Chapter 8

Linear Kinetics

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8.1 Overview

As studied in the previous chapter, kinematic analyses are concerned with the description of the geometric and time-dependent aspects of motion in terms of displacement, velocity, and acceleration without dealing with the factors causing the motion. The field of *kinetics*, on the other hand, is based on kinematics and incorporates into the analysis the effects of forces that cause the motion.

Based on the type of motion involved, the field of kinetics can be divided into linear (translational) and angular (rotational) kinetics. Translation is caused by the net force applied on an object, whereas rotation is the consequence of the net torque. An object will translate and rotate simultaneously (undergo a general motion) if there is both a net force and a net moment acting on it. In addition to classifying a motion as translational, rotational, or general, the field of kinetics can be further distinguished as the *kinetics of particles* and the *kinetics of rigid bodies*. Particle kinetics is easier to implement than rigid body kinetics that introduces the size and shape of the bodies into the analyses. The distinction between a particle and a rigid body is particularly important if the object is undergoing a rotational motion. If the object is sufficiently small or it is undergoing translational motion only, then the geometric characteristics of the object may be ignored and the object can be treated as a particle located at its center of gravity with a mass equal to the total mass of the object. For example, what is significant for a person pushing a block on a flat surface is the total mass of the block, not its size or shape.

Kinetic analyses utilize Newton's second law of motion which can be formulated in various ways. One way of representing Newton's second law of motion is in terms of the equations of motion, which are particularly suitable for solving problems requiring the analysis of acceleration. The use of the equations of motion for linear kinetics will be discussed next. Another way of formulating Newton's second law is through work and energy methods that will also be discussed in this chapter within the context of linear kinetics. There are also methods based on impulse and momentum that will be presented in Chap. 11.

8.2 Equations of Motion

A body accelerates if there is a non-zero net force acting on it. Newton's second law of motion states that the magnitude of the acceleration of a body is directly proportional to the magnitude of the resultant force and inversely proportional to its

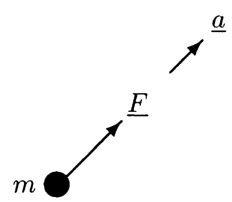

Fig. 8.1 *An object will accelerate in the direction of the applied force*

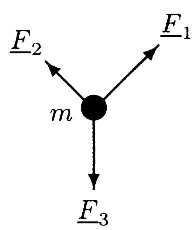

Fig. 8.2 *The net force is the vector sum of all forces acting on the object*

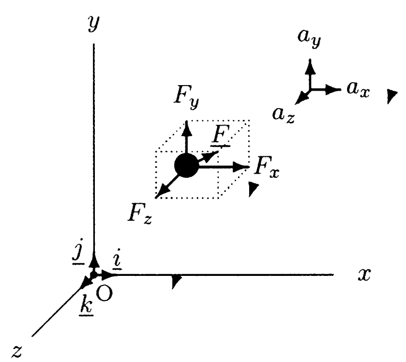

Fig. 8.3 *Rectangular components of the force and acceleration vectors*

mass. The direction of the acceleration is the same as the direction of the resultant force.

Consider a particle of mass m that is acted upon by a force $\underline{F}$ and let $\underline{a}$ be the resulting acceleration of the particle (Fig. 8.1). Newton's second law of motion can be expressed as:

$$\underline{F} = m\underline{a} \tag{8.1}$$

If there is more than one force acting on the particle (Fig. 8.2), then $\underline{F}$ in Eq. (8.1) must be replaced by the net or the resultant of all forces acting on it. The resultant of a system of forces can be determined by considering the vector sum of all forces. Therefore:

$$\sum \underline{F} = m\underline{a} \tag{8.2}$$

This is known as the *equation of motion*. Since force and acceleration are vector quantities, they can be expressed in terms of their components in reference to a chosen coordinate frame. For translational motion analyses, it is best to use the Cartesian (rectangular) coordinate system that consists of the x, y, and z axes with $\underline{i}$, $\underline{j}$, and $\underline{k}$ unit vectors indicating the positive x, y, and z directions, respectively (Fig. 8.3). If there is a single force acting on an object, then the force vector and the resulting acceleration vector can be expressed in terms of their components along the rectangular coordinate directions:

$$\underline{F} = F_x\underline{i} + F_y\underline{j} + F_z\underline{k} \tag{8.3}$$

$$\underline{a} = a_x\underline{i} + a_y\underline{j} + a_z\underline{k} \tag{8.4}$$

Substituting Eqs. (8.3) and (8.4) into Eq. (8.1):

$$F_x\underline{i} + F_y\underline{j} + F_z\underline{k} = ma_x\underline{i} + ma_y\underline{j} + ma_z\underline{k} \tag{8.5}$$

This vector equation is valid if the following conditions are satisfied:

$$\begin{aligned} F_x &= ma_x \\ F_y &= ma_y \\ F_z &= ma_z \end{aligned} \tag{8.6}$$

If there is more than one force acting on the object, then F_x, F_y, and F_z must be replaced by the sum of all forces acting in the x, y, and z directions, respectively:

$$\begin{aligned} \sum F_x &= ma_x \\ \sum F_y &= ma_y \\ \sum F_z &= ma_z \end{aligned} \tag{8.7}$$

Equations (8.7) state that the sum of all forces acting in one direction is equal to the mass times the acceleration of the body in that direction. Note that for one-dimensional motion analysis, only one of these equations need to be considered. For a two-dimensional case, two of the above equations are sufficient to analyze the problem.

8.3 Special Cases of Translational Motion

A force can be applied in various ways. For example, an applied force may be constant or it may vary over time. Applied forces can be measured in various ways as well. The magnitude of a force vector can be measured as a function of time, as a function of the relative position of the object upon which it is applied, or as a function of velocity. Some of these cases will be discussed next. To illustrate the methods of handling these cases in a concise manner, it will be assumed that the motion is along a straight line in the x direction and under the effect of only one applied force. Using the vectorial properties of the parameters involved, these methods can be easily expanded to analyze two- and three-dimensional translational motions under the action of more than one force.

Note that the derivations provided in this section are aimed to demonstrate that different cases can be handled through proper mathematical manipulations. The mathematics involved for the cases in which the applied force is a function of displacement may be beyond the scope of this text, and can be omitted without losing the continuity of the topics to be covered in the following sections.

8.3.1 Force Is Constant

If a force applied on an object has a constant magnitude and direction, the object will move with a constant acceleration in the direction of the applied force. Assume that a force with magnitude F_x is applied on an object with mass m. The magnitude a_x of the constant acceleration of the object in the x direction can be calculated using the equation of motion in the x direction:

$$a_x = \frac{F_x}{m} = \text{constant}$$

Once the acceleration of the object is determined, the kinematic equations can be utilized to calculate the velocity and displacement of the object as well:

$$v_x = v_{x_0} + \int_{t_0}^{t} a_x \mathrm{d}t = v_{x_0} + \frac{F_x}{m} t \tag{8.8}$$

$$x = x_0 + \int_{t_0}^{t} v_x \mathrm{d}t = x_0 + v_{x_0} t + \frac{1}{2} \frac{F_x}{m} t^2 \tag{8.9}$$

Here, v_{x_0} and x_0 are the initial speed and displacement of the object in the x direction at time $t = 0$. Note that these results can be used for a situation in which there is a second force with constant magnitude F_y acting on the object in the y direction simply by replacing x with y throughout the equations.

8.3.2 Force Is a Function of Time

If the magnitude of a force applied on an object is a function of time, then $F_x = F_x t$. The resulting acceleration of the object is also a function of time:

$$a_x(t) = \frac{F_x(t)}{m}$$

The velocity and displacement of the object can now be determined using the kinematic relationships:

$$v_x = v_{x_0} + \int_{t_0}^{t} a_x(t) \mathrm{d}t \tag{8.10}$$

$$x = x_0 + \int_{t_0}^{t} v_x(t) \mathrm{d}t \tag{8.11}$$

The function $F_x(t)$ must be provided so that the integral in Eq. (8.10) can be evaluated.

8.3.3 Force Is a Function of Displacement

Sometimes it is more convenient to express force as a function of displacement, in which case $F_x = F_x(x)$. By definition, acceleration is equal to the time rate of change of velocity. Therefore, the equation of motion in the x direction can be expressed as:

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{F_x(x)}{m}$$

Employing the chain rule of differentiation (see Appendix C.2.6), the time derivative of velocity can be expressed as:

$$\frac{dv_x}{d_t} = \frac{dv_x}{d_x}\frac{dx}{dt} = \frac{dv_x}{dx}v_x$$

Therefore, the equation of motion in the x direction is:

$$v_x\frac{dv_x}{dx} = \frac{F_x(x)}{m}$$

Multiplying both sides by dx:

$$v_x dv_x = \frac{F_x(x)}{m}dx$$

The left-hand side of this equation is a function of v_x only and can be integrated with respect to v_x, and the right-hand side is a function of x only and can be integrated with respect to x:

$$\int_{v_{x_0}}^{v_x} v_x dv_x = \int_{x_0}^{x} \frac{F_x(x)}{m}dx$$

Evaluating the integral on the left-hand side:

$$\frac{1}{2}\left(v_x{}^2 - v_{x_0}{}^2\right) = \frac{1}{m}\int_{x_0}^{x} F_x(x)dx$$

Rearranging the order of terms:

$$v_x{}^2 = v_{x_0}{}^2 + \frac{2}{m}\int_{x_0}^{x} F_x(x)dx \tag{8.12}$$

F_x must be provided as a function of x, so that the integral in Eq. (8.12) can be evaluated. For given $F_x(x)$, Eq. (8.12) will yield v_x as a function of x. Once F_x is known, the acceleration of the object can be determined using:

$$a_x = v_x\frac{dv_x}{d_x} \tag{8.13}$$

8.4 Procedure for Problem Solving in Kinetics

The procedure for analyzing the kinetic characteristics of objects undergoing translational motion using the equations of motion can be outlined as follows:

- Draw a simple, neat diagram of the system to be analyzed.
- Isolate the bodies of interest from their surroundings and draw their free-body diagrams by showing all external forces acting on them. Indicate the correct directions for the known forces. If the direction of a force vector is not known, assume a

positive direction for it. If that force appears to have a negative value in the solution, it would mean that the assumed direction for the force vector was incorrect.

• Designate the direction of motion of each object on the sidelines (not as parts of the free-body diagrams). It is particularly important to be consistent with the assumed direction of the motion throughout the analyses.

• Choose a convenient coordinate system. For two-dimensional cases, rectangular coordinates x and y are usually the most convenient.

• Apply the equations of motion. For two-dimensional motion analysis there are two governing equations, and therefore, the number of unknowns to be determined cannot be more than two. In linear kinetics, the unknowns are either forces or accelerations.

• Include the correct directions of forces and accelerations in the solution, along with their units.

• The kinematic relations between position, velocity, and acceleration can also be utilized if the information about the velocity and/or position of the object analyzed is given or required.

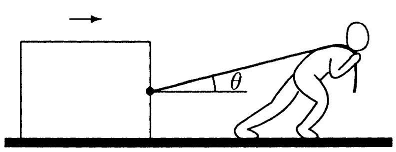

Fig. 8.4 *A block is being pulled on a horizontal surface*

Example 8.1 As illustrated in Fig. 8.4, consider a block of mass $m = 50$ kg which is being pulled on a rough, horizontal surface by a person using a rope. Assume that the person is applying a constant force of $T = 150$ N on the block, the rope makes an angle $\theta = 30°$ with the horizontal, and the coefficient of kinetic friction between the block and the horizontal surface is $\mu = 0.2$.

Determine the acceleration of the block if the bottom surface of the block remains in full contact with the floor throughout the motion.

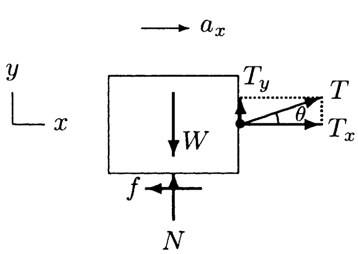

Fig. 8.5 *The free-body diagram of the block*

Solution: The free-body diagram of the block is shown in Fig. 8.5. The positive x direction is chosen in the direction of motion of the block. W is the weight of the block, f is the magnitude of the frictional force acting in the direction opposite to the direction of motion, and N is the magnitude of the reaction force applied by the floor on the block. $\underline{T}$ is the force exerted by the person which is transmitted to the block through the rope. The rope makes an angle $\theta = 30°$ with the horizontal. Therefore, $\underline{T}$ has components in the x and y directions:

$$T_x = T\cos\theta \quad (\rightarrow)$$

$$T_y = T\sin\theta \quad (\uparrow)$$

The weight of the block is due to the gravitational effect of Earth on the mass of the block, and can be expressed as:

$$W = mg \quad (\downarrow)$$

The magnitude of the frictional force is proportional to the magnitude of the normal force, and they are related through the coefficient of friction between the surfaces in contact:

$$f = \mu N \qquad (\leftarrow) \tag{i}$$

Equations of motion in the x and y directions can now be applied to determine an expression for the acceleration of the block. The block has no motion in the y direction, and therefore, the acceleration of the block in the y direction is zero ($a_y = 0$). The equation of motion in the y direction is:

$$\sum F_y = 0: \quad N + T_y - W = 0$$

Solving this equilibrium equation for force N will yield:

$$N = W - T_y = mg - T\sin\theta \tag{ii}$$

Substituting Eq. (ii) into Eq. (i) will yield:

$$f = \mu(mg - T\sin\theta) \tag{iii}$$

Now, the equation of motion in the x direction can be considered:

$$\sum F_x = ma_x: \quad T_x - f = ma_x$$

Solving this equation for a_x will yield

$$a_x = \frac{1}{m}(T_x - f) \tag{iv}$$

Substituting Eq. (iii) and $T_x = T\cos\theta$ into Eq. (iv):

$$a_x = \frac{1}{m}[T\cos\theta - \mu(mg - T\sin\theta)]$$

Substituting the numerical values $m = 50$ kg, $T = 150$ N, $\theta = 30°$, $\mu = 0.2$, and $g = 9.8$ m/s, and carrying out the calculations will yield $a_x = 0.94\ \text{m/s}^2\ (\rightarrow)$.

8.5 Work and Energy Methods

The fundamental method of analyzing the kinetic characteristics of bodies is based on the equations of motion which are mathematical representations of Newton's second law of motion. Using the equations of motion, one can determine accelerations. In some cases, particularly when the forces involved are not constant, the solution of equations of motion may be difficult. To handle such situations, alternative methods are developed that are based on the concepts of work and energy. These methods are also derived from Newton's laws, and can be applied to analyze the forces, velocities, and displacements involved in relatively complex systems without resorting to the equations of motion.

8.6 Mechanical Work

By definition, *mechanical work* is the product of force and corresponding displacement. Work is a scalar quantity. There is no direction associated with work.

8.6.1 Work Done by a Constant Force

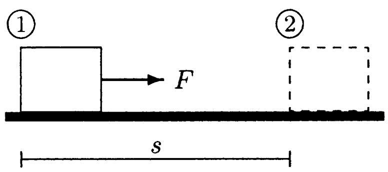

Fig. 8.6 *A constant force applied on the block displaces it from position 1 to position 2*

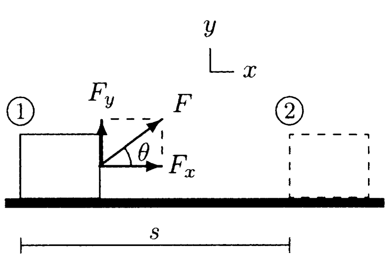

Fig. 8.7 *A constant force that makes an angle θ with the horizontal is applied on the block*

To explore the definition of work, consider the block in Fig. 8.6. Assume that a constant, horizontal force $\underline{F}$ is applied on the block so as to move it from position 1 to position 2, which are s distance apart. The work done, W, by force $\underline{F}$ on the block to move the block from position 1 to 2 is equal to the magnitude of the force vector times the displacement:

$$W = Fs \tag{8.14}$$

Consider the same block which is pulled from position 1 to 2 by another constant force $\underline{F}$ that makes an angle θ with the horizontal (Fig. 8.7). The work done by $\underline{F}$ on the block is equal to the magnitude of the force component in the direction of displacement times the displacement itself. Since the component of $\underline{F}$ along the horizontal is $F_x = F\cos\theta$, the work done by $\underline{F}$ to move the block from position 1 to 2 is:

$$W = F_x s = Fs\ \cos\theta \tag{8.15}$$

Note that Eqs. (8.14) and (8.15) are consistent with each other since $\cos\theta = 1$ when $\theta = 0°$.

For a force to do work, the body on which the force is applied must undergo a displacement and the force vector must have a non-zero component in the direction of displacement. For example, the vertical component, $F_y = F\sin\theta$, of the force vector in Fig. 8.7 does no work because the block is not displaced in the vertical direction.

Work done can be positive or negative. The work done by a force is positive if the force is applied in the same direction as the displacement. If the applied force and displacement have opposite directions, then the work done by that force is negative. A typical example of negative work is the one done by a frictional force. As illustrated in Fig. 8.8, assume that a block is pulled by a force $\underline{F}$ toward the right to displace the block by a distance s. The work done W_f by the frictional force $\underline{f}$ on the block while the block was displaced by a distance s is:

$$W_f = -fs \tag{8.16}$$

If there is more than one external force acting on a body in motion, then there is one work done for each force. The net work done is the algebraic sum of work done by individual forces. For example, the net work done for the case illustrated in Fig. 8.8 is:

$$W = F_s - fs$$

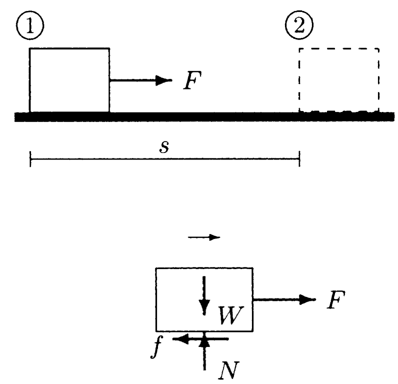

Fig. 8.8 *Frictional forces do negative work*

8.6.2 Work Done by a Varying Force

Equation (8.14) can only be used to calculate the work done by a constant force. If an applied force is a function of displacement, then the work done can be calculated by considering the integral of the force over the distance it is applied.

As illustrated in Fig. 8.9, consider a block pulled along the x direction by a force F_x that varies with the displacement of the block in the x direction. That is, $F_x = F_x(x)$. Assume that the block that was originally located at position 1 moves to position 2, which are s distance apart. Let x_1 and x_2 represent the initial and final positions of the block, respectively. If the variation of F_x with respect to x is known, then the work done by F_x to move the block from position 1 to 2 can be determined using:

$$W = \int_{x_1}^{x_2} F_x \mathrm{d}x \tag{8.17}$$

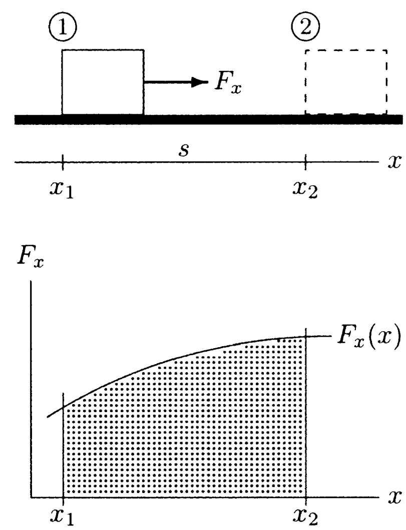

Fig. 8.9 *Work is equal to the area under the force versus displacement curve*

Note that the evaluation of the definite integral in Eq. (8.17) will yield the total area under the force versus position curve, the x axis, and vertical lines passing through $x = x_1$ and $x = x_2$ (the shaded area in Fig. 8.9). Also note that if the force F_x is constant, then the integration in Eq. (8.17) will yield $W = F_x(x_2 - x_1) = F_x s$, which is consistent with Eq. (8.14).

8.6.3 Work as a Scalar Product

For some applications, it may be convenient to utilize the definition of work as the dot (scalar) product of the force and displacement vectors. As discussed in Appendix B.14, the dot product of any two vectors is a scalar quantity equal to the product of magnitudes of the two vectors multiplied by the cosine of the smaller angle between the two. In the case of work done by a constant force $\underline{F}$ on a body whose displacement vector is given by $\underline{s}$:

$$W = \underline{F} \cdot \underline{s} \tag{8.18}$$

If θ is the smaller angle between vectors $\underline{F}$ and $\underline{s}$, then:

$$W = \underline{F} \cdot \underline{s} = F\,s\,\cos\theta \tag{8.19}$$

Force and displacement vectors can be expressed in terms of their rectangular components:

$$\underline{F} = F_x\underline{i} + F_y\underline{j} + F_z\underline{k} \tag{8.20}$$

$$\underline{s} = x\underline{i} + y\underline{j} + z\underline{k} \tag{8.21}$$

The dot product of unit vectors are such that $\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1$ and $\underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{k} = \underline{k} \cdot \underline{i} = 0$. Substituting Eqs. (8.20) and (8.21) into Eq. (8.18) and carrying out the dot products of unit vectors will yield:

$$W = F_x x + F_y y + F_z z \tag{8.22}$$

Equation (8.22) is significant in that it represents the total work done by the components of the force vector in the x, y, and z directions. For example, the work done in the x direction is equal to the magnitude of the force component in the x direction times the displacement in the same direction. Note that for a biaxial motion in the xy-plane, Eq. (8.22) reduces to $W = F_x x + F_y y$ and $W = F_x x$ for a uniaxial motion in the x direction.

8.7 Mechanical Energy

The term *energy* is used to describe the capacity of a system to do work on another system. Energy can take various forms such as mechanical, thermal, chemical, and nuclear. The field of mechanics is primarily concerned with the mechanical form of energy. Mechanical energy can be categorized as potential energy and kinetic energy. Energy is also a scalar quantity.

8.7.1 Potential Energy

The *potential energy* of a system is associated with its position or elevation. It is the energy stored in the system that can be converted into kinetic energy. The concept of potential energy comes from the perception that an object located at a height can do useful work if it is allowed to descend. The potential of an object to do work due to the relative height of its center of gravity is defined as *gravitational potential energy*. Consider the object with weight $W = mg$ shown in Fig. 8.10. The object is at position 1 which is located at a height h measured relative to position 2. The gravitational potential energy, $\mathcal{E}_P$, of the object at position 1 relative to position 2 is:

$$\mathcal{E}_P = Wh = mgh \tag{8.23}$$

Notice that Wh is essentially the work that the force of gravity would do on the object to move it from position 1 to position 2, which are h distance apart.

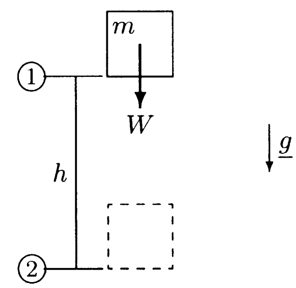

Fig. 8.10 *Gravitational potential energy*

8.7.2 Kinetic Energy

Kinetic energy is associated with motion. Every moving object has a kinetic energy. The kinetic energy, $\mathcal{E}_K$, of an object with mass m moving with a speed v is equal to the product of one half of the mass and the square of the speed of the object:

$$\mathcal{E}_K = \frac{1}{2} m v^2 \tag{8.24}$$

8.8 Work–Energy Theorem

There is a relationship between the kinetic energy and the work done. The net work done, W_{12}, on an object to displace the object from position 1 to position 2 is equal to the change in kinetic energy, $\Delta\mathcal{E}_K$, of the object between positions 1 and 2. This is known as the *work–energy theorem* and can be expressed as:

$$W_{12} = \Delta\mathcal{E}_K = \mathcal{E}_{K2} - \mathcal{E}_{K1} \tag{8.25}$$

8.9 Conservation of Energy Principle

Forces may be conservative and nonconservative. A force is conservative if the work done by that force to move an object between two positions is independent of the path taken. A typical example of conservative forces is the gravitational force. The frictional force, on the other hand, is a nonconservative force. Nonconservative forces dissipate energy as heat.

The network done on a system by conservative forces is converted into kinetic and potential energies in such a manner that the total energy of the system (sum of kinetic and potential energies) remains constant throughout the motion. This is known as the *principle of conservation of mechanical energy*, and between any two positions 1 and 2 it can be stated as:

$$\mathcal{E}_{K1} + \mathcal{E}_{P1} = \mathcal{E}_{K2} + \mathcal{E}_{P2} \tag{8.26}$$

8.10 Dimension and Units of Work and Energy

Mechanical work and energy have the same dimension and units. By definition, work done is force times displacement. Therefore, work has the dimension of force times the dimension of length.

$$[\text{Work}] = [\text{Force}][\text{Displacement}] = M\frac{L^2}{T^2}$$

The units of work and energy in different systems of units are provided in Table 8.1.

Table 8.1 *Units of work and energy*

System	Units of work and energy	Special name
SI	Newton-meter (Nm)	Joule (J)
c–g–s	Dyne-centimeter (dyn cm)	erg
British	Pound-foot (lb ft)	

8.11 Power

Power, $\mathcal{P}$, is defined as the time rate of work done:

$$\mathcal{P} = \frac{\mathrm{d}W}{\mathrm{d}t} \tag{8.27}$$

The work done by a constant force on an object can be determined by considering the dot product of the force and displacement vectors ($W = \underline{F} \cdot \underline{s}$):

$$\mathcal{P} = \frac{\mathrm{d}}{\mathrm{d}t}(\underline{F} \cdot \underline{s})$$

If the force vector $\underline{F}$ is constant, then:

$$\mathcal{P} = \underline{F} \cdot \frac{\mathrm{d}\underline{s}}{\mathrm{d}t} = (\underline{F} \cdot \underline{v}) \tag{8.28}$$

In Eq. (8.28), $\underline{v}$ is the velocity vector of the object. If the applied force is collinear with the velocity, then $\mathcal{P} = Fv$. Power is a scalar quantity, and has the dimension of force times velocity. The units of power are given in Table 8.2.

Table 8.2 *Units of power (1 hp = 550 lb ft = 746 W)*

System	Units of work and energy	Special name
SI	Nm/s = J/s	Watt (W)
c–g–s	dyn cm/s = erg/s	
British	lb ft/s	Horsepower (hp)

8.12 Applications of Energy Methods

The work–energy theorem stated in Eq. (8.23) and the principle of conservation of energy stated by Eq. (8.24) provide alternative methods of problem solving in dynamics. The work–energy theorem can be used to analyze problems involving nonconservative forces. On the other hand, the principle of conservation of energy is useful only when the forces involved are conservative. As compared to the applications of the equations of motion, these methods are easier to apply and are particularly useful when the information provided or to be determined is in terms of velocities rather than accelerations. Definitions of important concepts introduced in this chapter and various methods of analyses in kinetics are summarized in Table 8.3. The following examples will demonstrate some of the applications of these methods.

Table 8.3 *Summary of equations and formulas*

Work done by a varying force	$W = \int_{x_1}^{x_2} F_x \mathrm{d}x$
Work done by a constant force	$W = F_x(x_2 - x_1) = F_x s$
Potential energy	$\mathcal{E}_\mathrm{P} = mgh$
Kinetic energy	$\mathcal{E}_\mathrm{K} = \frac{1}{2}mv^2$
Conservation of energy principle	$\mathcal{E}_\mathrm{K1} + \mathcal{E}_\mathrm{P1} = \mathcal{E}_\mathrm{K2} + \mathcal{E}_\mathrm{P2}$
Work–energy theorem	$W_{12} = \mathcal{E}_\mathrm{K2} - \mathcal{E}_\mathrm{K1}$
Equation of motion	$\sum F_x = ma_x$

Example 8.2 A 20 kg block is pushed up a rough, inclined surface by a constant force of $P = 150$ N that is applied parallel to the incline (Fig. 8.11). The incline makes an angle $\theta = 30°$ with the horizontal and the coefficient of friction between the incline and the block is $\mu = 0.2$.

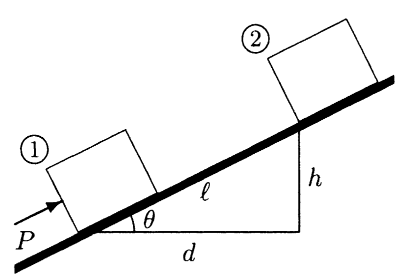

Fig. 8.11 *A block is pushed from position 1 to 2*

If the block is displaced by $l = 10$ m, determine the work done on the block by force $\underline{P}$, by the force of friction, and by the force of gravity. What is the net work done on the block?

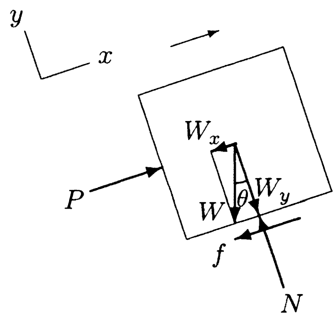

Fig. 8.12 *Free-body diagram of the block*

Solution: The free-body diagram of the block is shown in Fig. 8.12. W is the weight of the block, f is the frictional force at the bottom surface of the block, and N is the reaction force applied by the incline on the block. The x and y directions are chosen in such a manner that the motion occurs in the positive x direction, and there is no displacement of the block in the y direction. Force $\underline{P}$ is applied in the same direction as the displacement of the block. Therefore, the work done by $\underline{P}$ to displace the block by a distance of l along the incline is:

$$W_P = Pl \tag{i}$$

The weight $\underline{W}$ of the block has components along the x and y directions, such that $W_x = W \sin\theta$ and $W_y = W \cos\theta$. Since there is no motion in the y direction, the block is in equilibrium in the y direction. The equilibrium in the y direction requires that $N = W_y$, or since $W = mg$, $N = mg \cos\theta$. The relationship between the frictional force and the normal force at the surfaces of contact is such that $f = \mu N = \mu \cos\theta$. The frictional force acts in a direction parallel to the incline but opposite to that of the displacement of the block. Therefore, the work done by $\underline{f}$ on the block as the block is displaced by a distance l is:

$$W_f = -fl = -\mu mgl \cos\theta \tag{ii}$$

The force of gravity (weight) always acts downward. In this case, it has a component parallel to the incline in the negative x direction with magnitude $W_x = mg \sin\theta$. The work done by W_x on the block as it moves from position 1 to position 2 is:

$$W_g = -mgl \sin\theta \tag{iii}$$

Knowing the work done by individual forces acting on the block, we can determine the net work done on the block:

$$W = W_P + W_f + W_g = Pl - mgl\,(\sin\theta + \mu \cos\theta) \tag{iv}$$

Substituting the numerical values of the parameters involved into Eqs. (i) through (iv) and carrying out the calculations will yield:

$$W_P = (150)(10) = 1500 \text{ J}$$

$$W_f = -(0.2)(20)(9.8)(10)(\cos 30^\circ) = -340 \text{ J}$$

$$W_g = -(20)(9.8)(10)(\sin 30^\circ) = -980 \text{ J}$$

$$W = 1500 - 340 - 980 = 180 \text{ J}$$

Example 8.3 Figure 8.13 illustrates a pendulum with mass m and length l. The mass is pulled to position 1 that makes an angle θ with the vertical and is released to swing.

Assuming that frictional effects and air resistance are negligible, determine the speed v_2 of the mass when it is at position 2.

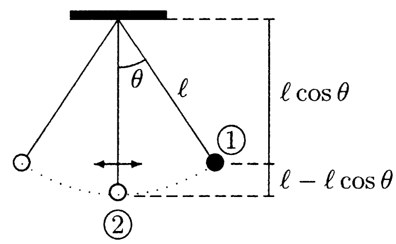

Fig. 8.13 *The pendulum*

Solution: The mass has zero speed and no kinetic energy at the instant of release (position 1). If we choose position 2 to be the datum from which heights are measured, then the mass is located at a height $h_1 = l(1 - \cos\theta)$ at position 1, and the height of the mass at position 2 is zero. Applying the conservation of energy principle between positions 1 and 2:

$$\mathcal{E}_{K1} + \mathcal{E}_{P1} = \mathcal{E}_{K2} + \mathcal{E}_{P2}$$

$$\frac{1}{2}m{v_1}^2 + mgh_1 = \frac{1}{2}m{v_2}^2 + mgh_2$$

Substituting $v_1 = 0$, $h_1 = l(1 - \cos\theta)$, and $h_2 = 0$ into this equation and solving it for the speed of the mass at position 2 will yield:

$$v_2 = \sqrt{2\,gl(1 - \cos\theta)}$$

Example 8.4 As illustrated in Fig. 8.14, consider a ski jumper moving down a track to acquire sufficient speed to accomplish the ski jumping task. The length of the track is $l = 25$ m and the track makes an angle $\theta = 45°$ with the horizontal.

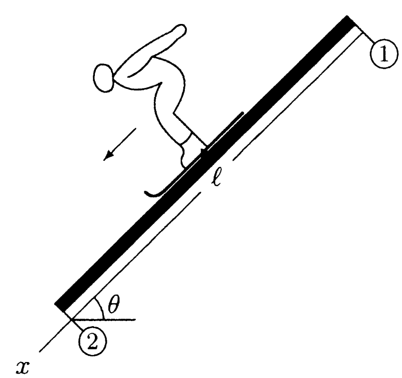

Fig. 8.14 *A ski jumper*

If the skier starts at the top of the track with zero initial speed, determine the takeoff speed of the skier at the bottom of the track using (a) the work–energy theorem, (b) the conservation of energy principle, and (c) the equation of motion along with the kinematic relationships. Assume that the effects of friction and air resistance are negligible.

***Solution (a)*: Work–Energy Method** The free-body diagram of the ski jumper is shown in Fig. 8.15. The forces acting on the ski jumper are the gravitational force $\underline{W}$ and the reaction force applied by the track on the skis in a direction perpendicular to the track. The x direction is chosen to coincide with the direction of motion and y is perpendicular to the track. Therefore, the weight of the ski jumper has components along the x and y directions, such that $W_x = W\sin\theta = mg\sin\theta$ and $W_y = W\cos\theta = mg\cos\theta$. On the other hand, $\underline{N}$ acts in the y direction. Note that W_x is the driving force for the skier.

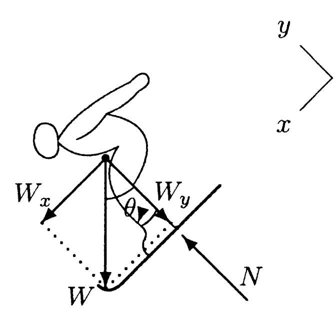

Fig. 8.15 *The free-body diagram of the ski jumper*

Since there is only one force component in the x direction, the work done by that force component is also the net work done on the ski jumper. Labeling the top and the bottom of the track as positions 1 and 2, the work done by W_x to move the skier from position 1 to 2 that are l distance apart is:

$$W_{12} = W_x l = mgl \sin\theta \quad \text{(i)}$$

According to the work–energy theorem, W_{12} must be equal to the change in kinetic energy of the skier between positions 1 and 2:

$$W_{12} = \mathcal{E}_{K2} - \mathcal{E}_{K1} = \frac{1}{2}mv_2{}^2 - \frac{1}{2}mv_1{}^2 \quad \text{(ii)}$$

The second term on the right-hand side of Eq. (ii) is zero because the initial speed of the skier is $v_1 = 0$. Substituting Eq. (i) into Eq. (ii), eliminating the repeated parameter m (the mass of the ski jumper), and solving Eq. (ii) for the takeoff speed v_2 of the ski jumper will yield:

$$v_2 = \sqrt{2\,gl \sin\theta} \quad \text{(iii)}$$

***Solution (b)*: Conservation of Energy Method** Since the effects of nonconservative forces due to friction and air resistance are assumed to be negligible, this problem can also be analyzed by utilizing the principle of conservation of energy. Between positions 1 and 2 of the ski jumper:

$$\mathcal{E}_{K1} + \mathcal{E}_{P1} = \mathcal{E}_{K2} + \mathcal{E}_{P2}$$

$$\frac{1}{2}mv_1{}^2 + mgh_1 = \frac{1}{2}mv_2{}^2 + mgh_2 \quad \text{(iv)}$$

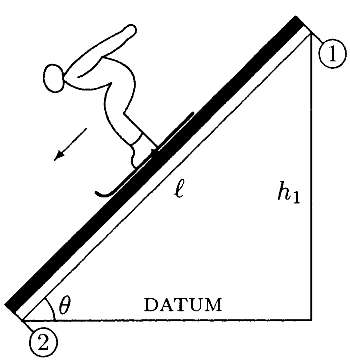

Fig. 8.16 *$h_1 = t \sin\theta$ and $h_2 = 0$*

In Eq. (iv), the first term on the left-hand side is zero since $v_1 = 0$. If we measure heights relative to the bottom of the track (or selecting 2 to be the datum as shown in Fig. 8.16), then height $h_2 = 0$ and the height of the top of the track is $h_1 = l \sin\theta$. Therefore, the second term on the right-hand side of Eq. (iv) is zero as well. Substituting $h_1 = l \sin\theta$ into Eq. (iv), eliminating the repeated parameter m, and solving Eq. (iv) for v_2 will again yield Eq. (iii).

***Solution (c)*: Using the Equation of Motion** The equation of motion in the direction of motion (x) is:

$$\sum F_x = m\,a_x : \qquad W_x = m\,a_x \quad \text{(v)}$$

Substituting $W_x = mg \sin\theta$ in Eq. (v), eliminating m, and solving Eq. (v) for the acceleration of the ski jumper in the x direction will yield:

$$a_x = g \sin\theta \quad \text{(vi)}$$

Since the acceleration of the ski jumper is due to gravity only, what we have is a one-dimensional motion with constant acceleration. By definition, acceleration is the time rate of change of velocity, or velocity is the integral of acceleration with respect to time. Since a_x is constant and the initial velocity of the ski jumper at position 1 is zero, we can write:

$$v_x = a_x t \tag{vii}$$

The kinematic relationship between the velocity and displacement is such that displacement is equal to the integral of velocity. If we measure the displacement relative to the initial position of the ski jumper, then the initial displacement is zero. Therefore, the equation relating displacement, acceleration, and time is:

$$x = \frac{1}{2} a_x t^2 \tag{viii}$$

Equation (vii) can be solved for time $t = v_x/a_x$, which can then be substituted into Eq. (viii) so as to eliminate t. This will yield:

$$x = \frac{1}{2} \frac{{v_x}^2}{a_x}$$

Solving this equation for v_x will give:

$$v_x = \sqrt{2xa_x} \tag{ix}$$

This is a general solution relating the acceleration, speed, and displacement of the ski jumper when the ski jumper is anywhere along the track. $x = l$ and $v_x = v_2$ when the ski jumper reaches the bottom of the track, and the acceleration of the ski jumper is always $a_x = g \sin\theta$. Substituting these parameters into Eq. (ix) will again yield Eq. (iii).

Finally, substituting the numerical values of $g = 9.8 \text{ m/s}^2$, $l = 25 \text{ m}$, and $\theta = 45°$ into Eq. (iii) and carrying out the calculations will yield $v_2 = 18.6 \text{ m/s}$.

Remarks

- It is clear that for problems involving displacement, speed, and force, applications of the methods based on the work–energy theorem and the conservation of energy principle are more straightforward as compared to the application of equations of motion. In general, one should try work–energy or conservation of energy methods first before resorting to the equations of motion.

- Since the effects of nonconservative forces due to friction and air resistance are neglected, the solution of the problem is independent of the shape of the track or how the skier covers the distance between the top and bottom of the track. The most

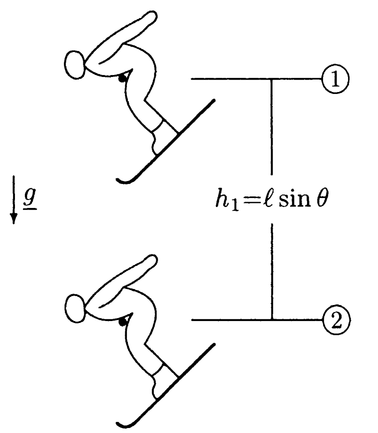

Fig. 8.17 *The solution of the problem is independent of the path of motion*

important parameter in this problem affecting the takeoff speed of the skier is the total vertical distance between locations 1 and 2. This implies that the problem could be simplified by noting that the skier undergoes a "free fall" between 1 and 2 which are $h_1 = l \sin\theta$ distance apart. This is illustrated in Fig. 8.17. Applying the principle of conservation of energy between locations 1 and 2 will again yield Eq. (iii).

8.13 Exercise Problems

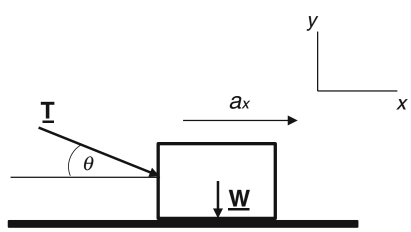

Fig. 8.18 *Problem 8.1*

Problem 8.1 As shown in Fig. 8.18, consider a block of mass m which is moving on a rough horizontal surface under the effect of externally applied force T. The line of action of the force makes an angle θ with the horizontal. If the coefficient of friction between the block and the surface is μ, determine an expression for the acceleration a_x of the block in the direction of motion.

Answer: $a_x = \frac{T(\cos\theta - \mu\sin\theta)}{m} - \mu g$

Problem 8.2 As shown in Fig. 8.4, consider a block of mass 45 kg which is being pulled on a rough horizontal surface by a person using a rope. The rope makes an angle $\theta = 35°$ with the horizontal. As the result of constant force $T = 190$ N applied by the person, the block moves with constant acceleration $a = 0.86$ m/s^2 in the direction of the applied force. Determine the coefficient of friction μ between the block and the surface if the bottom of the block remains in full contact with the ground surface during the motion.

Answer: $\mu = 0.35$

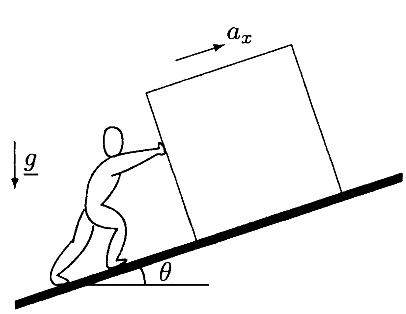

Fig. 8.19 *Problem 8.3*

Problem 8.3 Figure 8.19 shows a person pushing a block of mass m on a surface that makes an angle θ with the horizontal. The coefficient of kinetic friction between the block and the inclined surface is μ.

If the person is applying a force with constant magnitude P and in a direction parallel to the incline, show that the acceleration of the block in the direction of motion can be expressed as:

$$a_x = \frac{P}{m} - g(\mu\cos\theta + \sin\theta)$$

Problem 8.4 As shown in Fig. 8.20, consider a block moving down an incline as the result of externally applied force F parallel to the incline. The incline makes an angle α with the horizontal. If the coefficient of kinetic friction between the block and the incline is μ, determine an expression for the acceleration a_x of the block in the direction of motion.

Answer: $a_x = \frac{F}{m} + g(\sin\alpha - \mu\cos\alpha)$

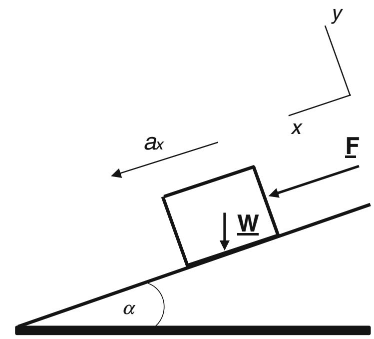

Fig. 8.20 *Problem 8.4*

Problem 8.5 As shown in Fig. 8.7, consider a block that is being pulled on a horizontal surface from position (1) to position (2). The magnitude of force applied on the block is $F = 85$ N and it makes an angle θ with the horizontal. If the work done on the block is $W = 1642$ J, determine the displacement S of the block in the direction of motion. Assume that the friction between the block and the surface is negligible.

Answer: $S = 20$ m

Problem 8.6 As shown in Fig. 8.21, consider a 15 kg block being pushed up the rough incline by a constant force of $P = 160$ N applied parallel to the horizontal. The incline makes an angle $\theta = 25°$ with the horizontal and the coefficient of friction between the block and the incline is $\mu = 0.35$. If the block is moved up the incline by $l = 9$ m, determine the work done on the block by,

(a) The externally applied force W_P
(b) The force of gravity W_g
(c) The frictional force W_f

Determine the net work W done on the block.

Answers: (a) $W_P = 1305$ J; (b) $W_g = 559$ J; (c) $W_f = 70.3$ J; $W = 675.7$ J

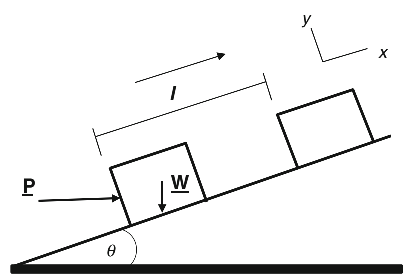

Fig. 8.21 *Problem 8.6*

Problem 8.7 Consider that a force with magnitude F_x that varies with displacement along the x direction is applied on an object. Assume that the variation of the force is as shown in Fig. 8.22 where force is measured in Newtons and displacement is measured in meters.

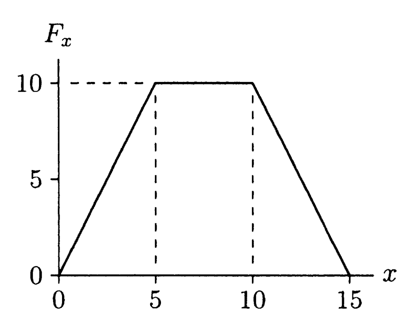

Fig. 8.22 *Problem 8.7*

Determine the work done by F_x on the object as the object moved from $x = 0$ to $x = 15$ m.

Answer: 100 J

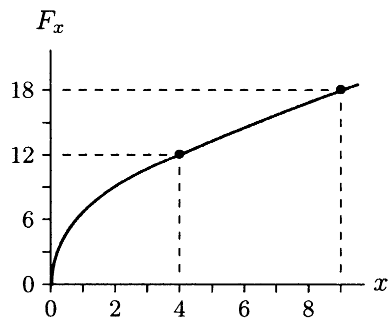

Fig. 8.23 *Problem 8.8*

Problem 8.8 A force with varying magnitude F_x is applied on an object and the displacement of the object is recorded in terms of x. The applied force is then plotted as a function of displacement and the curve shown in Fig. 8.23 is obtained. It is observed that between $x = 0$ and $x = 9$ m, the force is proportional to the square root of displacement:

$$F_x = c\sqrt{x}$$

Here, F_x is measured in Newtons and x in meters, and the constant of proportionality between F_x and x is estimated to be $c = 6$.

Determine the work done by F_x on the object to move the object from:

(a) $x = 0$ to $x = 4$ m
(b) $x = 0$ to $x = 9$ m
(c) $x = 4$ to $x = 9$ m

Answers: (a) 32 J, (b) 108 J, and (c) 76 J

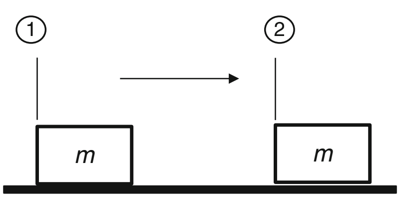

Fig. 8.24 *Problem 8.9*

Problem 8.9 As shown in Fig. 8.24, consider a 30 kg block that initially rested at position (1). Over time, the object has moved from position (1) to position (2). It is estimated that the speed of the object at position (2) was $V_2 = 1.5$ m/s. Neglecting the friction between the block and the ground, determine the work done W_{12} on the block to complete the move.

Answer: $W_{12} = 33.8$ J

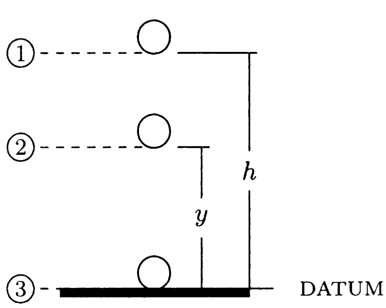

Fig. 8.25 *Problem 8.10*

Problem 8.10 As illustrated in Fig. 8.25, a ball is dropped from a height h measured from ground level. If the air resistance is neglected, show that the speed of the ball as a function of height y measured from ground level can be expressed as:

$$v = \sqrt{2g(h - y)}$$

Here, g is the magnitude of the gravitational acceleration.

Problem 8.11 Consider the 12 kg object located at position (1) in Fig. 8.26, which is at height h measured relative to position (3) at ground level. If the gravitational potential energy of the object at position (1) is $EP_1 = 588\ J$, determine:

(a) The vertical distance h between positions (1) and (3).
(b) The potential energy EP_2 of the object at position (2), located halfway between positions (1) and (3).

Answers: (a) $h = 5$ m; (b) $EP_2 = 294$ J

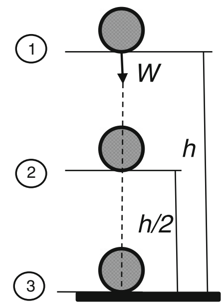

Fig. 8.26 *Problem 8.11*

Problem 8.12 The ski jumper in Fig. 8.27 is moving down a track to acquire sufficient speed to accomplish the jumping task. The length of the track is l, the track makes an angle θ with the horizontal, and the coefficient of friction between the track and the skis is μ.

If the ski jumper starts at the top of the track with zero initial speed, determine expressions for:

(a) The takeoff speed v_2 of the ski jumper at the bottom of the track using the work–energy theorem
(b) The acceleration a_x of the ski jumper using the equation of motion

Assume that effects of air resistance are negligible.

Answers:

(a) $v_2 = \sqrt{2lg(\sin\theta - \mu\cos\theta}$
(b) $a_x = g(\sin\theta - \mu\cos\theta)$

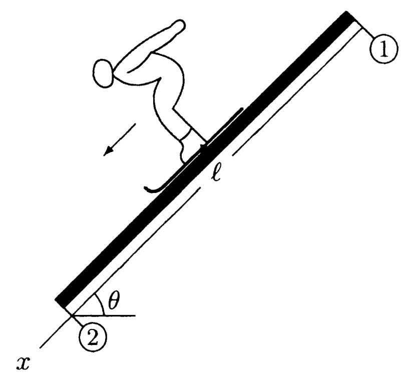

Fig. 8.27 *Problem 8.12*

Problem 8.13 As shown in Fig. 8.28, consider a 9 kg object initially rested at position (1), which is measured at distance h above the ground. The object falls with a constant gravitational acceleration of $g = 9.8\ \text{m/s}^2$ and after $t_2 = 2.5$ s hits the ground at position (2). If the air resistance is negligible, determine:

(a) The speed V_2 of the object at position (2).
(b) The vertical distance h between positions (1) and (2).
(c) The potential energy EP_1 of the object at position (1).
(d) The kinetic energy EK_2 of the object at position (2).

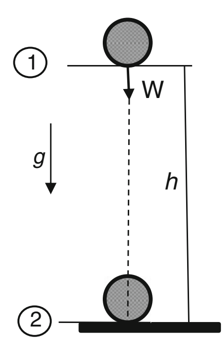

Fig. 8.28 *Problems 8.13 and 8.14*

Answers: (a) $V_2 = 24.5$ m/s; (b) $h = 30.6$ m; (c) $EP_1 = 2701$ J; (d) $EK_2 = 2701$ J

Problem 8.14 The same object in Fig. 8.28 is being dropped from a height of $h_1 = 6$ m measured above the ground. As the air resistance is negligible, calculate the speed V_2 of the object at the point of impact by using the principle of conservation of mechanical energy.

Answer: $V_2 = 10.8$ m/s