Chapter 9

Angular Kinematics

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9.1 Polar Coordinates

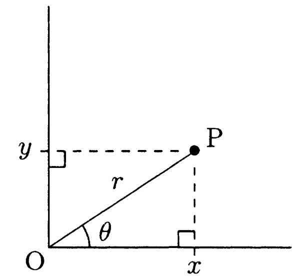

Fig. 9.1 *Rectangular and polar coordinates of point P*

Two-dimensional angular motions of bodies are commonly described in terms of a pair of parameters, r and θ (theta), which are called the *polar coordinates*. Polar coordinates are particularly well suited for analyzing motions restricted to circular paths. As illustrated in Fig. 9.1, let O and P be two points on a two-dimensional surface. The location of P with respect to O can be specified in many different ways. For example, in terms of rectangular coordinates, P is a point with coordinates x and y. Point P is also located at a distance r from point O with r making an angle θ with the horizontal. Both x and y, and r and θ specify the position of P with respect to O uniquely, and O forms the origin of both the rectangular and polar coordinate systems. Note that these pairs of coordinates are not mutually independent. If one pair is known, then the other pair can be calculated because they are associated with a right triangle: r is the hypotenuse, θ is one of the two acute angles, and x and y are the lengths of the adjacent and opposite sides of the right triangle with respect to angle θ. Therefore:

$$x = r\cos\theta$$
$$y = r\sin\theta \tag{9.1}$$

Expressing r and θ in terms of x and y:

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right) \tag{9.2}$$

9.2 Angular Position and Displacement

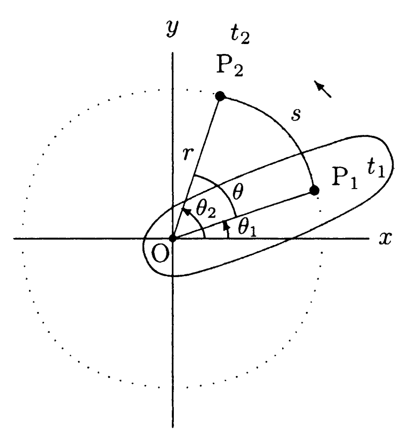

Fig. 9.2 *$\theta = \theta_2 - \theta_1$ is the angular displacement in the time interval between t_1 and t_2*

Consider an object undergoing a rotational motion in the xy-plane about a fixed axis. Let O be a point in the xy-plane along the axis of rotation of the object, and P be a fixed point on the object located at a distance r from O (Fig. 9.2). Point P will move in a circular path of radius r and center located at O. Assume that at some time t_1, the point is located at P_1 with OP_1 making an angle θ_1 with the horizontal. At a later time t_2, the point is at P_2, with OP_2 making an angle θ_2 with the horizontal. Angles θ_1 and θ_2 define the *angular positions* of the point at times t_1 and t_2, respectively. If θ denotes the change in angular position of the point in the time interval between t_1 and t_2, then $\theta = \theta_2 - \theta_1$ is called the *angular displacement* of the point in the same time interval. In the same time interval, the point travels a distance s measured along the circular path. The equation relating the radius r of the circle, angle θ, and arc length s is:

$$s = r\,\theta \quad \text{or} \quad \theta = \frac{s}{r} \tag{9.3}$$

Table 9.1 *Selected angles in degrees and radians*

DEGREES (°)	RADIANS (RAD)
30	$\pi/6 = 0.524$
45	$\pi/4 = 0.785$
60	$\pi/3 = 1.047$
90	$\pi/2 = 1.571$
180	$\pi = 3.142$
270	$3\pi/2 = 4.712$
360	$2\pi = 6.283$

In Eq. (9.3), angle θ must be measured in radians, rather than in degrees. As reviewed in Appendix C, radians and degrees are related in that there are 360° in a complete circle that must correspond to an arc length equal to the circumference, $s = 2\pi r$, of the circle, with $\pi = 3.14$ approximately. Therefore, $\theta = s/r = 2\pi r/r = 2\pi$ for a complete circle, or $360° = 2\pi$. One radian is then equal to $360°/2\pi = 57.3°$. The following formula can be used to convert angles given in degrees to corresponding angles in radians:

$$\theta\,(\text{radians}) = \frac{\pi}{180}\theta\,(\text{degrees})$$

Selected angles and their equivalents in radians are listed in Table 9.1.

9.3 Angular Velocity

The time rate of change of angular position is called *angular velocity*, and it is commonly denoted by the symbol ω (omega). If the angular position of an object is known as a function of time, its angular velocity can be determined by taking the derivative of the angular position with respect to time:

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta} \tag{9.4}$$

Angular velocity of the object is the first derivative of its angular position. The *average angular velocity* ($\overline{\omega}$) of an object in the time interval between t_1 and t_2 is defined by the ratio of change in angular position of the object divided by the time interval:

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_2 - \theta_1}{t_2 - t_1} \tag{9.5}$$

In Eq. (9.5), θ_1 and θ_2 are the angular positions of the object at times t_1 and t_2, respectively.

9.4 Angular Acceleration

The angular velocity of an object may vary during motion. The time rate of change of angular velocity is called *angular acceleration*, usually denoted by the symbol α (alpha). If the angular velocity of a body is given as a function of time, then its angular acceleration can be determined by considering the derivative of the angular velocity with respect to time:

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} \tag{9.6}$$

The *average angular acceleration*, $\overline{\alpha}$, is equal to the change in angular velocity over the time interval in which the change occurs. If ω_1 and ω_2 are the instantaneous angular velocities of a body measured at times t_1 and t_2, respectively, then the average angular acceleration of the body in the time interval between t_1 and t_2 is:

$$\overline{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_1}{t_1 - t_2} \tag{9.7}$$

Note that using the definition of angular velocity in Eq. (9.4), angular acceleration can alternatively be expressed in the following forms:

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \ddot{\theta} \tag{9.8}$$

Angular acceleration of the body is the second derivation of its angular position. Also note that Eqs. (9.4) and (9.6) are the kinematic equations relating angular quantities θ, ω, and α.

Angular displacement, velocity, and acceleration are vector quantities. Therefore, their directions must be stated as well as their magnitudes. For two-dimensional problems, the motion is either in the clockwise or in the counterclockwise direction. Angular displacement and velocity are positive in the direction of motion. Angular acceleration is positive when angular velocity is increasing over time, and it is negative when angular velocity is decreasing over time.

9.5 Dimensions and Units

From Eq. (9.3), the angular displacement θ of an object undergoing circular motion is equal to the ratio of the arc length s and radius r of the circular path. Both arc length and radius have the dimension of length. Therefore, the dimension of angular displacement is 1, or it is a *dimensionless* quantity:

$$[\text{ANGULAR DISPLACEMENT}] = \frac{L}{L} = 1$$

By definition, angular velocity is the time rate of change of angular position, and angular acceleration is the time rate of change of angular velocity. Therefore, angular velocity has the dimension of 1 over time, and angular acceleration has the dimension of angular velocity divided by time, or 1 over time squared.

Note that angular quantities θ, ω, and α differ dimensionally from their linear counterparts x, v, and a by a length factor.

The units of angular quantities in different unit systems are the same. Angular displacement is measured in radians (rad), angular velocity is measured in radians per second (rad/s) or s^{-1}, and angular acceleration is measured in radians per second squared (rad/s^2) or s^{-2}.

9.6 Definitions of Basic Concepts

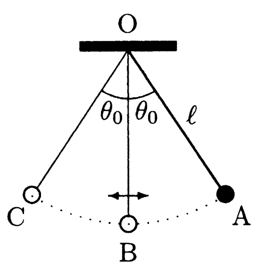

Fig. 9.3 *Pendulum*

To be able to define concepts common in angular motions, consider the simple pendulum illustrated in Fig. 9.3. The pendulum consists of a mass attached to a string. The string is fixed to the ceiling at one end and the mass is free to swing. Assume that l is the length of the string and it is attached to the ceiling at O. If the mass is simply released, it would stretch the string and come to a rest at B that represents the *neutral* or *equilibrium position* of the mass. If the mass is pulled to the side, to position A, so that the string makes an angle θ with the vertical and is then released, the mass will oscillate or swing back and forth about its neutral position in a circular arc path of radius l. Due to internal friction and air resistance, the oscillations will die out over time and eventually the pendulum will come to a stop at its neutral position. An analysis of the motion characteristics of this relatively simple system may give us considerable insight into the nature of other more complex dynamic systems.

For the sake of simplicity, we ignore the air resistance and frictional effects, and assume that once the pendulum is excited, it will oscillate forever. Also, assume that there is a roll of paper behind the pendulum that moves in a prescribed manner. (For example, 10 mm of paper rolls up in each second.) Furthermore, the mass has a dye on it that marks the position of the mass on the paper. In other words, as the mass swings back and forth, it draws its motion path on the paper that would look like the one illustrated in Fig. 9.4. In Fig. 9.4, θ represents the angle that the pendulum makes with the vertical and t is time. Angle θ is a measure of the instantaneous angular position of the pendulum.

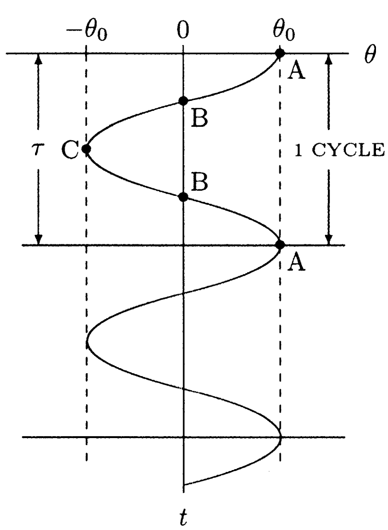

Fig. 9.4 *Simple harmonic motion*

The motion described in Fig. 9.4 is known as the *simple harmonic motion*. At time $t = 0$ that corresponds to the instant when the mass is first released, the mass is located at A which makes an angle θ_0 with the vertical. The mass swings, passes through B where $\theta = 0$, and reaches C where $\theta = -\theta_0$. Here, it is assumed that θ is positive between A and B, zero at B, and negative between B and C. At C, the mass momentarily stops and then reverses its direction of motion from clockwise to counterclockwise. It passes through B again and returns to A, thus completing one full *cycle* in a time interval of τ (tau) seconds, which is called the *period* of harmonic motion. The total angle covered by the pendulum between A and C is called the *range of motion* (ROM) and, in this case, it is equal to $2\theta_0$. Also, the entire motion of the pendulum is confined between $+\theta$ and $-\theta$ that set the limits of the range of motion. Half of the range of motion is called the *amplitude* of the oscillations measured in radians and here is equal to θ_0. Note that in this case, both the amplitude and period of the harmonic motion are constants. Also note that since the effects of friction and air resistance are neglected, the series of events between A, B, C, B, and A are repeated forever in τ time intervals.

From Fig. 9.4, it is clear that angular position θ is a function of time t. Furthermore, θ is a harmonic, cyclic function of t that must remind us of trigonometric functions. As discussed in Appendix C, the θ versus t graph in Fig. 9.4 can be compared to the graphs of known functions to establish the functions that relate θ and t. It can be shown that:

$$\theta = \theta_0 \cos(\varphi t)$$

In this equation, the parameter θ_0 multiplied with the cosine function is the amplitude of the harmonic motion and φ (phi) is called the *angular frequency* measured in radians per second (rad/s). The period and angular frequency are related:

$$\varphi = \frac{2\pi}{\tau} \qquad (\pi = 3.1416)$$

For oscillatory motions, the reciprocal of the period is called the *frequency*, f, measured in Hertz (Hz) that represents the total number of cycles occurring per second:

$$f = \frac{1}{\tau} = \frac{\varphi}{2\pi}$$

Note that for the simple harmonic motion discussed herein the parameters involved (range of motion, amplitude, period, and frequency) are constants. Also note that the validity of the function relating angular position and time can be checked by assigning values to t and calculating corresponding θ values. For example, at A: $t = 0$, $\varphi t = 0$, $\cos(0) = 1$, and $\theta = \theta_0$. At B: $t = \tau/4$, $\varphi t = \pi/2 = 90^\circ$, $\cos(90) = 0$, and $\theta = 0$. At C: $t = \tau/2$, $\varphi t = \pi = 180^\circ$, $\cos(180) = -1$, and $\theta = -\theta_0$. All of these are consistent with the observations in Fig. 9.4.

Now that we have defined most of the important parameters involved, we can also determine the angular velocity and angular acceleration of the pendulum. Utilizing Eqs. (9.4) and (9.6):

$$\omega = \frac{d\theta}{dt} = -\theta_0 \varphi \sin(\varphi t)$$

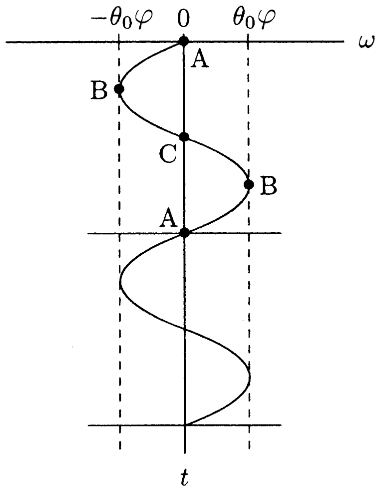

Fig. 9.5 *Angular velocity ω versus time t*

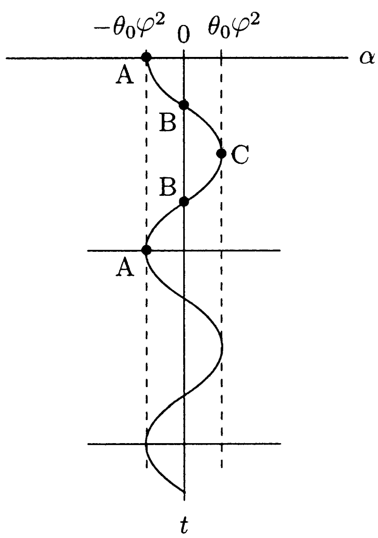

Fig. 9.6 *Angular acceleration α versus time t*

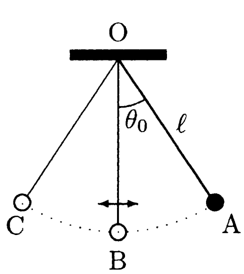

Fig. 9.7 *Pendulum under the effect of air resistance*

$$\alpha = \frac{d\omega}{dt} = -\theta_0\varphi^2 \cos(\varphi t)$$

Once we derived the functions relating angular velocity and angular acceleration with time, we then can determine both angular velocity and angular acceleration at any time by assigning different values to t and calculating corresponding values of ω and α.

For example, concerning ω, at point A: $t = 0$, $\varphi t = 0$, $\sin(0) = 0$, and $\omega = 0$. At point B: $t = \tau/4$, $\varphi t = \pi/2 = 90°$, $\sin(90°) = 1$, and $\omega = -\theta_0\varphi$. At point C: $t = \tau/2$, $\varphi t = \pi = 180°$, $\sin(180°) = 0$, and $\omega = 0$. With the pendulum swinging back from point C to point A, at point B: $t = 3\tau/4$, $\varphi t = 3\pi/2 = 270°$, $\sin(270°) = -1$, and $\omega = \theta_0\varphi$. At point A: $t = \tau$, $\varphi t = 2\pi = 360°$, $\sin(360°) = 0$, and $\omega = 0$. Furthermore, concerning α, at point A: $t = 0$, $\varphi t = 0$, $\cos(0) = 1$, and $\alpha = -\theta_0\varphi^2$. At point B: $t = \tau/4$, $\varphi t = \pi/2 = 90°$, $\cos(90°) = 0$, and $\alpha = 0$. At point C: $t = \tau/2$, $\varphi t = \pi = 180°$, $\cos(180°) = -1$, and $\alpha = \theta_0\varphi^2$. With the pendulum swinging back from point C to point A, at point B: $t = 3\tau/4$, $\varphi t = 3\pi/2 = 270°$, $\cos(270°) = 0$, and $\alpha = 0$. At point A: $t = \tau$, $\varphi t = 2\pi = 360°$, $\cos(360°) = 1$, and $\alpha = -\theta_0\varphi^2$.

These functions are plotted in Figs. 9.5 and 9.6. Notice that the amplitude of the angular velocity of the pendulum is $\theta_0\varphi$, and the amplitude of its angular acceleration is $\theta_0\varphi^2$. They are the terms multiplied by the sine and cosine functions. At A, the angular velocity is zero. Between A and B, the mass accelerates and the magnitude of its angular velocity increases in the clockwise direction. The angular velocity reaches a peak value of $\theta_0\varphi$ at B. The angular velocity is negative and the angular acceleration is positive between B and C. Therefore, the mass decelerates (its angular velocity decreases in the clockwise direction) between B and C. The angular velocity reduces to zero at C. In the meantime, the magnitude of the angular acceleration reaches its peak value of $\theta_0\varphi^2$. Between C and B, the mass accelerates in the counterclockwise direction, the magnitude of its angular velocity returns to a peak at B, slows down between B and A, and momentarily comes to rest at A. This series of events is repeated over time.

Next, consider that the mass is again pulled to A so that the pendulum makes an angle θ_0 with the vertical and is released (Fig. 9.7). The mass will oscillate about its neutral position in a circular arc path of radius l. Due to internal friction and air resistance, the oscillations will die out over time and eventually the pendulum will come to a rest at its neutral position, B. This type of motion is called *damped oscillations*. To help understand some aspects of damped oscillations, consider the angular position θ versus time t graph shown in Fig. 9.8. The pendulum completes four full cycles in t_f seconds before coming to a stop. The period of each cycle is equal, but the amplitude of the harmonic oscillations decreases linearly with time and to zero at time t_f. That is, we have a harmonic motion with a constant period but varying amplitude. For measured θ_0, τ, and t_f, the θ versus time graph shown in Fig. 9.8 can be represented as:

$$\theta = \theta_0 \left(1 - \frac{t}{t_f}\right) \cos(\varphi\, t)$$

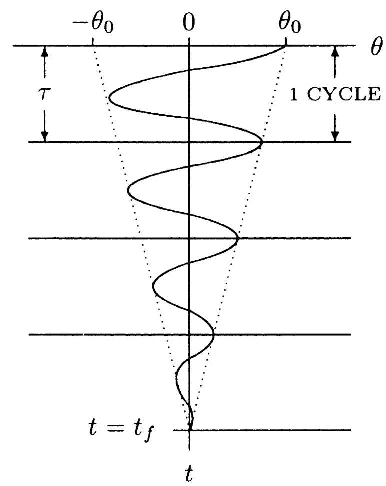

Fig. 9.8 *Damped oscillations*

Here, φ is again the angular frequency of harmonic oscillations and is equal to $2\pi/\tau$. What is different in this case is that the harmonic oscillations of the pendulum are confined between two converging straight lines that can be represented by the functions $\theta = \theta_0(1 - t/t_f)$ and $\theta = -\theta_0(1 - t/t_f)$, and that the oscillations of the pendulum are "damped-out" by friction and air resistance. Knowing the angular position of the pendulum as a function of time enables us to determine the angular velocity and acceleration of the pendulum. Using Eqs. (9.4) and (9.6), and applying the product and chain rules of differentiation (see Appendix C):

$$\omega = \frac{d\theta}{dt} = \frac{\theta_0}{t_f} \cos(\varphi t) - \theta_0\, \varphi \left(1 - \frac{t}{t_f}\right) \sin(\varphi t)$$

$$\alpha = \frac{d\omega}{dt} = \frac{2\theta_0\varphi}{t_f} \sin(\varphi t) - \theta_0\varphi^2 \left(1 - \frac{t}{t_f}\right) \cos(\varphi t)$$

These functions are relatively complex. Their graphs are shown in Fig. 9.9, which are obtained simply by assuming a value for τ, assigning values to t, calculating corresponding ω and α, and plotting them.

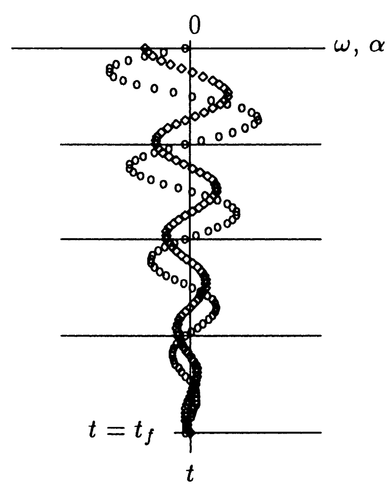

Fig. 9.9 *Angular velocity ω (open circles) and angular acceleration α (open diamonds) versus time t*

Example 9.1 *Shoulder abduction*

Figure 9.10 shows a person doing shoulder abduction in the frontal plane. O represents the axis of rotation of the shoulder joint in the frontal plane, line OA represents the position of the arm when it is stretched out parallel to the ground (horizontal), line OB represents the position of the arm when the hand is at its highest elevation, and line OC represents the position of the arm when the hand is closest to the body. In other words, for this activity, OB and OC are the arm's limits of range of motion. Assume that the angle between OA and OB is equal to the angle between OA and OC, which are represented by angle θ_0. The motion of the arm is symmetric with respect to line OA. Also assume that the time it takes for the arm to cover the angles between OA and OB, OB and OA, OA and OC, and OC and OA are approximately equal.

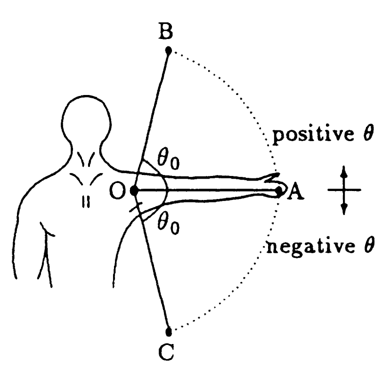

Fig. 9.10 *Shoulder abduction*

Derive expressions for the angular displacement, velocity, and acceleration of the arm. Take the period of angular motion of the arm to be 3 s and the angle θ_0 to be 80°.

Solution: Notice the similarities between the motion of the arm in this example and the simple harmonic motion of the pendulum discussed previously. In this case, angle θ_0 represents the amplitude of the angular displacement of the arm while undergoing a harmonic motion about line OA. The range of motion of the arm is equal to twice that of angle θ_0. The period of the angular motion is given as $\tau = 3$ s, and the angular frequency of harmonic oscillations of the arm about line OA (the horizontal) can be calculated as $\varphi = 2\pi/\tau = 2.09$ rad/s. If we let θ represent the angular displacement of the arm measured relative to the position defined by line OA, then θ can be written as a sine function of time:

$$\theta = \theta_0 \sin(\varphi t) \tag{i}$$

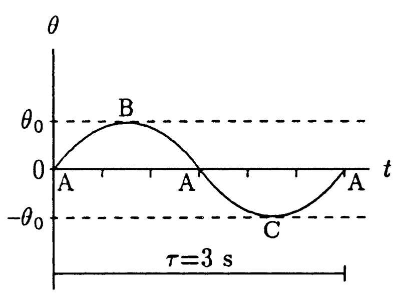

Fig. 9.11 *Graph of function $\theta = \theta_0 \sin(\varphi t)$ with $\varphi = 2\pi/t$*

The angular displacement of the arm as given in Eq. (i) is plotted as a function of time in Fig. 9.11. Notice that θ is zero when the arm is at position A. θ assumes positive values between A and B, and it is negative while the arm is between A and C. θ reaches its peak at B and C, and θ_0 is the amplitude of angular displacement of the arm. Since all of these are consistent with the information provided in the statement of the problem, Eq. (i) does represent the angular displacement of the arm.

To derive expressions for the angular velocity and acceleration of the arm, we have to consider time derivatives of the function given in Eq. (i). The time rate of change of angular displacement is defined as angular velocity:

$$\omega = \frac{d\theta}{dt} = \theta_0 \varphi \cos(\varphi t) \tag{ii}$$

The time rate of change of angular velocity is angular acceleration:

$$\alpha = \frac{d\omega}{dt} = -\theta_0 \varphi^2 \sin(\varphi t) \tag{iii}$$

Equations (ii) and (iii) can alternatively be written as:

$$\omega = \omega_0 \cos(\varphi t) \tag{iv}$$

$$\alpha = -\alpha_0 \sin(\varphi t) \tag{v}$$

Here, ω_0 is the amplitude of the angular velocity and α_0 is the amplitude of the angular acceleration of the arm, such that:

$$\omega_0 = \theta_0 \varphi = \theta_0 \frac{2\pi}{\tau}$$

$$\alpha_0 = \theta_0\,\varphi^2 = \theta_0\frac{4\pi^2}{\tau^2}$$

Notice that the amplitude of the angular velocity is a linear function of the angular frequency, and the amplitude of angular acceleration is a quadratic function of angular frequency. Angular frequency, on the other hand, is inversely proportional with the period of harmonic oscillations. Therefore, low period indicates high frequency, which indicates high angular velocity and acceleration amplitudes.

We can use the numerical values of $\theta_0 = 80^\circ = 1.40$ rad and $\varphi = 2.09$ rad/s to calculate ω_0 and α_0 as 2.93 rad/s and 6.12 rad/s^2, respectively. Equations (i), (iv), and (v) can now be expressed as:

$$\theta = 1.40\ \sin(2.09\ t) \qquad \text{(vi)}$$

$$\omega = 2.93\ \cos(2.09\ t) \qquad \text{(vii)}$$

$$\alpha = -6.12\ \sin(2.09\ t) \qquad \text{(viii)}$$

Equations (vi) through (viii) can be used to calculate the instantaneous angular position, velocity, and acceleration of the arm at any time t. These equations can also be used to plot θ, ω, and α versus t graphs for the arm by assigning values to time and calculating corresponding θ, ω, and α values which are provided in Table 9.2.

Table 9.2 *Values of θ, ω, and α*

t	(φt) (RAD)	(φt) (DEGREE)	SIN	COS	θ	ω	α
0	0	0	0	1.0	0	2.93	0
0.25	0.523	30	0.5	0.866	0.7	2.54	–3.06
0.5	1.045	60	0.866	0.5	1.2	1.465	–5.3
0.75	1.57	90	1.0	0	1.4	0	–6.12
1.0	2.09	120	0.866	–0.5	1.2	–1.465	–5.3
1.25	2.6	150	0.5	–0.866	0.7	–2.54	–3.06
1.5	3.14	180	0	–1.0	0	–2.93	0
1.75	3.66	210	–0.5	–0.866	–0.7	–2.54	3.06
2.0	4.18	240	–0.866	–0.5	–1.2	–1.465	5.3
2.25	4.7	270	–1.0	0	–1.4	0	6.12
2.5	5.23	300	–0.866	0.5	–1.2	1.465	5.3
2.75	5.75	330	–0.5	0.866	–0.7	2.54	3.06
3.0	6.27	360	0	1.0	0	2.93	0

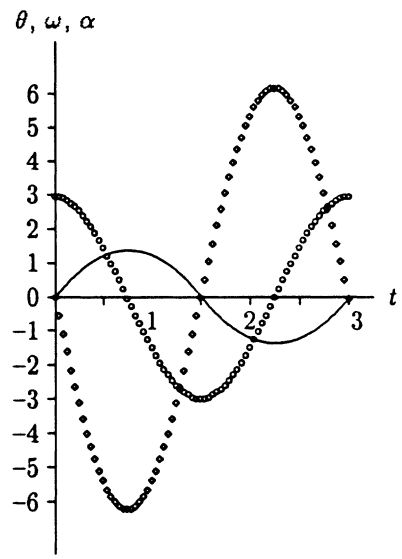

Fig. 9.12 *Angular position, velocity (open circles), and acceleration (open diamonds) versus time (θ in rad, ω in rad/s, and α in rad/s^2, and t in s)*

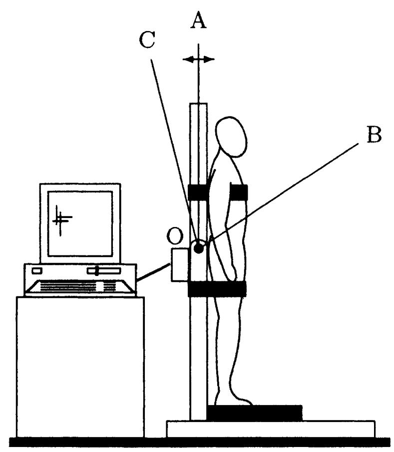

Fig. 9.13 *Dynamometer*

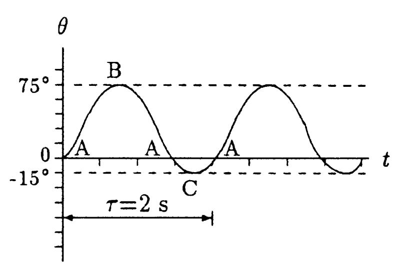

Fig. 9.14 *Angular position versus time graph*

Based on the obtained data, a set of sample graphs is shown in Fig. 9.12 for a single cycle.

Example 9.2 *Flexion–extension test*

Figure 9.13 illustrates a computer-controlled dynamometer that can be used to measure angular displacement, angular velocity, and torque output of the trunk. During a repetitive flexion–extension test in the sagittal plane (plane that passes through the chest and divides the body into right-hand and left-hand parts), a subject is placed in the dynamometer, positioned in the machine so that the subject's fifth lumbar vertebra (L5/S1) is aligned with the flexion–extension axis (indicated as O) of the machine, tied to the equipment firmly, and asked to perform trunk flexion and extension as long as possible, exerting as much effort as possible. The angular position of the subject's trunk relative to the upright position is measured and recorded. The data collected is then plotted to obtain an angular displacement θ versus time t graph. The curves obtained for this particular subject in different cycles are observed to be qualitatively and quantitatively similar except for the first and the last few cycles. A couple of sample cycles are provided in Fig. 9.14, in which the angular displacement of the trunk measured in degrees is plotted as a function of time measured in seconds.

The angular position measurements are made relative to the upright position in which the angular displacement of the trunk is zero. The subject flexes between A and B, and reaches a peak flexion at B. The extension phase is identified with the motion of the trunk from B toward A. The angular displacement of the trunk is positive between A and B. Between A and C, the trunk undergoes hyperextension and reaches a peak extension at C. In this range, the angular displacement of the trunk assumes negative values.

The purpose of this example is to demonstrate the means of analyzing experimentally collected data. The specific task is to find a function that can express the angular displacement of the subject's trunk as a function of time, from which we can derive expressions for the angular velocity and acceleration of the trunk.

Solution: The problem may be easier to visualize if we form an analogy between the upper body and a mechanical system called the *inverted pendulum*, shown in Fig. 9.15. An inverted pendulum consists of a concentrated mass m attached to a very light rod of length l that is hinged to the ground through an axis about which it is allowed to rotate. In this case, the concentrated mass represents the total mass of the upper body. The hinge corresponds to the disc between the fifth lumbar vertebra and the sacrum, about which the upper body rotation occurs in the sagittal plane. Length l is the distance between the fifth lumbar vertebra and the center of gravity of the upper body.

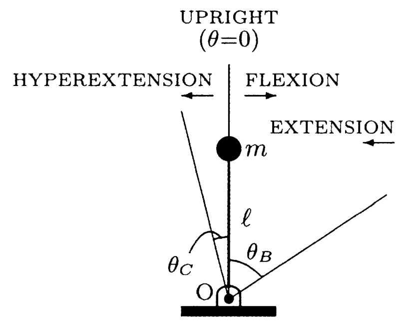

Fig. 9.15 *Inverted pendulum*

It is clear from Fig. 9.14 that θ is a harmonic (sine or cosine) function of time. From Fig. 9.14, it is possible to read the peak angles the trunk makes with the upright position during flexion and extension phases, and the period of harmonic motions. However, it is not easy to determine exactly how θ varies with time. To obtain a function relating θ and t, we must work through several steps.

Let τ be the period of harmonic oscillations, and θ_B and θ_C the peak angular displacements of the trunk in the flexion and extension phases, respectively. From Fig. 9.14 or using the experimentally obtained raw data, $\tau = 2$ s, $\theta_B = 75°$, and $\theta_C = -15°$. Knowing the period, the angular frequency of the harmonic oscillations can be determined:

$$\varphi = \frac{2\pi}{\tau} = \frac{2\pi}{2} = \pi \ \text{rad/s}$$

Using θ_B and θ_C, we can also calculate the range of motion of the trunk. By definition, range of motion is the total angle covered by the rotating object. Therefore:

$$\text{ROM} = \theta_B + \theta_C = 75° + 15° = 90°$$

Figure 9.14 is redrawn in Fig. 9.16 in which two sets of coordinates are used. In addition to θ and t, we have a second set of coordinates Θ (capital theta) and T that is obtained by translating the origin of the θ versus t coordinate system to a point with coordinates $t = t_M$ and $\theta = \theta_M$. Here, θ_M designates the mean angular displacement that can be calculated as:

$$\theta_M = \frac{\theta_B - \theta_C}{2} = \frac{75° - 15°}{2} = 30°$$

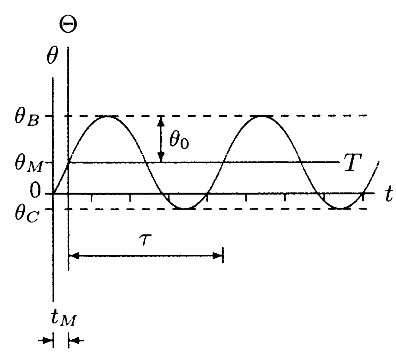

Fig. 9.16 *Translating the θ versus t coordinate frame to Θ versus T coordinate frame*

Time t_M corresponds to the time when $\theta = \theta_M$. t_M can be determined from the experimentally collected data. In this case, $t_M = 0.232$ s.

We define a second set of coordinates so that, with respect to Θ and T, the function representing the angular displacement versus time curve is simply a sine function:

$$\Theta = \theta_0 \sin(\varphi T) \qquad \text{(i)}$$

In Eq. (i), θ_0 is the amplitude of the harmonic oscillations and is equal to one-half of the range of motion:

$$\theta_0 = \frac{\text{ROM}}{2} = \frac{90°}{2} = 45° \quad \left(\frac{\pi}{4} \ \text{rad}\right)$$

We now have a function representing the experimentally obtained curve in terms of Θ and T. If we can relate Θ to θ and T to t, then we can derive a function in terms of θ and t. This can be achieved by employing *coordinate transformation*. Notice that $\Theta = 0$ when $\theta = \theta_M$. Therefore:

$$\Theta = \theta - \theta_M \tag{ii}$$

Also notice that $T = 0$ when $t = t_M$. Hence:

$$T = t - t_M \tag{iii}$$

Substituting Eqs. (ii) and (iii) into Eq. (i) will yield:

$$\theta = \theta_M + \theta_0 \sin[\varphi(t - t_M)] \tag{iv}$$

In Eq. (iv), the angular displacement of the trunk is defined as a function of time, representing the experimentally obtained curve shown in Fig. 9.14. We can also obtain expressions for the angular velocity and acceleration of the trunk by considering the time derivatives of θ in Eq. (iv):

$$\omega = \frac{d\theta}{dt} = \theta_0\, \varphi\, \cos[\varphi(t - t_M)] \tag{v}$$

$$\alpha = \frac{d\omega}{dt} = -\theta_0\, \varphi^2\, \sin[\varphi(t - t_M)] \tag{vi}$$

The numerical values of θ_M, θ_0, t_M, and φ can be substituted into the above equations to obtain:

$$\theta = \frac{\pi}{6} + \frac{\pi}{4}\, \sin[\pi(t - 0.232)] \tag{vii}$$

$$\omega = \frac{\pi^2}{4}\, \cos[\pi(t - 0.232)] \tag{viii}$$

$$\alpha = -\frac{\pi^3}{4}\, \sin[\pi(t - 0.232)] \tag{ix}$$

These functions are plotted in Fig. 9.17 to obtain angular displacement, velocity, and acceleration versus time graphs for the trunk.

Note that the validity of Eq. (vii) can be checked by assigning values to t and calculating corresponding θ values using Eq. (vii). For example, $\theta = 0$ when $t = 0$ and $t = \tau = 2$ s, and $\theta = \pi/6 = 0.52$ rad or 30° when $t = t_M = 0.232$ s. These are consistent with the initial data presented in Fig. 9.14.

Also note that angular velocity is a cosine function of time. The amplitude of the ω versus t curve shown in Fig. 9.17 is equal to the coefficient $\pi^2/4 = 2.47$ rad/s in front of the cosine function in Eq. (viii). Similarly, the amplitude of the angular acceleration is $\pi^3/4 = 7.75\,\text{rad/s}^2$.

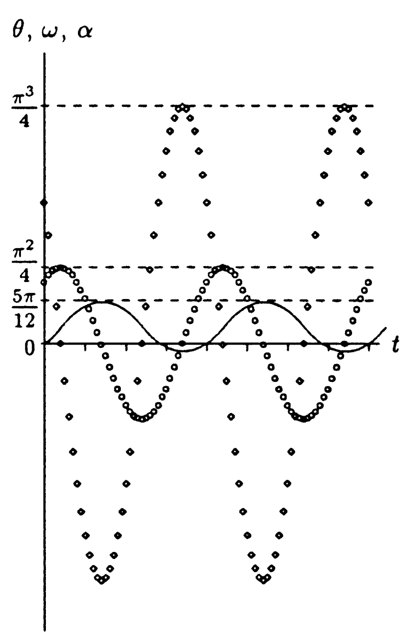

Fig. 9.17 *Angular position, velocity (open circles), and acceleration (open diamonds) versus time*

9.7 Rotational Motion About a Fixed Axis

Consider the arbitrarily shaped object in Fig. 9.18. Assume that the object is undergoing a rotational motion in the *xy*-plane about a fixed axis that is perpendicular to the *xy*-plane. Let O and P be two points in the *xy*-plane, such that O is along the axis of rotation of the object and P is a fixed point on the rotating object located at a distance *r* from point O. Due to the rotation of the object, point P will experience a circular motion with *r* being the radius of its circular path.

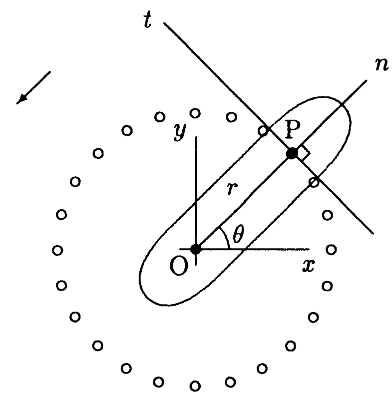

Fig. 9.18 *n and t are the normal (radial) and tangential directions at point P*

To describe circular motions, it is usually convenient to define velocity and acceleration vectors with respect to two mutually perpendicular directions normal (radial) and tangential to the circular path of motion. These directions are indicated as n and t in Fig. 9.18, and are also known as *local coordinates*. By definition, the velocity vector $\underline{v}$ is always tangent to the path of motion. Therefore, for a circular motion, the velocity vector can have only one component tangent to the circular path of motion (Fig. 9.19). $\underline{v}$ is called the *tangential* or *linear velocity*. The magnitude v of the velocity vector can be determined by considering the time rate of change of relative position of point P along the circular path:

$$v = \frac{ds}{dt} \tag{9.9}$$

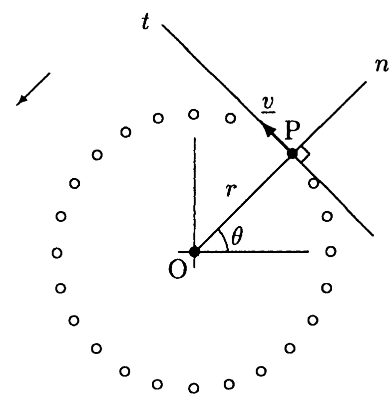

Fig. 9.19 *Velocity vector $\underline{v}$ is always tangent to the path of motion*

For a circular motion, the acceleration vector can have both tangential and normal components (Fig. 9.20). The *tangential acceleration* $\underline{a}_t$ is related to the change in magnitude of the velocity vector and has a magnitude:

$$a_t = \frac{dv}{dt} \tag{9.10}$$

The *normal acceleration* $\underline{a}_n$ is related to the change in direction of the velocity vector and has a magnitude:

$$a_n = \frac{v^2}{r} \tag{9.11}$$

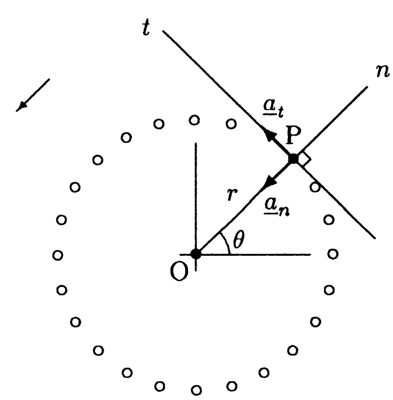

Fig. 9.20 *$\underline{a}_1$ and $\underline{a}_n$ are the tangential and normal components of the acceleration vector*

For an object undergoing a rotational motion, a_t is zero if the object is rotating with constant v. On the other hand, a_n is always present because it is associated with the direction of $\underline{v}$ that changes continuously throughout the motion.

The direction of $\underline{a}_t$ is the same as the direction of $\underline{v}$ if v is increasing, or opposite to that of $\underline{v}$ if v is decreasing over time. The normal component of the acceleration vector is also known as *radial* or *centripetal* (center-seeking), and it is always directed toward the center of rotation of the body.

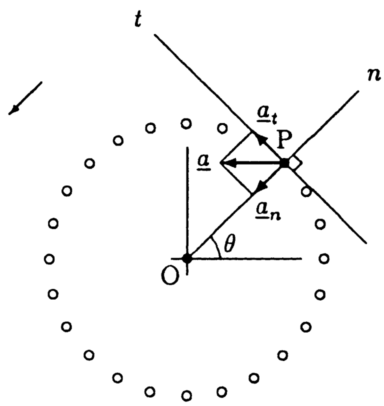

Fig. 9.21 *$\underline{a}$ is the resultant linear acceleration vector*

If the tangential and normal acceleration components are known, then the net or resultant acceleration of a point on a body rotating about a fixed axis can also be determined (Fig. 9.21). If $\underline{t}$ and $\underline{n}$ are unit vectors indicating positive tangential and normal directions, respectively, then the resultant acceleration vector can be expressed as:

$$\underline{a} = \underline{a}_t + \underline{a}_n = a_t\underline{t} + a_n\underline{n} \tag{9.12}$$

The magnitude of the resultant acceleration vector can be determined as:

$$a = \sqrt{{a_t}^2 + {a_n}^2} \tag{9.13}$$

On the other hand, the velocity vector can be expressed as:

$$\underline{v} = v\underline{t} \tag{9.14}$$

Note that v and a are linear quantities. v has the dimension of length divided by time, and both a_t and a_n have the dimension of length divided by time squared. Also note that it is customary to take the positive normal direction (the direction of $\underline{n}$) to be outward (from the center of rotation toward the rim), and the positive tangential direction (the direction of $\underline{t}$) to be counterclockwise.

9.8 Relationships Between Linear and Angular Quantities

Recall from Eq. (9.3) that $s = r\theta$. For a circular motion, radius r is constant and Eq. (9.9) can be evaluated as follows:

$$v = \frac{d}{dt}(r\,\theta) = r\frac{d\theta}{dt}$$

By definition, time rate of change of angular displacement is angular velocity. Therefore:

$$v = r\omega \tag{9.15}$$

Equation (9.15) states that the magnitude of the linear velocity of a point in a body that is undergoing a rotational motion about a fixed axis is equal to the distance of that point from the center of rotation multiplied by the angular velocity of the body. Notice that at a given instant, every point on the body has the same angular velocity but may have different linear velocities. The magnitude of the linear velocity increases with increasing radial distance, or as one moves outward from the center of rotation toward the rim.

Using the relationship given in Eq. (9.15), Eq. (9.10) can be evaluated for a motion in a circular path as follows:

$$a_t = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt}$$

By definition, time rate of change of angular velocity is angular acceleration. Therefore:

$$a_t = r\alpha \tag{9.16}$$

Similarly, substituting Eq. (9.15) into Eq. (9.11) will yield:

$$a_n = r\omega^2 \tag{9.17}$$

Equations (9.15), (9.16), and (9.17) relate linear quantities v, a_t, and a_n to angular quantities r, ω, and α. Equation (9.16) states that the tangential component of linear acceleration of a point on a body rotating about a fixed axis is equal to the distance of that point from the axis of rotation times the angular acceleration of the body.

9.9 Uniform Circular Motion

Uniform circular motion occurs when the angular velocity of an object undergoing a rotational motion about a fixed axis is constant. When angular velocity is constant, angular acceleration is zero. Therefore, for a point located at a radial distance r from the center of rotation of an object undergoing uniform circular motion:

$$\begin{aligned} v &= r\,\omega \qquad (\text{constant}) \\ a_t &= 0 \\ a_n &= r\,\omega^2 \qquad (\text{constant}) \end{aligned}$$

9.10 Rotational Motion with Constant Acceleration

In Chap. 7, a set of kinematic equations [Eqs. (7.11) through (7.14)] were derived to analyze the motion characteristics of bodies undergoing translational motion with constant acceleration. Similar equations can also be derived for rotational motion about a fixed axis with constant angular acceleration:

$$\omega = \omega_0 + \alpha_0\, t \tag{9.18}$$

$$\theta = \theta_0 + \omega_0\, t + \frac{1}{2}\,\alpha_0\, t^2 \tag{9.19}$$

$$\theta = \theta_0 + \frac{1}{2}\,(\omega + \omega_0)t \tag{9.20}$$

$$\omega^2 = {\omega_0}^2 + 2\alpha_0(\theta - \theta_0) \tag{9.21}$$

In Eqs. (9.18) through (9.21), α_0 is the constant angular acceleration, and θ_0 and ω_0 are the initial angular position and velocity of the object at time $t_0 = 0$, respectively.

9.11 Relative Motion

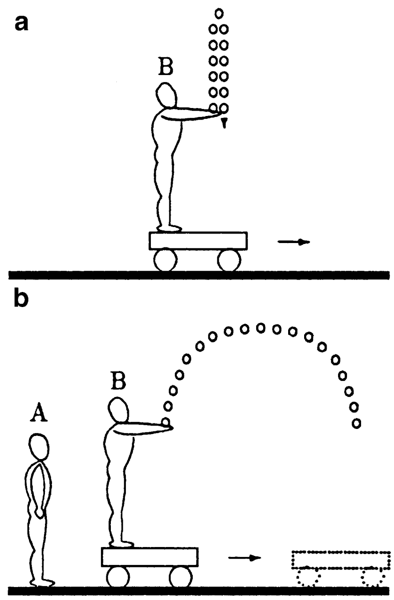

Fig. 9.22 *A motion observed by different observers in different reference frames may be different*

A motion observed in different frames of reference may be different. For example, the motion of a train observed by a stationary person would be different than the motion of the same train observed by a passenger in a moving car. The motion of a ball thrown up into the air by a person riding in a moving vehicle would have a vertical path as observed by the person riding in the same vehicle (Fig. 9.22a), but a curved path for a second, stationary person watching the ball (Fig. 9.22b). The general approach in analyzing such physical situations requires defining the motion of the moving body with respect to a convenient moving coordinate frame, defining the motion of this frame with respect to a fixed coordinate frame, and combining the two.

Assume that the motion of a point P in a moving body is to be analyzed. Let XYZ and xyz refer to two coordinate frames with origins at A and B, respectively (Fig. 9.23). Assume that the XYZ frame is fixed (stationary) and the xyz frame is moving, such that the respective coordinate directions (for example, x and X) remain parallel throughout the motion. This implies that the xyz coordinate frame is undergoing a translational motion only, and that the same set of unit vectors $\underline{i}$, $\underline{j}$, and $\underline{k}$ can be used in both reference frames. The motion of the moving xyz frame can be identified by specifying the motion of its origin B. If $\underline{r}_B$ denotes the position vector of B with respect to the fixed coordinate frame, then the velocity and acceleration vectors of B with respect to the XYZ coordinate frame are:

$$\underline{v}_B = \frac{d}{dt}\left(\underline{r}_B\right) = \dot{\underline{r}}_B \qquad \underline{a}_B = \frac{d}{dt}\left(\underline{v}_B\right) = \ddot{\underline{r}}_B$$

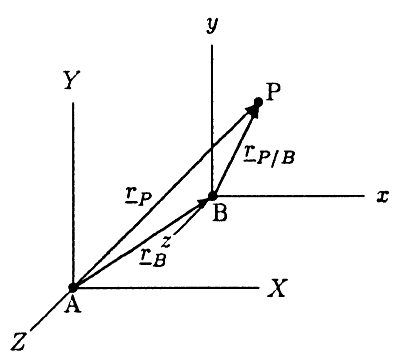

Fig. 9.23 *XYZ is a fixed and xyz is a moving coordinate frame*

Similarly, the motion of point P with respect to the moving coordinate frame xyz can be defined by the position vector $\underline{r}_{P/B}$ of point P relative to the origin B of the xyz frame. The first and second time derivatives of $\underline{r}_{P/B}$ will yield the velocity and acceleration vectors of point P relative to the xyz frame:

$$\underline{v}_{P/B} = \frac{d}{dt}\left(\underline{r}_{P/B}\right) = \dot{\underline{r}}_{P/B} \qquad \underline{a}_{P/B} = \frac{d}{dt}\left(\underline{v}_{P/B}\right) = \ddot{\underline{r}}_{P/B}$$

Finally, the position vector $\underline{r}_P$, velocity vector $\underline{v}_P$, and acceleration vector $\underline{a}_P$ of point P with respect to the fixed coordinate frame XYZ can be obtained by superposition:

$$\underline{r}_{\mathrm{P}} = \underline{r}_{\mathrm{B}} + \underline{r}_{\mathrm{P/B}} \tag{9.22}$$

$$\underline{v}_{\mathrm{P}} = \underline{v}_{\mathrm{P}} + \underline{v}_{\mathrm{P/B}} \tag{9.23}$$

$$\underline{a}_{\mathrm{P}} = \underline{a}_{\mathrm{P}} + \underline{a}_{\mathrm{P/B}} \tag{9.24}$$

The motion of point B (which happens to be the origin of the moving coordinate frame *xyz*) with respect to the fixed *XYZ* coordinate frame is called the *absolute motion* of B and is denoted by the subscript B. Similarly, the motion of point P observed relative to the *XYZ* frame is the absolute motion of P. The motion of point P with respect to the moving coordinate frame is called the *relative motion* of P and is denoted by the subscript P/B. Note here that the position vector $\underline{r}_{\mathrm{P/B}}$ refers to a vector drawn from point B to point P. Also note that the position vector of point P relative to the *XYZ* coordinate frame could also be expressed as $\underline{r}_{\mathrm{P/A}}$. However, by convention, $\underline{r}_{\mathrm{P}}$ implies that the position vector is defined relative to the fixed coordinate frame.

Example 9.3 Consider the motion described in Fig. 9.22. A person (B) riding on a vehicle that is moving toward the right by a constant speed of 2 m/s throws a ball straight up into the air with an initial speed of 10 m/s.

Describe the motion of the ball as observed by a stationary person (A) in the time interval between when the ball is first released and when it reaches its maximum elevation.

Solution: This is a two-dimensional problem and can be analyzed in three steps. First, let *x* and *y* represent a coordinate frame moving with the vehicle. With respect to the *xy* frame, the ball thrown up into the air will undergo one-dimensional linear motion (translation) in the *y* direction (Fig. 9.24). Because of the constant downward gravitational acceleration, the ball will decelerate in the positive *y* direction, reach its maximum elevation, change its direction of motion, and begin to descend. With respect to the *xy* coordinate frame moving with the vehicle, or as observed by person B moving with the vehicle, the speed of the ball in the *y* direction between the instant of release and when the ball reaches its peak elevation can be determined from (see Chap. 7):

$$v_y = v_{y_0} - gt$$

Here, $v_{y0} = 10$ m/s is the initial speed of the ball and $g \approx 10\,\mathrm{m/s^2}$ is the magnitude of gravitational acceleration. This equation is valid in the time interval between $t = 0$ (the instant of release) and $t = v_{y_0}/g = \frac{10}{10} = 1$ s (the time it takes for the ball to reach its maximum elevation where $v_y = 0$). As observed by person B,

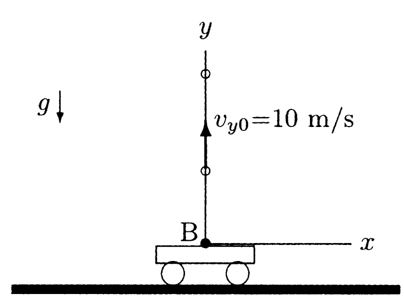

Fig. 9.24 *Relative to the xy frame, the ball is undergoing a translational motion in the y direction*

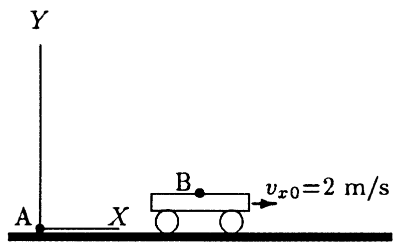

Fig. 9.25 *Relative to the XY frame, the ball is undergoing a translational motion in the X direction with constant velocity*

the ball has no motion in the x direction. Therefore, the velocity $\underline{v}_{P/B}$ of the ball relative to person B can be expressed as:

$$\underline{v}_{P/B} = v_y \underline{j}$$

Next, let X and Y represent a coordinate frame fixed to the ground. With respect to the XY frame, or with respect to the stationary person A, the vehicle is moving in the positive X direction with a constant speed of $v_{x_0} = 2\,\text{m/s}$ (Fig. 9.25). Therefore:

$$\underline{v}_B = v_{x_0} \underline{i}$$

Finally, to determine the velocity of the ball relative to person A, we have to add velocity vectors $\underline{v}_B$ and $\underline{v}_{P/B}$ together:

$$\underline{v}_P = \underline{v}_B + \underline{v}_{P/B} = v_{x0} \underline{i} + v_y \underline{j}$$

Or, by substituting the known parameters:

$$\underline{v}_P = 2\underline{i} + (10 - 10)\underline{j}$$

For example, half a second after the ball is released, the ball has a velocity:

$$\underline{v}_P = 2\underline{i} + 5\underline{j}$$

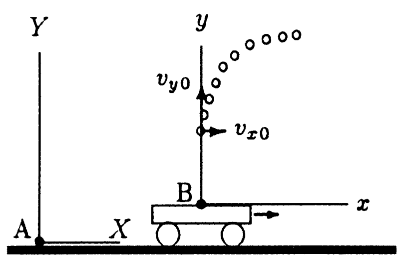

Fig. 9.26 *Relative to the XY frame, the ball moves both in the X and Y directions*

That is, according to person A or relative to the XY coordinate frame, the ball is moving to the right with a speed of 2 m/s and upward with a speed of 5 m/s (Fig. 9.26). At this instant, the magnitude of the net velocity of the ball is $v_P = \sqrt{(2)^2 + (5)^2} = 5.4\,\text{m/s}$.

9.12 Linkage Systems

A *linkage system* is composed of several parts connected to each other and/or to the ground by means of hinges or joints, such that each part constituting the system can undergo motion relative to the other segments. An example of such a system is the *double pendulum* shown in Fig. 9.27. A double pendulum consists of two bars hinged together and to the ground. Linkage systems are also known as *multi-link systems*.

If the angular velocity and acceleration of individual parts are known, then the principles of relative motion can be applied to analyze the motion characteristics of each part constituting the multi-link system. The following example will illustrate the procedure of analyzing the motion of a double pendulum. However, the procedure to be introduced can be generalized to analyze any multi-link system.

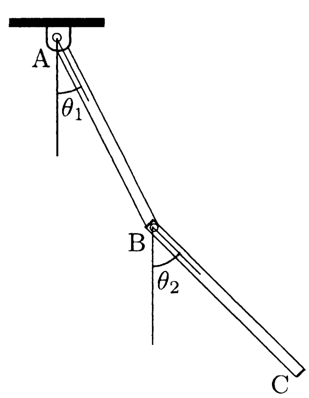

Fig. 9.27 *Double pendulum*

An important concept associated with linkage systems is the number of independent coordinates necessary to describe the motion characteristics of the parts constituting the system. The number of independent parameters required defines the *degrees of freedom* of the system. For example, the two-dimensional motion characteristics of the simple pendulum shown in Fig. 9.28 can be fully described by θ that defines the location of the pendulum uniquely. Therefore, a simple pendulum has one degree of freedom. On the other hand, parameters θ_1 and θ_2 are necessary to analyze the coplanar motion of bar BC of the double pendulum shown in Fig. 9.27, and therefore, a double pendulum has two degrees of freedom.

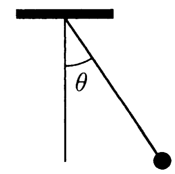

Fig. 9.28 *Pendulum*

Example 9.4 *Double pendulum*

Assume that arms AB and BC of the double pendulum shown in Fig. 9.29 are undergoing coplanar motion. Let $l_1 = 0.3$ m and $l_2 = 0.3$ m be the lengths of arms AB and BC, and θ_1 and θ_2 be the angles arms AB and BC make with the vertical. The angular velocity and acceleration of arm AB are measured as $\omega_1 = 2$ rad/s (counterclockwise) and $\alpha_1 = 0$ relative to point A. The angular velocity and acceleration of arm BC is measured as $\omega_2 = 4$ rad/s (counterclockwise) and $\alpha_2 = 0$ relative to point B.

Determine the linear velocity and acceleration of point B on arm AB and point C on arm BC at an instant when $\theta_1 = 30°$ and $\theta_2 = 45°$.

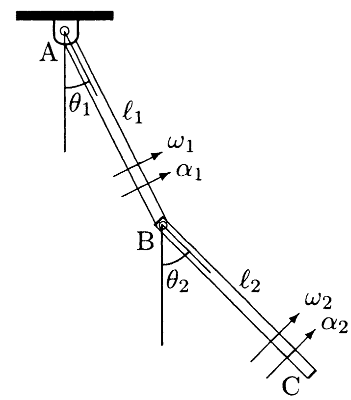

Fig. 9.29 *Double pendulum*

Solution: Let X and Y refer to a set of rectangular coordinates with origin located at A, and x and y be a second set of rectangular coordinates with origin at B. The XY coordinate frame is stationary, while the xy frame can move as point B moves. Since the angular velocity and acceleration of arm AB are given relative to point A, the motion characteristics of any point on arm AB can be determined with respect to the XY coordinate frame. Similarly, the motion of any point on arm BC can easily be analyzed relative to the xy coordinate frame.

Motion of point B as observed from point A:

Every point on arm AB undergoes a rotational motion about a fixed axis passing through point A with constant angular velocity of $\omega_1 = 2$ rad/s. Every point on arm AB experiences a uniform circular motion in the counterclockwise direction. As illustrated in Fig. 9.30, point B moves in a circular path of radius l_1. Magnitudes of linear velocity in the tangential direction and linear acceleration in the normal direction of point B can be determined using:

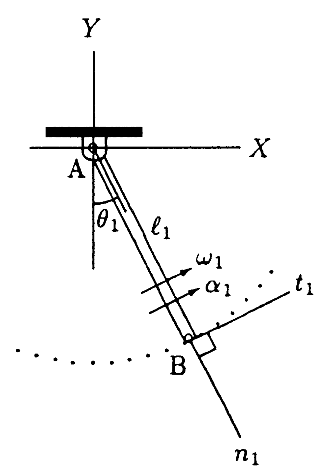

Fig. 9.30 *Circular motion of B as observed from point A*

$$v_B = l_1\,\omega_1$$

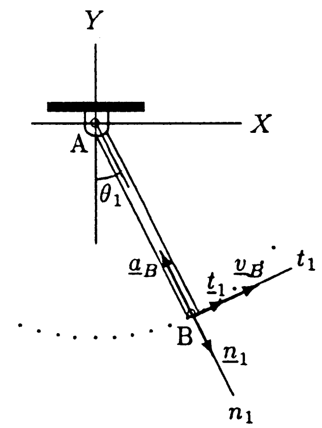

Fig. 9.31 *Tangential velocity and normal acceleration of B*

$$a_B = l_1\,\omega_1{}^2$$

The magnitude of the tangential component of the acceleration vector is zero since ω_1 is constant or since $\alpha_1 = 0$. Therefore, v_B and a_B are essentially the magnitudes of the resultant linear velocity and acceleration vectors. To express these quantities in vector forms, let n_1 and t_1 represent the normal and tangential directions to the circular path of point B when arm AB makes an angle θ_1 with the horizontal (Fig. 9.31). Also let $\underline{n}_1$ and $\underline{t}_1$ be unit vectors in the positive n_1 and t_1 directions, such that the positive $\underline{n}_1$ direction is outward (i.e., from A toward B) and positive $\underline{t}_1$ direction is pointing in the direction of motion (i.e., counterclockwise). The normal (centripetal) acceleration is always directed toward the center of motion, and is acting in the negative $\underline{n}_1$ direction:

$$\underline{v}_B = v_B\underline{t}_1 = l_1\omega_1\underline{t}_1$$

$$\underline{a}_B = -a_B\underline{n}_1 = -l_1\omega_1{}^2\underline{n}_1$$

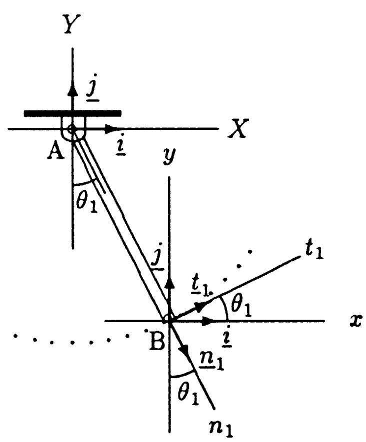

Fig. 9.32 *Expressing unit vectors $\underline{n}_1$ and $\underline{t}_1$ in terms of Cartesian unity vectors $\underline{I}$ and $\underline{j}$*

Notice that directions defined by unit vectors $\underline{n}_1$ and $\underline{t}_1$ change continuously as point B moves along its circular path. That is, $\underline{n}_1$ and $\underline{t}_1$ define a set of *local coordinate* directions that vary in time. By employing proper coordinate transformations, we can express these unit vectors in terms of Cartesian unit vectors $\underline{i}$ and $\underline{j}$. Cartesian coordinate directions, which are *global* as opposed to local, are not influenced by the motion of point B. The coordinate transformation can be done by expressing unit vectors $\underline{n}_1$ and $\underline{t}_1$ in terms of Cartesian unit vectors $\underline{i}$ and $\underline{j}$. It can be observed from the geometry of the problem that (Fig. 9.32):

$$\underline{n}_1 = \sin\theta_1\underline{i} - \cos\theta_1\underline{j}$$

$$\underline{t}_1 = \cos\theta_1\underline{i} + \sin\theta_1\underline{j}$$

Therefore, the velocity and acceleration vectors of point B with respect to the *XY* coordinate frame and in terms of Cartesian unit vectors are:

$$\underline{v}_B = l_1\omega_1\left(\cos\theta_1\underline{i} + \sin\theta_1\underline{j}\right)$$

$$\underline{a}_B = -l_1\omega_1{}^2\left(\sin\theta_1\underline{i} - \cos\theta_1\underline{j}\right)$$

If we substitute the numerical values of $l_1 = 0.3$ m, $\theta_1 = 30°$, and $\omega_1 = 2$ rad/s, and carry out the necessary calculations we obtain:

$$\underline{v}_B = 0.52\,\underline{i} + 0.30\,\underline{j} \qquad \text{(i)}$$

$$\underline{a}_B = -0.60\,\underline{i} + 1.04\,\underline{j} \qquad \text{(ii)}$$

Motion of point C as observed from point B:

The motion of point C as observed from point B is similar to the motion of point B as observed from point A. Point C rotates with a constant angular velocity of ω_2 in a circular path of radius l_2 about point B (Fig. 9.33). Therefore, the derivation of velocity and acceleration vectors for point C relative to the *xy* coordinate frame follows the same procedure outlined for the derivation of velocity and acceleration vectors for point B relative to the *XY* coordinate frame. The magnitudes of the tangential velocity and normal acceleration vectors of point C relative to B are:

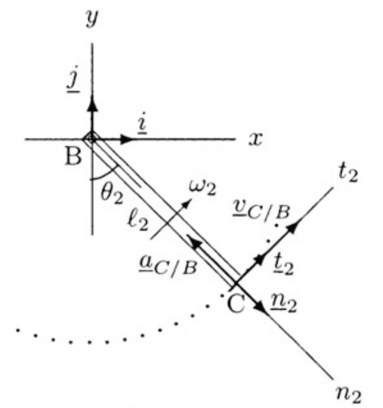

Fig. 9.33 *Circular motion of C as observed from point B*

$$v_{C/B} = l_2\, \omega_2$$

$$a_{C/B} = l_2\, {\omega_2}^2$$

If $\underline{n}_2$ and $\underline{t}_2$ are unit vectors in the normal and tangential directions to the circular path of C when arm BC makes an angle θ_2 with the vertical, then:

$$\underline{v}_{C/B} = v_{C/B}\underline{t}_2 = l_2\omega_2\underline{t}_2$$

$$\underline{a}_{C/B} = -a_{C/B}\underline{n}_2 = -l_2{\omega_2}^2\underline{n}_2$$

From Fig. 9.34, unit vectors $\underline{n}_2$ and $\underline{t}_2$ can be expressed in terms of Cartesian unit vectors $\underline{i}$ and $\underline{j}$ as:

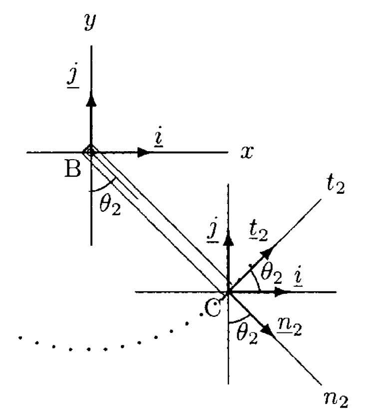

Fig. 9.34 *Expressing unit vectors $\underline{n}_1$ and $\underline{t}_1$ in terms of Cartesian unit vectors $\underline{i}$ and $\underline{j}$*

$$\underline{n}_2 = \sin\theta_2\underline{i} - \cos\theta_2\underline{j}$$

$$\underline{t}_2 = \cos\theta_2\underline{i} + \sin\theta_2\underline{j}$$

Therefore, the velocity and acceleration vectors of point C relative to the *xy* coordinate frame can be written as:

$$\underline{v}_{C/B} = l_2\omega_2\left(\cos\theta_2\underline{i} + \sin\theta_2\underline{j}\right)$$

$$\underline{a}_{C/B} = -l_2{\omega_2}^2\left(\sin\theta_2\underline{i} - \cos\theta_2\underline{j}\right)$$

Substituting the numerical values of $l_2 = 0.3$ m, $\theta_2 = 45°$, and $\omega_2 = 4$ rad/s, and carrying out the necessary calculations we obtain:

$$\underline{v}_{C/B} = 0.85\ \underline{i} + 0.85\ \underline{j} \qquad \text{(iii)}$$

$$\underline{a}_{C/B} = -3.39\ \underline{i} + 3.39\ \underline{j} \qquad \text{(iv)}$$

Motion of point C as observed from point A:

We determined the velocity and acceleration of point C relative to B, and velocity and acceleration of point B with respect to A. Now, we can apply the principles of relative motion to determine the velocity and acceleration of point C as observed from point A or with respect to the *XY* coordinate frame:

$$\underline{v}_C = \underline{v}_B + \underline{v}_{C/B} \quad \text{(v)}$$

$$\underline{a}_C = \underline{a}_B + a_{C/B} \quad \text{(vi)}$$

Since we have already expressed $\underline{v}_B$, $\underline{v}_{C/B}$, $\underline{a}_B$, and $a_{C/B}$ in terms of Cartesian unit vectors, we can simply substitute Eqs. (i) and (iii) into Eq. (v), and Eqs. (ii) and (iv) into Eq. (vi):

$$\underline{v}_C = \left(0.52\,\underline{i} + 0.30\,\underline{j}\right) + \left(0.85\,\underline{i} + 0.85\,\underline{j}\right)$$
$$\underline{a}_C = \left(-0.60\,\underline{i} + 1.04\,\underline{j}\right) + \left(-3.39\,\underline{i} + 3.39\,\underline{j}\right)$$

Collecting the horizontal and vertical components together:

$$\underline{v}_C = 1.37\,\underline{i} + 1.15\,\underline{j}$$
$$\underline{a}_C = -3.39\,\underline{i} + 4.4\bar{3}\,\underline{j}$$

The magnitudes of the velocity and acceleration vectors are:

$$\underline{v}_C = \sqrt{(1.37)^2 + (1.15)^2} = 1.79 \ \text{m/s}$$
$$\underline{a}_C = \sqrt{(3.99)^2 + (4.43)^2} = 5.96 \ \text{m/s}^2$$

9.13 Exercise Problems

Problem 9.1 As shown in Fig. 9.10, consider a person doing shoulder abduction in the frontal plane. Point O designates the axis of rotation of the shoulder joint in the frontal plane. Line OA represents the position of the arm when it is stretched out parallel to the horizontal. Line OB represents the position of the arm when the hand is at its highest elevation, and line OC represents the position of the arm when the hand is closest to the body. For this activity, lines OB and OC are the limits of ROM for the arms. For this system, assume that the angle between lines OA and OB is equal to the angle between lines OA and OC, which are represented by an angle θ_0. Furthermore, the motion of the arm is symmetrical with respect to the line OA. Also assume that the time it takes for the arm to cover the angles between lines OA and OB, OB and OA, OA and OC, and OC and OA is approximately equal. If the period of the angular motion of the arm is $\tau = 3.5$ s and the angle $\theta_0 = 75°$:

(a) Derive expressions for the angular displacement θ, angular velocity ω, and angular acceleration α of the arm; (b) determine the angular displacement, angular velocity, and angular acceleration of the arm at $t_1 = 0.5$ s, $t_2 = 1.0$ s, and $t_3 = 1.25$ s after the rotational motion began.

Answers:

(a) $\theta = 1.31 \sin(1.79\ t), \omega = 2.34 \cos(1.79\ t), \alpha = -4.2 \sin(1.79\ t)$
(b) $\theta_1 = 1.02$ rad, $\omega_1 = 1.46$ rad/s, $\alpha_1 = -3.28$ rad/s^2
$\theta_2 = 1.28$ rad, $\omega_2 = -0.51$ rad/s, $\alpha_2 = -4.1$ rad/s^2
$\theta_3 = 1.03$ rad, $\omega_3 = -1.45$ rad/s, $\alpha_3 = -3.29$ rad/s^2

Problem 9.2 Consider an object undergoing a rotational motion in the xy-plane about a fixed axis perpendicular to the plane of motion (Fig. 9.35). Let O be a point in the xy-plane along the axis of rotation, and P is a fixed point on the object. Due to the rotation of the object, point P will move in a circular path with a radius $r = 0.8$ m. The relative position of point P along its circular path is given as a function of time $S = 0.45\ t^{\frac{4}{3}}$. Determine the distance (S) traveled by point P along its path, and the magnitude of its linear velocity (v), tangential (a_t), normal (a_n), and net accelerations 3 s after the motion began.

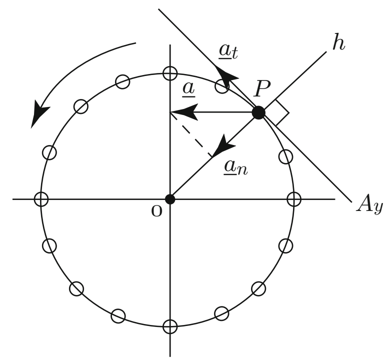

Fig. 9.35 *Problems 9.2 and 9.3*

Answers: $S = 1.95$ m; $v = 0.86$ m/s; $a_t = 0.1$ m/s^2; $a_n = 0.92$ m/s^2; $a = 0.93$ m/s^2

Problem 9.3 Consider an object undergoing a rotational motion in the xy-plane about a fixed axis perpendicular to the plane of motion (Fig. 9.35). Let O be a point in the xy-plane along the axis of rotation and P is a fixed point on the object. Due to the rotation of the object, point P will experience a circular motion with the radius of its circular path $r = 0.6$ m.

Assume that at some point in time, the angular acceleration of the point P is $\alpha = 5$ rad/s and an angle between the vectors of its tangential and net acceleration is $\beta = 30°$.

Determine the magnitudes of linear velocity (v) of point P and the magnitude of its tangential (a_t), normal (a_n), and net (a) acceleration vectors.

Answers:
$v = 1.02$ m/s; $a_t = 3$ m/s^2; $a_n = 1.72$ m/s^2; $a = 3.46$ m/s^2

Problem 9.4 As shown in Fig. 9.18, consider the arbitrarily shaped object undergoing a uniform circular motion in the xy-plane about a fixed axis that is perpendicular to the xy-plane. Point O is located in the xy-plane along the axis of rotation and

P is a fixed point on the object located at a distance r from the point O. If the magnitude of velocity vector of point P is $v = 1.3$ m/s, and the radius of its circular path is $r = 0.65$ m, determine:

(a) The angular velocity ω of point P
(b) The magnitude of acceleration vector a of point P

Answers: (a) $\omega = 1.85$ rad/s; (b) $a = 2.22$ m/s^2

Problem 9.5 Consider an arbitrarily shaped object undergoing a rotational motion in the xy-plane (Fig. 9.36). Let O be a point in the xy-plane along the axis of rotation, and P, N, and M are fixed points on the object located at distances r, 0.5 r, and 0.25 r, respectively. Due to the rotational motion of the object, points P, N, and M will move along their circular path.

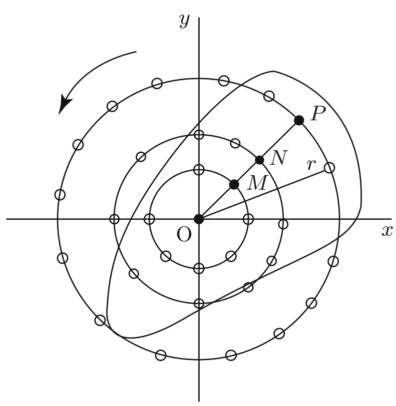

Fig. 9.36 *Problem 9.5*

If the linear velocity of point P is $v_p = 4.5$ m/s and it is located at distance $r = \text{OP} = 1.2$ m from the axis of rotation, determine the magnitude of angular velocity of points P, N, and M, and the magnitude of the linear velocity of points N and M.

Answers: $\omega_\text{P} = \omega_\text{N} = \omega_\text{M} = 3.75$ rad/s; $v_\text{N} = 2.25$ m/s; $v_\text{M} = 1.125$ m/s

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Problem 9.6 Consider an object initially at rest that begins a uniform rotational motion with constant angular acceleration in the xy-plane about a fixed axis that is perpendicular to the xy-plane. If the magnitude of the constant angular acceleration is $\alpha = 0.95$ rad/s^2, determine: (a) the angular velocity ω of the object at $t_1 = 1.5$ s, $t_2 = 2.0$ s, and $t_3 = 3.5$ s after the rotational motion began; (b) the angular position θ of the object with respect to its initial position at $t_1 = 1.5$ s, $t_2 = 2.0$ s, and $t_3 = 3.5$ s after the rotational motion began; and (c) convert angles obtained in radians into corresponding angles in degrees.

Answers:

(a) $\omega_1 = 1.43$ rad/s, $\omega_2 = 1.9$ rad/s, $\omega_3 = 3.33$ rad/s
(b) $\theta_1 = 1.07$ rad, $\theta_2 = 1.9$ rad, $\theta_3 = 5.82$ rad
(c) $\theta_1 = 61.3°$, $\theta_2 = 108.9°$, $\theta_3 = 333.6°$

Problem 9.7 Consider a gymnast doing giant circles around a high bar (Fig. 9.37). Assume that the center of gravity of the gymnast is located at distance r from the bar and is undergoing a uniform circular motion with a linear velocity of $v = 5.0$ m/s. After completing several cycles, the gymnast releases the bar at the instant when his center of gravity is directly beneath the bar, and then he undergoes a projectile motion and lands on the floor at point P. The distance between point P and the projection of the point of release on the floor (point O′) is $l = 4.0$ m.

Determine the time elapsed between the instant of release and landing (t), the height of the gymnast's center of gravity above the floor at the point of release (h), and the height of the bar above the floor (H).

Answers: $t = 0.8$ s; $h = 3.14$ m; $H = 4.39$ m

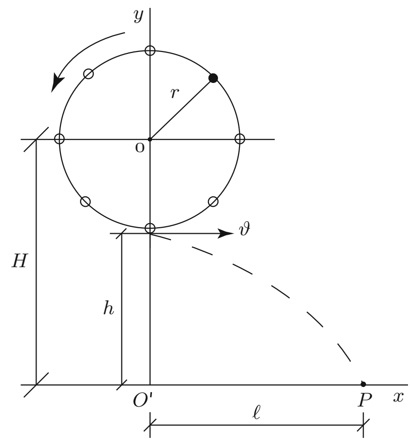

Fig. 9.37 *Problem 9.7*