^h \Gamma(0), \qquad h = 1, 2, \ldots \tag{12.3}$$

Given Φ and Σ, we can compute $\Gamma(0)$. For $h = 1$, the second equation above implies $\Gamma(1) = \Phi\Gamma(0)$. Inserting this expression in the first equation and using the fact that $\Gamma(-1) = \Gamma(1)'$, we get an equation in $\Gamma(0)$:

$$\Gamma(0) = \Phi\Gamma(0)\Phi' + \Sigma.$$

This equation can be solved for $\Gamma(0)$:

$$\begin{aligned} \mathrm{vec}\Gamma(0) &= \mathrm{vec}(\Phi\Gamma(0)\Phi') + \mathrm{vec}\Sigma \\ &= (\Phi \otimes \Phi)\mathrm{vec}\Gamma(0) + \mathrm{vec}\Sigma, \end{aligned}$$

where $\otimes$ and "vec" denote the Kronecker-product and the vec-operator, respectively.[3] Thus,

$$\mathrm{vec}\Gamma(0) = (I_{n^2} - \Phi \otimes \Phi)^{-1}\mathrm{vec}\Sigma. \tag{12.4}$$

The assumption that $\{X_t\}$ is causal with respect to $\{Z_t\}$ guarantees that the eigenvalues of $\Phi \otimes \Phi$ are strictly smaller than one in absolute value, implying that $I_{n^2} - \Phi \otimes \Phi$ is invertible.[4]

If the process is a causal VAR(p) process the covariance function can be found in two ways. The first one rewrites the process in companion form as a VAR(1) process and applies the procedure just outlined. The second way relies on the Yule-Walker equation. This equation is obtained by multiplying the stochastic difference equation from the left by X_t' and then successively by X_{t-h}', $h > 0$, and taking expectations:

$$\begin{aligned} \Gamma(0) &= \Phi_1\Gamma(-1) + \ldots + \Phi_p\Gamma(-p) + \Sigma, \\ &= \Phi_1\Gamma(1)' + \ldots + \Phi_p\Gamma(p)' + \Sigma, \\ \Gamma(h) &= \Phi_1\Gamma(h-1) + \ldots + \Phi_p\Gamma(h-p). \end{aligned} \tag{12.5}$$

The second equation can be used to compute $\Gamma(h)$, $h \geq p$, recursively taking $\Phi_1, \ldots, \Phi_p$ and the starting values $\Gamma(p-1), \ldots, \Gamma(0)$ as given. The starting value can be retrieved by transforming the VAR(p) model into the companion form and proceeding as explained above.

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[3]The vec-operator stacks the column of a $n \times m$ matrix to a column vector of dimension nm. The properties of $\otimes$ and vec can be found, e.g. in Magnus and Neudecker (1988).

[4]If the eigenvalues of Φ are λ_i, $i = 1, \ldots, n$, then the eigenvalues of $\Phi \otimes \Phi$ are $\lambda_i\lambda_j$, $i, j = 1, \ldots, n$ (see Magnus and Neudecker (1988)).

Example

We illustrate the computation of the covariance function using the same example as in Sect. 12.3. First, we transform the model into the companion form:

$$Y_t = \begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{1,t-1} \\ X_{2,t-1} \end{pmatrix} = \begin{pmatrix} 0.8 & -0.5 & -0.3 & -0.3 \\ 0.1 & -0.5 & -0.2 & 0.3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_{1,t-1} \\ X_{2,t-1} \\ X_{1,t-2} \\ X_{2,t-2} \end{pmatrix} + \begin{pmatrix} Z_{1,t} \\ Z_{2,t} \\ 0 \\ 0 \end{pmatrix}.$$

Equation (12.4) implies that $\Gamma_Y(0)$ is given by:

$$\text{vec}\Gamma_Y(0) = \text{vec}\begin{pmatrix} \Gamma_X(0) & \Gamma_X(1) \\ \Gamma_X(1)' & \Gamma_X(0) \end{pmatrix} = (I_{16} - \Phi \otimes \Phi)^{-1}\text{vec}\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$$

so that $\Gamma_X(0)$ and $\Gamma_X(1)$ become:

$$\Gamma_X(0) = \begin{pmatrix} 2.4201 & 0.5759 \\ 0.5759 & 3.8978 \end{pmatrix} \qquad \Gamma_X(1) = \begin{pmatrix} 1.3996 & -0.5711 \\ -0.4972 & -2.5599 \end{pmatrix}.$$

The other covariance matrices can then be computed recursively according to Eq. (12.5):

$$\Gamma_X(2) = \Phi_1\Gamma_X(1) + \Phi_2\Gamma_X(0) = \begin{pmatrix} 0.4695 & -05191 \\ 0.0773 & 2.2770 \end{pmatrix},$$

$$\Gamma_X(3) = \Phi_1\Gamma_X(2) + \Phi_2\Gamma_X(1) = \begin{pmatrix} 0.0662 & -0.6145 \\ -0.4208 & -1.8441 \end{pmatrix}.$$

Appendix: Autoregressive Final Form

Definition 12.1 defined the VARMA process $\{X_t\}$ as a solution to the corresponding multivariate stochastic difference equation (12.1). However, as pointed out by Zellner and Palm (1974) there is an equivalent representation in the form of n univariate ARMA processes, one for each X_{it}. Formally, these representations, also called *autoregressive final form* or *transfer function form* (Box and Jenkins 1976), can be written as

$$\det \Phi(\mathrm{L})X_{it} = \left[\Phi^*(\mathrm{L})\Theta(\mathrm{L})\right]_{i\bullet} Z_t$$

where the index $i\bullet$ indicates the i-th row of $\Phi^*(L)\Theta(L)$. Thereby $\Phi^*(L)$ denotes the adjugate matrix of $\Phi(L)$.[5] Thus each variable in X_t may be investigated separately as an univariate ARMA process. Thereby the autoregressive part will be the same for each variable. Note, however, that the moving-average processes will be correlated across variables.

The disadvantage of this approach is that it involves rather long AR and MA lags as will become clear from the following example.[6] Take a simple two-dimensional VAR of order one, i.e. $X_t = \Phi X_{t-1} + Z_t$, $Z_t \sim \text{WN}(0, \Sigma)$. Then the implied univariate processes will be ARMA(2,1) processes. After some straightforward manipulations we obtain:

$$(1 - (\phi_{11} + \phi_{22})L + (\phi_{11}\phi_{22} - \phi_{12}\phi_{21})L^2)X_{1t} = Z_{1t} - \phi_{22}Z_{1,t-1} + \phi_{12}Z_{2,t-1},$$
$$(1 - (\phi_{11} + \phi_{22})L + (\phi_{11}\phi_{22} - \phi_{12}\phi_{21})L^2)X_{2t} = \phi_{21}Z_{1,t-1} + Z_{2t} - \phi_{11}Z_{2,t-1}.$$

It can be shown by the means given in Sects. 1.4.3 and 1.5.1 that the right hand sides are observationally equivalent to MA(1) processes.

[5]The elements of the adjugate matrix A^* of some matrix A are given by $[A^*]_{ij} = (-1)^{i+j}M_{ij}$ where M_{ij} is the minor (minor determinant) obtained by deleting the i-th column and the j-th row of A (Meyer 2000, p. 477).

[6]The degrees of the AR and the MA polynomial can be as large as np and $(n-1)p+q$, respectively.