Chapter 10

Angular Kinetics

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10.1 Kinetics of Angular Motion

The kinetic characteristics of objects undergoing translational motion were discussed in Chap. 8. Kinetic analyses utilize Newton's second law of motion that can be formulated in terms of the equations of motion and work and energy methods. Similar methods can be employed to analyze the kinetic characteristics of objects undergoing rotational motion.

Consider an object undergoing a rotational motion in the xy-plane about a fixed point O (Fig. 10.1). Let P be a point on the object located at a distance r from point O. As the object rotates, point P moves in a circular path of radius r and center located at point O. To be able to analyze the kinetic characteristics of point P using the equations of motion, forces acting on the object and the acceleration of point P can be expressed in terms of their components normal and tangential to the circular path of motion. If n and t designate the normal (radial) and tangential directions at point P, then the equations of motion can be expressed as:

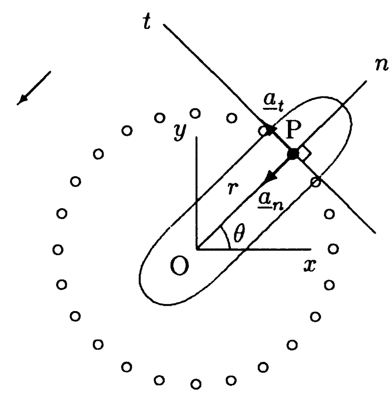


Fig. 10.1 *Rotational motion*

$$\sum F_n = ma_n \tag{10.1}$$

$$\sum F_t = ma_t \tag{10.2}$$

Here, $\sum F_n$ is the net force acting in the normal direction, $\sum F_t$ is the net force acting in the tangential direction, a_n is the magnitude of the centripetal acceleration (always directed toward the center of rotation), and a_t is the magnitude of tangential acceleration. While applying Eq. (10.1), the forces acting toward the center of rotation (centripetal forces) must be taken to be positive, and the forces directed outward (centrifugal forces) must be negative. For rotational motion about a fixed axis, the motion characteristics are completely known if the linear velocity (with magnitude v and direction tangent to the circular path) and the radius r of the circular path are known. Since $a_n = v^2/r$ and $a_t = dv/dt$, Eqs. (10.1) and (10.2) can alternatively be written as:

$$\sum F_n = m\frac{v^2}{r} \tag{10.3}$$

$$\sum F_t = m\frac{dv}{dt} \tag{10.4}$$

Note that $v = r\omega$ and $dv/dt = r\alpha$. Therefore, if the angular velocity and angular acceleration are known, then the kinetic characteristics of the problem can be analyzed using:

$$\sum F_n = mr\omega^2 \tag{10.5}$$

$$\sum F_t = mr\alpha \tag{10.6}$$

Note that for rotational motion there is always a normal component of the acceleration vector, and therefore, a force acting in the normal direction, but the tangential components of the force and acceleration vectors may or may not exist.

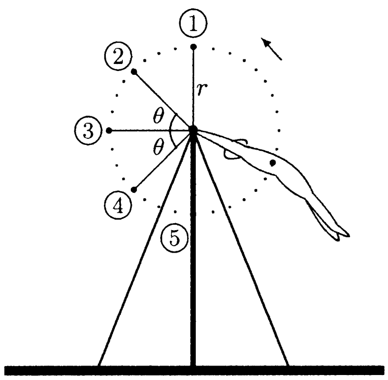


Fig. 10.2 *A gymnast on the high bar*

Example 10.1 Figure 10.2 illustrates a 60 kg gymnast swinging on a high bar. The rotational motion of the gymnast may be simplified by modeling the gymnast as a particle attached to a string such that the mass of the particle is equal to the mass of the gymnast and the length of the string is equal to the distance between the high bar and the center of gravity of the gymnast. As the gymnast moves, the center of gravity undergoes a circular motion.

Assume that the center of gravity of the gymnast is located at a distance $r = 1\,\text{m}$ from the high bar, the speed of the center of gravity at position 1 is almost zero, and that the effects of air resistance are negligible. Position 1 is directly above the high bar and it represents the highest elevation reached by the center of gravity of the gymnast.

(a) By using the conservation of energy principle, calculate the speeds of the gymnast's center of gravity at positions 2, 3, 4, and 5. As shown in Fig. 10.2, positions 2 and 4 make an angle $\theta = 45°$ with the horizontal, position 3 is along the same horizontal line as the high bar, and position 5 is directly under the high bar.
(b) Calculate the angular velocities of the gymnast at positions 1, 2, 3, 4, and 5.
(c) Calculate the normal component of the linear accelerations of the gymnast's center of gravity at positions 1, 2, 3, 4, and 5.
(d) Calculate the forces applied on gymnast's arms at positions 1, 2, 3, 4, and 5.
(e) Calculate the tangential component of the linear accelerations of the gymnast's center of gravity at positions 1, 2, 3, 4, and 5.
(f) Calculate the angular accelerations of the gymnast at positions 1, 2, 3, 4, and 5.

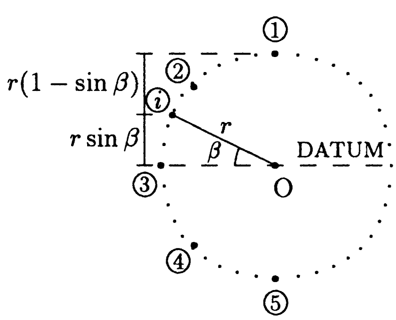


Fig. 10.3 *For different β values, i represents positions 1, 2, 3, 4, and 5*

Solution

(a) Different positions of the gymnast's center of gravity are illustrated in Fig. 10.3. Point O in Fig. 10.3 represents the high bar. The conservation of energy principle states that if an object is moving under the effect of conservative forces, then the total energy (sum of potential and kinetic energies) will remain constant throughout the motion. Between any two positions 1 and 2:

$$\mathcal{E}_{P1} + \mathcal{E}_{K1} = \mathcal{E}_{P2} + \mathcal{E}_{K2}$$

$$mgh_1 + \frac{1}{2}\,m{v_1}^2 = mgh_2 + \frac{1}{2}m{v_2}^2$$

Here, $m = 60\,\text{kg}$ is the total mass of the gymnast, $g = 9.8\,\text{m/s}^2$ is the magnitude of gravitational acceleration, $v_1 = 0$ is the speed of the gymnast at position 1, v_2 is the unknown speed of the gymnast at position 2, and h_1 and h_2 are the heights of positions 1 and 2 relative to a datum. Once a datum is chosen, h_1 and h_2 can be calculated and the above equation can be solved for the unknown parameter v_2. To calculate v_3, v_4, and v_5, the above procedure must be repeated.

Instead of carrying out the same procedure four times, consider the position of the gymnast labeled as "i" in Fig. 10.3. Assume that the line connecting point O and position i makes an angle β with the horizontal that passes through O such that $\beta = 90°$ at position 1, $\beta = 45°$ at position 2, $\beta = 0°$ at position 3, $\beta = -45°$ at position 4, and $\beta = -90°$ at position 5. We can apply the conservation of energy principle between positions 1 and i:

$$mgh_1 + \frac{1}{2}m{v_1}^2 = mgh_i + \frac{1}{2}m{v_i}^2$$

Since $v_1 = 0$, the second term on the left-hand side of this equation is zero. Also, m is a common parameter in all of the terms and can be eliminated. To calculate heights h_1 and h_i, we need to choose a datum. If we choose the datum to coincide with the level of high bar, then from the geometry of the problem $h_1 = r$ and $h_i = r\sin\beta$. We can substitute h_1 and h_i into the above equation and solve it for v_i:

$$v_i = \sqrt{2gr\,(1 - \sin\beta)}$$

This is a general solution valid for any position on the circular path of motion of the gymnast's center of gravity. For example, when angle $\beta = 45°$, $i = 2$ and $v_2 = \sqrt{2(9.8)(1)(1 - \sin 45)} = 2.39\,\text{m/s}$. When $\beta = 0°$, $i = 3$ and $v_3 = \sqrt{2(9.8)(1)(1 - \sin 0)} = 4.43\,\text{m/s}$. Similarly, $v_4 = 5.77\,\text{m/s}$ and $v_5 = 6.26\,\text{m/s}$. These results are used to plot a speed versus angular position (measured in terms of angle β) graph in Fig. 10.4.

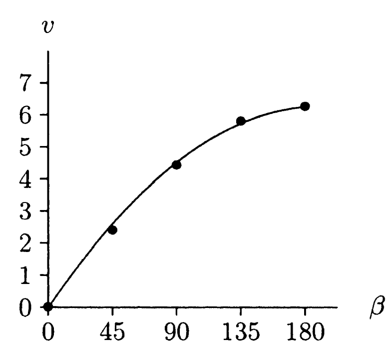


Fig. 10.4 *Speed measured in m/s versus angle β in degrees*

(b) The relationship between the angular velocity and linear velocity of a point on an object undergoing a rotational motion about a fixed axis is:

$$\omega = \frac{v}{r}$$

For example, for position 2, $\omega_2 = v_2/r = 2.39/1 = 2.39\,\text{rad/s}$. Similarly, $\omega_1 = 0$, $\omega_3 = 4.43\,\text{rad/s}$, $\omega_4 = 5.57\,\text{rad/s}$, and $\omega_5 = 6.26\,\text{rad/s}$.

(c) The normal (radial) component of the acceleration of the gymnast's center of gravity can be calculated using:

$$a_n = \frac{v^2}{r} = r\omega^2$$

For example, $a_{n2} = r\omega_2{}^2 = (1)(2.39)^2 = 5.74\,\text{m/s}^2$ for position 2. On the other hand, $a_{n1} = 0$, $a_{n3} = 19.62\,\text{m/s}^2$, $a_{n4} = 33.41\,\text{m/s}^2$ and $a_{n5} = 39.19\,\text{m/s}^2$.

(d) To calculate the forces applied on the gymnast's arms, consider Fig. 10.5 that shows the free-body diagrams of the gymnast's center of gravity. The forces acting on the gymnast are those applied by gravity and on the gymnast's arms by the high bar. The force applied by gravity is always directed vertically downward. The force applied by the high bar is in the radial direction. The equation of motion in the normal direction is:

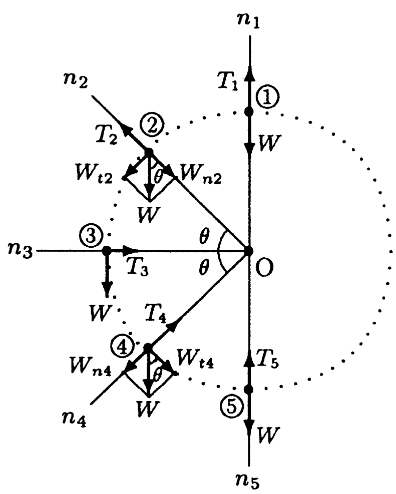


Fig. 10.5 *Forces acting on the gymnast's center of gravity*

$$\sum F_n = ma_n$$

At position 1, $a_{n1} = 0$ and the gymnast is in equilibrium. Therefore, since the weight of the gymnast is acting downward, the force, T_1, applied by the high bar on the gymnast's arms must be upward. Applying the equation of equilibrium:

$$\sum F_{n1} = 0: \quad W - T_1 = 0$$
$$T_1 = W = mg$$
$$T_1 = (60)(9.8) = 588\,\text{N}$$

At position 2, forces acting in the normal direction are the radial component, W_{n2}, of the gymnast's weight and T_2 applied by the high bar on the gymnast's arms. From the geometry of the problem, $W_{n2} = W\cos(45)$ and the direction of $\underline{W}_{n2}$ is toward the center of rotation. At this point, we do not know the direction of $\underline{T}_2$. We can assume that it is centrifugal. Now, we can apply the equation of motion in the normal direction:

$$\sum F_{n2} = m\,a_{n2}: \quad W_{n2} - T_2 = ma_{n2}$$
$$T_2 = W_{n2} - ma_{n2}$$
$$T_2 = mg\ \sin(45) - ma_{n2}$$
$$T_2 = (60)(9.8)(\sin 45) - (60)(5.74) = 71\,\text{N}$$

Since we calculated a positive value for T_2, the direction we assumed for $\underline{T}_2$ was correct. In other words, the effect of the high bar is such that it is "pushing" the arms of the gymnast at position 2.

At position 3, the gymnast's weight has no component in the radial direction. The force acting in the normal direction is T_3 applied by the high bar. In this case, assume that $\underline{T}_3$ is centripetal (toward O). Writing the equation of motion for position 3:

$$\sum F_{n3} = ma_{n3}: \quad T_3 = ma_{n3}$$
$$T_3 = (60)(19.62) = 1177\,\text{N}$$

Again, since we calculated a positive value for T_3, the direction we assumed for $\underline{T}_3$ was correct. At position 3, the tendency of the gymnast is to move away from the center of rotation and what is holding the gymnast in the circular path of motion is the "pulling" effect of the high bar.

At position 4, forces acting in the radial direction are T_4 applied by the high bar and $W_{n4} = W \sin(45)$ component of the gymnast's weight. From the geometry of the problem, $\underline{W}_{n4}$ is centrifugal. Assuming that $\underline{T}_4$ is centripetal and applying the equation of motion:

$$\sum F_{n4} = ma_{n4}: \quad T_4 - W_{n4} = ma_{n4}$$
$$T_4 = W_{n4} + ma_{n4}$$
$$T_4 = mg \sin(45) + ma_{n4}$$
$$T_4 = (60)(9.8)(\sin 45) + (60)(33.41)$$
$$T_4 = 2420\,\text{N}$$

Similarly at position 5:

$$\sum F_{n5} = ma_{n5}: \quad T_5 - W = ma_{n5}$$
$$T_5 = mg + ma_{n5}$$
$$T_5 = (60)(9.8) + (60)(39.19) = 2939\,\text{N}$$

In Fig. 10.6, these results are used to plot a force applied by the high bar on the arms of the gymnast versus angular position (measured in terms of angle β in Fig. 10.3) graph. Note that between positions 1 and 2, the high bar has a pushing effect on the arms. In other words, the force applied by the high bar on the arms is compressive. Just after position 2, the force applied by the high bar is zero, and thereafter it has a pulling or tensile effect on the arms.

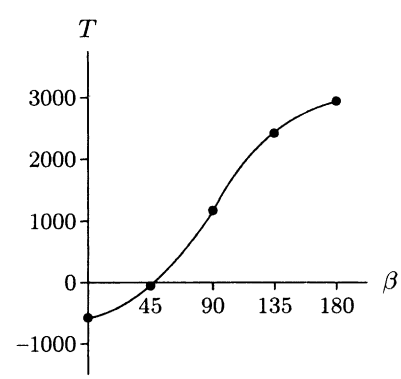


Fig. 10.6 *Force applied by the high bar on the gymnast's arms (measured in Newtons) versus angle β in degrees*

(e) The equation of motion in the tangential direction is:

$$\sum F_t = ma_t \quad \text{or} \quad a_t = \frac{\sum F_t}{m}$$

Forces acting in the tangential direction for different positions of the gymnast's center of gravity are shown in Fig. 10.5. At position 1, there is no force in the tangential direction, and therefore, $a_{t1} = 0$. At position 2, $W_{t2} = W \cos(45)$ is the only tangential force. Therefore:

$$a_{t1} = \frac{W_{t2}}{m} = \frac{mg \cos(45)}{m} = g \cos(45) = 6.93\,\text{m/s}^2$$

At position 3, $W = mg$ is the only tangential force:

$$a_{t3} = \frac{W_{t3}}{m} = \frac{mg}{m} = g = 9.80\,\text{m/s}^2$$

At position 4, $W_{t4} = W\cos(45)$ is the only tangential force:

$$a_{t4} = \frac{W_{t4}}{m} = \frac{mg\,\cos(45)}{m} = g\,\cos(45) = 6.93\,\text{m/s}^2$$

There is no tangential force at position 5, and therefore, $a_{t5} = 0$.

(f) Now that we calculated the tangential components of the acceleration vector, we can also calculate the angular acceleration of the gymnast using:

$$\alpha = \frac{a_t}{r}$$

Note that $r = 1\,\text{m}$. Therefore, $\alpha_1 = 0$, $\alpha_2 = 6.93\,\text{rad/s}^2$, $\alpha_3 = 9.80\,\text{rad/s}^2$, $\alpha_4 = 6.93\,\text{rad/s}^2$, and $\alpha_5 = 0$.

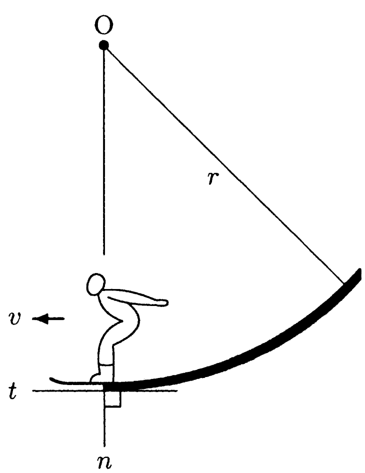


Fig. 10.7 *Circular end region of a ski jump track*

Example 10.2 Figure 10.7 illustrates the circular end region of a ski jump track. The radius of curvature of the track at this region is $r = 50\,\text{m}$. At the end of the track, the direction normal to the track coincides with the vertical and the direction tangential to the track coincides with the horizontal.

Consider a 70 kg ski jumper who is decelerating at a rate of $1.5\ \text{m/s}^2$ due to air resistance. If the friction on the track is negligible and the ski jumper reaches the end of the track with a horizontal velocity of $v = 20\,\text{m/s}$, determine the forces applied on the skier by the air resistance and the track.

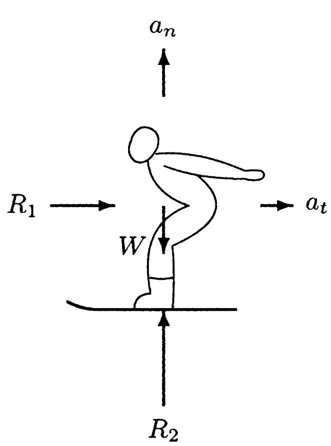


Fig. 10.8 *Free-body diagram of the ski jumper before takeoff*

Solution The free-body diagram of the ski jumper at the very end of the track (just before takeoff) is shown in Fig. 10.8. The forces acting on the ski jumper are $\underline{W}$ due to gravity, $\underline{R}_1$ due to air resistance, and the reaction force $\underline{R}_2$ applied by the track on the skis. Note that $\underline{R}_2$ is applied in the vertical direction or direction normal to the track. It is assumed that $\underline{R}_1$ due to air resistance is applied in the horizontal direction or direction tangent to the track.

At the circular end region of the track, the ski jumper undergoes a motion in a circular path with radius $r = 50\,\text{m}$. The speed of the ski jumper at the very end of the track is $v = 20\,\text{m/s}$. Therefore, the magnitude of the ski jumper's acceleration in the normal direction is:

$$a_n = \frac{v^2}{r} = \frac{(20)^2}{50} = 8.0\,\text{m/s}^2$$

Due to air resistance, the ski jumper is decelerating at a rate of 1.5 m/s^2 in the direction of motion (toward the left). Or, the ski jumper is accelerating at a rate of 1.5 m/s^2 in the direction opposite to the direction of motion (toward the right). Therefore, the tangential acceleration of the ski jumper toward the right is:

$$a_t = 1.5\,\mathrm{m/s^2}$$

Now we can utilize the equations of motion. In the tangential direction:

$$\sum F_t = ma_t: \quad R_1 = ma_t$$
$$R_1 = (70)(1.5) = 105\ \mathrm{N}$$

Equation of motion in the normal direction:

$$\sum F_n = ma_n: \quad R_2 = W = ma_n$$
$$R_2 = W = ma_n = mg + ma_n$$
$$R_2 = (70)(9.8) + (70)(8.0) = 1246\ \mathrm{N}$$

Therefore, at the very end of the ski jump track, air resistance is applying a horizontal force of $R_1 = 105\,\mathrm{N}$ to retard the motion of the skier and the track is applying a vertical force of $R_2 = 1246\,\mathrm{N}$ on the skis. Note that R_2 includes the effects of the weight W and rotational inertia ma_n of the ski jumper.

10.2 Torque and Angular Acceleration

Torque is the quantitative measure of the ability of a force to rotate an object. The mathematical definition of torque is the same as that of moment, studied in detail in Chap. 3. Consider the bolt and wrench arrangement illustrated in Fig. 10.9. Force $\underline{F}$ applied on the wrench rotates the wrench, which advances the bolt into the wall by rotating it in the clockwise direction. The magnitude of torque $\underline{M}$ due to force $\underline{F}$ about point O is:

$$M = rF_t = rF\ \sin\varphi \qquad (10.7)$$

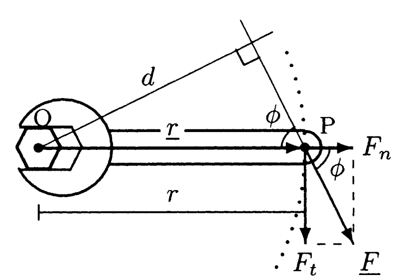


Fig. 10.9 *Force $\underline{F}$ applied on the wrench produces a clockwise torque about the centerline of the bolt*

The line of action of $\underline{M}$ is perpendicular to the plane of rotation and its direction can be determined by using the right-hand rule (in this case, clockwise).

An object would rotate about an axis if the rotational motion of the object is not constrained and if there is a net torque acting on the object about that axis. The angular acceleration of an object undergoing a rotational motion is directly proportional to the resultant torque acting on it. To derive the relationship between torque and angular acceleration, consider a particle of mass m undergoing a rotational motion about a fixed axis. Let O be a point on this axis, r be the radius of the circular path of motion, and $\underline{F}$ be the tangential force causing the rotational motion (Fig. 10.10). The equation of motion in the tangential direction can be written as:

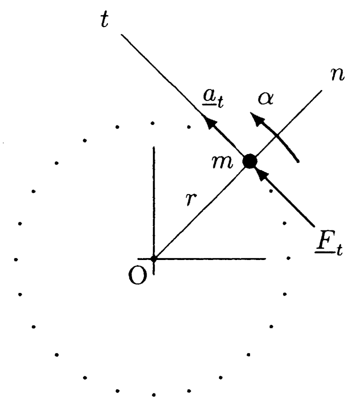


Fig. 10.10 $F_1 = ma_1$ *and* $M_O = I_O\alpha$

$$F_t = ma_t \tag{10.8}$$

In Eq. (10.8), a_t is the magnitude of the tangential acceleration of the particle. If the angular acceleration, α, of the particle is known, then $a_t = r\alpha$. Replacing a_t by $r\alpha$ and multiplying both sides of Eq. (10.8) by r will yield:

$$r\,F_t = (mr^2)\alpha \tag{10.9}$$

Note that the left-hand side of Eq. (10.9) is the magnitude M_o of the torque generated by force $\underline{F}_t$ about O. The term mr^2 on the right-hand side is known as the *mass moment of inertia* of the particle about O. Denoting the mass moment of inertia with I_o, Eq. (10.9) can also be written as:

$$M_o = I_o\alpha \tag{10.10}$$

If there is more than one torque-generating force applied to the particle, then M_o in Eq. (10.10) represents the net torque acting on the particle about O. The general form of Eq. (10.10) can be obtained by representing the torque and angular acceleration as vector quantities:

$$\underline{M} = I\underline{\alpha} \tag{10.11}$$

This is the rotational analogue of Newton's second law of motion, which states that angular acceleration is directly proportional to the net torque and inversely proportional to the mass moment of inertia.

10.3 Mass Moment of Inertia

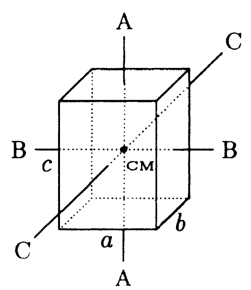
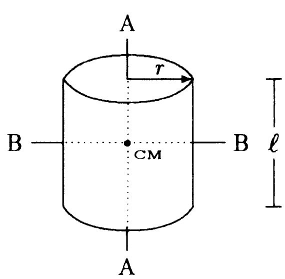
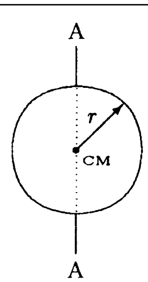
In general, the term inertia implies resistance to change. When a rotation-causing force is applied to a pivoted body, its tendency to resist angular acceleration depends on its mass moment of inertia. The larger the mass moment of inertia of a body, the more difficult it is to accelerate it in rotation. For a particle of mass m, the mass moment of inertia about an axis is defined as the mass times the square of the shortest distance, r, between the particle and the axis about which the mass moment of inertia is to be determined:

$$I = mr^2 \tag{10.12}$$

A rigid body that is not a particle can be assumed to consist of many particles, the sum of masses of which is equal to the total mass of the body itself. The mass moment of inertia of the entire body can be determined by considering the sum of the mass of each particle of the body multiplied by the square of its distance from the axis of rotation.

Notice that the mass moment of inertia of a rigid body is proportional to its mass, which is a function of its density and volume. Therefore, the mass moment of inertia of a body depends upon its material and geometric properties as well as the location and orientation of the axis about which it is to be determined. The ability of a body to resist changes in its angular velocity is dependent not only upon the mass of the body, but also upon the distribution of the mass. The greater the concentration of mass at the periphery, the greater the mass moment of inertia and the more difficult it is to change the angular velocity. For a rigid body with a simple, symmetrical geometry and homogeneous composition, the mass moment of inertia about an axis coinciding with an axis of symmetry, called a *centroidal axis*, can be calculated relatively easily. In Table 10.1, moments

Table 10.1 *Moments of inertia of homogeneous rigid bodies with different geometries about their centroidal axes*

	*Rectangular Prism* $I_{AA} = \frac{1}{12} m\left(a^2 + b^2\right)$ $I_{BB} = \frac{1}{12} m\left(b^2 + c^2\right)$ $I_{CC} = \frac{1}{12} m(c^2 + a^2)$ $V = abc$
	*Solid Cylinder or Disc* $I_{AA} = \frac{1}{2} mr^2$ $I_{BB} = \frac{1}{12} m(3r^2 + \ell^2)$ $V = \pi r^2$
	*Solid Sphere* $I_{AA} = \frac{2}{5} mr^2 V = \frac{4}{3} \pi r^3$

Their volumes are also provided

of inertia for some geometric shapes are provided about their centroidal axes.

The mass moment of inertia is a scalar quantity. It has a dimension $[M][L^2]$, and is measured in terms of kg m^2 in SI.

10.4 Parallel-Axis Theorem

If the mass moment of inertia of a body about a centroidal axis is known, then the mass moment of inertia of the same body about any other axis parallel to that centroidal axis can be determined using the *parallel-axis theorem*. This theorem can be stated as:

$$I = I_c + m{r_c}^2 \tag{10.13}$$

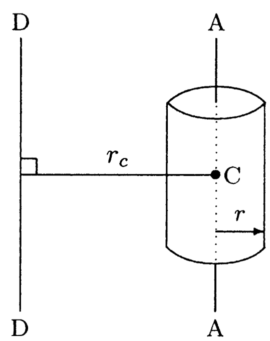


Fig. 10.11 *According to the parallel-axis theorem,* $I_{DD} = I_{AA} + m{r_c}^2$

In Eq. (10.13), m is the total mass of the body, I_c is the mass moment of inertia of the body about one of its centroidal axes, I is the required mass moment of inertia about an axis parallel to the centroidal axis, and r_c is the shortest distance between the two axes. For example, consider the solid cylinder shown in Fig. 10.11. From Table 10.1, the mass moment of inertia of the cylinder about AA is $I_{AA} = \frac{1}{2}mr^2$. The mass moment of inertia of the same cylinder about DD, which is parallel to AA and located at a distance r_c from AA, is:

$$I_{DD} = I_{AA} + m{r_c}^2 = \frac{1}{2}mr^2 + m{r_c}^2$$

Note that in the case of human body segments, each segment or limb rotates about the joints at either end of the moving segment rather than about its mass center or centroidal axes. Furthermore, mass moment of inertia measurements can only be made about a joint center. If needed, the parallel-axis theorem can be utilized to determine the mass moment of inertia of a segment about its mass center.

10.5 Radius of Gyration

Consider a rigid body of mass m. Let I be the mass moment of inertia of the rigid body about a given axis AA. Also consider a point mass m located at a distance ρ (rho) from the same axis such that its mass moment of inertia $m\rho^2$ about AA is equal to the mass moment of inertia I of the rigid body about AA. That is:

$$\rho = \sqrt{\frac{I}{m}} \tag{10.14}$$

ρ is called the *radius of gyration*, and for rotational motion analysis, the rigid body can be treated as a point with the mass equal to the total mass of the body and located at a distance ρ from the axis of rotation.

10.6 Segmental Motion Analysis

The information provided in the previous sections can be utilized to develop mathematical models for analyzing the motion characteristics of human body segments. Here, the general procedure for developing a dynamic model of a body segment will be outlined, and then applied to analyze the rotational motion of the lower leg about the knee joint.

The first step of a dynamic model analysis involves defining the forces acting on the body segment. These may include the gravitational (weight), external, inertial, muscle, and joint reaction forces. The weight of the body segment can be assumed to act at its center of gravity, and therefore, the center of gravity of the segment must be known. The magnitude, point of application, and direction of any external force present must be specified. Inertial forces are those present due to the dynamics of the problem under consideration. One way of incorporating inertial effects into the model is through the use of the radius of gyration. Muscle and joint reaction forces are the unknowns to be determined as a result of these analyses. It is important to draw the free-body diagram of the segment to be analyzed, and to identify all the known and unknown forces acting on it.

The next step of dynamic analysis is the identification of measurable quantities. In general, both the angular displacement of the moving body segment and the net torque generated about its axis of rotation can be measured as functions of time over the range of segmental motion. The angular displacement measurement techniques include goniometric, dynamometric, and photogrammetric methods. Through the use of kinematic relationships, the angular displacement data can be used to calculate the angular velocity and angular acceleration of the moving segment. If the angular displacement θ is known as a function of time t, then the angular velocity ω and angular acceleration α can be determined by considering the first and second derivatives of θ with respect to t:

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

If it is not possible to find a function representing the relationship between the angular displacement and time, then numerical differentiation techniques can be employed. Once the angular acceleration is determined, the relationship between torque, mass moment of inertia, and angular acceleration can be used to calculate the net torque produced about the joint center, provided that the mass moment of inertia (or the radius of gyration) of the segment about the joint center is known.

For a two-dimensional (planar) motion analysis of a segment about its joint center, if the net torque M produced about the joint axis is measured as a function of time and the mass moment of inertia I of the moving segment about the same axis is known, then the angular acceleration of the segment can be determined from:

$$\alpha = \frac{M}{I}$$

If needed, the angular velocity and displacement of the moving segment can also be determined by considering the integral of the function representing angular acceleration with respect to time.

It is clear from this discussion that anthropometric information about the moving segment must be available. For this purpose, anthropometric data tables listing average segmental weights, lengths, and radius of gyration can be utilized. Another important consideration is that the instantaneous center of rotation of the moving segment must also be known. Note however that the instantaneous center of rotation about a given joint may vary.

The final step of dynamic model analyses involves the computation of muscle and joint reaction forces. Note that the net torque measured or calculated includes the effects of all forces acting on the moving segment. The torque generated by the muscles crossing the joint can be determined by subtracting the effects of external and gravitational forces from the net torque measured or calculated about the joint center. If the forces generated by individual muscles are required, then additional factors must be considered. For example, the locations of muscle attachments and the lines of action (lines of pull) of the muscle forces must be known. The distribution of forces among different muscles must also be specified. If the segmental motion is achieved primarily by a single muscle group, then the rotational component of the muscle force can be determined by applying Newton's second law of motion. Since the line of action of the muscle force is assumed to be known, the magnitude of the muscle force can also be determined.

Example 10.3 *Knee extension*

The angular motion of the lower leg about the knee joint, and the forces and torques produced by the muscles crossing the knee joint during knee flexions and extensions have been investigated by a number of researchers utilizing different experimental techniques. One of these techniques is discussed here.

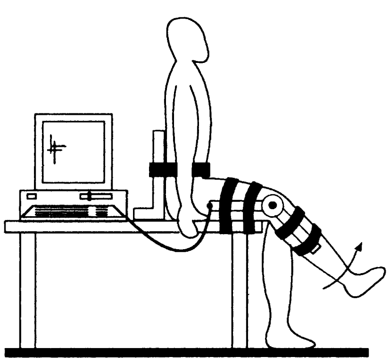


Fig. 10.12 *Knee extension*

Consider the person illustrated in Fig. 10.12. The test subject is sitting on a table, with the back placed against a back rest and the lower legs free to rotate about the knee joint. The subject's torso is strapped to the back rest and the right thigh is strapped firmly to the table. A well-padded sawhorse is placed in front of the subject to prevent hyperextension at the knee joint (not shown in Fig. 10.12). An electrogoniometer is attached to the subject's right leg. The arms of the goniometer are aligned with the estimated long axes of the thigh and shank, and the axis of rotation of the goniometer is aligned with the estimated axis of rotation of the knee joint. The subject is then asked to extend the lower leg as rapidly as possible. The signals received from the electrogoniometer's potentiometer are stored in a computer, and are used to calculate the angular displacement θ of the lower leg as measured from its initial vertical position. Using a finite difference (numerical differentiation) technique, the angular velocity ω and angular acceleration α of the lower leg are also computed.

Some of the forces acting on the lower leg are shown in Fig. 10.13, along with the geometric parameters of the model under consideration. This model is based on the assumption that the quadriceps muscle is the primary muscle group responsible for knee extension. Point O represents the instantaneous center of rotation of the knee joint. The patellar tendon is attached to the tibia at A. For the position of the lower leg relative to the upper leg shown in Fig. 10.13, it is estimated that the line of pull of the patellar tendon force $\underline{F}_m$ makes an angle β with the long axis of the tibia. The lever arm of $\underline{F}_m$ relative to O can be represented by a distance a that changes as the lower leg moves up through the range of motion. The total weight of the lower leg is W and its center of gravity is located at B, which is at a distance b from O measured along the long axis of the tibia. The intended direction of motion is counterclockwise (extension).

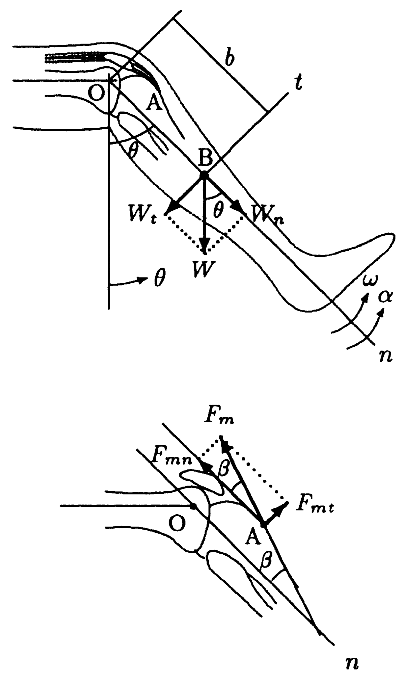


Fig. 10.13 *Some of the forces acting on the lower leg*

At an instant when $\theta = 60°$, $\omega = 5\,\text{rad/s}$, and $\alpha = 200\,\text{rad/s}^2$, and assuming that $W = 50\,\text{N}$, $a = 4$ cm, $b = 22$ cm, $\beta = 24°$, and the mass moment of inertia of the lower leg about the knee joint is $I_o = 0.25\,\text{kg}\,\text{m}^2$, determine:

(a) The net torque produced about the knee joint
(b) The tension in the patellar tendon
(c) The reaction force at the knee joint

Solution

(a) From Newton's second law of motion, the net torque M_o generated about the knee joint is:

$$M_o = I_o\,\alpha = (0.25)(200) = 50\,\text{Nm} \quad \text{(ccw)}$$

(b) Note that M_o is the magnitude of the net torque about the knee joint and it includes the rotational effects of all of the external forces acting on the lower leg. Relative to the knee joint, there are two external forces with rotational effects: patellar tendon force $\underline{F}_m$ and weight of the lower leg $\underline{W}$. For the position of the lower leg that makes an angle $\theta = 60°$ with the vertical, the lever arm of the patellar tendon force relative to O is estimated to be $a = 0.04\,\text{m}$. Therefore, the torque generated by $\underline{F}_m$ relative to O is $M_m = aF_m$ (counterclockwise). On the other hand, $\underline{W}$ acts downward. The torque generated by $\underline{W}$ about O is $M_w = W_t b = \sin\theta b$ (clockwise). Therefore, the magnitude of the net torque, M_o, about the knee joint due to $\underline{F}_m$ and $\underline{W}$ is:

$$M_o = M_m - M_w = a\,F_m - b\,W\,\sin\theta$$

This equation can be solved for the unknown force, F_m:

$$F_m = \frac{M_o + b\,W\,\sin\theta}{a}$$

Substituting the known parameters and carrying out the calculations will yield:

$$F_m = \frac{50 + (0.22)(50)(\sin 60°)}{0.04} = 1488\,\text{N}$$

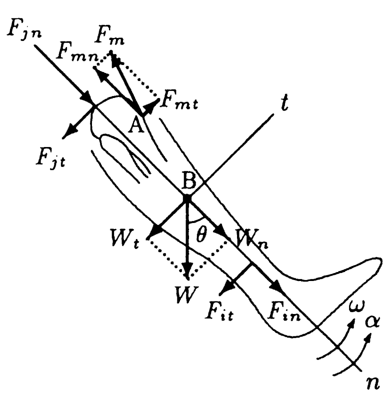


Fig. 10.14 *Free-body diagram of the lower leg*

(c) The free-body diagram of the lower leg is shown in Fig. 10.14. W is the weight of the lower leg, F_m is the magnitude of the patellar tendon force applied on the tibia at A, F_{jn} is the component of the tibiofemoral joint reaction force along the long axis of the tibia, and F_{jt} is the component of the tibiofemoral joint reaction force in a direction perpendicular to the long axis of the tibia. The components of $\underline{W}$ and $\underline{F}_m$ along the long axis of the tibia and in the perpendicular direction are also shown in Fig. 10.14, as well as the components of inertial force F_{i}.

One way of taking into account the inertial effects of a moving body, in this case a rotating body segment, is by means of something known as d'Alembert's principle. If the mass m, distance r between the mass center and the axis of rotation, angular velocity ω, and angular acceleration α of the rotating body are known, then the magnitudes of inertial forces F_{in} and F_{it} which are normal and tangent to the path of motion can be calculated using Eqs. (10.5) and (10.6) provided in Sect. 10.1:

$$F_{\text{in}} = ma_{\text{n}} = mr\omega^2$$

$$F_{\text{it}} = ma_{\text{t}} = mr\alpha$$

In this case, we have $m = W/g = 50/9.8 = 5.1\,\text{kg}$, $r = b = 0.22\,\text{cm}$, $\omega = 5\,\text{rad/s}$, and $\alpha = 200\ \text{rad/s}^2$. a_{n} and a_{t} are the components of the acceleration vector of the lower leg in the directions normal and tangential to its path of motion. $\underline{a}_{\mathrm{n}}$ is always toward the center of rotation, and since the motion is counterclockwise, $\underline{a}_{\mathrm{t}}$ is counterclockwise. Therefore, the inertial forces $\underline{F}_{\mathrm{in}}$ and $\underline{F}_{\mathrm{it}}$ are such that $\underline{F}_{\mathrm{in}}$ is centripetal (toward the center of rotation) and $\underline{F}_{\mathrm{it}}$ is trying to rotate the leg in the counterclockwise direction. As illustrated in Fig. 10.14, d'Alembert's principle can be applied by assuming that $\underline{F}_{\mathrm{in}}$ is a centrifugal force (rather than centripetal) trying to pull the leg outward, $\underline{F}_{\mathrm{it}}$ is trying to rotate the lower leg in the clockwise direction (rather than in the counterclockwise direction), and the system is in static equilibrium. The conditions of the equilibrium of the system can be represented by the following equations valid along the directions normal and tangential to the path of motion:

$$\sum F_{\mathrm{n}} = 0: \quad F_{\mathrm{jn}} - F_{\mathrm{mn}} + F_{\mathrm{in}} + W_{\mathrm{n}} = 0$$

$$\sum F_{\mathrm{t}} = 0: \quad F_{\mathrm{jt}} - F_{\mathrm{mt}} + F_{\mathrm{it}} + W_{\mathrm{t}} = 0$$

Solving these equations for the components of the joint reaction force will yield:

$$F_{\mathrm{jn}} = F_{\mathrm{mn}} - F_{\mathrm{in}} - W_{\mathrm{n}}$$

$$F_{\mathrm{jt}} = F_{\mathrm{mt}} - F_{\mathrm{it}} - W_{\mathrm{t}}$$

Substituting the known parameters by their mathematical expressions will yield:

$$F_{\mathrm{jn}} = F_m \cos\beta - mb\omega^2 - W\cos\theta$$

$$F_{\mathrm{jt}} = F_m \sin\beta - mb\alpha - W\sin\theta$$

Now, substituting the numerical values and carrying out the calculations will yield:

$$F_{\mathrm{jn}} = (1488)(\cos 24) - (5.1)(0.22)(5)^2 - (50)(\cos 60) = 1306\,\mathrm{N}$$

$$F_{\mathrm{jt}} = (1488)(\sin 24) - (5.1)(0.22)(200) - (50)(\sin 60) = 338\,\mathrm{N}$$

Therefore, the magnitude of the resultant force applied by the femur on the tibia is $F_{\mathrm{j}} = \sqrt{(F_{\mathrm{jn}})^2 + (F_{\mathrm{jt}})^2} = 1349\,\mathrm{N}$.

10.7 Rotational Kinetic Energy

Assume that the rigid body shown in Fig. 10.15 is composed of many small particles and that the body rotates about a fixed axis with an angular velocity ω. If m_i and v_i are the mass and the

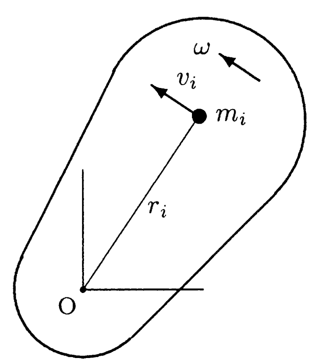


Fig. 10.15 *Rotational motion of a body about a fixed axis*

speed of the ith particle in the body, respectively, then the kinetic energy of the particle is:

$$\mathcal{E}_{Ki} = \frac{1}{2} m_i\, v_i^2$$

At any instant, every particle in the body has the same angular velocity ω, but the linear velocity of each particle depends on its distance measured from the axis of rotation. If r_i is the perpendicular distance between the ith particle and the axis of rotation (i.e., radius of the circular motion path of the ith particle), then $v_i = r_i\omega$ and its kinetic energy is $\mathcal{E}_{Ki} = \frac{1}{2} m_i r_i^2 \omega^2$. Each particle in the body has a kinetic energy, and the total kinetic energy, $\mathcal{E}_K$, of the rotating body is the sum of the kinetic energies of the individual particles in the body. That is,

$$\mathcal{E}_K = \sum_{i=1}^{n} \mathcal{E}_{Ki} = \frac{1}{2}\left(\sum_{i=1}^{n} m_i r_i^2\right)\omega^2$$

The quantity in parentheses is the mass moment of inertia I of the body. Therefore:

$$\mathcal{E}_K = \frac{1}{2}\, I\omega^2 \qquad (10.15)$$

Equation (10.15) defines the rotational kinetic energy of a body in terms of the mass moment of inertia and angular velocity of the body, and it is analogous to the kinetic energy $\mathcal{E}_K = \frac{1}{2} m v^2$ associated with linear motion.

10.8 Angular Work and Power

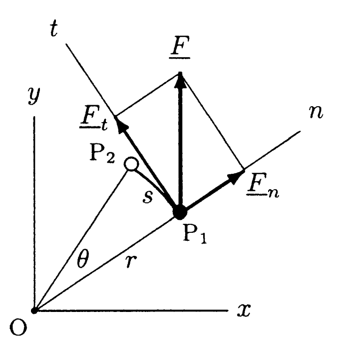


Fig. 10.16 *A particle located at P_1 is displaced by an angle θ or arc length s to position P_2*

By definition, the work done by a force is equal to the magnitude of the force times the corresponding displacement. The *angular work done* by a force applied on a rotating body is related to the angular displacement of the body. Consider a body rotating about a fixed axis at O due to an applied force $\underline{F}$. As illustrated in Fig. 10.16, let P_1 and P_2 represent the positions of a point in the body at times t_1 and t_2, respectively. In the time interval between t_1 and t_2, the body rotates through an arc of length s or angle θ. The work done by $\underline{F}$ on the body is equal to the magnitude of the component of the force vector in the direction of motion (tangential component, F_t), times the displacement s:

$$W = F_t s$$

The arc length is related to the angular displacement through the radius of the circular path of motion as $s = r\theta$. Therefore:

$$W = F_t r\theta$$

By definition, $F_t r$ is the magnitude M of the torque generated by force $\underline{F}$ about O. Hence:

$$W = M\theta \tag{10.16}$$

In other words, the work done by a rotation-producing force is equal to the torque generated by the force times the angular displacement of the body. Notice that the normal (radial) component of the force vector does not work on a body undergoing rotational motion because there is no motion in the normal direction.

It must be pointed out here that the relationship between angular work done, torque, and angular displacement given in Eq. (10.16) is valid when the torque is constant. The work done by a torque, which is a function of angular displacement, on a body to rotate the body from position 1 to 2 is:

$$W = \int_{\theta_1}^{\theta_2} M \, d\theta \tag{10.17}$$

Here, θ_1 and θ_2 are the angular displacements of the body at positions 1 and 2, respectively. Equation (10.17) can also be written in terms of the change in angular velocity by noting that $M = I\alpha$ and $\alpha = d\omega/dt$. Using the chain rule of differentiation:

$$M = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\,\frac{d\theta}{dt} = I\omega\frac{d\omega}{d\theta}$$

Substituting this into Eq. (10.17):

$$W = \int_{\omega_1}^{\omega_2} I\omega \; d\omega = \frac{1}{2}I{\omega_2}^2 - \frac{1}{2}I{\omega_1}^2 \tag{10.18}$$

In Eq. (10.18), ω_1 and ω_2 are the angular velocities of the body at positions 1 and 2, respectively. Equation (10.18), known as the *work-energy theorem in rotational motion*, states that the net angular work done on a rigid body in rotating the body about a fixed axis is equal to the change in the body's rotational kinetic energy.

The rate at which work is done is known as power. The *angular power* describes the rate at which angular work is done. For a constant torque:

$$P = \frac{dW}{dt} = M\frac{d\theta}{dt} = M\omega \tag{10.19}$$

That is, the angular power is equal to the product of the applied torque and the angular velocity of the body.

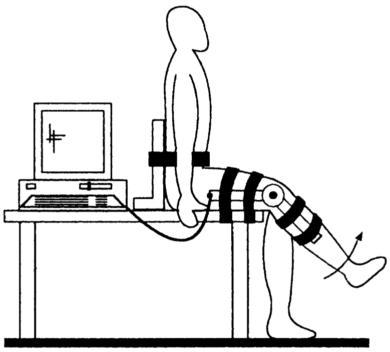


Fig. 10.17 *Knee extension*

Example 10.4 Consider the knee extension problem analyzed in Example 10.3. As illustrated in Fig. 10.17, the person is seated on a table. The upper body is strapped to a back rest and the right thigh is strapped firmly on the table with the lower leg hanging vertically downward. The person is then asked to extend the right lower leg. The angular displacement of the lower leg during knee extension is determined via a goniometer attached to the leg. After a series of computations, it is determined that the lower leg was extended from $\theta = 0°$ to $90°$ in a time period of 0.5 s with an average angular velocity of 3 rad/s by producing an average extensor muscle torque of 90 Nm.

Assuming that the mass moment of inertia of the lower leg about the center of rotation of the knee joint is 92 kg m^2, calculate the average angular kinetic energy produced, angular work done, and angular power generated by the knee extensor muscles to extend the lower leg from $\theta = 0°$ to $90°$.

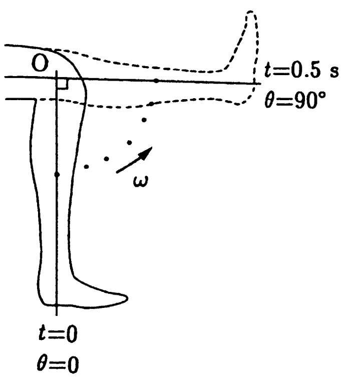


Fig. 10.18 *Movement of the lower leg*

Solution: The range of motion of the lower leg is $\Delta\theta = 90°$, which is covered in a time period of $\Delta t = 0.5\,\text{s}$ (Fig. 10.18). The mass moment of inertia of the lower leg about the knee joint is given as $I_o = 92\,\text{kg}\,\text{m}^2$. The average angular velocity of the lower leg is calculated to be $\overline{\omega} = 3\,\text{rad/s}$ and the average torque produced by the knee extensors is $\overline{M} = 90\,\text{Nm}$. Therefore, the average angular kinetic energy produced by the knee extensor muscles is:

$$\overline{\varepsilon}_\text{k} = \frac{1}{2} I_o \overline{\omega}^2 = \frac{1}{2}(92)(3)^2 = 414\,J$$

The average work done by the muscles to extend the lower leg at an angle of $90°$ or $90 \times \pi/180 = 1.57\,\text{rad}$ is:

$$\overline{W} = \overline{M}\Delta\theta = (90)(1.57) = 141.3\ \text{J}$$

The average power generated by the extensors is:

$$\overline{P} = \overline{M}\overline{\omega} = (90)(3) = 270\,\text{W}$$

10.9 Exercise Problems

Problem 10.1 As illustrated in Fig. 10.12, consider the person performing extension/flexion movements of the lower leg about the knee joint (point O) to investigate the forces and torques produced by muscles crossing the knee joint. The setup of the experiment is described in Example 10.3 above.

The geometric parameters of the model under investigation, some of the forces acting on the lower leg and its free-body diagrams are shown in Figs. 10.13 and 10.14. For this system, the angular displacement, angular velocity, and angular acceleration of the lower leg were computed using data obtained during the experiment such that at an instant when $\theta = 65°$, $\omega = 4.5$ rad/s, and $\alpha = 180$ rad/s^2. Furthermore, for this system assume that $a = 4.0$ cm, $b = 23$ cm, $\beta = 25°$, and the net torque generated about the knee joint is $M_0 = 55$ Nm. If the torque generated about the knee joint by the weight of the lower leg is $M_w = 11.5$ Nm, determine:

(a) The mass moment of inertia of the lower leg about the knee joint
(b) The weight of the lower leg
(c) The tension in the patellar tendon
(d) The reaction force at the knee joint

Answers: (a) $I_0 = 0.3$ kg m^2, (b) $W = 55.2$ N, (c) $F_m = 1662.5$ N, $F_{\mathrm{j}} = 1517.3$ N

Problem 10.2 As shown in Fig. 10.17, consider the person performing extension movements of the lower leg about the knee joint. The setup of the experiment is described in Example 10.4. After a series of computations it is determined that the lower leg was extended from $\theta = 0°$ to $\theta = 85°$ with an average angular velocity of $\omega = 3.3$ rad/s. For this system it is also estimated that the angular power generated by the knee extensor muscles during the experiment is $P = 290$ W. Assuming that the mass moment of inertia of the lower leg about the center of rotation of the knee joint is $I_0 = 93$ kg m^2, determine:

(a) The torque produced by the knee extensor muscles
(b) The work done by the muscles to extend the lower leg
(c) The kinetic energy generated by the knee extensor muscles

Answers: $M = 87.9$ Nm, $W = 130$ J, $E_{\mathrm{k}} = 506.4$ J

Problem 10.3 Consider the 15 kg solid cylinder shown in Fig. 10.19, undergoing a rotational motion under the effect of externally applied force. If the mass moment of inertia of the cylinder about its centroidal axis is $I_{\mathrm{AA}} = 0.5$ kg m^2, determine the radius of the cylinder.

Answer: $r = 0.26$ m

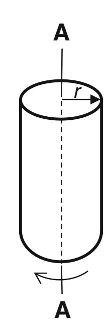


Fig. 10.19 *Problems 10.3 and 10.4*

Problem 10.4 As shown in Fig. 10.19, consider a solid cylinder undergoing a rotational motion about its centroidal axis AA with angular velocity of $\omega = 3.5$ rad/s. For this system it is estimated that the rotational kinetic energy of the cylinder is $E_k = 30$ J. If the radius of the cylinder is $r = 0.5$ m, determine:

(a) The mass moment of inertia of the cylinder about its centroidal axis
(b) The mass of the cylinder

Answers: (a) $I_{AA} = 4.9$ kg m^2, (b) $m = 19.6$ kg

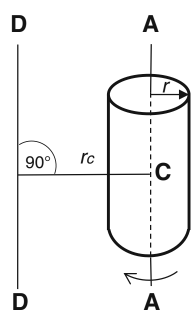


Fig. 10.20 *Problem 10.5*

Problem 10.5 In Fig. 10.20, a solid cylinder is undergoing a rotational motion about its centroidal axis AA. If the mass of the cylinder is $m = 12$ kg and its radius is $r = 0.3$ m, determine:

(a) The mass moment of inertia of the cylinder about the centroidal axis
(b) The mass moment of inertia of the cylinder about an axis DD which is located at a distance $r_c = 0.2$ m parallel to the centroidal axis

Answers: (a) $I_{AA} = 0.54$ kg m^2, (b) $I_{DD} = 1.02$ kg m^2