



## Chapter 13

# Stress and Strain

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### 13.1 Basic Loading Configurations

An object subjected to an external force will move in the direction of the applied force. The object will deform if its motion is constrained in the direction of the applied force. Deformation implies relative displacement of any two points within the object. The extent of deformation will be dependent upon many factors including the magnitude, direction, and duration of the applied force, the material properties of the object, the geometry of the object, and environmental factors such as heat and humidity.

In general, materials respond differently to different loading configurations. For a given material, there may be different mechanical properties that must be considered while analyzing its response to, for example, tensile loading as compared to loading that may cause bending or torsion. Figure 13.1 is drawn to illustrate different loading conditions, in which an L-shaped beam is subjected to forces  $\underline{F}_1$ ,  $\underline{F}_2$ , and  $\underline{F}_3$ . The force  $\underline{F}_1$  subjects the arm AB of the beam to tensile loading. The force  $\underline{F}_2$  tends to bend the arm AB. The force  $\underline{F}_3$  has a bending effect on arm BC and a twisting (torsional) effect on arm AB. Furthermore, all of these forces are subjecting different sections of the beam to shear loading.

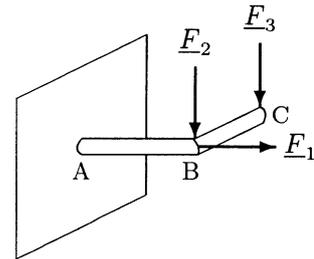


Fig. 13.1 Loading modes

### 13.2 Uniaxial Tension Test

The mechanical properties of materials are established by subjecting them to various experiments. The mechanical response of materials under tensile loading is analyzed by the *uniaxial* or *simple tension test* that will be discussed next. The response of materials to forces that cause bending and torsion will be reviewed in the following chapter.

The experimental setup for the uniaxial tension test is illustrated in Fig. 13.2. It consists of one fixed and one moving head with attachments to grip the test specimen. A specimen is placed and firmly fixed in the equipment, a tensile force of known magnitude is applied through the moving head, and the corresponding elongation of the specimen is measured. A general understanding of the response of the material to tensile loading is obtained by repeating this test for a number of specimens made of the same material, but with different lengths, cross-sectional areas, and under tensile forces with different magnitudes.

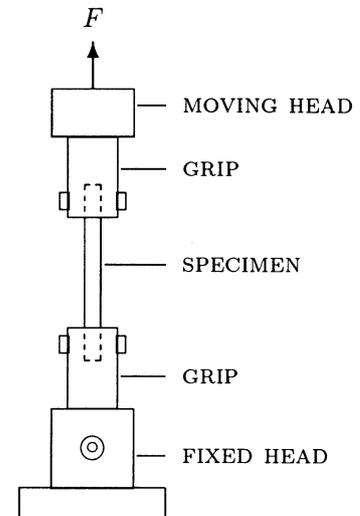


Fig. 13.2 Uniaxial tension test

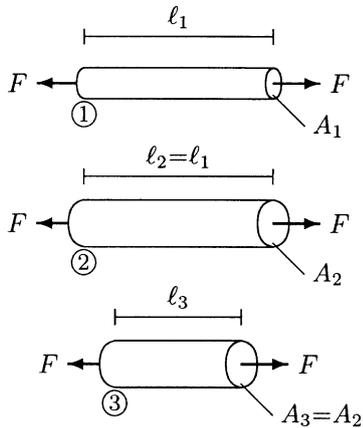


Fig. 13.3 Specimens

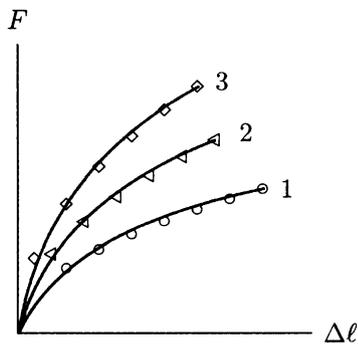


Fig. 13.4 Load-elongation diagrams

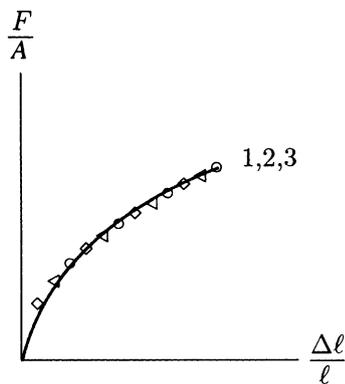


Fig. 13.5 Load over area versus load over length diagram

### 13.3 Load-Elongation Diagrams

Consider the three bars shown in Fig. 13.3. Assume that these bars are made of the same material. The first and the second bars have the same length but different cross-sectional areas, and the second and third bars have the same cross-sectional area but different lengths. Each of these bars can be subjected to a series of uniaxial tension tests by gradually increasing the applied forces and measuring corresponding increases in their lengths. If  $F$  is the magnitude of the applied force and  $\Delta l$  is the increase in length, then the data collected can be plotted to obtain a load versus elongation diagram for each specimen. Effects of geometric parameters (cross-sectional area and length) on the load-bearing ability of the material can be judged by drawing the curves obtained for each specimen on a single graph (Fig. 13.4) and comparing them. At a given force magnitude, the comparison of curves 1 and 2 indicates that the larger the cross-sectional area, the more difficult it is to deform the specimen in a simple tension test, and the comparison of curves 2 and 3 indicates that the longer the specimen, the larger the deformation in tension.

Note that instead of applying a series of tensile forces to a single specimen, it is preferable to have a number of specimens with almost identical geometries and apply one force to one specimen only once. As will be discussed later, a force applied on an object may alter its mechanical properties.

Another method of representing the results obtained in a uniaxial tension test is by first dividing the magnitude of the applied force  $F$  with the cross-sectional area  $A$  of the specimen, normalizing the amount of deformation by dividing the measured elongation with the original length of the specimen, and then plotting the data on a  $F/A$  versus  $\Delta l/l$  graph as shown in Fig. 13.5. The three curves in Fig. 13.4, representing three specimens made of the same material, are represented by a single curve in Fig. 13.5. It is obvious that some of the information provided in Fig. 13.4 is lost in Fig. 13.5. That is why the representation in Fig. 13.5 is more advantageous than that in Fig. 13.4. The single curve in Fig. 13.5 is unique for a particular material, independent of the geometries of the specimens used during the experiments. This type of representation eliminates geometry as one of the variables, and makes it possible to focus attention on the mechanical properties of different materials. For example, consider the curves in Fig. 13.6, representing the mechanical behavior of materials A and B in simple tension. It is clear that material B can be deformed more easily than material A in a uniaxial tension test, or material A is “stiffer” than material B.

### 13.4 Simple Stress

Consider the cantilever beam shown in Fig. 13.7a. The beam has a circular cross-section, cross-sectional area  $A$ , welded to the wall at one end, and is subjected to a tensile force with magnitude  $F$  at the other end. The bar does not move, so it is in equilibrium. To analyze the forces induced within the beam, the method of sections can be applied by hypothetically cutting the beam into two pieces through a plane ABCD perpendicular to the centerline of the beam. Since the beam as a whole is in equilibrium, the two pieces must individually be in equilibrium as well. This requires the presence of an internal force collinear with the externally applied force at the cut section of each piece. To satisfy the condition of equilibrium, the internal forces must have the same magnitude as the external force (Fig. 13.7b). The internal force at the cut section represents the resultant of a force system distributed over the cross-sectional area of the beam (Fig. 13.7c). The intensity of the internal force over the cut section (force per unit area) is known as the *stress*. For the case shown in Fig. 13.7, since the force resultant at the cut section is perpendicular (normal) to the plane of the cut, the corresponding stress is called a *normal stress*. It is customary to use the symbol  $\sigma$  (sigma) to refer to normal stresses. The intensity of this distributed force may or may not be uniform (constant) throughout the cut section. Assuming that the intensity of the distributed force at the cut section is uniform over the cross-sectional area  $A$ , the normal stress can be calculated using:

$$\sigma = \frac{F}{A} \quad (13.1)$$

If the intensity of the stress distribution over the area is not uniform, then Eq. (13.1) will yield an *average normal stress*. It is customary to refer to normal stresses that are associated with tensile loading as *tensile stresses*. On the other hand, *compressive stresses* are those associated with compressive loading. It is also customary to treat tensile stresses as positive and compressive stresses as negative.

The other form of stress is called *shear stress*, which is a measure of the intensity of internal forces acting parallel or tangent to a plane of cut. To get a sense of shear stresses, hold a stack of paper with both hands such that one hand is under the stack while the other hand is above it. First, press the stack of papers together. Then, slowly slide one hand in the direction parallel to the surface of the papers while sliding the other hand in the opposite direction. This will slide individual papers relative to one another and generate frictional forces on the surfaces of individual papers. The shear stress is comparable to the intensity of the frictional force over the surface area upon which it is applied. Now, consider the cantilever beam

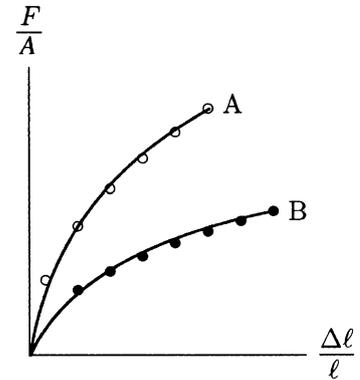


Fig. 13.6 Material A is stiffer than material B

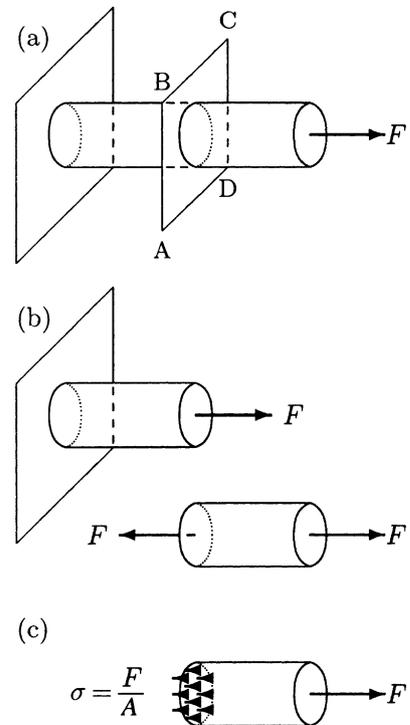


Fig. 13.7 Normal stress

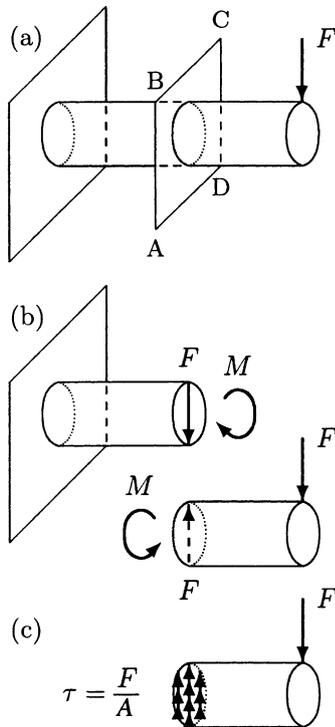


Fig. 13.8 Shear stress

Table 13.1 Units of stress

SYSTEM	UNITS OF STRESS	SPECIAL NAME
SI	N/m <sup>2</sup>	Pascal (Pa)
c-g-s	dyn/cm <sup>2</sup>	
British	lb/ft <sup>2</sup> or lb/in. <sup>2</sup>	psf or psi

illustrated in Fig. 13.8a. A downward force with magnitude  $F$  is applied to its free end. To analyze internal forces and moments, the method of sections can be applied by fictitiously cutting the beam through a plane ABCD that is perpendicular to the centerline of the beam. Since the beam as a whole is in equilibrium, the two pieces thus obtained must individually be at equilibrium as well. The free-body diagram of the right-hand piece of the beam is illustrated in Fig. 13.8b along with the internal force and moment on the left-hand piece. For the equilibrium of the right-hand piece, there has to be an upward force resultant and an internal moment at the cut surface. Again for the equilibrium of this piece, the internal force must have a magnitude  $F$ . This is known as the *internal shearing force* and is the resultant of a distributed load over the cut surface (Fig. 13.8c). The intensity of the shearing force over the cut surface is known as the *shear stress*, and is commonly denoted with the symbol  $\tau$  (tau). If the area of this surface (in this case, the cross-sectional area of the beam) is  $A$ , then:

$$\tau = \frac{F}{A} \tag{13.2}$$

The underlying assumption in Eq. (13.2) is that the shear stress is distributed uniformly over the area. For some cases this assumption may not be true. In such cases, the shear stress calculated by Eq. (13.2) will represent an average value.

The dimension of stress can be determined by dividing the dimension of force  $[F] = [M][L]/[T^2]$  with the dimension of area  $[L^2]$ . Therefore, stress has the dimension of  $[M]/[L][T^2]$ . The units of stress in different unit systems are listed in Table 13.1. Note that stress has the same dimension and units as pressure.

### 13.5 Simple Strain

Generally, strain is defined as the relative change in the shape or size of an object due to an externally applied force. In biomechanics, strain, which is also known as unit deformation, is a measure of the degree or intensity of deformation. Consider the bar in Fig. 13.9. Let A and B be two points on the bar located at a distance  $l_1$ , and C and D be two other points located at a distance  $l_2$  from one another, such that  $l_1 > l_2$ .  $l_1$  and  $l_2$  are called *gage lengths*. The bar will elongate when it is subjected to tensile loading. Let  $\Delta l_1$  be the amount of elongation measured between A and B, and  $\Delta l_2$  be the increase in length between C and D.  $\Delta l_1$  and  $\Delta l_2$  are certainly some measures of deformation. However, they depend on the respective gage lengths, such that  $\Delta l_1 > \Delta l_2$ . On the other hand, if the ratio of the amount of elongation to the gage length is calculated for each case and compared, it would

be observed that  $((\Delta l_1/\Delta l_1) \simeq (\Delta l_2/\Delta l_2))$ . Elongation per unit gage length is known as *strain* and is a more fundamental means of measuring deformation.

As in the case of stress, two types of strains can be distinguished. The *normal* or *axial strain* is associated with axial forces and defined as the ratio of the change (increase or decrease) in length,  $\Delta l$ , to the original gage length,  $l$ , and is denoted with the symbol  $\epsilon$  (epsilon):

$$\epsilon = \frac{\Delta l}{l} \tag{13.3}$$

When a body is subjected to tension, its length increases, and both  $\Delta l$  and  $\epsilon$  are positive. The length of a specimen under compression decreases, and both  $\Delta l$  and  $\epsilon$  become negative.

The second form of strain is related to distortions caused by shearing forces. Consider the rectangle (ABCD) shown in Fig. 13.10, which is acted upon by a pair of shearing forces. Shear forces deform the rectangle into a parallelogram (A'B'C'D). Here shear can be defined as the change in an angle between two initially perpendicular lines AB and AD. If the relative horizontal displacement of the top of the rectangle is  $d$  and the height of the rectangle is  $l$ , then the *average shear strain* is defined as the ratio of  $d$  and  $l$  which is equal to the tangent of angle  $\gamma$  (gamma). This angle is usually very small. For small angles, the tangent of the angle is approximately equal to the angle itself. Hence, the average shear strain is equal to angle  $\gamma$  (measured in radians), which can be calculated using:

$$\gamma \cong \tan(\gamma) \cong \frac{d}{l} \tag{13.4}$$

Strains are calculated by dividing two quantities having the dimension of length. Therefore, they are dimensionless quantities and there is no unit associated with them. For most applications, the deformations and consequently the strains involved are very small, and the precision of the measurements taken is very important. To indicate the type of measurements taken, it is not unusual to attach units such as cm/cm or mm/mm next to a strain value. Strains can also be given in percent. In engineering applications, the strains involved are of the order of magnitude 0.1% or 0.001.

Figure 13.11 is drawn to compare the effects of tensile, compressive, and shear loading. Figure 13.11a shows a square object (in a two-dimensional sense) under no load. The square object is ruled into 16 smaller squares to illustrate different modes of deformation. In Fig. 13.11b, the object is subjected to a pair of tensile forces. Tensile forces distort squares into rectangles such that the dimension of each square in the direction of applied force (axial dimension) increases while its dimension perpendicular to the direction of the applied force (transverse dimension) decreases. In Fig. 13.11c, the object is subjected to a pair of

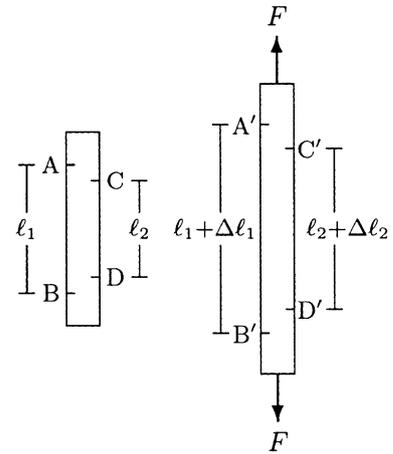


Fig. 13.9 Normal strain

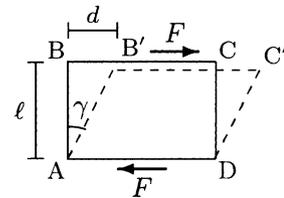


Fig. 13.10 Shear strain

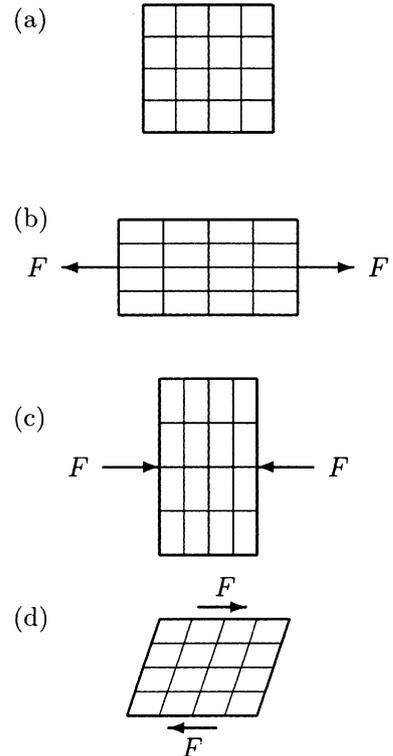


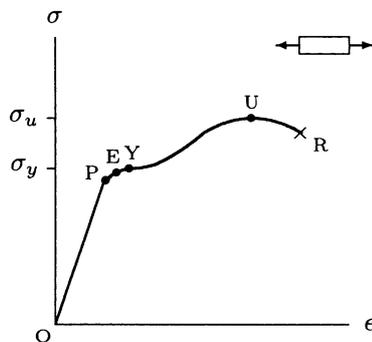
Fig. 13.11 Distortions of a square object (a) under tensile (b), compressive (c), and shear (d) loading

compressive forces that distort squares into rectangles such that the axial dimension of each square decreases while its transverse dimension increases. In Fig. 13.11d, the object is subjected to a pair of shear forces that distort the squares into diamonds.

### 13.6 Stress–Strain Diagrams

It was demonstrated in Sect. 13.3 that the results of uniaxial tension tests can be used to obtain a unique curve representing the relationship between the applied load and corresponding deformation for a material. This can be achieved by dividing the applied load with the cross-sectional area ( $F/A$ ) of the specimen, dividing the amount of elongation measured with the gage length ( $\Delta l/l$ ), and plotting a  $F/A$  versus  $\Delta l/l$  graph. Notice however that for a specimen under tension,  $F/A$  is the average tensile stress  $\sigma$  and ( $\Delta l/l$ ) is the average tensile strain  $\epsilon$ . Therefore, the  $F/A$  versus ( $\Delta l/l$ ) graph of a material is essentially the *stress–strain diagram* of that material.

Different materials demonstrate different stress–strain relationships, and the stress–strain diagrams of two or more materials can be compared to determine which material is relatively stiffer, harder, tougher, more ductile, and/or more brittle. Before explaining these concepts related to the strength of materials, it is appropriate to first analyze a typical stress–strain diagram in detail.



**Fig. 13.12** Stress–strain diagram for axial loading

Consider the stress–strain diagram shown in Fig. 13.12. There are six distinct points on the curve that are labeled as O, P, E, Y, U, and R. Point O is the origin of the  $\sigma - \epsilon$  diagram, which corresponds to the initial no load, no deformation stage. Point P represents the *proportionality limit*. Between O and P, stress and strain are linearly proportional, and the  $\sigma - \epsilon$  curve is a straight line. Point E represents the *elastic limit*. The stress corresponding to the elastic limit is the greatest stress that can be applied to the material without causing any permanent deformation within the material. The material will not resume its original size and shape upon unloading if it is subjected to stress levels beyond the elastic limit. Point Y is the *yield point*, and the stress  $\sigma_y$  corresponding to the yield point is called the *yield strength* of the material. At this stress level, considerable elongation (yielding) can occur without a corresponding increase of load. U is the highest stress point on the  $\sigma - \epsilon$  curve. The stress  $\sigma_u$  is the *ultimate strength* of the material. For some materials, once the ultimate strength is reached, the applied load can be decreased and continued yielding may be observed. This is due to the phenomena called *necking* that will be discussed later. The last point on the  $\sigma - \epsilon$  curve is R, which represents the *rupture* or *failure point*. The stress at which the rupture occurs is called the *rupture strength* of the material.

For some materials, it may not be easy to determine or distinguish the elastic limit and the yield point. The yield strength of such materials is determined by the *offset method*, illustrated in Fig. 13.13. The offset method is applied by drawing a line parallel to the linear section of the stress–strain diagram and passing through a strain level of about 0.2% (0.002). The intersection of this line with the  $\sigma - \epsilon$  curve is taken to be the yield point, and the stress corresponding to this point is called the *apparent yield strength* of the material.

Note that a given material may behave differently under different load and environmental conditions. If the curve shown in Fig. 13.12 represents the stress–strain relationship for a material under tensile loading, there may be a similar but different curve representing the stress–strain relationship for the same material under compressive or shear loading. Also, temperature is known to alter the relationship between stress and strain. For a given material and fixed mode of loading, different stress–strain diagrams may be obtained under different temperatures. Furthermore, the data collected in a particular tension test may depend on the rate at which the tension is applied on the specimen. Some of these factors affecting the relationship between stress and strain will be discussed later.

### 13.7 Elastic Deformations

Consider the partial stress–strain diagram shown in Fig. 13.14. Y is the yield point, and in this case, it also represents the proportionality and elastic limits.  $\sigma_y$  is the yield strength and  $\epsilon_y$  is the corresponding strain. (The  $\sigma - \epsilon$  curve beyond the elastic limit is not shown.) The straight line in Fig. 13.14 represents the stress–strain relationship in the elastic region. *Elasticity* is defined as the ability of a material to resume its original (stress free) size and shape upon removal of applied loads. In other words, if a load is applied on a material such that the stress generated in the material is equal to or less than  $\sigma_y$ , then the deformations that took place in the material will be completely recovered once the applied loads are removed (the material is unloaded).

An elastic material whose  $\sigma - \epsilon$  diagram is a straight line is called a *linearly elastic* material. For such a material, the stress is linearly proportional to strain, and the constant of proportionality is called the *elastic* or *Young's modulus* of the material. Denoting the elastic modulus with  $E$ :

$$\sigma = E\epsilon \quad (13.5)$$

The elastic modulus,  $E$ , is equal to the slope of the  $\sigma - \epsilon$  diagram in the elastic region, which is constant for a linearly elastic material.  $E$  represents the *stiffness* of a material, such that the higher the elastic modulus, the stiffer the material.

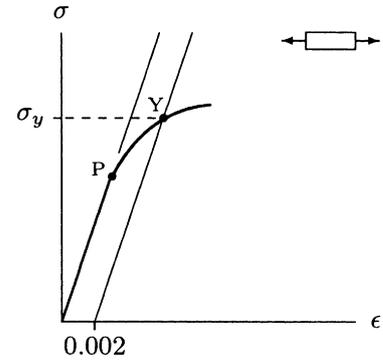


Fig. 13.13 Offset method

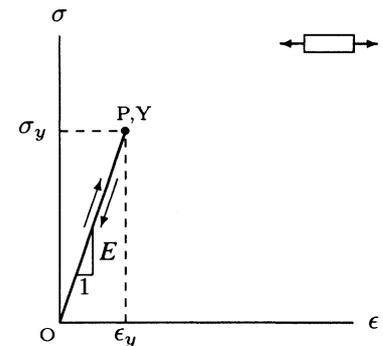


Fig. 13.14 Stress–strain diagram for a linearly elastic material ( $\nearrow$ : loading;  $\nwarrow$ : unloading)

The distinguishing factor in linearly elastic materials is their elastic moduli. That is, different linearly elastic materials have different elastic moduli. If the elastic modulus of a material is known, then the mathematical definitions of stress and strain ( $\sigma = F/A$  and  $\epsilon = \Delta l/l$ ) can be substituted into Eq. (13.5) to derive a relationship between the applied load and corresponding deformation:

$$\Delta l = \frac{Fl}{EA} \tag{13.6}$$

In Eq. (13.6),  $F$  is the magnitude of the tensile or compressive force applied to the material,  $E$  is the elastic modulus of the material,  $A$  is the area of the surface that cuts the line of action of the applied force at right angles,  $l$  is the length of the material measured along the line of action of the applied force, and  $\Delta l$  is the amount of elongation or shortening in  $l$  due to the applied force. For a given linearly elastic material (or any material for which the deformations are within the linearly elastic region of the  $\sigma - \epsilon$  diagram) and applied load, Eq. (13.6) can be used to calculate the corresponding deformation. This equation can be used when the object is under a tensile or compressive force.

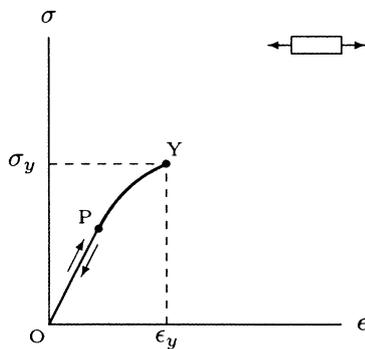


Fig. 13.15 Stress–strain diagram for a nonlinearly elastic material

Not all elastic materials demonstrate linear behavior. As illustrated in Fig. 13.15, the stress–strain diagram of a material in the elastic region may be a straight line up to the proportionality limit followed by a curve. A curve implies varying slope and nonlinear behavior. Materials for which the  $\sigma - \epsilon$  curve in the elastic region is not a straight line are known as *nonlinear elastic materials*. For a nonlinear elastic material, there is not a single elastic modulus because the slope of the  $\sigma - \epsilon$  curve is not constant throughout the elastic region. Therefore, the stress–strain relationships for nonlinear materials take more complex forms. Note however that even nonlinear materials may have a linear elastic region in their  $\sigma - \epsilon$  diagrams at low stress levels (the region between points O and P in Fig. 13.15).

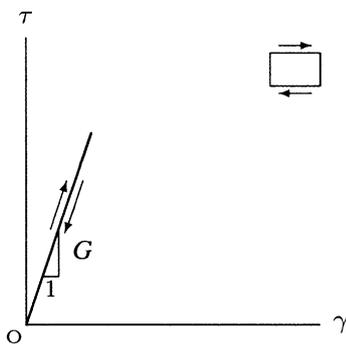


Fig. 13.16 Shear stress versus shear strain diagram for a linearly elastic material

Some materials may exhibit linearly elastic behavior when they are subjected to shear loading (Fig. 13.16). For such materials, the shear stress,  $\tau$ , is linearly proportional to the shear strain,  $\gamma$ , and the constant of proportionality is called the *shear modulus* or the *modulus of rigidity*, which is commonly denoted with the symbol  $G$ :

$$\tau = G\gamma \tag{13.7}$$

The shear modulus of a given linear material is equal to the slope of the  $\tau - \gamma$  curve in the elastic region. The higher the shear modulus, the more rigid the material.

Note that Eqs. (13.5) and (13.6) relate stresses to strains for linearly elastic materials, and are called *material functions*. Obviously, for a

given material, there may exist different material functions for different modes of deformation. There are also *constitutive equations* that incorporate all material functions.

### 13.8 Hooke's Law

The load-bearing characteristics of elastic materials are similar to those of springs, which was first noted by Robert Hooke. Like springs, elastic materials have the ability to store potential energy when they are subjected to externally applied loads. During unloading, it is the release of this energy that causes the material to resume its undeformed configuration. A linear spring subjected to a tensile load will elongate, the amount of elongation being linearly proportional to the applied load (Fig. 13.17). The constant of proportionality between the load and the deformation is usually denoted with the symbol  $k$ , which is called the *spring constant* or *stiffness* of the spring. For a linear spring with a spring constant  $k$ , the relationship between the applied load  $F$  and the amount of elongation  $d$  is:

$$F = kd \quad (13.8)$$

By comparing Eqs. (13.5) and (13.8), it can be observed that stress in an elastic material is analogous to the force applied to a spring, strain in an elastic material is analogous to the amount of deformation of a spring, and the elastic modulus of an elastic material is analogous to the spring constant of a spring. This analogy between elastic materials and springs is known as *Hooke's Law*.

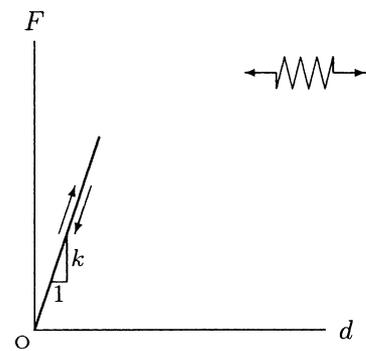


Fig. 13.17 Load-elongation diagram for a linear spring

### 13.9 Plastic Deformations

We have defined elasticity as the ability of a material to regain completely its original dimensions upon removal of the applied forces. Elastic behavior implies the absence of permanent deformation. On the other hand, *plasticity* implies permanent deformation. In general, materials undergo plastic deformations following elastic deformations when they are loaded beyond their elastic limits or yield points.

Consider the stress-strain diagram of a material shown in Fig. 13.18. Assume that a specimen made of the same material is subjected to a tensile load and the stress,  $\sigma$ , in the specimen is brought to such a level that  $\sigma > \sigma_y$ . The corresponding strain in the specimen is measured as  $\epsilon$ . Upon removal of the applied load, the material will recover the plastic deformation that had taken place by following an unloading path parallel to the initial linearly elastic region (straight line between points O and P). The point where this path cuts the strain axis is called the plastic

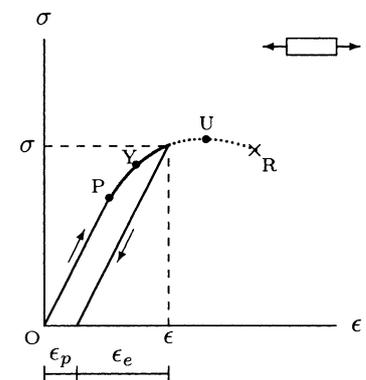


Fig. 13.18 Plastic deformation

strain,  $\epsilon_p$ , that signifies the extent of permanent (unrecoverable) shape change that has taken place in the specimen.

The difference in strains between when the specimen is loaded and unloaded ( $\epsilon - \epsilon_p$ ) is equal to the amount of elastic strain,  $\epsilon_e$ , that had taken place in the specimen and that was recovered upon unloading. Therefore, for a material loaded to a stress level beyond its elastic limit, the total strain is equal to the sum of the elastic and plastic strains:

$$\epsilon = \epsilon_e + \epsilon_p \tag{13.9}$$

The elastic strain,  $\epsilon_e$ , is completely recoverable upon unloading, whereas the plastic strain,  $\epsilon_p$ , is a permanent residue of the deformations.

### 13.10 Necking

As defined in Sect. 13.6, the largest stress a material can endure is called the ultimate strength of that material. Once a material is subjected to a stress level equal to its ultimate strength, an increased rate of deformation can be observed, and in most cases, continued yielding can occur even by reducing the applied load. The material will eventually fail to hold any load, and rupture. The stress at failure is called the rupture strength of the material, which may be lower than its ultimate strength. Although this may seem to be unrealistic, the reason is due to a phenomenon called *necking* and because of the manner in which stresses are calculated.

Stresses are usually calculated on the basis of the original cross-sectional area of the material. Such stresses are called *conventional stresses*. Calculating a stress by dividing the applied force with the original cross-sectional area is convenient but not necessarily accurate. The *true* or *actual stress* calculations must be made by taking the cross-sectional area of the deformed material into consideration. As illustrated in Fig. 13.19, under a tensile load a material may elongate in the direction of the applied load but contract in the transverse directions. At stress levels close to the breaking point, the elongation may occur very rapidly and the material may narrow simultaneously. The cross-sectional area at the narrowed section decreases, and although the force required to further deform the material may decrease, the force per unit area (stress) may increase. As illustrated with the dotted curve in Fig. 13.20, the actual stress–strain curve may continue having a positive slope, which indicates increasing strain with increasing stress rather than a negative slope, which implies increasing strain with decreasing stress. Also the rupture point and the point corresponding to the ultimate strength of the material may be the same.

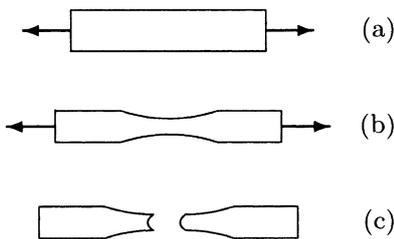


Fig. 13.19 Necking

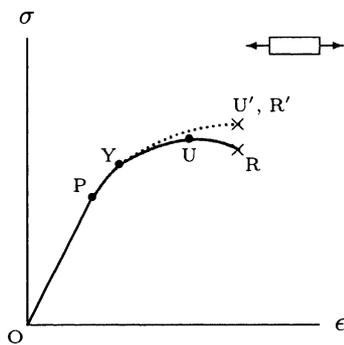
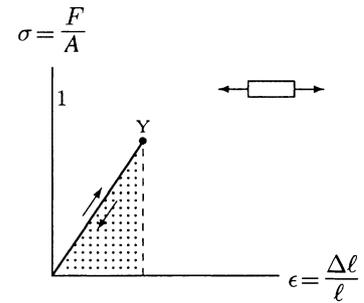


Fig. 13.20 Conventional (solid curve) and actual (dotted curve) stress–strain diagrams

### 13.11 Work and Strain Energy

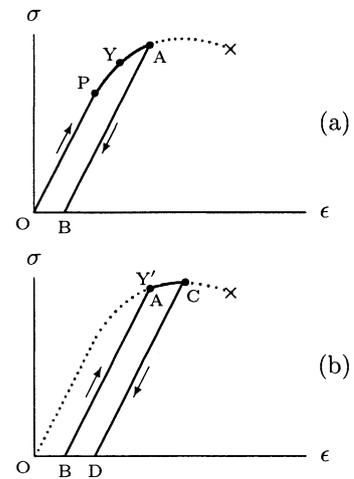
In dynamics, *work done* is defined as the product of force and the distance traveled in the direction of applied force, and *energy* is the capacity of a system to do work. Stress and strain in deformable body mechanics are respectively related to force and displacement. Stress multiplied by area is equal to force, and strain multiplied by length is displacement. Therefore, the product of stress and strain is equal to the work done on a body per unit volume of that body, or the *internal work* done on the body by the externally applied forces. For an elastic body, this work is stored as an internal *elastic strain energy*, and it is the release of this energy that brings the body back to its original shape upon unloading. The maximum elastic strain energy (per unit volume) that can be stored in a body is equal to the total area under the  $\sigma - \epsilon$  diagram in the elastic region (Fig. 13.21). There is also a *plastic strain energy* that is dissipated as heat while deforming the body.



**Fig. 13.21** Internal work done and elastic strain energy per unit volume

### 13.12 Strain Hardening

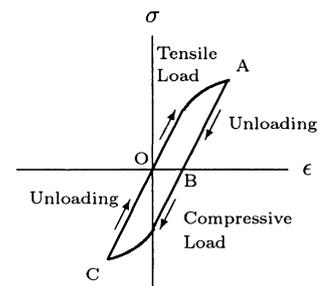
Figure 13.22 represents the  $\sigma - \epsilon$  diagram of a material. Assume that the material is subjected to a tensile force such that the stress generated is beyond the elastic limit (yield point) of the material. The stress level in the material is indicated with point A on the  $\sigma - \epsilon$  diagram. Upon removal of the applied force, the material will follow the path AB which is almost parallel to the initial, linear section OP of the  $\sigma - \epsilon$  diagram. The strain at B corresponds to the amount of plastic deformation in the material. If the material is reloaded, it will exhibit elastic behavior between B and A, the stress at A being the new yield strength of the material. This technique of changing the yield point of a material is called *strain hardening*. Since the stress at A is greater than the original yield strength of the material, strain hardening increases the yield strength of the material. Upon reloading, if the material is stressed beyond A, then the material will deform according to the original  $\sigma - \epsilon$  curve for the material.



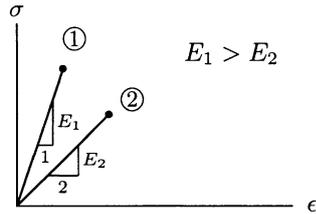
**Fig. 13.22** Strain hardening

### 13.13 Hysteresis Loop

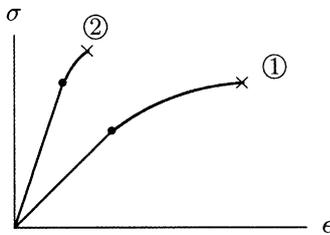
Consider the  $\sigma - \epsilon$  diagram shown in Fig. 13.23. Between points O and A, a tensile force is applied on the material and the material is deformed beyond its elastic limit. At A, the tensile force is removed, and the line AB represents the unloading path. At B the material is reloaded, this time with a compressive force. At C, the compressive force applied on the material is removed. Between C and O, a second unloading occurs, and



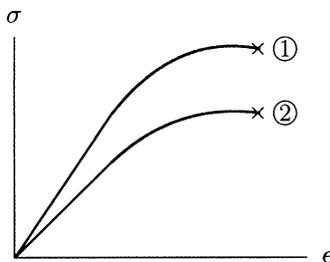
**Fig. 13.23** Hysteresis loop



**Fig. 13.24** Material 1 is stiffer than material 2



**Fig. 13.25** Material 1 is more ductile and less brittle than material 2



**Fig. 13.26** Material 1 is tougher than material 2

finally the material resumes its original shape. The loop OABCO is called the *hysteresis loop*, and the area enclosed by this loop is equal to the total strain energy dissipated as heat to deform the body in tension and compression.

### 13.14 Properties Based on Stress–Strain Diagrams

As defined earlier, the elastic modulus of a material is equal to the slope of its stress–strain diagram in the elastic region. The elastic modulus is a relative measure of the *stiffness* of one material with respect to another. The higher the elastic modulus, the stiffer the material and the higher the resistance to deformation. For example, material 1 in Fig. 13.24 is stiffer than material 2.

A *ductile* material is one that exhibits a large plastic deformation prior to failure. For example, material 1 in Fig. 13.25 is more ductile than material 2. A *brittle* material, on the other hand, shows a sudden failure (rupture) without undergoing a considerable plastic deformation. Glass is a typical example of a brittle material.

*Toughness* is a measure of the capacity of a material to sustain permanent deformation. The toughness of a material is measured by considering the total area under its stress–strain diagram. The larger this area, the tougher the material. For example, material 1 in Fig. 13.26 is tougher than material 2.

The ability of a material to store or absorb energy without permanent deformation is called the *resilience* of the material. The resilience of a material is measured by its *modulus of resilience* which is equal to the area under the stress–strain curve in the elastic region. The modulus of resilience is equal to  $\sigma_y \epsilon_y / 2$  or  $\sigma_y^2 / 2E$  for linearly elastic materials.

Although they are not directly related to the stress–strain diagrams, there are other important concepts used to describe material properties. A material is called *homogeneous* if its properties do not vary from location to location within the material. A material is called *isotropic* if its properties are independent of direction or orientation. A material is called *incompressible* if it has a constant density.

### 13.15 Idealized Models of Material Behavior

Stress–strain diagrams are most useful when they are represented by mathematical functions. The stress–strain diagrams of materials may come in various forms, and it may not be possible to find a single mathematical function to

represent them. For the sake of mathematical modeling and the analytical treatment of material behavior, these diagrams can be simplified. Some of these diagrams representing certain idealized material behavior are illustrated in Fig. 13.27.

A *rigid* material is one that cannot be deformed even under very large loads (Fig. 13.27a). A *linearly elastic* material is one for which the stress and strain are linearly proportional, with the modulus of elasticity being the constant of proportionality (Fig. 13.27b). A *rigid-perfectly plastic* material does not exhibit any elastic behavior, and once a critical stress level is reached, it will deform continuously and permanently until failure (Fig. 13.27c). After a linearly elastic response, a *linearly elastic-perfectly plastic* material is one that deforms continuously at a constant stress level (Fig. 13.27d). Figure 13.27e represents the stress–strain diagram for *rigid-linearly plastic* behavior. The stress–strain diagram of a *linearly elastic-linearly plastic* material has two distinct regions with two different slopes (bilinear), in which stresses and strains are linearly proportional (Fig. 13.27f).

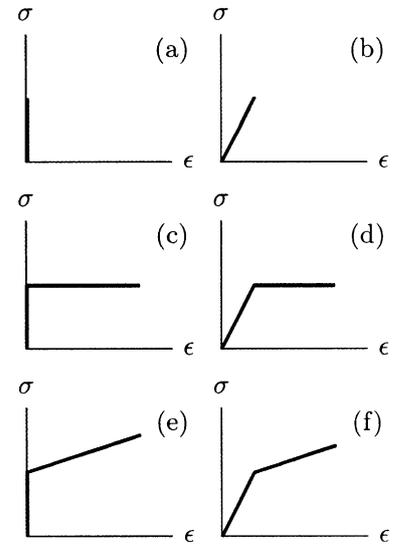


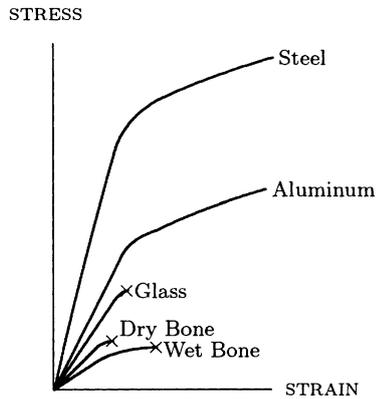
Fig. 13.27 Idealized models of material behavior

### 13.16 Mechanical Properties of Materials

Table 13.2 lists properties of selected materials in terms of their tensile yield strengths ( $\sigma_y$ ), tensile ultimate strengths ( $\sigma_u$ ), elastic moduli ( $E$ ), shear moduli ( $G$ ), and Poisson’s ratios ( $\nu$ ). The significance of the Poisson’s ratio will be discussed in the following chapter. Note that mechanical properties of a material can vary depending on many factors including its content (for example, in the case of steel, its carbon content) and the

Table 13.2 Average mechanical properties of selected materials

MATERIAL	YIELD STRENGTH $\sigma_y$ (MPa)	ULTIMATE STRENGTH $\sigma_u$ (MPa)	ELASTIC MODULUS $E$ (MPa)	SHEAR MODULUS GPa	POISSON’S RATIO $\nu$
Muscle	–	0.2	–	–	0.49
Tendon	–	70	0.4	–	0.40
Skin	–	8	0.5	–	0.49
Cortical bone	80	130	17	3.3	0.40
Glass	35–70	–	70–80	–	–
Cast iron	40–260	140–380	100–190	42–90	0.29
Aluminum	60–220	90–390	70	28	0.33
Steel	200–700	400–850	200	80	0.30
Titanium	400–800	500–900	100	45	0.34

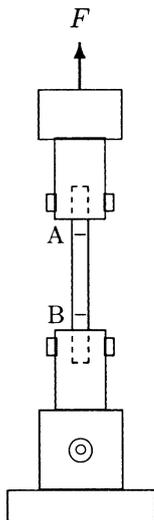


**Fig. 13.28** Gross comparison of stress–strain diagrams of selected materials

processes used to manufacture the material (for example, hard rolled, strain hardened, or heat treated). As will be discussed in later chapters, biological materials exhibit time-dependent properties. That is, their response to external forces depends on the rate at which the forces are applied. Bone, for example, is an anisotropic material. Its response to tensile loading in different directions is different, and different elastic moduli are established to account for its response in different directions. Therefore, the values listed in Table 13.2 are some averages and ranges, and are aimed to provide a sense of the orders of magnitude of numbers involved (Fig. 13.28).

### 13.17 Example Problems

The following examples will demonstrate some of the uses of the concepts introduced in this chapter.



**Fig. 13.29** Example 13.1

**Example 13.1** A circular cylindrical rod with radius  $r = 1.26\text{ cm}$  is tested in a uniaxial tension test (Fig. 13.29). Before applying a tensile force of  $F = 1000\text{ N}$ , two points A and B that are at a distance  $l_0 = 30\text{ cm}$  (gauge length) are marked on the rod. After the force is applied, the distance between A and B is measured as  $l_1 = 31.5\text{ cm}$ .

Determine the tensile strain and average tensile stress generated in the rod.

**Solution:** By definition, the tensile strain is equal to the ratio of the amount of elongation to the original length. The amount of elongation,  $\Delta l$ , is the difference between the gage lengths before and after deformation:

$$\Delta l = l_1 - l_0 = 31.5 - 30.0 = 1.5\text{ cm}$$

Therefore, the tensile strain  $\epsilon$  generated in the rod is:

$$\epsilon = \frac{\Delta l}{l_0} = \frac{1.5}{30} = 0.05\text{ cm/cm}$$

The bar has a circular cross-section with radius  $r = 1.26\text{ cm}$  or  $r = 0.0126\text{ m}$ . The cross-sectional area  $A$  of the bar is:

$$A = \pi r^2 = (3.1416)(0.0126)^2 = 5 \times 10^{-4}\text{ m}^2$$

The average tensile stress is equal to the applied force per unit area of the surface that cuts the line of action of the force at right angles. In this case, it is the cross-sectional area of the rod. Therefore:

$$\sigma = \frac{F}{A} = \frac{1000}{0.0005} = 2,000,000 = 2 \times 10^6\text{ Pa} = 2\text{ MPa}$$

**Example 13.2** Two specimens made of two different materials are tested in a uniaxial tension test by applying a force of  $F = 20\text{ kN}$  ( $20 \times 10^3\text{ N}$ ) on each specimen (Fig. 13.30). Specimen 1 is an aluminum bar with elastic modulus  $E_1 = 70\text{ GPa}$  ( $70 \times 10^9\text{ Pa}$ ) and a rectangular cross-section ( $1\text{ cm} \times 2\text{ cm}$ ). Specimen 2 is a steel rod with elastic modulus  $E_2 = 200\text{ GPa}$  ( $200 \times 10^9\text{ Pa}$ ) and a circular cross-section (radius  $1\text{ cm}$ ).

Calculate tensile stresses developed in each specimen. Assuming that the tensile stress in each specimen is below the proportionality limit of the material, calculate the tensile strain for each specimen. Also, if the original length of each specimen was  $30\text{ cm}$ , what are their lengths after deformation?

**Solution:** The tensile stress is equal to the ratio of the applied force and the cross-sectional area of the specimen. The cross-sectional areas of the specimens are:

$$A_1 = (1\text{ cm})(2\text{ cm}) = 2\text{ cm}^2 = 2 \times 10^{-4}\text{ m}^2$$

$$A_2 = \pi(1\text{ cm})^2 = 3.14\text{ cm}^2 = 3.14 \times 10^{-4}\text{ m}^2$$

Therefore, the tensile stresses developed in each specimen are:

$$\sigma_1 = \frac{F}{A_1} = \frac{20 \times 10^3}{2 \times 10^{-4}} = 100 \times 10^6\text{ Pa} = 100\text{ MPa}$$

$$\sigma_2 = \frac{F}{A_2} = \frac{20 \times 10^3}{3.14 \times 10^{-4}} = 63.7 \times 10^6\text{ Pa} = 63.7\text{ MPa}$$

That is, the aluminum bar (specimen 1) is stressed more than the steel rod (specimen 2).

To calculate the tensile strains corresponding to tensile stresses  $\sigma_1$  and  $\sigma_2$ , we can assume that the deformations are elastic and that the stresses  $\sigma_1$  and  $\sigma_2$  are below the proportionality limits for aluminum and steel. In other words, the stresses are linearly proportional to strains and the elastic moduli  $E_1$  and  $E_2$  are the constants of proportionality. Hence:

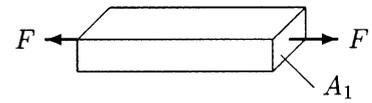
$$\epsilon_1 = \frac{\sigma_1}{E_1} = \frac{100 \times 10^6}{70 \times 10^9} = 1.43 \times 10^{-3}$$

$$\epsilon_2 = \frac{\sigma_2}{E_2} = \frac{63.7 \times 10^6}{200 \times 10^9} = 0.32 \times 10^{-3}$$

These results suggest that the aluminum bar is stretched more than the steel rod.

The original length of each specimen was  $l_0 = 30\text{ cm}$ . After deformation, assume that the aluminum bar is elongated by  $\Delta l_1$  to length  $l_1$ , and the steel rod is elongated by  $\Delta l_2$  to length  $l_2$ . By definition, tensile strain is  $\epsilon = \Delta l/l_0$ , or the amount of elongation is  $\Delta l = \epsilon l_0$ . On the other hand, the length of the

SPECIMEN 1: ALUMINUM BAR



SPECIMEN 2: STEEL ROD

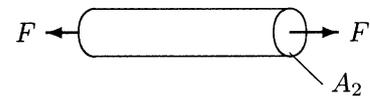


Fig. 13.30 Example 13.2

specimen after deformation is equal to the original length plus the amount of elongation. For example, for the aluminum bar,  $l_1 = l_0 + \Delta l$ , or  $l_1 = l_0(1 + \epsilon_1)$ . Therefore:

$$l_1 = l_0(1 + \epsilon_1) = 30(1 + 0.00143) = 30.0429 \text{ cm}$$

$$l_2 = l_0(1 + \epsilon_2) = 30(1 + 0.000320) = 30.00960 \text{ cm}$$

In other words, the increase in length of the aluminum bar and the steel rod is less than 1 mm.

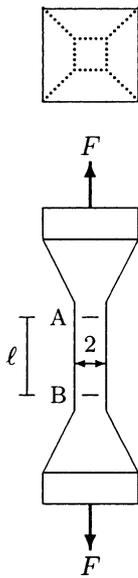


Fig. 13.31 The top and side views of the specimen

**Example 13.3** An experiment was designed to determine the elastic modulus of the human bone (cortical) tissue. Three almost identical bone specimens were prepared. The specimen size and shape used is shown in Fig. 13.31, which has a square ( $2 \times 2$  mm) cross-section. Two sections, A and B, are marked on each specimen at a fixed distance apart. Each specimen was then subjected to tensile loading of varying magnitudes, and the lengths between the marked sections were again measured electronically. The following data was obtained:

APPLIED FORCE, $F$ (N)	MEASURED GAGE LENGTH, $l$ (mm)
0	5.000
240	5.017
480	5.033
720	5.050

Determine the tensile stresses and strains developed in each specimen, plot a stress–strain diagram for the bone, and determine the elastic modulus ( $E$ ) for the bone.

**Solution:** The cross-sectional area of each specimen is  $A = 4 \text{ mm}^2$  or  $4 \times 10^{-6} \text{ m}^2$ . When the applied load is zero, the gage length is 5 mm, which is the original (undeformed) gage length,  $l_0$ . Therefore, the stress and strain developed in each specimen can be calculated using:

$$\sigma = \frac{F}{A} \quad \epsilon = \frac{l - l_0}{l_0}$$

The following table lists stresses and strains calculated using the above formulas:

$F$ (N)	$\sigma \times 10^6$ (Pa)	$l$ (mm)	$\epsilon$ (mm/mm)
0	0	5.000	0.0
240	60	5.017	0.0034
480	120	5.033	0.0066
720	180	5.050	0.0100

In Fig. 13.32, the stress and strain values computed are plotted to obtain a  $\sigma - \epsilon$  graph for the bone. Notice that the relationship between the stress and strain is almost linear, which is indicated in Fig. 13.32 by a straight line.

Recall that the elastic modulus of a linearly elastic material is equal to the slope of the straight line representing the  $\sigma - \epsilon$  relationship for that material. Therefore:

$$E = \frac{\sigma}{\epsilon} = \frac{180 \times 10^6}{0.0100} = 18 \times 10^9 \text{ Pa} = 18 \text{ GPa}$$

**Example 13.4** Figure 13.33 illustrates a fixation device consisting of a plate and two screws, which can be used to stabilize fractured bones. During a single leg stance, a person can apply his/her entire weight to the ground via a single foot. In such situations, the total weight of the person is applied back on the person through the same foot, which has a compressive effect on the leg, its bones, and joints. In the case of a patient with a fractured leg bone (in this case, the femur), this force is transferred from below to above (distal to proximal) the fracture through the screws of the fixation device.

If the diameter of the screws is  $D = 5 \text{ mm}$  and the weight of the patient is  $W = 700 \text{ N}$ , determine the shear stress exerted on the screws of a two-screw fixation device during a single leg stance on the leg with a fractured bone.

**Solution:** Free-body diagrams of the fixation device and the screws are shown in Fig. 13.34. Note that the screw above the fracture is pushing the plate downward, whereas the screw below the fracture is pushing the plate upward. Each screw is applying a force on the plate equal to the weight of the person. The same magnitude force is also acting on the screws but in the opposite directions. For example, for the screw above the fracture, the plate is exerting an upward force on the head of the screw and the bone is applying a downward force. The effects of the forces applied on the screws are such that they are trying to shear the screws in a plane perpendicular to the centerline of the screws. With respect to the cross-sectional areas of the screws, these are shearing forces.

The shear stress  $\tau$  generated in the screws can be calculated by using the following relationship between the shear force  $F$  and area  $A$  over which the shear stress is to be determined:

$$\tau = \frac{F}{A}$$

In this case,  $F$  is equal to the weight ( $W = 700 \text{ N}$ ) of the patient. Since the diameters of the screws are given, the cross-sectional

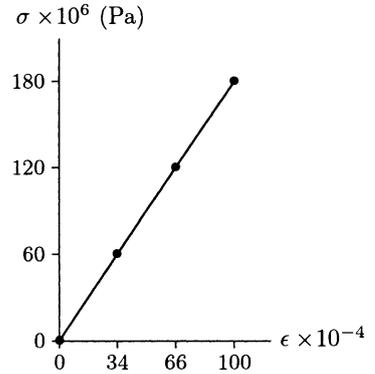


Fig. 13.32 Stress-strain diagram for the bone

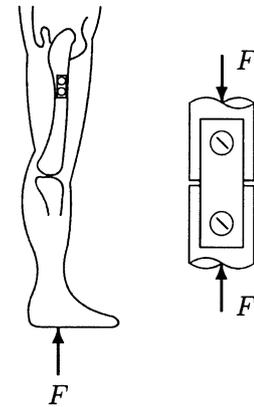


Fig. 13.33 Example 13.4

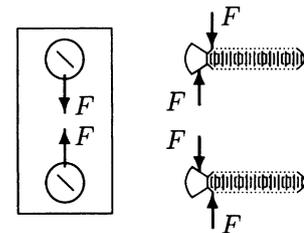
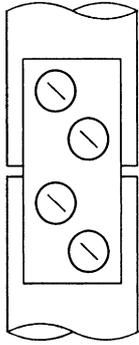
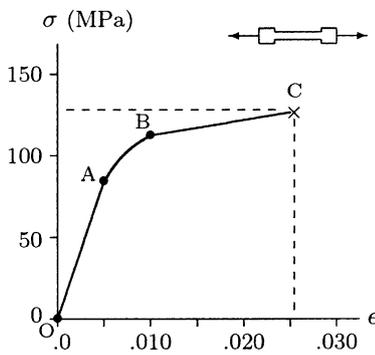


Fig. 13.34 Forces applied and the plate and screws



**Fig. 13.35** A four-screw fixation device



**Fig. 13.36** Tensile stress–strain diagram for cortical bone (1 MPa = 10<sup>6</sup> Pa)

area of each screw can be calculated as  $A = \pi D^2/4 = 19.6 \text{ mm}^2$  or  $A = 19.6 \times 10^{-6} \text{ m}^2$ . Substituting the numerical value of  $A$  and  $F = W = 700 \text{ N}$  into the above formula and carrying out the computation will yield  $\tau = 35.7 \times 10^6 \text{ Pa}$ .

Note that if we had a four-screw rather than a two-screw fixation device as shown in Fig. 13.35, then each screw would be subjected to a shearing force equal to half of the total weight of the patient.

**Example 13.5** Specimens of human cortical bone tissue were subjected to simple tension test until fracture. The test results revealed a stress–strain diagram shown in Fig. 13.36, which has three distinct regions. These regions are an initial linearly elastic region (between O and A), an intermediate nonlinear elasto-plastic region (between A and B), and a final linearly plastic region (between B and C). The average stresses and corresponding strains at points O, A, B, and C are measured as:

POINT	STRESS $\sigma$ (MPa)	STRAIN $\epsilon$ (mm/mm)
O	0	0.0
A	85	0.005
B	114	0.010
C	128	0.026

Using this information, determine the elastic and strain hardening moduli of the bone tissue in the linear regions of its  $\sigma - \epsilon$  diagram. Note that the *strain hardening modulus* is the slope of the  $\sigma - \epsilon$  curve in the plastic region. Also, find mathematical expressions relating stresses to strains in the linearly elastic and linearly plastic regions.

**Solution:** The elastic modulus  $E$  is the slope of the  $\sigma - \epsilon$  curve in the elastic region. Between O and A, the bone exhibits linearly elastic material behavior, and the  $\sigma - \epsilon$  curve is a straight line. The slope of this line is:

$$E = \frac{\sigma_A - \sigma_O}{\epsilon_A - \epsilon_O} = \frac{85 \times 10^6 - 0}{0.005 - 0.0} = 17 \times 10^9 \text{ Pa} = 17 \text{ GPa}$$

The strain hardening modulus  $E'$  is equal to the slope of the  $\sigma - \epsilon$  curve in the plastic region. Between B and C, the bone exhibits a linearly plastic material behavior, and its  $\sigma - \epsilon$  curve is a straight line. Therefore:

$$E' = \frac{\sigma_C - \sigma_B}{\epsilon_C - \epsilon_B} = \frac{128 \times 10^6 - 114 \times 10^6}{0.026 - 0.010} = 0.875 \times 10^9 \text{ Pa}$$

The stress–strain relationship between O and A is:

$$\sigma = E\epsilon \quad \text{or} \quad \epsilon = \frac{\sigma}{E}$$

The relationship between  $\sigma$  and  $\epsilon$  in the linearly plastic region between B and C can be expressed as:

$$\sigma = \sigma_B + E'(\epsilon - \epsilon_B) \quad \text{or} \quad \epsilon = \epsilon_B + \frac{1}{E'}(\sigma - \sigma_B)$$

For example, the strain corresponding to a tensile stress of  $\sigma = 120 \text{ MPa}$  can be calculated as:

$$\epsilon = 0.010 + \frac{120 \times 10^6 - 114 \times 10^6}{0.875 \times 10^9} = 0.017$$

**Example 13.6** Consider the structure shown in Fig. 13.37. The horizontal beam AB has a length  $l = 4 \text{ m}$ , weight  $W = 500 \text{ N}$ , and is hinged to the wall at A. The beam is supported by two identical steel bars of length  $h = 3 \text{ m}$ , cross-sectional area  $A = 2 \text{ cm}^2$ , and elastic modulus  $E = 200 \text{ GPa}$ . The steel bars are attached to the beam at B and C, where C is equidistant from both ends of the beam.

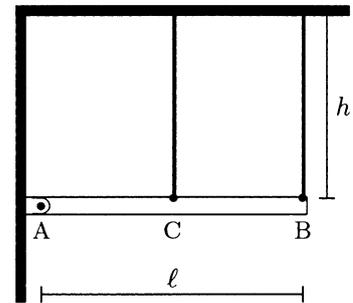
Determine the forces applied by the beam on the steel bars, the reaction force at A, the amount of elongation in each steel bar, and the stresses generated in each bar. Assume that the beam material is much stiffer than the steel bars.

**Solution:** This problem will be analyzed in three stages.

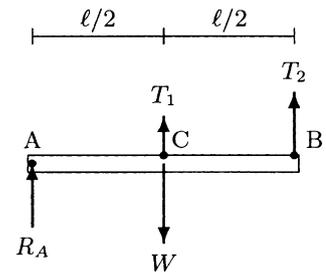
### Static Analysis

The free-body diagram of the beam is shown in Fig. 13.38. The total weight,  $W$ , of the beam is assumed to act at its geometric center located at C.  $R_A$  is the magnitude of the ground reaction force applied on the beam through the hinge joint at A, and  $T_1$  and  $T_2$  are the forces applied by the steel bars on the beam. Note that only a vertical reaction force is considered at A since there is no horizontal force acting on the beam.

We have a coplanar force system with three unknowns:  $R_A$ ,  $T_1$ , and  $T_2$ . There are three equations of equilibrium available from statics. Since there is no horizontal force, the horizontal equilibrium of the beam is automatically satisfied. Therefore, we have two equations but three unknowns. In other words, we have a statically indeterminate system and the equations of equilibrium are not sufficient to fully analyze this problem. We need an additional equation that will be derived by taking into consideration the deformability of the parts constituting the system.



**Fig. 13.37** A statically indeterminate system



**Fig. 13.38** The free-body diagram of the beam

The vertical equilibrium of the beam requires that:

$$\sum F_y = 0 : T_1 + T_2 + R_A = W \tag{i}$$

For the rotational equilibrium of the beam about point A:

$$\sum M_A = 0 : \frac{1}{2}T_1 + T_2 = \frac{1}{2}W \tag{ii}$$

**Geometric Compatibility**

The horizontal beam is hinged to the wall at A, and the weight of the beam tends to rotate the beam about A in the clockwise direction. Because of the weight of the beam, which is applied as tensile forces  $T_1$  and  $T_2$  on the bars, the steel bars will deflect (elongate) and the beam will slightly swing about A (Fig. 13.39). Because of the forces acting on the beam, the beam may bend a little as well. Since it is stated that the beam material is much stiffer than the steel bars, we can ignore the deformability of the beam and assume that it maintains its straight configuration. If  $\delta_1$  and  $\delta_2$  refer to the amount of deflections in the steel bars, then from Fig. 13.39:

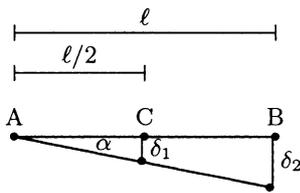


Fig. 13.39 Deflection of the beam

$$\tan \alpha \simeq \frac{\delta_1}{l/2} \simeq \frac{\delta_2}{l} \tag{iii}$$

Note that this relationship is correct when deflections ( $\delta_1$  and  $\delta_2$ ) are small for which angle  $\alpha$  is small.

Next we need take into consideration the relationship between applied forces and corresponding deformations.

**Stress–Strain (Force-Deformation) Analyses**

The steel bars with length  $h$  elongate by  $\delta_1$  and  $\delta_2$ . Therefore, the tensile strains in the bars are:

$$\epsilon_1 = \frac{\delta_1}{h} \quad \epsilon_2 = \frac{\delta_2}{h} \tag{iv}$$

The bars are subjected to tensile forces  $T_1$  and  $T_2$ . The cross-sectional area of each bar is given as  $A$ . Therefore, the tensile stresses exerted by the beam on the steel rods are:

$$\sigma_1 = \frac{T_1}{A} \quad \sigma_2 = \frac{T_2}{A} \tag{v}$$

Assuming that the stresses involved are within the proportionality limit for steel, we can apply the Hooke’s law to relate stresses to strains:

$$\sigma_1 = E\epsilon_1 \quad \sigma_2 = E\epsilon_2 \tag{vi}$$

Now, we can substitute Eqs. (iv) and (v) into Eq. (vi) so as to eliminate stresses and strains. This will yield:

$$\delta_1 = \frac{T_1 h}{EA} \quad \delta_2 = \frac{T_2 h}{EA} \tag{vii}$$

Note that from Eq. (iii):

$$\delta_2 = 2\delta_1 \quad (\text{viii})$$

Substituting Eqs. (vii) into Eq. (viii) will yield:

$$T_2 = 2T_1 \quad (\text{ix})$$

Now, we have a total of three equations, Eqs. (i), (ii), and (ix), with three unknowns,  $R_A$ ,  $T_1$ , and  $T_2$ . Solving these equations simultaneously will yield:

$$T_1 = \frac{1}{5}W = 100 \text{ N}$$

$$T_2 = \frac{2}{5}W = 200 \text{ N}$$

$$R_A = \frac{2}{5}W = 200 \text{ N}$$

Once tensile forces  $T_1$  and  $T_2$  are determined, Eq. (vii) can be used to calculate the amount of elongations, Eq. (v) can be used to calculate the tensile stresses, and Eq. (iv) can be used to calculate the tensile strains developed in the steel bars:

$$\sigma_1 = \frac{T_1}{A} = 0.5 \times 10^6 \text{ Pa} = 0.5 \text{ MPa}$$

$$\sigma_2 = \frac{T_2}{A} = 1.0 \times 10^6 \text{ Pa} = 1.0 \text{ MPa}$$

$$\epsilon_1 = \frac{\sigma_1}{E} = 2.5 \times 10^{-6}$$

$$\epsilon_2 = \frac{\sigma_2}{E} = 5.0 \times 10^{-6}$$

Note that the calculated strains are very small. Correspondingly, the deformations are very small as predicted earlier while deriving the relationship in Eq. (iii). Also note that the stresses developed in the steel bars are much lower than the proportionality limit for steel. Therefore, the assumption made to relate stresses and strains in Eq. (vi) was correct as well.

### 13.18 Exercise Problems

*Answers to all problems in this section are provided at the end of the chapter.*

**Problem 13.1** Complete the following definitions with appropriate expressions.

- (a) Unit deformation of a material as a result of an applied load is called \_\_\_\_\_.

- (b) The internal resistance of a material to deformation due to externally applied forces is called \_\_\_\_\_.
- (c) \_\_\_\_\_ is a measure of the intensity of internal forces acting parallel or tangent to a plane of cut, while \_\_\_\_\_ are associated with the intensity of internal forces that are perpendicular to the plane of cut.
- (d) On the stress–strain diagram, the stress corresponding to the \_\_\_\_\_ is the highest stress that can be applied to the material without causing permanent deformation.
- (e) On the stress–strain diagram, the highest stress level corresponds to the \_\_\_\_\_ of the material.
- (f) For some materials, it may not be easy to distinguish the yield point. The yield strength of such materials may be determined by the \_\_\_\_\_.
- (g) \_\_\_\_\_ is defined as the ability of a material to resume its original (stress-free) size and shape upon removal of applied loads.
- (h) For linearly elastic materials, stress is linearly proportional to strain and the constant of proportionality is called the \_\_\_\_\_ of the material.
- (i) The distinguishing factor in linearly elastic materials is their \_\_\_\_\_.
- (j) Materials for which the stress–strain curve in the elastic region is not a straight line are known as \_\_\_\_\_ materials.
- (k) \_\_\_\_\_ is the constant of proportionality between shear stress and shear strain for linearly elastic materials.
- (l) A mathematical equation that relates stresses to strains is called a \_\_\_\_\_.
- (m) The analogy between elastic materials and springs is known as \_\_\_\_\_.
- (n) \_\_\_\_\_ implies permanent (unrecoverable) deformations.
- (o) The area under the stress–strain diagram in the elastic region corresponds to the \_\_\_\_\_ energy stored in the material while deforming the material.
- (p) \_\_\_\_\_ energy is dissipated as heat while deforming the material.
- (q) The area enclosed by the \_\_\_\_\_ signifies the total strain energy dissipated as heat while loading and unloading a material.
- (r) The technique of changing the yield point of a material by loading the material beyond its yield point is called \_\_\_\_\_.
- (s) The elastic modulus of a material is a relative measure of the \_\_\_\_\_ of one material with respect to another.
- (t) A \_\_\_\_\_ material is one that exhibits a large plastic deformation prior to failure.
- (u) A \_\_\_\_\_ material is one that shows a sudden failure (rupture) without undergoing a considerable plastic deformation.

- (v) \_\_\_\_\_ is a measure of the capacity of a material to sustain permanent deformation. The toughness of a material is measured by considering the total area under its stress-strain diagram.
- (w) The ability of a material to store or absorb energy without permanent deformation is called the \_\_\_\_\_ of the material.
- (x) If the mechanical properties of a material do not vary from location to location within the material, then the material is called \_\_\_\_\_.
- (y) If a material has constant density, then the material is called \_\_\_\_\_.
- (z) If the mechanical properties of a material are independent of direction or orientation, then the material is called \_\_\_\_\_.

**Problem 13.2** Curves in Fig. 13.40 represent the relationship between tensile stress and tensile strain for five different materials. The “dot” on each curve indicates the yield point and the “cross” represents the rupture point. Fill in the blank spaces below with the correct number referring to a material.

- Material \_\_\_\_\_ has the highest elastic modulus.
- Material \_\_\_\_\_ is the most ductile.
- Material \_\_\_\_\_ is the most brittle.
- Material \_\_\_\_\_ has the lowest yield strength.
- Material \_\_\_\_\_ has the highest strength.
- Material \_\_\_\_\_ is the toughest.
- Material \_\_\_\_\_ is the most resilient.
- Material \_\_\_\_\_ is the most stiff.

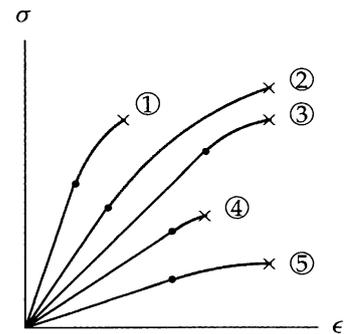


Fig. 13.40 Problem 13.2

**Problem 13.3** Consider two bars, 1 and 2, made of two different materials. Assume that these bars were tested in a uniaxial tension test. Let  $F_1$  and  $F_2$  be the magnitudes of tensile forces applied on bars 1 and 2, respectively.  $E_1$  and  $E_2$  are the elastic moduli and  $A_1$  and  $A_2$  are the cross-sectional areas perpendicular to the applied forces for bar 1 and 2, respectively. For the conditions indicated below, determine the correct symbol relating tensile stresses  $\sigma_1$  and  $\sigma_2$  and tensile strains  $\epsilon_1$  and  $\epsilon_2$ . Note that “>” indicates greater than, “<” indicates less than, “=” indicates equal to, and “?” indicates that the information provided is not sufficient to make a judgement.

- (a) If  $A_1 > A_2$  and  $F_1 = F_2$  then  $\sigma_1 > = ? < \sigma_2$  and  $\epsilon_1 > = ? < \epsilon_2$ .
- (b) If  $E_1 > E_2$ ,  $A_1 = A_2$  and  $F_1 = F_2$  then  $\sigma_1 > = ? < \sigma_2$  and  $\epsilon_1 > = ? < \epsilon_2$

**Problem 13.4** As illustrated in Fig. 13.29, consider a circular cylindrical rod tested in a uniaxial tension test. Two points A and B located at a distance  $l_0 = 32$  cm from each other are marked on the rod and a tensile force of  $F = 980$  N is applied on the rod. If the tensile strain and tensile stress generated in the rod were  $\epsilon = 0.06$  cm/cm and  $\sigma = 2.2$  MPa, determine:

- (a) The radius,  $r$ , of the rod after the application of the force
- (b) The total elongation of the rod,  $\Delta l$ , after the application of the force

**Problem 13.5** Consider an aluminum rod of radius  $r = 1.5$  cm subjected to a uniaxial tension test by force of  $F = 23$  kN. If the elastic modulus of the aluminum rod is  $E = 70$  GPa, determine:

- (a) The tensile stress,  $\sigma$ , developed in the rod
- (b) The tensile strain,  $\epsilon$ , developed in the rod

**Problem 13.6** Figure 13.41 illustrates a bone specimen with a circular cross-section. Two sections, A and B, that are  $l_0 = 6$  mm distance apart are marked on the specimen. The radius of the specimen in the region between A and B is  $r_0 = 1$  mm.

This specimen was subjected to a series of uniaxial tension tests until fracture by gradually increasing the magnitude of the applied force and measuring corresponding deformations. As a result of these tests, the following data were recorded:

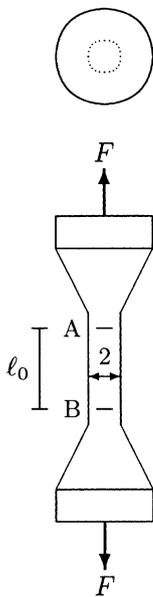


Fig. 13.41 Problem 13.6

RECORD #	FORCE, $F$ (N)	DEFORMATION, $\Delta l$ (mm)
1	94	0.009
2	190	0.018
3	284	0.027
4	376	0.050
5	440	0.094

If record 3 corresponds to the end of the linearly elastic region and record 5 corresponds to fracture point, carry out the following:

- (a) Calculate average tensile stresses and strains for each record.
- (b) Draw the tensile stress–strain diagram for the bone specimen.
- (c) Calculate the elastic modulus,  $E$ , of the bone specimen.

- (d) What is the ultimate strength of the bone specimen?
- (e) What is the yield strength of the bone specimen? (use the offset method)

**Problem 13.7** Consider a fixation device consisting of a plate and two screws that was used to stabilize a fractured femoral bone of a patient (Fig. 13.33). The weight of the patient was  $W = 833 \text{ N}$ . If the shear stress generated in the screws was  $\tau = 35.12 \times 10^6 \text{ Pa}$ , determine the diameter,  $d$ , of the screws.

**Problem 13.8** Human femur bone was subjected to a uniaxial tension test. As a result of a series of experiments, the stress-strain curve shown in Fig. 13.42 was obtained. Based on the graph in Fig. 13.42, determine the following parameters:

- (a) Elastic modulus,  $E$ .
- (b) Apparent yield strength,  $\sigma_y$  (use the offset method).
- (c) Ultimate strength,  $\sigma_u$ .
- (d) Strain,  $\epsilon_1$ , corresponding to yield stress.
- (e) Strain,  $\epsilon_2$ , corresponding to ultimate stress.
- (f) Strain energy when stress is at the proportionality limit.

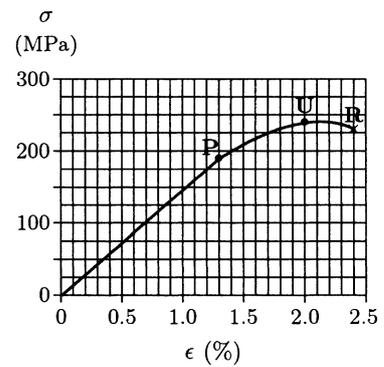


Fig. 13.42 Problem 13.8

**Problem 13.9** As illustrated in Fig. 13.43, consider a structure including the horizontal beam AB hinged to the wall at point A and supported by steel bar at point B. The length of the beam is  $l = 3.7 \text{ m}$  and its weight is  $W = 450 \text{ N}$ . The length of the steel bar is  $h = 2.5 \text{ m}$  and its radius is  $r = 1.2 \text{ cm}$ . If the elastic modulus of the steel bar is  $E = 200 \text{ GPa}$ , determine:

- (a) The magnitude of reaction force,  $R_A$ , at point A
- (b) The tensile stress,  $\sigma$ , exerted by the beam on the steel bar
- (c) The tensile strain,  $\epsilon$ , developed in the bar

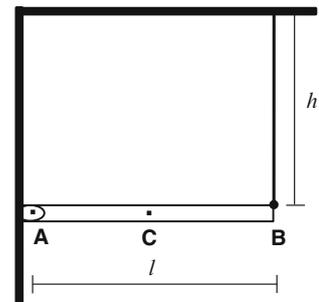


Fig. 13.43 Problem 13.9

**Problem 13.10** Consider the uniform, horizontal beam shown in Fig. 13.44a. The beam is hinged to a wall at A and a weight  $W_2 = 400 \text{ N}$  is attached on the beam at B. Point C represents the center of gravity of the beam, which is equidistant from A and B. The beam has a weight  $W_1 = 100 \text{ N}$  and length  $l = 4 \text{ m}$ . The beam is supported by two vertical rods, 1 and 2, attached to the beam at D and E. Rod 1 is made of steel with elastic modulus  $E_1 = 200 \text{ GPa}$  and a cross-sectional area  $A_1 = 500 \text{ mm}^2$ , and rod 2 is bronze with elastic modulus  $E_2 = 80 \text{ GPa}$  and  $A_2 = 400 \text{ mm}^2$ . The original (undeformed) lengths of both rods is  $h = 2 \text{ m}$ . The

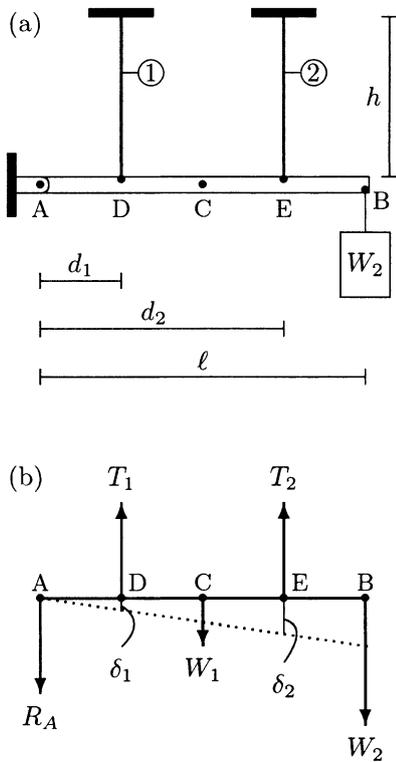


Fig. 13.44 Problem 13.10

distance between A and D is  $d_1 = 1\text{ m}$  and the distance between A and E is  $d_2 = 3\text{ m}$ .

The free-body diagram of the beam and its deflected orientation is shown in Fig. 13.44b, where  $T_1$  and  $T_2$  represent the forces exerted by the rods on the beam. Symbols  $\delta_1$  and  $\delta_2$  represent the amount of deflection the steel and bronze rods undergo, respectively. Note that the beam material is assumed to be very stiff (almost rigid) as compared to the rods so that it maintains its straight shape.

- (a) Calculate tensions  $T_1$  and  $T_2$ , and the reactive force  $R_A$  on the beam at A.
- (b) Calculate the average tensile stresses  $\sigma_1$  and  $\sigma_2$  generated in the rods.

**Answers**

Answers to Problem 13.1:

(a) strain	(n) Plasticity
(b) stress	(o) elastic strain
(c) Shear stress, normal stress	(p) Plastic strain
(d) elastic limit	(q) hysteresis loop
(e) ultimate strength	(r) strain hardening
(f) offset method	(s) stiffness
(g) Elasticity	(t) ductile
(h) elastic (Young's) modulus	(u) brittle
(i) elastic (Young's) modulus	(v) Toughness
(j) nonlinear elastic	(w) resilience
(k) Shear modulus	(x) homogeneous
(l) material function	(y) incompressible
(m) Hooke's Law	(z) isotropic

Answers to Problem 13.2: 1, 2, 4, 5, 2, 2, 3, 1

Answers to Problem 13.3:

- (a)  $\sigma_1 < \sigma_2, \epsilon_1 > \epsilon_2$
- (b)  $\sigma_1 = \sigma_2, \epsilon_1 < \epsilon_2$

Answers to Problem 13.4: (a)  $r = 1.2\text{ cm}$ , (b)  $\Delta l = 1.92\text{ cm}$

Answers to Problem 13.5: (a)  $\sigma = 32.6\text{ MPa}$ , (b)  $\epsilon = 0.47 \times 10^{-3}$

*Answers to Problem 13.6:*

- (c)  $E = 20 \text{ GPa}$
- (d)  $\sigma_u = 140 \text{ MPa}$
- (e)  $\sigma_y = 118 \text{ MPa}$

*Answer to Problem 13.7:*  $d = 5.5 \text{ mm}$

*Answers to Problem 13.8:*

- (a)  $E = 14.6 \text{ GPa}$
- (b)  $\sigma_y = 235 \text{ MPa}$
- (c)  $\sigma_u = 240 \text{ MPa}$
- (d)  $\epsilon_2 = 0.018$
- (e)  $\epsilon_2 = 0.02$
- (f)  $1.235 \text{ MPa}$

*Answers to Problem 13.9:* (a)  $R_A = 225 \text{ N}$ , (b)  $\sigma = 0.5 \text{ MPa}$ ,  
(c)  $\epsilon = 2.5 \times 10^{-6}$

*Answers to Problem 13.10:*

- (a)  $T_1 = 464 \text{ N}$ ,  $T_2 = 445 \text{ N}$ ,  $R_A = 409 \text{ N}$
  - (b)  $\sigma_1 = 0.928 \text{ MPa}$ ,  $\sigma_2 = 1.113 \text{ MPa}$
-