

Chapter 7

Linear Kinematics

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7.1 Uniaxial Motion

Uniaxial motion is one in which the motion occurs only in one direction, and it is the simplest form of linear or translational motion. A car traveling on a straight highway, an elevator going up and down in a shaft, and a sprinter running a 100-m race are examples of uniaxial motion.

Kinematic analyses utilize the relationships between the position, velocity, and acceleration vectors. For uniaxial motion analyses, it is usually more practical to define a direction, such as x , to coincide with the direction of motion, define kinematic parameters in that direction, and carry out the analyses as if displacement, velocity, and acceleration are scalar quantities.

7.2 Position, Displacement, Velocity, and Acceleration

Consider the car illustrated in Fig. 7.1. Assume that the car is initially stationary and located at 0. At time t_0 , the car starts moving to the right on a straight horizontal path. At some time t_1 , the car is observed to be at 1 and at a later time t_2 it is located at 2. 0, 1, and 2 represent *positions* of the car at different times, and 0 also represents the *initial position* of the car. It is a common practice to start measuring time beginning with the instant when the motion starts, in which case $t_0 = 0$.

The position of the car at different times must be measured with respect to a point in space. Let x be a measure of horizontal distances relative to the initial position of the car. If x_0 represents the initial position of the car, then $x_1 = 0$. If 1 and 2 are located at x_1 and x_2 distances away from 0, then x_1 and x_2 define the relative positions of the object at times t_1 and t_2 , respectively. Since the relative position of the car is changing with time, x is a function of time t , or $x = f(t)$. In the time interval between t_1 and t_2 , the position of the car changed by an amount $\Delta x = x_2 - x_1$, where Δ (capital delta) implies change. This change in position is the displacement of the car in the time interval $\Delta t = t_2 - t_1$.

During a uniaxial horizontal motion, the car may be located on the right or the left of the origin 0 of the x axis. Assuming that the positive x axis is toward the right, the position of the car is positive if it is located on the right of 0 and negative if it is on the left of 0. Similarly, the displacement of the car is positive if it is moving toward the right, and it is negative if the car is moving toward the left.

Velocity is defined as the time rate of change of relative position. If the position of an object moving in the x direction is known as a function of time, then the instantaneous velocity, v , of the

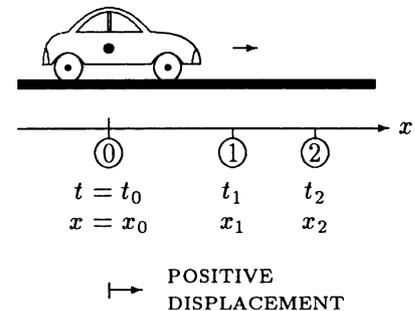


Fig. 7.1 The car is located at positions 0, 1, and 2 at times t_0 , t_1 , and t_2 , respectively

object can be determined by considering the derivative of x with respect to t :

$$v = \frac{dx}{dt} \quad (7.1)$$

If required, the average velocity, \bar{v} , of the object in any time interval can be determined by considering the ratio of change in position (displacement) of the object and the time it takes to make that change. For example, the average velocity of the car in Fig. 7.1 in the time interval between t_1 and t_2 is:

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \quad (7.2)$$

In Eq. (7.2), the “bar” over v indicates average, and x_1 and x_2 are the relative positions of the car at times t_1 and t_2 , respectively.

Velocity is a vector quantity and may take positive and negative values, indicating the direction of motion. The velocity is positive if the object is moving away from the origin in the positive x direction, and it is negative if the object is moving in the negative x direction. The magnitude of the velocity vector is called *speed*, which is always a positive quantity.

The instantaneous velocity of an object may vary during a particular motion. In other words, velocity may be a function of time, or $v = f(t)$. Acceleration is defined as the time rate of change of velocity. If the velocity of an object is known as a function of time, then its instantaneous acceleration, a , can be determined by considering the derivative of v with respect to t :

$$a = \frac{dv}{dt} \quad (7.3)$$

In general, the acceleration of a moving object may vary with time. In other words, acceleration may be a function of time, or $a = a(t)$.

There is also *average acceleration*, \bar{a} , that can be determined by considering the ratio of the change in velocity of the object and the time elapsed during that change. For example, if the instantaneous velocities v_1 and v_2 of the car in Fig. 7.2 at times t_1 and t_2 are known, then the average acceleration of the car in the time interval between t_1 and t_2 can be calculated:

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1} \quad (7.4)$$

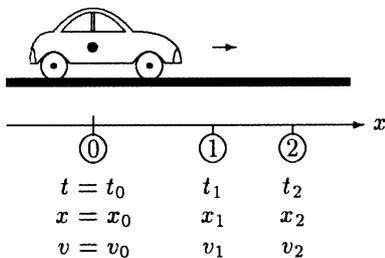


Fig. 7.2 v_0 is the initial velocity, and v_1 and v_2 are the velocities of the car at times t_1 and t_2 , respectively

Acceleration is a vector quantity and may be positive or negative. Positive acceleration does not always mean that the object is speeding up and negative acceleration does not always imply that the object is slowing down. At a given instant, if the velocity and acceleration are both positive or negative, then the object

is said to be speeding up or accelerating. For a uniaxial motion in the x direction, if both the velocity and acceleration are positive, then the object is moving in the positive x direction with an increasing speed. If both the velocity and acceleration are negative, then the object is moving in the negative x direction with an increasing speed. On the other hand, if the velocity and acceleration have opposite signs, then the object is slowing down or decelerating. For example, for a uniaxial motion in the x direction, if the velocity is positive and acceleration is negative, then the object is moving in the positive x direction with a decreasing speed. If the velocity is negative and acceleration is positive, then the object is moving in the negative x direction with a decreasing speed. Finally, if the acceleration is zero, then the object is said to have a constant or uniform velocity. All of these possibilities are summarized in Table 7.1.

Table 7.1 Acceleration, deceleration, and constant velocity conditions

	v	A
Increasing speed Or acceleration	+	+
	-	-
Decreasing speed Or deceleration	+	-
	-	+
Constant speed	\pm	0

Acceleration is derived from velocity, which is itself derived from position. Therefore, there must be a way to relate acceleration and position directly. This relationship can be derived by substituting Eq. (7.1) into Eq. (7.3):

$$v = \frac{dx}{dt} = \dot{x}$$

$$a = \frac{dv}{dt} = \frac{d}{dt} \left(\frac{dx}{dt} \right) = \frac{d^2x}{dt^2} = \ddot{x}$$

The “dots” over x in the above equations indicate differentiation with respect to time. One dot signifies the first derivative with respect to time and two dots imply the second derivative.

7.3 Dimensions and Units

The relative position is measured in units of length. By definition, displacement is equal to the change of position, velocity is the time rate of change of relative position, and acceleration is the time rate of change of velocity. Therefore, relative position and displacement have the dimension of length, velocity has the dimension of length divided by time, and acceleration has the dimension of velocity divided by time:

$$[\text{POSITION}] = L$$

$$[\text{DISPLACEMENT}] = L$$

$$[\text{VELOCITY}] = \frac{[\text{DISPLACEMENT}]}{[\text{TIME}]} = \frac{L}{T}$$

$$[\text{ACCELERATION}] = \frac{[\text{VELOCITY}]}{[\text{TIME}]} = \frac{L/T}{T} = \frac{L}{T^2}$$

Based on these dimensions, the units of displacement, velocity, and acceleration in different unit systems can be determined. Some of these units are listed in Table 7.2.

Table 7.2 Units of displacement, velocity, and acceleration

UNIT SYSTEM	DISPLACEMENT	VELOCITY	ACCELERATION
SI	Meter (m)	m/s	m/s ²
c-g-s	Centimeter (cm)	cm/s	cm/s ²
British	Foot (ft)	ft/s	ft/s ²

7.4 Measured and Derived Quantities

In practice, it is possible to measure position, velocity, and acceleration over time. From any one of the three, the other two quantities can be determined by employing proper differentiation and/or integration, or through the use of graphical and numerical techniques. If the position of an object undergoing uniaxial motion in the x direction is measured and recorded, then the position can be expressed as a function of time, $x = f(t)$. Once the function representing the position of the object is established, the velocity and acceleration of the object at different times can be calculated using:

$$v = \frac{dx}{dt} \quad (7.5)$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \quad (7.6)$$

If the velocity of an object undergoing uniaxial motion in the x direction is measured and expressed as a function of time, $v = f(t)$, then the position of the object relative to its initial position and instantaneous acceleration of the object can be calculated using:

$$x = x_0 + \int_{t_0}^t v \, dt \quad (7.7)$$

$$a = \frac{dv}{dt} \quad (7.8)$$

The lower limit of integration, t_0 , in Eq. (7.7) corresponds to the time at which the first measurements are taken, and the upper limit corresponds to any time t . x_0 is the initial position of the object at time t_0 . For practical purposes, t_0 can be taken to be zero. This would mean that all time measurements are made

relative to the instant when the motion began. Also, $x_0 = 0$ if all position measurements are made relative to the initial position of the object.

If the acceleration of an object is measured and expressed as a function of time, $a = f(t)$, then the instantaneous velocity and position of the object relative to its initial velocity and position can be calculated using:

$$v = v_0 + \int_{t_0}^t a \, dt \quad (7.9)$$

$$x = x_0 + \int_{t_0}^t v \, dt \quad (7.10)$$

In Eqs. (7.9) and (7.10), x_0 and v_0 correspond to the initial position and initial velocity of the object at time t_0 . Note that these equations relate change of position and velocity relative to the initial position and velocity of the moving object. However, these equations are valid relative to the position and velocity of the object at any time. For example, if x_1 and v_1 represent the position and velocity of the object at time t_1 , then Eqs. (7.9) and (7.10) can also be expressed as:

$$v = v_1 + \int_{t_1}^t a \, dt$$

$$x = x_1 + \int_{t_1}^t v \, dt$$

7.5 Uniaxial Motion with Constant Acceleration

A common type of uniaxial motion occurs when the acceleration is constant. If a_0 represents the constant acceleration of an object, v_0 is its initial velocity, and x_0 is its initial position at time $t_0 = 0$, then Eqs. (7.9) and (7.10) will yield:

$$v = v_0 + a_0 t \quad (7.11)$$

$$\begin{aligned} x &= x_0 + \int_{t_0}^t (v_0 + a_0 t) \, dt \\ &= x_0 + \int_{t_0}^t v_0 \, dt + \int_{t_0}^t a_0 t \, dt \\ &= x_0 + v_0 t + \frac{1}{2} a_0 t^2 \end{aligned}$$

$$x = x_0 + v_0t + \frac{1}{2}a_0t^2 \tag{7.12}$$

For a given initial position, initial velocity, and constant acceleration of an object undergoing uniaxial motion in the x direction, Eqs. (7.11) and (7.12) can be used to determine the velocity and position of the object as functions of time relative to its initial velocity and position. Note that Eqs. (7.11) and (7.12) can be expressed relative to any other time and position. For example, if x_1 and v_1 represent the known position and velocity of the object at time t_1 , then:

$$v = v_1 + a_0(t - t_1)$$

$$x = x_1 + v_1(t - t_1) + \frac{1}{2}a_0(t - t_1)^2$$

Figure 7.3 shows an acceleration versus time graph for an object moving with constant acceleration, a_0 . According to Eq. (7.11), velocity is a linear function of time. As illustrated in Fig. 7.4, the velocity versus time graph is a straight line with constant slope that is equal to the magnitude of the constant acceleration. This is consistent with the fact that the slope of a function is equal to the derivative of that function, and that the derivative of velocity with respect to time is equal to acceleration. In Eq. (7.12), displacement is a quadratic function of time, and as illustrated in Fig. 7.5, the graph of this function is a parabola. At any given time, the slope of this function is equal to the velocity of the object at that instant.

For a uniaxial motion with constant acceleration, it is also possible to derive an expression between velocity, displacement, and time by solving Eq. (7.11) for a_0 and substituting it into Eq. (7.12). This will yield:

$$x = x_0 + \frac{1}{2}(v + v_0)t \tag{7.13}$$

Similarly, an expression between velocity, displacement, and acceleration can be derived by solving Eq. (7.11) for t and substituting it into Eq. (7.12):

$$v^2 = v_0^2 + 2a_0(x - x_0) \tag{7.14}$$

Caution. Equations (7.11) through (7.14) are valid if the acceleration is constant. Furthermore, the direction of the parameters involved must be handled properly. For example, if the direction of acceleration is opposite to that of the positive x direction, then the “plus” sign in front of the terms carrying acceleration must be changed to a “minus” sign.

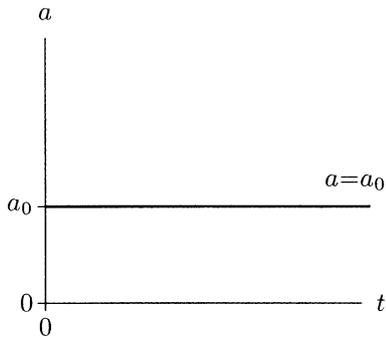


Fig. 7.3 Constant (uniform) acceleration

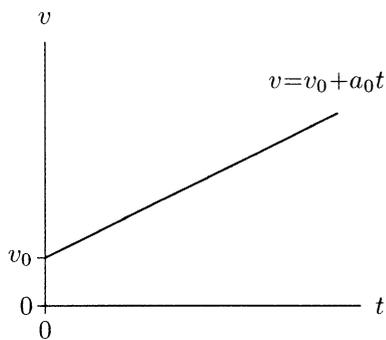


Fig. 7.4 When acceleration is constant, velocity is a linear function of time

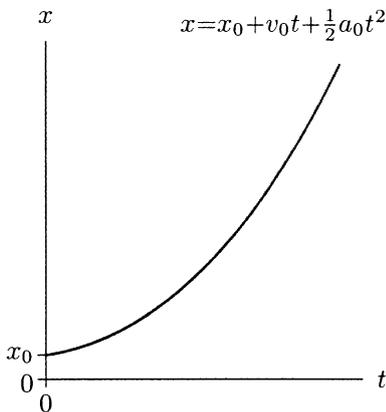
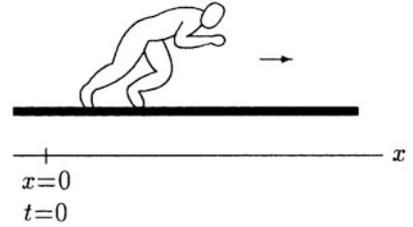


Fig. 7.5 When acceleration is constant, change of position is a quadratic function of time

7.6 Examples of Uniaxial Motion

The following examples are aimed to demonstrate the use of the kinematic equations (7.5) through (7.10).



Example 7.1 The short distance runner illustrated in Fig. 7.6 completed a 100-m race in 10 s. The time it took for the runner to reach the first 10 m and each successive 10 m mark were recorded by 10 observers using stopwatches. The data collected were then plotted to obtain the position versus time graph shown in Fig. 7.6. It is suggested that the data may be represented with the following function:

$$x = 0.46t^{7/3}$$

Here, change of position x is measured in meters, and time t is measured in seconds.

Determine the velocity and acceleration of the runner as functions of time, and the instantaneous velocity and acceleration of the runner 5 s after the start.

Solution: Since the function representing the position of the runner is known, it can be differentiated with respect to time once to determine the velocity, and twice to determine the acceleration:

$$v = \frac{dx}{dt} = \frac{d}{dt} (0.46t^{7/3}) = 1.07t^{4/3}$$

$$a = \frac{dv}{dt} = \frac{d}{dt} (1.07t^{4/3}) = 1.43t^{1/3}$$

The graphs of these functions are shown in Fig. 7.7.

To evaluate the velocity and acceleration of the runner 5 s after the start, substitute $t = 5$ s in the above equations and carry out the calculations. This will yield:

$$v = 9.15 \text{ m/s}$$

$$a = 2.45 \text{ m/s}^2$$

Example 7.2 The speedometer reading of a car driven on a straight highway is recorded for a total time interval of 3 min. The data collected are represented with a speed versus time diagram shown in Fig. 7.8. The dotted curve in Fig. 7.8 represents the actual measurements that are approximated by

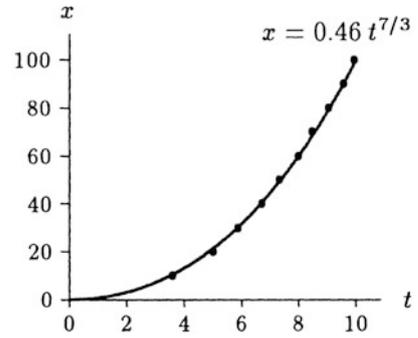


Fig. 7.6 Relative position, x , measured in meters versus time, t , measured in seconds

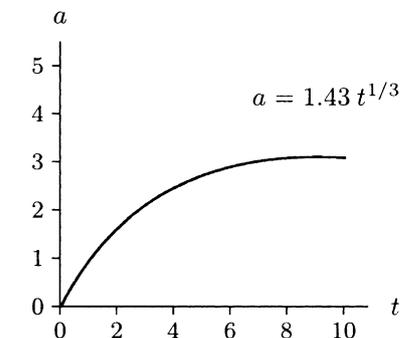
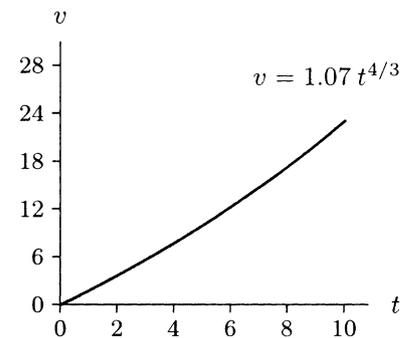


Fig. 7.7 Speed, v (m/s), and acceleration, a (m/s^2), versus time, t (s), curves for the runner

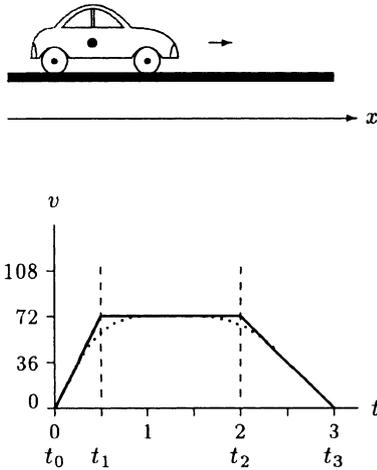


Fig. 7.8 Speed, v (km/h), versus time, t (min), diagram for the car

three straight lines (solid lines in Fig. 7.8). According to the information presented in Fig. 7.8, the speed of the car increases linearly from $v_0 = 0$ to $v_1 = 72$ km/h between times $t_0 = 0$ and $t_1 = 30$ s. Between times $t_1 = 30$ s and $t_2 = 120$ s, the speed of the car is constant at 72 km/h. Beginning at time $t_2 = 120$ s, the driver applies the brakes, decreases the speed of the car linearly with time, and brings the car to a stop in 60 s.

Determine expressions for the speed, displacement, and acceleration of the car as functions of time. Calculate the total distance traveled by the car in 3 min.

Solution: The speed measurements were made in kilometers per hour (km/h) that need to be converted to meters per second (m/s). This can be achieved by noting that 1 km is equal to 1000 m and that there are 3600 s in 1 h. Therefore, 72 km/h is equal to 20 m/s, which is calculated as:

$$72 \frac{\text{km}}{\text{h}} = 72 \times \frac{1000}{3600} = 20 \text{ m/s}$$

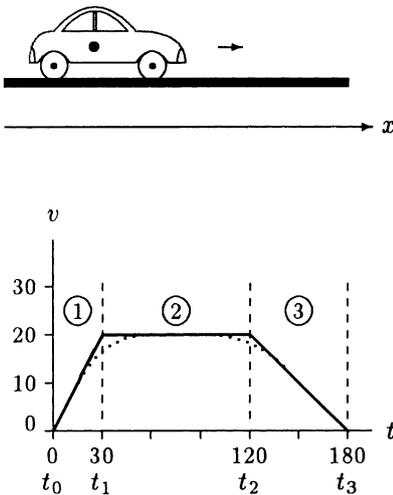


Fig. 7.9 Speed, v (m/s), versus time, t (s), diagram for the car

The speed versus time graph in Fig. 7.8 is redrawn in Fig. 7.9, in which speed is expressed in meters per second and time in seconds.

Because of the approximations made, the speed versus time graph in Fig. 7.9 has three distinct regions and there is not a single function that can represent the entire graph. Therefore, this problem should be analyzed in three phases.

Phase 1

Between $t_0 = 0$ and $t_1 = 30$ s, the speed of the car increased linearly with time from 0 to 20 m/s. As discussed in Appendix C, all linear functions can be represented as $X = A + BY$. In this expression, Y is the independent variable, X is the dependent variable, and A and B are some constant coefficients. In this case, we have time as the independent variable and speed is the dependent variable. Since the relationship between the speed of the car and time in phase 1 is linear, we can write:

$$v = A + Bt \tag{i}$$

The function given in Eq. (i) is a general expression between v and t because coefficients A and B are not yet determined. We need two conditions to calculate A and B (two unknowns). These conditions can be obtained from Fig. 7.9. When the car first started to move, $t = 0$ and $v = 0$, and $v = 20$ m/s when $t = 30$ s. Substituting the initial condition ($v = 0$ when $t = 0$) into Eq. (i) will yield $A = 0$, and substituting the second condition ($v = 20$ m/s when $t = 30$ s) will yield $B = 0.667$. Substituting $A = 0$ and $B = 0.667$ back into Eq. (i) will yield the function relating the speed of the car and time in phase 1:

$$v = 0.667t \quad (\text{ii})$$

Note that since we already converted speed measurements into meter per second and time into seconds, the speed in Eq. (ii) is in meters per second and time is in seconds.

Now, Eqs. (7.7) and (7.8) can be utilized to determine the displacement and acceleration of the car in phase 1. If we measure displacements relative to the starting point, then the initial position of the car was $x_0 = 0$. Therefore:

$$x = x_0 + \int_0^t v \, dt = \int_0^t (0.667t) \, dt = 0.667 \left[\frac{t^2}{2} \right]_0^t = 0.333t^2 \quad (\text{iii})$$

$$a = \frac{dv}{dt} = \frac{d}{dt}(0.667t) = 0.667 \quad (\text{iv})$$

From Eq. (iv), the acceleration of the car in phase 1 was constant at 0.667 m/s^2 . The total distance traveled by the car at the end of phase 1 can be determined by substituting $t = 30\text{s}$ into Eq. (iii). This will yield:

$$x_1 = 0.333t^2 = 0.333(30)^2 = 300\text{m}$$

Phase 2

Phase 2 starts when time is $t_1 = 30\text{s}$ and ends when it is $t_2 = 120\text{s}$. In phase 2, the speed of the car was constant at 20 m/s . Therefore, the function representing the speed in phase 2 is:

$$v = 20 \quad (\text{v})$$

The total distance traveled by the car in phase 1 was computed as $x_1 = 300\text{m}$. $x_1 = 300\text{m}$ also represents the initial position of the car in phase 2. Phase 1 ended when time was $t_1 = 30\text{s}$. Therefore, phase 2 began when time $t_1 = 30\text{s}$. We can now write Eq. (7.7) relative to t_1 and x_1 :

$$\begin{aligned} x &= x_1 + \int_{t_1}^{t_2} v \, dt = 300 + \int_{30}^{t_2} 20 \, dt = 300 + 20[t]_{t_1}^{t_2} \\ x &= 300 + 20(t_2 - t_1) \end{aligned} \quad (\text{vi})$$

The acceleration of the car in phase 2 can be determined using Eq. (7.8):

$$a = \frac{dv}{dt} = 0 \quad (\text{vii})$$

From Eq. (vii), the acceleration of the car in phase 2 is zero. The total distance traveled by the car at the end of phase 2 can be determined by substituting $t = 120\text{s}$ into Eq. (vi). This will yield:

$$x_2 = 300 + 20(120 - 30) = 300 + 1800 = 2100\text{m}$$

Phase 3

Between $t_2 = 120\text{s}$ and $t_3 = 180\text{s}$, the speed of the car decreased linearly with time and to zero in 60 s. The function representing the relationship between speed of the car and time in phase 3 can be determined using Eq. (i). The coefficients A and B in Eq. (i) can be calculated by taking into consideration two conditions related to phase 3. For example, $v = 20\text{m/s}$ when $t = 120\text{s}$ and $v = 0$ when $t = 180\text{s}$. Substituting the second condition in Eq. (i) will yield $A + 180B = 0$ or $A = -180B$. Substituting the first condition and $A = -180B$ into Eq. (i) will yield $B = -0.333$. Since $A = -180B$ and $B = -0.333$, $A = 60$. Therefore, the function that relates the speed of the car and time in phase 3 is:

$$v = 60 - 0.333t \tag{viii}$$

Phase 3 begins when time is $t_2 = 120\text{s}$ and the initial position of the car at phase 3 is $x_2 = 2100\text{m}$. Using Eq. (7.7):

$$\begin{aligned} x_3 &= x_2 + \int_{t_2}^{t_3} v dt = 2100 + \int_{120}^{180} (60 - 0.333t) dt \\ &= 2100 + \int_{120}^{180} 60 dt - \int_{120}^{180} 0.333t dt \\ &= 2100 + [t]_{120}^{180} - \frac{0.333}{2} [t^2]_{120}^{180} \\ x_3 &= 2100 + 60[t]_{120}^{180} - 0.167[t^2]_{120}^{180} \end{aligned} \tag{ix}$$

Using Eq. (7.8):

$$a = \frac{dv}{dt} = \frac{d}{dt} (60 - 0.333t) = -0.333 \tag{x}$$

The total distance traveled by the car can be determined by solving Eq. (ix). This will yield:

$$\begin{aligned} x_3 &= 2100 + 60(180 - 120) - 0.167(180^2 - 120^2) \\ &= 2100 + 3600 - 3006 = 2694\text{m} \end{aligned}$$

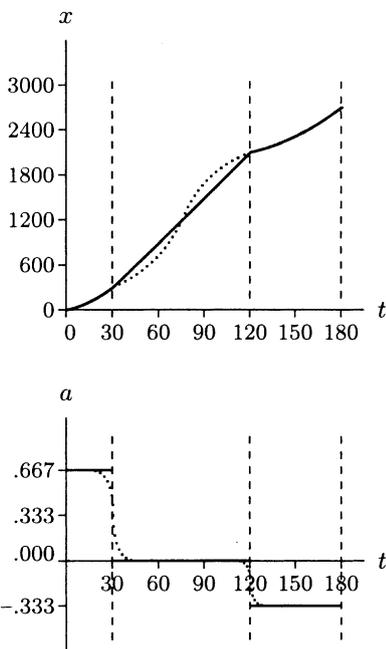


Fig. 7.10 Displacement, x (m), and acceleration, a (m/s^2), versus time, t (s), graphs for the car

In Fig. 7.10, the functions derived for the displacement and acceleration of the car in different phases are used to plot displacement and acceleration versus time graphs (solid curves). In all phases, the car is moving in the positive x direction. Therefore, the displacement of the car is positive throughout. In phase 1, the acceleration of the car is positive, indicating increasing speed in the positive x direction. In phase 2, the acceleration of the car is zero and the speed is constant. In phase 3, the acceleration of the car is negative, indicating deceleration in the positive x direction.

Note that the displacement and acceleration versus time graphs (solid curves) in Fig. 7.10 are not continuous. For example, the slope of the x versus t curve at the end of phase 1 is not necessarily equal to the slope of the x versus t curve at the beginning of phase 2. The discontinuity is more significant for the a versus t graph. This is due to the fact that we approximated the actual speed versus time graph of the car with three straight lines in three regions. In reality, as illustrated by the dotted curves in Fig. 7.10, the variations in the slopes of these curves would be less marked and more continuous.

Example 7.3 Consider the skier illustrated in Fig. 7.11 descending a straight slope. Assume that the skier is moving down the slope at a constant acceleration of 2 m/s^2 and that the speed of the skier at position 0 is observed to be 10 m/s .

Calculate the speed v_1 of the skier when the skier is at position 1, which is at a distance $l = 100 \text{ m}$ from position 0 measured parallel to the slope. Also, calculate the time t_1 it took for the skier to cover the distance between positions 0 and 1.

Solution: Since the skier is moving with a constant acceleration, Eqs. (7.11) and (7.12) can be used to analyze this problem. In Fig. 7.12, the direction parallel to the slope or the direction in which the skier is moving is identified by the x axis. For the sake of simplicity, the origin of the x axis is placed to coincide with position 0 so that $x_0 = 0$. Furthermore, we can make all time measurements relative to the instant when the skier was at position 0. That is, $t_0 = 0$. As indicated in Fig. 7.12, the speed of the skier at position 0 is $v_0 = 10 \text{ m/s}$. What we know about position 1 is the fact that it is located at a distance $l = 100 \text{ m}$ from position 0. Therefore, the position of the skier at 1 is $x_1 = l = 100 \text{ m}$. The time t_1 it took for the skier to cover the distance between positions 0 and 1, and the speed v_1 of the skier at position 1 are unknowns to be determined.

Use Eq. (7.12) first. Writing this equation between positions 0 and 1:

$$x_1 = x_0 + v_0 t_1 + \frac{1}{2} a_0 t_1^2$$

Here, $x_0 = 0$, $v_0 = 10 \text{ m/s}$, and $a_0 = 2 \text{ m/s}^2$ is the constant acceleration of the skier. Substituting these parameters into the above equation and rearranging the order of terms will yield:

$$t_1^2 + 10t_1 - 100 = 0$$

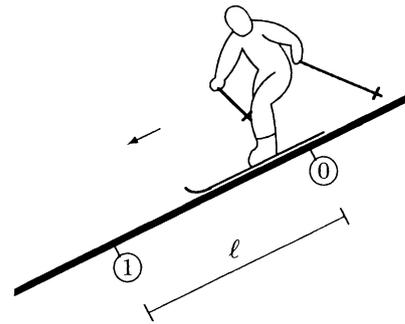


Fig. 7.11 A skier is moving down the slope with a constant acceleration of $a_0 = 2 \text{ m/s}^2$

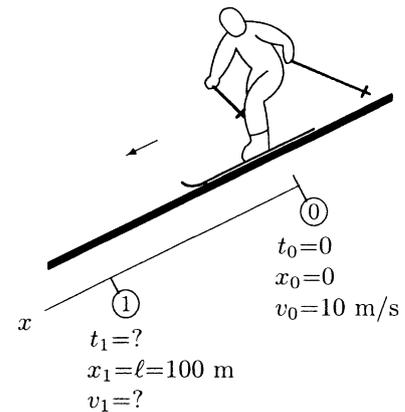


Fig. 7.12 Conditions at positions 0 and 1

Note that this is a quadratic equation. Solutions of quadratic equations are discussed in Appendix C.5. This equation has two solutions for t_1 , one positive and one negative. Since negative time does not make any sense, the positive solution must be adopted, which is $t_1 = 6.18\text{s}$. (For the validity of $t_1 = 6.18\text{s}$, substitute it back into the quadratic equation and check whether the equilibrium is satisfied.) In other words, it took 6.18 s for the skier to cover the distance between positions 0 and 1.

Now, we can use Eq. (7.11) to calculate the speed of the skier at position 1:

$$v_1 = v_0 + a_0 t_1$$

Substituting $v_0 = 10\text{m/s}$, $a_0 = 2\text{m/s}^2$, and $t_1 = 6.18\text{s}$ into the above equation and carrying out the calculations will yield $v_1 = 22.36\text{m/s}$.

Example 7.4 One of the most common examples of uniformly accelerated motion is that of an object allowed to fall vertically downward, which is called free fall. Free fall is a consequence of the effect of gravitational acceleration on the mass of the object. If the possible effects of air resistance are ignored (assuming that the motion occurs in vacuum), then the object released from a height would move downward with a constant acceleration equal to the magnitude of the gravitational acceleration, which is about 9.8m/s^2 .

As illustrated in Fig. 7.13, consider a person holding a ball at a height $h = 1.5\text{m}$ above the ground level. If the ball is released to descend, how much time it would take for the ball to hit the ground and what would be its impact velocity?

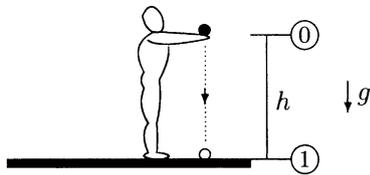


Fig. 7.13 Free fall

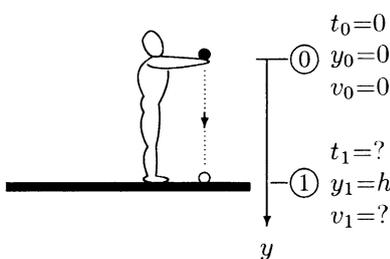


Fig. 7.14 Conditions at positions 0 and 1

Solution: This is another example of uniaxial motion with constant acceleration. Once the ball is released, it moves downward with constant acceleration $a_0 = 9.8\text{m/s}^2$, which is the magnitude of the gravitational acceleration. In Fig. 7.14, the direction of motion of the ball is identified with the y axis such that positive y direction is downward. Since the direction of gravitational acceleration is also downward, the acceleration of the ball is positive.

In Fig. 7.14, the origin of the y axis is chosen to coincide with the initial position, 0, of the ball. Therefore, the initial position of the ball is $y_0 = 0$. The initial time $t_0 = 0$ because all time measurements are made relative to the instant when the ball was released. The initial speed of the ball is $v_0 = 0$ because the ball was initially at rest. The ground level is identified as position 1, which is at a vertical distance $y_1 = h = 1.5\text{m}$ away from position 0. The task is to determine the time t_1 it took for the

ball to cover the vertical distance between positions 0 and 1, and the impact speed v_1 of the ball.

Since the acceleration is constant at $a_0 = 9.8\text{ m/s}^2$, we can use Eqs. (7.11) and (7.12). However, we must replace x in these equations with y . Writing Eq. (7.12) between positions 0 and 1:

$$y_1 = y_0 + v_0 t_1 + \frac{1}{2} a_0 t_1^2$$

Substituting $y_1 = 1.5\text{ m}$, $y_0 = 0$, $v_0 = 0$, and $a_0 = 9.8\text{ m/s}^2$ into this equation and solving it for t_1 will yield $t_1 = 0.55\text{ s}$. That is, in the absence of air resistance, it would take only 0.55 s for the ball to hit the ground when it is released from a height of 1.5 m from the ground level.

The impact speed can be calculated using Eq. (7.11):

$$v_1 = v_0 + a_0 t_1$$

Substituting $v_0 = 0$, $a_0 = 9.8\text{ m/s}^2$, and $t_1 = 0.55\text{ s}$ into the above equation and solving it for v_1 will yield $v_1 = 5.39\text{ m/s}$.

7.7 Biaxial Motion

Biaxial or two-dimensional motion is one in which the movement occurs on a plane surface. One-dimensional linear motion characteristics of an object are completely known if, for example, the position of the object in the direction of motion is known as a function of time. The concepts introduced earlier for uniaxial motion analysis can be expanded to analyze two- and three-dimensional linear movements. This can be achieved by considering the properties of displacement, velocity, and acceleration as vector quantities.

7.8 Position, Velocity, and Acceleration Vectors

For one-dimensional problems, the position of an object is defined by using a single coordinate axis. For plane problems, two coordinates must be specified to define the position of an object uniquely. As shown in Fig. 7.15, let x and y represent the usual Cartesian (rectangular) coordinate directions with unit vectors \underline{i} and \underline{j} indicating positive x and y directions, respectively. The origin of the coordinate system is located at O. The position vector \underline{r} of a point P in this xy -plane is a vector drawn from O toward P. The position vector can be represented in terms of x and y coordinates of point P:

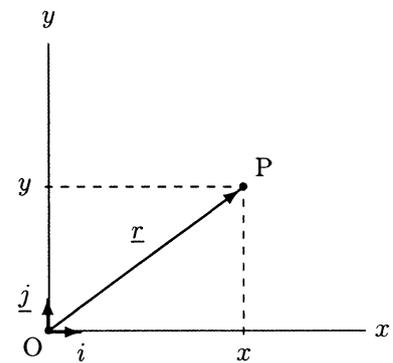


Fig. 7.15 \underline{r} is the position vector of point P

$$\underline{r} = x\underline{i} + y\underline{j} \tag{7.15}$$

The magnitude r of the position vector is equal to the length of the line connecting points O and P, which can be calculated as:

$$r = \sqrt{x^2 + y^2} \tag{7.16}$$

If point P represents the position of a moving object at some time t , then \underline{r} represents the *instantaneous position* of that object at that time. This implies that \underline{r} can change with time or x and y coordinates of the moving object are functions of time.

By definition, velocity is the time rate of change of position. Therefore, the *velocity vector* is equal to the derivative of the position vector with respect to time:

$$\underline{v} = \frac{d}{dt}(\underline{r}) = \frac{d}{dt}(x\underline{i} + y\underline{j}) = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} \tag{7.17}$$

For two-dimensional problems, the velocity vector may have up to two components (Fig. 7.16). If v_x and v_y refer to the scalar components of \underline{v} in the x and y directions, respectively, then the velocity vector can also be expressed as:

$$\underline{v} = v_x\underline{i} + v_y\underline{j} \tag{7.18}$$

By comparing Eqs. (7.17) and (7.18), we can conclude that:

$$v_x = \frac{dx}{dt} = \dot{x} \tag{7.19}$$

$$v_y = \frac{dy}{dt} = \dot{y} \tag{7.20}$$

v_x and v_y are also known as the *rectangular components* of \underline{v} , and they indicate how fast the object is moving in the x and y directions, respectively. If the components v_x and v_y of the velocity vector are known, then the magnitude v of their resultant can also be determined:

$$v = \sqrt{v_x^2 + v_y^2} \tag{7.21}$$

Note that v is a scalar quantity also known as the *speed*. As illustrated in Fig. 7.16, it is very important to remember that the direction of the velocity vector is always tangent to the path of the motion and pointing in the direction of motion.

The direction of the velocity vector can also be determined if its scalar components in the horizontal and vertical directions are known: $\tan \alpha = \frac{v_y}{v_x}$, then $\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right)$, where α is an angle that the velocity vector makes with the horizontal axis.

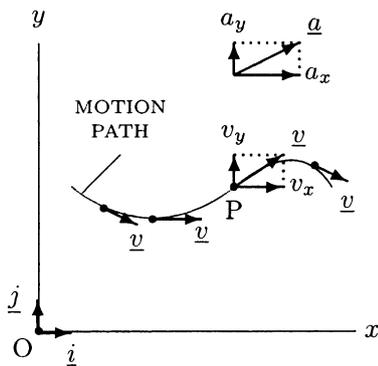


Fig. 7.16 Velocity vector is always tangent to the path of motion

By definition, acceleration is the time rate of change of velocity. Therefore, if the velocity vector of an object is known as a function of time, then its acceleration vector \underline{a} can also be determined by considering the derivative of \underline{v} with respect to time:

$$\underline{a} = \frac{d}{dt}(\underline{v}) = \frac{d}{dt}(v_x \underline{i} + v_y \underline{j}) = \frac{dv_x}{dt} \underline{i} + \frac{dv_y}{dt} \underline{j} \quad (7.22)$$

The acceleration vector can also be expressed in terms of its components in the x and y directions (Fig. 7.16):

$$\underline{a} = a_x \underline{i} + a_y \underline{j} \quad (7.23)$$

By comparing Eqs. (7.22) and (7.23), the rectangular components of the acceleration vector can alternatively be written as:

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = \ddot{x} \quad (7.24)$$

$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} = \ddot{y} \quad (7.25)$$

If needed, the magnitude a of the acceleration vector can be calculated as:

$$a = \sqrt{a_x^2 + a_y^2} \quad (7.26)$$

The extension of these concepts to analyze three-dimensional movements is straightforward. For three-dimensional motion analyses, there is a need for a third dimension, namely z . For example, the position vector of a point in space can be expressed as:

$$\underline{r} = x \underline{i} + y \underline{j} + z \underline{k}$$

Here, \underline{k} is the unit vector indicating the positive z direction. Similarly, the velocity and acceleration vectors in space can be expressed as:

$$\underline{v} = v_x \underline{i} + v_y \underline{j} + v_z \underline{k}$$

$$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$$

For example, v_z is the speed of the object in the z direction and is equal to the time rate of change of position in the z direction, and a_z is the scalar component of the acceleration vector in the z direction and is equal to the time rate of change of the speed of the object in the z direction.

7.9 Biaxial Motion with Constant Acceleration

Two-dimensional linear motion of an object in the xy -plane can be analyzed in two stages by first considering its motion in the x and y directions separately and then combining the results using the vectorial properties of the parameters involved. The parameters defining the motion in the x direction are x , its first time derivative v_x , and its second time derivative a_x . Similarly, y , v_y , and a_y are the parameters that define the motion of the object in the y direction. If the acceleration of an object undergoing two-dimensional linear motion is constant, then a_x and a_y must be constants. The details of uniaxial motion with constant acceleration were analyzed in the previous sections. The results of these analyses can readily be adopted to analyze two-dimensional motions with constant acceleration.

In the x direction, Eqs. (7.11) and (7.12) can be rewritten in the following more specific forms:

$$v_x = v_{x_0} + a_{x_0}t \quad (7.27)$$

$$x = x_0 + v_{x_0}t + \frac{1}{2}a_{x_0}t^2 \quad (7.28)$$

Similarly, in the y direction:

$$v_y = v_{y_0} + a_{y_0}t \quad (7.29)$$

$$y = y_0 + v_{y_0}t + \frac{1}{2}a_{y_0}t^2 \quad (7.30)$$

Here, x_0 and y_0 are the initial coordinates of the object, v_{x_0} and v_{y_0} are the initial velocity components in the x and y directions, and a_{x_0} and a_{y_0} are the constant components of the acceleration vector in the x and y directions, respectively. For given x_0 , y_0 , v_{x_0} , v_{y_0} , a_{x_0} , and a_{y_0} , Eqs. (7.27) through (7.30) can be used to calculate the relative position of the moving object and its velocity components at any time t .

Furthermore, these equations are valid if acceleration is constant and its direction coincides with the direction of motion. If the direction of acceleration is opposite the direction of motion, then the sign in front of the terms carrying the acceleration must be changed into minus (-). In the x direction, Eqs. (7.11) and (7.12) can be rewritten in the following forms:

$$v_x = v_{x_0} - a_{x_0}t$$

$$x = x_0 + v_{x_0}t - \frac{1}{2}a_{x_0}t^2$$

Similarly, in the y direction:

$$v_y = v_{y_0} - a_{y_0}t$$

$$y = y_0 + v_{y_0}t - \frac{1}{2}a_{y_0}t^2$$

7.10 Projectile Motion

When an object is thrown into the air in any direction other than the vertical, it will move in a curved path under the influence of gravity and air resistance. The gravity of Earth will pull the object downward with a constant gravitational acceleration of about 9.8 m/s^2 , and the air resistance will retard its motion in a direction opposite to the direction of motion. This very common form of motion, called projectile motion, is relatively simple to analyze once the effect of air resistance is ignored.

Projectile motion is a particular form of two-dimensional linear motion with constant acceleration. To be able to define the basic parameters involved in all projectile motions, consider the motion of a cannonball fired into the air (Fig. 7.17). Assume that the cannonball leaves the barrel and lands on the ground at the same elevation. As illustrated in Fig. 7.18, the cannonball ascends, reaches a peak, starts descending, and finally lands on the ground. The curved flight path of the cannonball is called the trajectory of motion. 0 represents the initial position of the cannonball, 1 is the peak it reaches, and 2 is the location of landing. v_0 is the magnitude of the initial velocity of the cannonball, which is called the *speed of release* or *takeoff speed*. θ is the angle the initial velocity vector makes with the horizontal and is called the *angle of release*. The vertical distance h between 0 and 1 is the maximum height that the cannonball reaches, and the horizontal distance l between 0 and 2 is called the *horizontal range of motion*. The total time the cannon ball remains in the air is called the *time of flight*.

The equations necessary to analyze projectile motions can be derived from Eqs. (7.27) through (7.30). For example, if the speed and angle of release of the projectile are known, then components of the velocity vector along the horizontal (x) and vertical (y) directions can be calculated:

$$\begin{aligned}v_{x_0} &= v_0 \cos \theta \\v_{y_0} &= v_0 \sin \theta\end{aligned}$$

Assuming that the air resistance on the cannonball is negligible, the acceleration of the cannonball in the x direction is zero throughout the motion. That is, $a_{x_0} = 0$. The gravitational acceleration g acts downward. Assuming that the y axis is positive upward, the gravitational acceleration acts in the negative y direction. To account for the negative direction of gravitational acceleration, the plus signs in front of the terms carrying

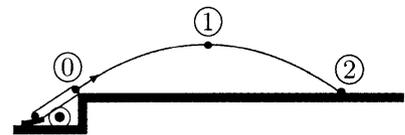


Fig. 7.17 A cannonball fired into the air will undergo a projectile motion

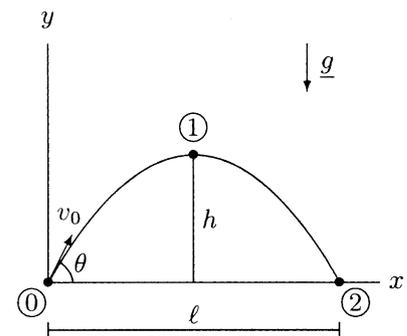


Fig. 7.18 Projectile motion

a_{y_0} in Eqs. (7.29) and (7.30) must be changed to minus. Under these considerations, Eqs. (7.27) through (7.30) take the following special forms for projectile motion:

$$x = x_0 + (v_0 \cos \theta)t \tag{7.31}$$

$$y = y_0 + (v_0 \sin \theta)t - \frac{1}{2}gt^2 \tag{7.32}$$

$$v_x = v_0 \cos \theta \tag{7.33}$$

$$v_y = v_0 \sin \theta - gt \tag{7.34}$$

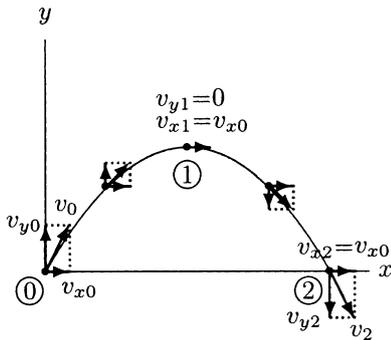


Fig. 7.19 Horizontal component of the velocity remains constant through the motion. Velocity vector is always tangent to the trajectory of motion

Here, if the origin of the xy coordinate frame is chosen to coincide with the initial position of the ball, then $x_0 = 0$ and $y_0 = 0$. Notice from Eq. (7.33) that the magnitude of the horizontal component of the velocity vector is not a function of time. Therefore, $v_x = v_{x_0}$ remains constant throughout the projectile motion (Fig. 7.19). The magnitude of the vertical component of the velocity vector is a linear function of time. It is positive (upward) initially, decreases in time as the object ascends, and drops to zero at the peak. After reaching the peak, the vertical component of the velocity vector changes its direction from upward to downward, while its magnitude increases until it lands on the ground. At any instant during the flight, the resultant velocity vector is tangent to the trajectory of the projectile motion.

Another important aspect of the projectile motion is that if both the takeoff and landing occur at the same elevation, then the motion is symmetric with respect to a plane that passes through the peak and cuts the plane of motion at right angles. In other words, the time it takes for the object to ascend is equal to the time it takes to descend.

In some cases, the objective of the projectile motion may be to increase the horizontal range of motion to a maximum. This is particularly true for a ski jumper, for example. Other situations may require a control over the height, which is the case for a high jumper. Therefore, it may be useful to derive some expressions for the horizontal range and maximum height of the projectile motion. Such derivations will be performed within the context of the following example problem.

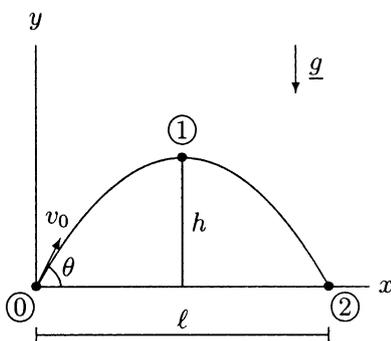


Fig. 7.20 h is the maximum elevation and is the horizontal range of motion

Example 7.5 Consider Fig. 7.20 which shows the trajectory of a projectile motion along with some of the parameters involved. $x_0 = 0$ and $y_0 = 0$ because the origin of the xy coordinate frame is chosen to coincide with the initial position of the object. Let t_1 be the time it takes to reach the peak, and t_2 be the total time of flight. At the peak, $y_1 = h$, $v_{y1} = 0$, and $x_1 = l/2$ from the symmetry of the motion. At the location where the object

lands, $x_2 = l$ and $y_2 = 0$. The maximum height h reached by the object can be determined by noting that $v_{y_1} = 0$ at the peak. Writing Eq. (7.34) between 0 and 1:

$$\begin{aligned}v_{y_1} &= v_0 \sin \theta - gt_1 \\ 0 &= v_0 \sin \theta - gt_1\end{aligned}$$

Solving this equation for t_1 will yield:

$$t_1 = \frac{v_0 \sin \theta}{g} \quad (\text{i})$$

Between 0 and 1, Eq. (7.32) will take the following form:

$$h = 0 + (v_0 \sin \theta)t_1 - \frac{1}{2}gt_1^2$$

Substituting Eq. (i) into this equation will yield:

$$h = \frac{v_0^2 \sin^2 \theta}{2g} \quad (\text{ii})$$

Again from the symmetry of motion, the time lapses during the ascent must be equal to the time lapses during the descent. In other words, $t_2 = 2t_1$. This can be proven by writing Eq. (7.32) between 0 and 2, and solving it for t_2 :

$$\begin{aligned}y_2 &= y_0 + (v_0 \sin \theta)t_2 - \frac{1}{2}gt_2^2 \\ 0 &= 0 + (v_0 \sin \theta)t_2 - \frac{1}{2}gt_2^2 \\ 0 &= v_0 \sin \theta - \frac{1}{2}gt_2\end{aligned}$$

Solving this equation for t_2 will yield:

$$t_2 = \frac{2v_0 \sin \theta}{g} \quad (\text{iii})$$

By comparing Eq. (iii) with Eq. (i), one can conclude that:

$$t_2 = 2t_1$$

Between 0 and 2, Eq. (7.31) can be written as:

$$l = 0 + (v_0 \cos \theta)t_2$$

Substituting Eq. (iii) into the above equation, and noting that $2 \cos \theta \sin \theta = \sin(2\theta)$ (see Appendix C.4):

$$l = \frac{v_0^2 \sin(2\theta)}{g} \quad (\text{iv})$$

Equations (i) through (iv) are special forms of more general equations for projectile motions as given in Eqs. (7.31) through

Table 7.3 Variations of $\sin^2\theta$ and $\sin(2\theta)$ for $0^\circ < \theta \leq 90^\circ$

θ	$\sin^2\theta$	$\sin(2\theta)$
0°	0.000	0.000
15°	0.067	0.500
30°	0.250	0.866
45°	0.500	1.000
60°	0.750	0.866
75°	0.933	0.500
90°	1.000	0.000

The maximum height h of the projectile is a function of $\sin^2\theta$, and range l depends on $\sin(2\theta)$

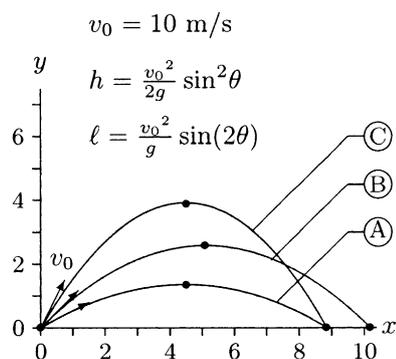


Fig. 7.21 Trajectories for $v_0 = 10 \text{ m/s}$ and $\theta = 30^\circ, 45^\circ, 60^\circ$ represented as A, B, and C (both x and y are in meters)

(7.34). From Eqs. (ii) and (iv), it is clear that for a given v_0 , the maximum height and the horizontal range of motion of the projectile are functions of the angle of release θ . To see the variations of h and l with θ , $\sin^2\theta$, and $\sin(2\theta)$ are computed for θ between 0° and 90° . The values calculated are listed in Table 7.3. In Fig. 7.21, a value of 10 m/s is assigned to v_0 and corresponding trajectories are calculated for $\theta = 30^\circ, 45^\circ$, and 60° . The significance of these results is that, for given v_0 , the range of motion l is maximum when $\theta = 45^\circ$. Therefore, if the purpose is to maximize the horizontal range of motion of the projectile, then the angle of takeoff must be close to 45° .

Sometimes it is easier to measure the range of motion l and the maximum height h of the projectile. In such cases, the unknowns are the takeoff speed v_0 and the angle of release θ . The relationship between these parameters can be derived as:

$$\theta = \arctan\left(\frac{4h}{l}\right) \tag{7.35}$$

$$v_0 = \frac{\sqrt{2gh}}{\sin\theta} \tag{7.36}$$

Note that the results obtained in this example are valid for cases in which takeoff and landing occur at the same elevation. Also, these results are not meant to be “memorized.” One of the objectives of this example was to demonstrate how Eqs. (7.31) through (7.34) that govern projectile motions could be manipulated.

7.11 Applications to Athletics

The concept of projectile motion may have many applications in athletics and sports mechanics. These applications include the motion analyses of athletes doing long jumping, high jumping, ski jumping, diving, and gymnastics, and the motion analyses of the discus, javelin, shot, baseball, basketball, football, and golf ball. The following examples are aimed to illustrate some of these applications. It should be reiterated here that we are ignoring the fundamental approximation in projectile motion analyses that possible effects of air resistance can influence the motion characteristics.

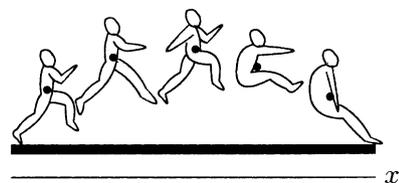


Fig. 7.22 Long jumper

Example 7.6 Based on the assumption that the air resistance is negligible, it is suggested that the overall motion characteristic of a long jumper may be analyzed by assuming that the center of gravity of the athlete undergoes a projectile motion (Fig. 7.22).

Consider an athlete who jumps a horizontal distance of 8 m. If the athlete was airborne for 1 s, calculate the takeoff speed, angle of release, and the maximum height the athlete's center of gravity reached.

Solution: Figure 7.23 illustrates the motion of the center of gravity of the long jumper. 0 represents takeoff, 1 represents the peak, and 2 is landing. Note that the origin of the xy coordinate system is placed at 0 so that all horizontal distance and elevation measurements are relative to the position of the center of gravity of the athlete at takeoff. $x_2 = l = 8\text{ m}$ is the horizontal range of motion and $t_2 = 1\text{ s}$ is the total time the athlete was airborne. The task is to calculate the takeoff speed v_0 , takeoff angle θ , and maximum height h . Note that since t_2 is known and the motion is symmetric with respect to the peak, the time it took for the athlete to reach to the peak is $t_1 = t_2/2 = 0.5\text{ s}$.

We know l and t_2 . The equation that governs the displacement in the horizontal (x) direction is Eq. (7.31). Writing this equation for position 2 relative to position 0 and substituting the known parameters:

$$\begin{aligned}x_2 &= x_0 + v_{x_0}t_2 \\l &= 0 + v_{x_0}t_2 \\8 &= v_{x_0}(1)\end{aligned}$$

Solving this equation for the unknown parameter will yield the horizontal component of the takeoff velocity, $v_{x_0} = 8.0\text{ m/s}$.

We also know that at landing the elevation of the center of gravity is zero. That is, $y_2 = 0$. The equation that governs the displacement in the vertical (y) direction is Eq. (7.32). Writing this equation for position 2 relative to position 0 and substituting the known parameters:

$$\begin{aligned}y_2 &= y_0 + v_{y_0}t_2 - \frac{1}{2}g t_2^2 \\0 &= 0 + v_{y_0}(1) - \frac{1}{2}(9.8)(1)^2 \\0 &= v_{y_0} - 4.9\end{aligned}$$

Solving this equation for the vertical component of the takeoff velocity will yield $v_{y_0} = 4.9\text{ m/s}$.

Now that we know the magnitudes of the horizontal and vertical components of the takeoff velocity, we can calculate the takeoff speed:

$$v_0 = \sqrt{v_{x_0}^2 + v_{y_0}^2} = \sqrt{(8.0)^2 + (4.9)^2} = 9.4\text{ m/s}$$

Note that $v_{x_0} = v_0 \cos \theta$. Since we know v_{x_0} and v_0 , we can calculate the takeoff angle:

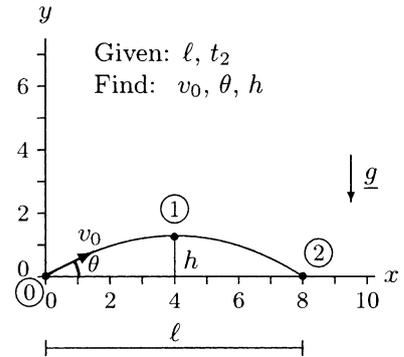


Fig. 7.23 Trajectory of the center of gravity of the athlete (both x and y are in meters)

$$\theta = \cos^{-1}\left(\frac{v_{x_0}}{v_0}\right) = \cos^{-1}\left(\frac{8.0}{9.4}\right) = \cos^{-1}(0.85) = 31.7^\circ$$

To calculate the maximum height reached, we can again utilize Eq. (7.32). Writing this equation for position 1 relative to 0 and solving it for h will yield:

$$y_1 = y_0 + v_{y_0}t_1 - \frac{1}{2}gt_1^2$$

$$h = 0 + (4.9)(0.5) - \frac{1}{2}(9.8)(0.5)^2$$

$$h = 1.2\text{m}$$

Therefore, at the peak, the center of gravity of the athlete was 1.2 m above the level it was at the takeoff.

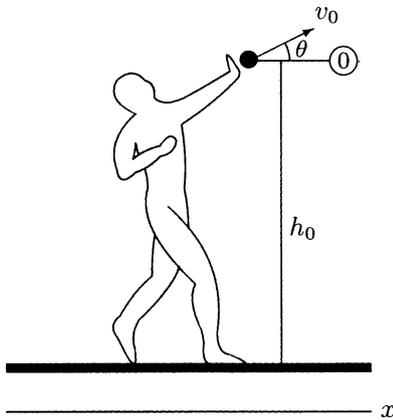


Fig. 7.24 Shot-putter

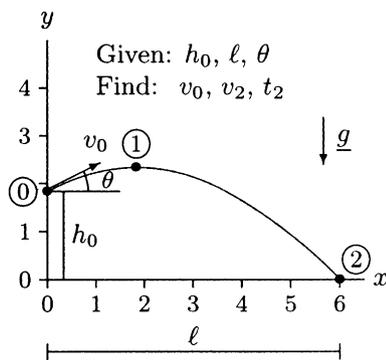


Fig. 7.25 Trajectory of the shot (x and y are in meters)

Example 7.7 During a practice, a shot-putter puts the shot at a distance $l = 6$ m. At the instant the athlete releases the shot, the elevation of the shot is $h_0 = 1.8$ m as measured from the ground level, and the angle of release is $\theta = 30^\circ$ (Fig. 7.24).

Determine the speed at which the athlete released the shot, the landing speed of the shot, and the total time the shot was in the air.

Solution: Equations (7.31) and (7.32) can be utilized to solve this problem. In Fig. 7.25, the origin of the xy coordinate frame is located at the ground level directly under the point of release which is designated as 0. The shot ascends, reaches a peak at 1, and lands on the field at 2. With respect to the coordinate frame adopted, the initial and landing coordinates of the shot are: $x_0 = 0, y_0 = h_0, x_2 = l$, and $y_2 = 0$. If t_2 refers to the total time the shot was in the air, then Eq. (7.31) can be written between 0 and 2 as:

$$x_2 = x_0 + (v_0 \cos \theta)t_2$$

$$l = 0 + (v_0 \cos \theta)t_2$$

Solving this equation for t_2 and substituting the known parameters:

$$t_2 = \frac{l}{v_0 \cos \theta} = \frac{6}{v_0 \cos 30^\circ} = \frac{6.93}{v_0} \tag{i}$$

Similarly, writing Eq. (7.32) between 0 and 2 and substituting the known parameters:

$$y_2 = y_0 + (v_0 \sin \theta)t_2 - \frac{1}{2}gt_2^2$$

$$0 = h_0 + (v_0 \sin \theta)t_2 - \frac{1}{2}gt_2^2$$

$$0 = 1.8 + (v_0 \sin 30^\circ)t_2 - \frac{1}{2}(9.8)t_2^2$$

$$0 = 1.8 + 0.5v_0t_2 = 4.9t_2^2 \quad (\text{ii})$$

Substituting Eq. (i) into Eq. (ii):

$$0 = 1.8 + 0.5v_0\left(\frac{6.93}{v_0}\right) - 4.9\left(\frac{6.93}{v_0}\right)^2$$

Simplifying the second term on the right-hand side of this equation by eliminating v_0 , carrying out the calculations, and solving this equation for v_0 will yield:

$$v_0 = 6.69 \text{ m/s}$$

Knowing v_0 , time t_2 can be calculated using Eq. (i). This will yield $t_2 = 1.04 \text{ s}$.

We can utilize Eqs. (7.33) and (7.34) to calculate the landing speed v_2 of the shot. Since the horizontal component of the velocity vector is constant throughout the projectile motion:

$$v_{x_2} = v_{x_0} = v_0 \cos \theta = (6.69)(\cos 30^\circ) = 5.79 \text{ m/s}$$

Writing Eq. (7.34) between 0 and 2:

$$v_{y_2} = v_{y_0} - gt_2 = v_0 \sin \theta - gt_2$$

Substituting the known parameters and carrying out the calculations will yield:

$$v_{y_2} = (6.69)(\sin 30^\circ) - (9.8)(1.04) = -6.85 \text{ m/s}$$

Note that we obtained a negative value for a scalar quantity, which is not permitted. Here, the negative sign implies direction. We adopted the upward direction to be positive for the y axis. The negative value calculated above indicates that the direction of the vertical component of the landing velocity is downward (opposite to that of positive y axis). Now, we can rewrite v_{y_2} as:

$$v_{y_2} = 6.85 \text{ m/s} \quad (\downarrow)$$

Knowing the magnitudes of the horizontal and vertical components of the landing velocity enables us to calculate the landing speed:

$$v_2 = \sqrt{v_{x_2}^2 + v_{y_2}^2} = \sqrt{(5.79)^2 + (6.85)^2} = 8.97 \text{ m/s}$$

Example 7.8 The diver illustrated in Fig. 7.26 undergoes both translational and rotational, or general motion. The overall general motion of the diver can be analyzed by observing the

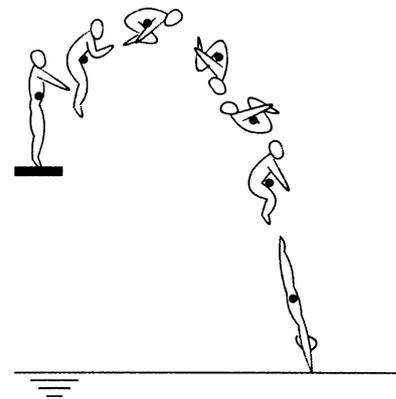


Fig. 7.26 A diver

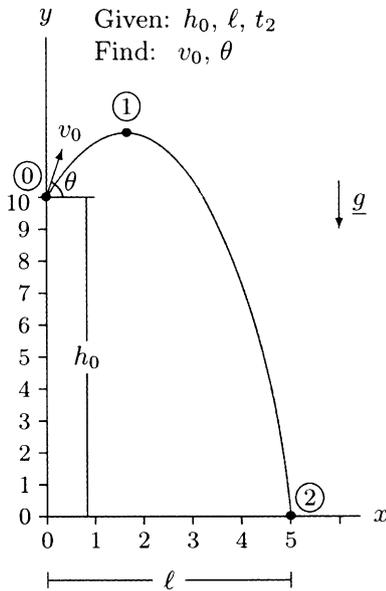


Fig. 7.27 Trajectory of the center of gravity of the diver (both x and y are in meters)

trajectory of the diver's center of gravity which can be assumed to undergo a projectile motion.

Consider a case in which a diver takes off from a diving board located at a height $h_0 = 10\text{ m}$ above the water level and enters the water at a horizontal distance $\ell = 5\text{ m}$ from the end of the board. If the total time the diver remains in the air is $t_2 = 2.5\text{ s}$, calculate the speed and angle of takeoff of the diver's center of gravity.

Solution: The trajectory of the center of gravity of the diver is shown in Fig. 7.27. In this case, speed and angle of takeoff (v_0 and θ) are not known, but are to be determined. In Fig. 7.27, 0, 1, and 2 represent the takeoff, peak, and entry into the water stages of motion, respectively. The origin of the xy coordinate frame is located at the water level directly under 0. Therefore, the coordinates of position 0 are: $x_0 = 0$ and $y_0 = h_0 = 10\text{ m}$. We know the coordinates of position 2 as well: $x_2 = \ell = 5\text{ m}$ and $y_2 = 0$. We know another parameter associated with position 2, which is $t_2 = 2.5\text{ s}$.

We can utilize Eqs. (7.31) and (7.32). Writing Eq. (7.31) between 0 and 2, and substituting the known parameters:

$$\begin{aligned}x_2 &= x_0 + (v_0 \cos \theta)t_2 \\5 &= 0 + (v_0 \cos \theta)(2.5)\end{aligned}$$

Solving this equation for $v_0 \cos \theta$:

$$v_0 \cos \theta = \frac{5}{2.5} = 2 \quad (\text{i})$$

Similarly, writing Eq. (7.32) between 0 and 2, and substituting the known parameters:

$$\begin{aligned}y_2 &= y_0 + (v_0 \sin \theta)t_2 - \frac{1}{2}gt_2^2 \\0 &= 10 + (v_0 \sin \theta)(2.5) - \frac{1}{2}(9.8)(2.5)^2\end{aligned}$$

Solving this equation for $v_0 \sin \theta$:

$$v_0 \sin \theta = 8.25 \quad (\text{ii})$$

Noting that $v_0 \sin \theta$ over $v_0 \cos \theta$ is equal to $\tan \theta$, divide Eq. (ii) by Eq. (i):

$$\tan \theta = \frac{8.25}{2} = 4.125$$

Considering the inverse tangent of the value calculated above will yield:

$$\theta = 76.4^\circ$$

The speed of takeoff can now be determined from Eq. (i):

$$v_0 = \frac{2}{\cos \theta} = \frac{2}{\cos 76.4^\circ} = 8.5 \text{ m/s}$$

7.12 Exercise Problems

Problem 7.1 As illustrated in Fig. 7.11, consider the skier descending a straight slope. Assume that the skier is moving down the slope at a constant acceleration a_0 . Moreover, the speed of the skier at positions (0) and (1) is observed to be $V_0 = 12 \text{ m/s}$ and $V_1 = 21 \text{ m/s}$, respectively. Furthermore, the time it took for the skier to cover the distance l between points (0) and (1) is measured to be $t_1 = 6.18 \text{ s}$. Calculate the constant acceleration of the skier and the distance l between points (0) and (1) he covered while descending the slope.

Answers: $a_0 = 1.94 \text{ m/s}^2$; $l = 111.2 \text{ m}$

Problem 7.2 As shown in Fig. 7.13, consider the person holding a ball at a certain height above the ground. Once the ball is released, it descends and hits the ground at the speed of $V_1 = 5.8 \text{ m/s}$. Assume that while the ball is descending, the air resistance was negligible. Calculate the time t_1 it takes for the ball to reach the ground and the height h of the ball at its initial position above the ground.

Answers: $t_1 = 0.59 \text{ s}$; $h = 1.7 \text{ m}$

Problem 7.3 Consider a person throwing a ball upward into the air with an initial speed of $v_0 = 10 \text{ m/s}$ (Fig. 7.28). Assume that at the instant when the ball is released, the person's hand is at a height $h_0 = 1.5 \text{ m}$ above the ground level.

Neglecting the possible effects of air resistance, determine the maximum height h_1 that the ball reached, the total time t_2 it took for the ball to ascend and descend, and the speed v_2 of the ball just before it hit the ground.

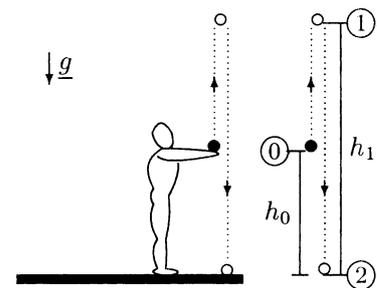


Fig. 7.28 Problem 7.3

Note that this problem must be handled in two phases: ascent and descent. Also note that the speed of the ball at the peak was zero.

Answers: $h_1 = 6.6 \text{ m}$, $t_2 = 1.16 \text{ s}$, $v_2 = 11.4 \text{ m/s}$

Problem 7.4 As shown in Fig. 7.28, assume that in another trial, a person has thrown the ball upward and it took 1.06 s to reach its maximum height h_1 . Once the ball reached its peak, it began descending and hit the ground. The total time for the ball to ascend and descend was $t = 2.24 \text{ s}$. Neglecting the possible effects of air resistance, calculate the initial speed V_0 with which the ball was thrown into the air, the maximum height h_1 it reached, the vertical distance h_2 the ball traveled between the peak and the ground, the speed of the ball V_2 just before the landing, and the initial height h_0 of the ball above the ground. Consider solving this problem in two phases: ascent and descent. Furthermore, consider that the speed of the ball at the peak was zero.

Answers: $V_0 = 10.4 \text{ m/s}$; $h_1 = 5.5 \text{ m}$; $h_2 = 6.8 \text{ m}$; $V_2 = 11.6 \text{ m/s}$; $h_0 = 1.3 \text{ m}$

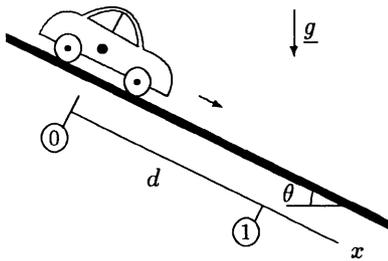


Fig. 7.29 Problems 7.5 and 7.6

Problem 7.5 Consider the car shown in Fig. 7.29. At position 0, the car is stationary on a hill that makes an angle θ with the horizontal. Assume that the gear of the car is at “neutral” and that at time $t_0 = 0$ the brakes of the car are released. Under the effect of the gravitational acceleration g , the car will start moving down the hill. After some time, the car will be at position 1, which is at a distance d from position 0 measured parallel to the hill.

Show that time t_1 to cover the distance between positions 0 and 1, and speed v_1 of the car at position 1 can be expressed as:

$$t_1 = \sqrt{\frac{2d}{g \sin \theta}}$$

$$v_1 = \sqrt{2gd \sin \theta}$$

Problem 7.6 At position (0), the car in Fig. 7.29 is stationary on a hill that makes an angle θ with the horizontal. At time $t_0 = 0$ the brakes of the car are released and the car starts moving down the hill under the effect of gravitational acceleration. As the air resistance is assumed to be negligible, at some time the car is

observed at position (1), which is at a distance $d = 50$ m from position (0) measured parallel to the hill. Furthermore, it took 3.8 s for the car to cover this distance. Determine the angle θ that the hill makes with the horizontal and the speed V_1 of the car at position (1).

Answers: $\theta = 45^\circ$; $V_1 = 26.3$ m/s

Problem 7.7 Based on the assumption that the air resistance is negligible, it is suggested that the overall motion characteristic of a long jumper may be analyzed by assuming that the center of gravity of the athlete undergoes a projectile motion (Fig. 7.30).

Consider an athlete who jumps a horizontal distance of 9 m after reaching a maximum height of 1.5 m. What was the takeoff speed v_0 of the athlete? Discuss how the athlete can improve his/her performance.

Answer: $v_0 = 9.8$ m/s

Problem 7.8 The ski jumper shown in Fig. 7.31 leaves the ramp with a horizontal speed of v_0 and lands on a slope that makes an angle $\beta = 45^\circ$ with the horizontal.

Neglecting air resistance (the effect of which may be quite significant), determine the takeoff speed v_0 , landing speed v_1 , and the total time t_1 , that the ski jumper was airborne if the skier touched down at a distance $d = 50$ m from the ramp measured parallel to the slope.

Answers: $v_0 = 13.2$ m/s, $v_1 = 29.6$ m/s, $t_1 = 2.7$ s

Problem 7.9 Assume that in another trial, the ski jumper in the previous problem manages to maintain the takeoff speed at $v_0 = 13.2$ m/s, but leaves the ramp at an angle $\theta = 10^\circ$ with the horizontal (Fig. 7.32).

How far from the ramp would the ski jumper land on the slope? Discuss whether the ski jumper improved his/her performance as compared to the trial in Problem 7.4. If yes, by how much?

Answers: $d = 57$ m, 14% improvement

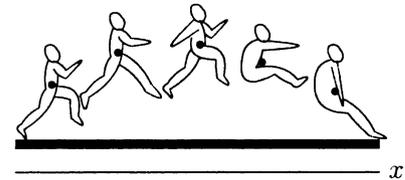


Fig. 7.30 Problem 7.7

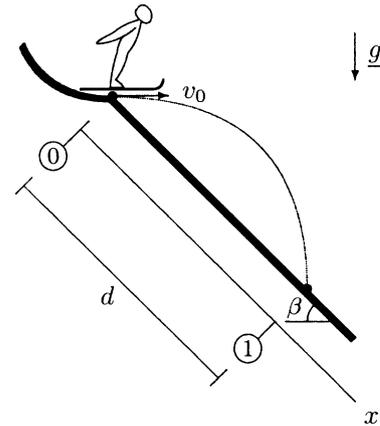


Fig. 7.31 Problem 7.8

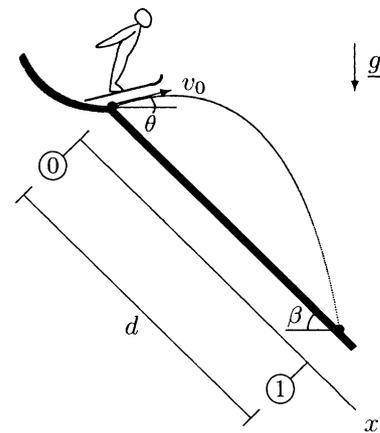


Fig. 7.32 Problem 7.9

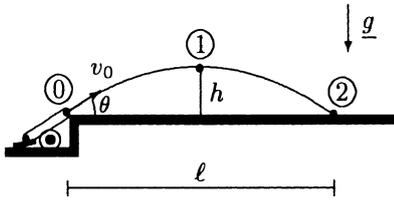


Fig. 7.33 Problems 7.10 and 7.11

Problem 7.10 Figure 7.33 illustrates the trajectory of a cannonball. Assume that the cannonball was fired into the air with an initial speed of $v_0 = 100\text{ m/s}$ at position 0. The cannonball landed at position 2 that is at a horizontal distance $l = 1000\text{ m}$ measured from position 0.

Calculate the angle of takeoff θ , time of flight t_2 , and maximum height h , that the cannonball reached.

Note that $2 \sin \theta \cos \theta = \sin(2\theta)$ and take $g = 10\text{ m/s}^2$.

Answers: $\theta = 45^\circ$, $t_2 = 14\text{ s}$, $h = 250\text{ m}$

Problem 7.11 Assume that once fired into the air, the cannonball in Fig. 7.33 underwent a projectile motion and hit the ground at point (2). The total time of the flight was $t_2 = 15\text{ s}$. Furthermore, the angle of release of the cannonball was $\theta = 45^\circ$ and its horizontal range of motion was $l = 1250\text{ m}$. Calculate the speed V_0 of the cannonball at the point of release (0) and the maximum height h_1 it reached during the flight at point (1).

Answers: $V_1 = 117.9\text{ m/s}$; $h_1 = 349.5\text{ m}$