1 Mathematics in Civilization

Key Topics

Babylonian Mathematics
Egyptian Civilisation
Greek and Roman Civilisation
Islamic Civilisation
Counting and Numbers
Solving Practical Problems
Syllogistic Logic
Algorithm
Early Ciphers

1.1 Introduction

It is difficult to think of western society today without modern technology. The last decades of the twentieth century have witnessed a proliferation of high-tech computers, mobile phones, text messaging, the Internet and the World Wide Web. Software is now pervasive, and it is an integral part of automobiles, airplanes, televisions and mobile communication. The pace of change as a result of all this new technology has been extraordinary. Today consumers may book flights over the World Wide Web as well as keep in contact with the family members in any part of the world via e-mail or mobile phone. In previous generations, communication often involved writing letters that took months to reach the recipient.

G. O'Regan, *Guide to Discrete Mathematics*, Texts in Computer Science,
DOI 10.1007/978-3-319-44561-8_1

Communication improved with the telegrams and the telephone in the late nineteenth century. Communication today is instantaneous with text messaging, mobile phones and e-mail, and the new generation probably views the world of their parents and grandparents as being old fashioned.

The new technologies have led to major benefits[1] to society and to improvements in the standard of living for many citizens in the western world. It has also reduced the necessity for humans to perform some of the more tedious or dangerous manual tasks, as computers may now automate many of these. The increase in productivity due to the more advanced computerized technologies has allowed humans, at least in theory, the freedom to engage in more creative and rewarding tasks.

Early societies had a limited vocabulary for counting: e.g. 'one, two, three, many' is associated with some primitive societies, and indicates primitive computation and scientific ability. It suggests that there was no need for more sophisticated arithmetic in the primitive culture as the problems dealt with were elementary. These early societies would typically have employed their fingers for counting, and as humans have five fingers on each hand and five toes on each foot then the obvious bases would have been 5, 10 and 20. Traces of the earlier use of the base 20 system are still apparent in modern languages such as English and French. This includes phrases such as 'three score' in English and '*quatre vingt*' in French.

The decimal system (base 10) is used today in western society, but the base 60 was common in computation *circa* 1500 B.C. One example of the use of base 60 today is the subdivision of hours into 60 min, and the subdivision of minutes into 60 s. The base 60 system (i.e. the sexagesimal system) is inherited from the Babylonians [1]. The Babylonians were able to represent arbitrarily large numbers or fractions with just two symbols. The binary (base 2) and hexadecimal (base 16) systems play a key role in computing (as the machine instructions that computers understand are in binary code).

The achievements of some of these ancient societies were spectacular. The archaeological remains of ancient Egypt such as the pyramids at Giza and the temples of Karnak and Abu Simbel are impressive. These monuments provide an indication of the engineering sophistication of the ancient Egyptian civilization. The objects found in the tomb of Tutankhamun[2] are now displayed in the Egyptian museum in Cairo, and demonstrate the artistic skill of the Egyptians.

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[1]Of course, it is essential that the population of the world moves towards more sustainable development to ensure the long-term survival of the planet for future generations. This involves finding technological and other solutions to reduce greenhouse gas emissions as well as moving to a carbon neutral way of life. The solution to the environmental issues will be a major challenge for the twenty first century.

[2]Tutankhamun was a minor Egyptian pharaoh who reigned after the controversial rule of Akenaten. Tutankamun's tomb was discovered by Howard Carter in the Valley of the Kings, and the tomb was intact. The quality of the workmanship of the artefacts found in the tomb is extraordinary and a visit to the Egyptian museum in Cairo is memorable.

The Greeks made major contributions to western civilization including contributions to Mathematics, Philosophy, Logic, Drama, Architecture, Biology and Democracy.[3] The Greek philosophers considered fundamental questions such as ethics, the nature of being, how to live a good life, and the nature of justice and politics. The Greek philosophers include Parmenides, Heraclitus, Socrates, Plato and Aristotle. The Greeks invented democracy and their democracy was radically different from today's representative democracy.[4] The sophistication of Greek architecture and sculpture is evident from the Parthenon on the Acropolis, and the Elgin marbles[5] that are housed today in the British Museum, London.

The Hellenistic[6] period commenced with Alexander the Great and led to the spread of Greek culture throughout most of the known world. The city of Alexandria became a centre of learning and knowledge during the Hellenistic period. Its scholars included Euclid who provided a systematic foundation for geometry. His work is known as 'The Elements', and consists of 13 books. The early books are concerned with the construction of geometric figures, number theory and solid geometry.

There are many words of Greek origin that are part of the English language. These include words such as psychology that is derived from two Greek words: *psyche* (ψυχε) and *logos* (λογος). The Greek word '*psyche*' means mind or soul, and the word '*logos*' means an account or discourse. Other examples are anthropology derived from '*anthropos*' (αντροπος) and '*logos*' (λογος).

The Romans were influenced by Greeks culture. The Romans built aqueducts, viaducts, and amphitheatres. They also developed the Julian calendar, formulated laws (*lex*); and maintained peace throughout the Roman Empire (*pax Romano*). The ruins of Pompeii and Herculaneum demonstrate their engineering capability. Their

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[3]The origin of the word "democracy' is from demos (δημος) meaning people and kratos (κρατος) meaning rule. That is, it means rule by the people. It was introduced into Athens following the reforms introduced by Cleisthenes. He divided the Athenian city state into thirty areas. Twenty of these areas were inland or along the coast and ten were in Attica itself. Fishermen lived mainly in the ten coastal areas; farmers in the ten inland areas; and various tradesmen in Attica. Cleisthenes introduced ten new clans where the members of each clan came from one coastal area, one inland area on one area in Attica. He then introduced a Boule (or assembly) which consisted of 500 members (50 from each clan). Each clan ruled for 1/10 th of the year.

[4]The Athenian democracy involved the full participations of the citizens (i.e., the male adult members of the city state who were not slaves) whereas in representative democracy the citizens elect representatives to rule and represent their interests. The Athenian democracy was chaotic and could also be easily influenced by individuals who were skilled in rhetoric. There were teachers (known as the Sophists) who taught wealthy citizens rhetoric in return for a fee. The origin of the word 'sophist' is the Greek word σοφος meaning wisdom. One of the most well known of the sophists was Protagorus. The problems with the Athenian democracy led philosophers such as Plato to consider alternate solutions such as rule by philosopher kings. This totalitarian utopian state is described in Plato's Republic.

[5]The Elgin marbles are named after Lord Elgin who moved them from the Parthenon in Athens to London in 1806. The marbles show the Pan-Athenaic festival that was held in Athens in honour of the goddess Athena after whom Athens is named.

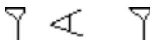
[6]The origin of the word Hellenistic is from Hellene ('Ελλην) meaning Greek.

numbering system is still employed in clocks and for page numbering in documents. However, it is cumbersome for serious computation. The collapse of the Roman Empire in Western Europe led to a decline in knowledge and learning in Europe. However, the eastern part of the Roman Empire continued at Constantinople until it was sacked by the Ottomans in 1453.

1.2 The Babylonians

The Babylonian[7] civilization flourished in Mesopotamia (in modern Iraq) from about 2000 B.C., until about 300 B.C. Various clay cuneiform tablets containing mathematical texts were discovered and later deciphered in the nineteenth century [2]. These included tables for multiplication, division, squares, cubes and square roots and the measurement of area and length. Their calculations allowed the solution of a linear equation and one root of a quadratic equation to be determined. The late Babylonian period (c. 300 B.C.) includes work on astronomy.

They recorded their mathematics on soft clay using a wedge shaped instrument to form impressions of the *cuneiform* numbers. The clay tablets were then baked in an oven or by the heat of the sun. They employed just two symbols (1 and 10) to represent numbers, and these symbols were then combined to form all other numbers. They employed a positional number system[8] and used the base 60 system. The symbol representing 1 could also (depending on the context) represent 60, 60^2, 60^3, etc. It could also mean 1/60, 1/3600, and so on. There was no zero employed in the system and there was no decimal point (no 'sexagesimal point'), and therefore the context was essential.



The example above illustrates the cuneiform notation and represents the number 60 + 10 + 1 = 71. The Babylonians used the base 60 system, and this base is still in use today in the division of hours into minutes and the division of minutes into seconds. One possible explanation for the use of the base 60 notation is the ease of dividing 60 into parts. It is divisible by 2,3,4,5,6,10,12,15,20 and 30. They were able to represent large and small numbers and had no difficulty in working with fractions (in base 60) and in multiplying fractions. The Babylonians maintained tables of reciprocals (i.e. $1/n$, $n = 1, \ldots 59$) apart from numbers like 7, 11, etc., which cannot be written as a finite sexagesimal expansion (i.e. 7, 11, etc., are not of the form $2^{\alpha}3^{\beta}5^{\gamma}$).

[7]The hanging gardens of Babylon were one of the seven wonders of the ancient world.

[8]A positional numbering system is a number system where each position is related to the next by a constant multiplier. The decimal system is an example: e.g., $546 = 5 * 10^2 + 4 * 10^1 + 6$.

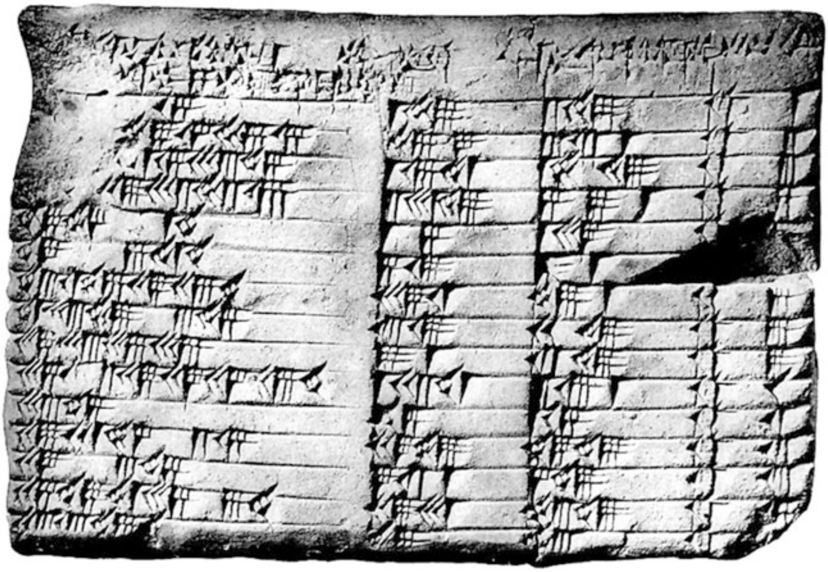


Fig. 1.1 The Plimpton 322 Tablet

The modern sexagesimal notation [1] 1;24,51,10 represents the number 1 + 24/60 + 51/3600 + 10/216,000 = 1 + 0.4 + 0.0141666 + 0.0000462 = 1.4142129. This is the Babylonian representation of the square root of 2. They performed multiplication as follows: e.g. consider 20 * sqrt(2) = (20) * (1;24,51,10)

$$20 * 1 = 20$$
$$20 * ;24 = 20 * 24/60 = 8$$
$$20 * 51/3600 = 51/180 = 17/60 = ;17$$
$$20 * 10/216{,}000 = 3/3600 + 20/216{,}000 = ;0,3,20$$

Hence, the product 20 * sqrt (2) = 20; + 8; +;17 +;0,3,20 = 28;17,3,20

The Babylonians appear to have been aware of Pythagoras's Theorem about 1000 years before the time of Pythagoras. The Plimpton 322 tablet (Fig. 1.1) records various Pythagorean triples, i.e. triples of numbers (*a, b, c*) where $a^2 + b^2 = c^2$. It dates from approximately 1700 B.C.

They developed an algebra to assist with problem solving, and their algebra allowed problems involving length, breadth and area to be discussed and solved. They did not employ notation for representation of unknown values (e.g. let *x* be the length and *y* be the breadth), and instead they used words like 'length' and 'breadth'. They were familiar with and used square roots in their calculations, and they were familiar with techniques that allowed one root of a quadratic equation to be solved.

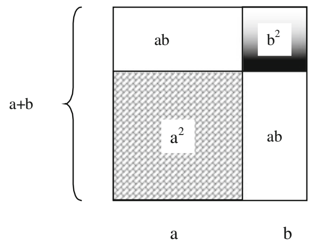


Fig. 1.2 Geometric representation of $(a + b)^2 = (a^2 + 2ab + b^2)$

They were familiar with various mathematical identities such as $(a + b)^2 = (a^2 + 2ab + b^2)$ as illustrated geometrically in Fig. 1.2. They also worked on astronomical problems, and they had mathematical theories of the cosmos to make predictions of when eclipses and other astronomical events would occur. They were also interested in astrology, and they associated various deities with the heavenly bodies such as the planets, as well as the sun and moon. They associated various cluster of stars with familiar creatures such as lions, goats and so on.

The Babylonians used counting boards to assist with counting and simple calculations. A counting board is an early version of the abacus, and it was usually made of wood or stone. The counting board contained grooves that allowed beads, or stones could be moved along the groove. The abacus differs from counting boards in that the beads in abaci contain holes that enable them to be placed in a particular rod of the abacus.

1.3 The Egyptians

The Egyptian Civilization developed along the Nile from about 4000 B.C. and the pyramids were built around 3000 B.C. They used mathematics to solve practical problems such as measuring time, measuring the annual Nile flooding, calculating the area of land, book keeping and accounting and calculating taxes. They developed a calendar circa 4000 B.C., which consisted of 12 months with each month having 30 days. There were then five extra feast days to give 365 days in a year. Egyptian writing commenced around 3000 B.C., and is recorded on the walls of temples and tombs.[9] A reed like parchment termed ‘papyrus’ was used for writing, and three Egyptian writing scripts were employed. These were hieroglyphics, the hieratic script, and the demotic script.

Hieroglyphs are little pictures and are used to represent words, alphabetic characters as well as syllables or sounds. Champollion deciphered hieroglyphics with his work on the Rosetta stone. This object was discovered during the

[9]The decorations of the tombs in the Valley of the Kings record the life of the pharaoh including his exploits and successes in battle.

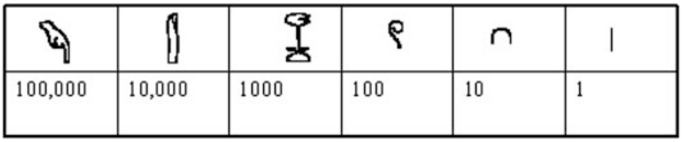


Fig. 1.3 Egyptian numerals

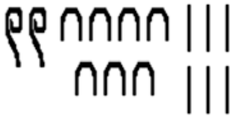


Fig. 1.4 Egyptian representation of a number

Napoleonic campaign in Egypt, and it is now in the British Museum in London. It contains three scripts: Hieroglyphics, Demotic script and Greek. The key to its decipherment was that the Rosetta stone contained just one name 'Ptolemy' in the Greek text, and this was identified with the hieroglyphic characters in the cartouche[10] of the hieroglyphics. There was just one cartouche on the Rosetta stone, and Champollion inferred that the cartouche represented the name 'Ptolemy'. He was familiar with another multilingual object that contained two names in the cartouche. One he recognized as Ptolemy and the other he deduced from the Greek text as 'Cleopatra'. This led to the breakthrough in the translation of the hieroglyphics [1].

The Rhind Papyrus is a famous Egyptian papyrus on mathematics. The Scottish Egyptologist, Henry Rhind, purchased it in 1858, and it is a copy created by an Egyptian scribe called Ahmose[11] around 1832 B.C. It contains examples of many kinds of arithmetic and geometric problems, and students may have used it as a textbook to develop their mathematical knowledge. This would allow them to participate in the pharaoh's building programme.

The Egyptians were familiar with geometry, arithmetic and elementary algebra. They had techniques to find solutions to problems with one or two unknowns. A base 10 number system was employed with separate symbols for one, ten, a hundred, a thousand, a ten thousand, a hundred thousand, and so on. These hieroglyphic symbols are represented in Fig. 1.3.

For example, the representation of the number 276 in Egyptian Hieroglyphics is described in Fig. 1.4.

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[10] The cartouche surrounded a group of hieroglyphic symbols enclosed by an oval shape. Champollion's insight was that the group of hieroglyphic symbols represented the name of the Ptolemaic pharaoh 'Ptolemy'.

[11] The Rhind papyrus is sometimes referred to as the Ahmes papyrus in honour of the scribe who wrote it in 1832 B.C.

Fig. 1.5 Egyptian representation of a fraction

The addition of two numerals is straightforward and involves adding the individual symbols, and where there are ten copies of a symbol it is then replaced by a single symbol of the next higher value. The Egyptian employed unit fractions (e.g. $1/n$ where n is an integer). These were represented in hieroglyphs by placing the symbol representing a 'mouth' above the number. The symbol 'mouth' represents part of the number. For example, the representation of the number 1/276 is described in Fig. 1.5.

The problems on the papyrus included the determination of the angle of the slope of the pyramid's face. They were familiar with trigonometry including sine, cosine, tangent and cotangent, and they knew how to build right angles into their structures by using the ratio 3:4:5. The Rhind papyrus also considered problems such as the calculation of the number of bricks required for part of a building project. Multiplication and division was cumbersome in Egyptian mathematics as they could only multiply and divide by two.

Suppose they wished to multiply a number n by 7. Then $n * 7$ is determined by $n * 2 + n * 2 + n * 2 + n$. Similarly, if they wished to divide 27 by 7 they would note that $7 * 2 + 7 = 21$ and that $27 - 21 = 6$ and that therefore the answer was 3(6/7). Egyptian mathematics was cumbersome and the writing of their mathematics was long and repetitive. For example, they wrote a number such as 22 by 10 + 10 + 1 + 1.

The Egyptians calculated the approximate area of a circle by calculating the area of a square 8/9 of the diameter of a circle. That is, instead of calculating the area in terms of our familiar πr^2 their approximate calculation yielded $(8/9 * 2r)^2 = (256/81)\ r^2$ or 3.16 r^2. Their approximation of π was 256/81 or 3.16. They were able to calculate the area of a triangle and volumes. The Moscow papyrus includes a problem to calculate the volume of the frustum. The formula for the volume of a frustum of a square pyramid[12] was given by $V = (1/3)\ h(b_1^2 + b_1 b_2 + b_2^2)$ and when b_2 is 0 then the well-known formula for the volume of a pyramid is given: i.e. $1/3\ hb_1^2$.

1.4 The Greeks

The Greeks made major contributions to western civilization including mathematics, logic, astronomy, philosophy, politics, drama and architecture. The Greek world of 500 B.C. consisted of several independent city-states such as Athens and Sparta, and various city-states in Asia Minor. The Greek polis (πολισ) or city-state

[12]The length of a side of the bottom base of the pyramid is b_1 and the length of a side of the top base is b_2.

tended to be quite small, and consisted of the Greek city and a certain amount of territory outside the city-state. Each city-state had political structures for its citizens, and some were oligarchs where political power was maintained in the hands of a few individuals or aristocratic families. Others were ruled by tyrants (or sole rulers), who sometimes took power by force, but who often had a lot of support from the public. The tyrants included people such as Solon, Peisistratus and Cleisthenes in Athens.

The reforms by Cleisthenes led to the introduction of the Athenian democracy. Power was placed in the hands of the citizens who were male (women or slaves did not participate in the Athenian democracy). It was an extremely liberal democracy where citizens voted on all important issues. Often, this led to disastrous results as speakers who were skilled in rhetoric could exert significant influence. This later led to Plato to advocate rule by philosopher kings rather than by democracy.[13]

Early Greek mathematics commenced approximately 500–600 B.C., with work done by Pythagoras and Thales. Pythagoras was a philosopher and mathematician who had spent time in Egypt becoming familiar with Egyptian mathematics. He lived on the island of Samos, and formed a secret society known as the Pythagoreans. They included men and women and believed in the transmigration of souls, and that number was the essence of all things. They discovered the mathematics for harmony in music with the relationship between musical notes being expressed in numerical ratios of small whole numbers. Pythagoras is credited with the discovery of Pythagoras's Theorem, although the Babylonians probably knew this theorem about 1000 years earlier. The Pythagorean society was dealt a major blow[14] by the discovery of the incommensurability of the square root of 2: i.e. there are no numbers p, q such that $\sqrt{2} = p/q$.

Thales was a sixth century (B.C.) philosopher from Miletus in Asia Minor who made contributions to philosophy, geometry and astronomy. His contributions to philosophy are mainly in the area of metaphysics, and he was concerned with questions on the nature of the world. His objective was to give a natural or scientific explanation of the cosmos, rather than relying on the traditional supernatural explanation of creation in Greek mythology. He believed that there was single substance that was the underlying constituent of the world, and he believed that this substance was water.

He also contributed to mathematics [3], and a well-known theorem in Euclidean geometry is named after him. It states that if A, B and C are points on a circle, and where the line AC is a diameter of the circle, then the angle $<ABC$ is a right angle.

The rise of Macedonia led to the Greek city-states being conquered by Philip of Macedonia in the fourth century B.C. His son, Alexander the Great, defeated the Persian Empire and extended his empire to include most of the known world. This

[13]Plato's Republic describes his utopian state, and seems to be based on the austere Spartan model.

[14]The Pythagoreans took a vow of silence with respect to the discovery of incommensurable numbers. However, one member of the society is said to have shared the secret result with others outside the sect, and an apocryphal account is that he was thrown into a lake for his betrayal and drowned. The Pythagoreans obviously took Mathematics seriously back then.

led to the Hellenistic Age with Greek language and culture spread throughout the known world. Alexander founded the city of Alexandra, and it became a major centre of learning. However, Alexander's reign was very short as he died at the young age of 33 in 323 B.C.

Euclid lived in Alexandria during the early Hellenistic period and he is considered as the father of geometry and the deductive method in mathematics. His systematic treatment of geometry and number theory is published in the 13 books of the Elements [4]. It starts from five axioms, five postulates and twenty-three definitions to logically derive a comprehensive set of theorems. His method of proof was often *constructive* in that, as well as demonstrating the truth of a theorem the proof would often include the construction of the required entity. He also used *indirect proof* as to show that there are an infinite number of primes

1. Suppose there are a finite number of primes (say n primes).
2. Multiply all n primes together and add 1 to form N.

$$(N = p_1 * p_2 * \ldots * p_n + 1)$$

3. N is not divisible by $p_1, p_2, \ldots, p_n$ as dividing by any of these gives a remainder of one.
4. Therefore, N must either be prime or divisible by some other prime that was not included in the list.
5. Therefore, there must be at least $n + 1$ primes.
6. This is a contradiction as it was assumed that there was a finite number of primes n.
7. Therefore, the assumption that there are a finite number of primes is false.
8. Therefore, there are an infinite number of primes.

Euclidean geometry included the parallel postulate (or Euclid's fifth postulate). This postulate generated interest, as many mathematicians believed that it was unnecessary and could be proved as a theorem. It states that:

Definition 1.1 (*Parallel Postulate*) If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

This postulate was later proved to be independent of the other postulates, with the development of non-Euclidean geometries in the nineteenth century. These include the *hyperbolic geometry* discovered independently by Bolyai and Lobachevsky, and *elliptic geometry* developed by Riemann. The standard model of Riemannian geometry is the sphere where lines are great circles.

Euclid's Elements is a systematic development of geometry starting from the small set of axioms, postulates and definitions, leading to theorems logically derived from the axioms and postulates. Euclid's deductive method influenced later mathematicians and scientists. There are some jumps in reasoning and the German mathematician, David Hilbert, later added extra axioms to address this.

The Elements contain many well-known mathematical results such as Pythagoras's Theorem, Thales Theorem, Sum of Angles in a Triangle, Prime Numbers, Greatest Common Divisor and Least Common Multiple, Euclidean Algorithm, Areas and Volumes, Tangents to a point and Algebra.

The Euclidean algorithm is one of the oldest known algorithms and is employed to produce the greatest common divisor of two numbers. It is presented in the Elements but was known well before Euclid. The algorithm to determine the gcd of two natural numbers, a and b, is given by

1. Check if b is zero. If so, then a is the gcd.
2. Otherwise, the gcd (a, b) is given by gcd $(b, a \bmod b)$.

It is also possible to determine integers p and q such that $ap + bq = \gcd(a, b)$.

The proof of the Euclidean algorithm is as follows. Suppose a and b are two positive numbers whose gcd has to be determined, and let r be the remainder when a is divided by b.

1. Clearly $a = qb + r$ where q is the quotient of the division.
2. Any common divisor of a and b is also a divisor or r (since $r = a - qb$).
3. Similarly, any common divisor of b and r will also divide a.
4. Therefore, the greatest common divisor of a and b is the same as the greatest common divisor of b and r.
5. The number r is smaller than b and we will reach $r = 0$ in finitely many steps.
6. The process continues until $r = 0$.

Comment 1.1

Algorithms are fundamental in computing as they define the procedure by which a problem is solved. A computer program implements the algorithm in some programming language.

Eratosthenes was a Hellenistic mathematician and scientist who worked at the library in Alexandria, which was the largest library in the ancient world. It was built during the Hellenistic period in the third century B.C. and destroyed by fire in 391 A.D.

Eratosthenes devised a system of latitude and longitude, and became the first person to estimate of the size of the circumference of the Earth (Fig. 1.6). His calculation proceeded as follows:

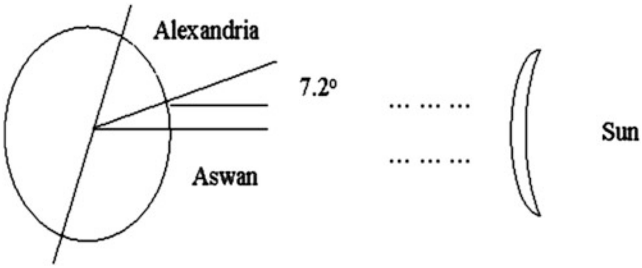

Fig. 1.6 Eratosthenes measurement of the circumference of the earth

1. On the summer solstice at noon in the town of Aswan[15] on the Tropic of Cancer in Egypt the Sun appears directly overhead.
2. Eratosthenes believed that the Earth was a sphere.
3. He assumed that rays of light came from the Sun in parallel beams and reached the Earth at the same time.
4. At the same time in Alexandria he had measured that the sun would be 7.2° south of the zenith.
5. He assumed that Alexandria was directly North of Aswan.
6. He concluded that the distance from Alexandria to Aswan was 7.2/360 of the circumference of the Earth.
7. Distance between Alexandria and Aswan was 5000 stadia (approximately 800 km).
8. He established a value of 252,000 stadia or approximately 40,320 km.

Eratosthenes's calculation was an impressive result for 200 B.C. The errors in his calculation were due to

1. Aswan is not exactly on the Tropic of Cancer but it is actually 55 km North of it.
2. Alexandria is not exactly North of Aswan and there is a difference of 3° longitude.
3. The distance between Aswan and Alexandria is 729 km not 800 km.
4. Angles in antiquity could not be measured with a high degree of precision.
5. The angular distance is actually 7.08° and not 7.2°.

Eratosthenes also calculated the approximate distance to the Moon and Sun and he also produced maps of the known world. He developed a very useful algorithm for determining all of the prime numbers up to a specified integer. The method is known as the Sieve of Eratosthenes and the steps are as follows:

[15]The town of Aswan is famous today for the Aswan high dam, which was built in the 1960s. There was an older Aswan dam built by the British in the late nineteenth century. The new dam led to a rise in the water level of Lake Nasser and flooding of archaeological sites along the Nile. Several archaeological sites such as Abu Simbel and the temple of Philae were relocated to higher ground.

1. Write a list of the numbers from 2 to the largest number that you wish to test for primality. This first list is called A.
2. A second list, called as B, is created to list the primes. It is initially empty.
3. The number 2 is the first prime number and is added to the list of primes in B.
4. Strike off (or remove) 2 and all multiples of 2 from List A.
5. The first remaining number in List A is a prime number and this prime number is added to List B.
6. Strike off (or remove) this number and all multiples of this number from List A.
7. Repeat steps 5 through 7 until no more numbers are left in List A.

Comment 1.2

The Sieve of Eratosthenes method is a well-known algorithm for determining prime numbers.

Archimedes was a Hellenistic mathematician, astronomer and philosopher who lived in Syracuse in the third century B.C. He discovered the law of buoyancy known as Archimedes's principle:

> The buoyancy force is equal to the weight of the displaced fluid.

He is believed to have discovered the principle while sitting in his bath He was so overwhelmed with his discovery that he rushed out onto the streets of Syracuse shouting '*Eureka*', but forgot to put on his clothes to announce the discovery.

The weight of the displaced liquid will be proportional to the volume of the displaced liquid. Therefore, if two objects have the same mass, the one with greater volume (or smaller density) has greater buoyancy. An object will float if its buoyancy force (i.e. the weight of liquid displaced) exceeds the downward force of gravity (i.e. its weight). If the object has exactly the same density as the liquid, then it will stay still, neither sinking nor floating upwards.

For example, a rock is generally a very dense material and will generally not displace its own weight. Therefore, a rock will sink to the bottom as the downward weight exceeds the buoyancy weight. However, if the weight of the object is less than the liquid it would displace then it floats at a level where it displaces the same weight of liquid as the weight of the object.

Archimedes (Fig. 1.7) was born in Syracuse[16] in the third century B.C. He was a leading scientist in the Greco-Roman world, and he is credited with designing several innovative machines.

His inventions include the 'Archimedes Screw' which was a screw pump that is still used today in pumping liquids and solids. Another of his inventions was the 'Archimedes Claw', which was a weapon used to defend the city of Syracuse. It was also known as the 'ship shaker' and it consisted of a crane arm from which a large metal hook was suspended. The claw would swing up and drop down on the attacking ship. It would then lift it out of the water and possibly sink it. Another of

[16]Sysacuse is located on the island of Sicily in Southern Italy.

Fig. 1.7 Archimedes in thought by Fetti

his inventions was said to be the 'Archimedes Heat Ray'. This device is said to have consisted of a number of mirrors that allowed sunlight to be focused on an enemy ship thereby causing it to go on fire.

He made good contributions to mathematics including developing a good approximation to π, as well as contributions to the positional numbering system, geometric series, and to maths physics. He also solved several interesting problems: e.g. the calculation of the composition of cattle in the herd of the Sun god by solving a number of simultaneous Diophantine equations. The herd consisted of bulls and cows with one part of the herd consisting of white, second part black, third spotted and the fourth brown. Various constraints were then expressed in Diophantine equations and the problem was to determine the precise composition of the herd. Diophantine equations are named after Diophantus who worked on number theory in the third century.

There is a well-known anecdote concerning Archimedes and the crown of King Hiero II. The king wished to determine whether his new crown was made entirely of solid gold, and that the goldsmith had not added substitute silver. Archimedes was required to solve the problem without damaging the crown, and as he was taking a bath he realized that if the crown was placed in water that the water displaced would give him the volume of the crown. From this he could then determine the density of the crown and therefore whether it consisted entirely of gold.

Archimedes also calculated an upper bound of the number of grains of sands in the known universe. The largest number in common use at the time was a myriad myriad (100 million), where a myriad is 10,000. Archimedes' numbering system goes up to $8 * 10^{16}$ and he also developed the laws of exponents: i.e. $10^a\ 10^b = 10^{a+b}$. His calculation of the upper bound includes not only the grains of sand on each beach but on the earth filled with sand and the known universe filled with sand. His final estimate of the upper bound for the number of grains of sand in a filled universe was 10^{64}.

Fig. 1.8 Plato and Aristotle

It is possible that he may have developed the odometer,[17] and this instrument could calculate the total distance travelled on a journey. An odometer is described by the Roman engineer Vitruvius around 25 B.C. It employed a wheel with a diameter of 4 feet, and the wheel turned 400 times in every mile.[18] The device included gears and pebbles and a 400-tooth cogwheel that turned once every mile and caused one pebble to drop into a box. The total distance travelled was determined by counting the pebbles in the box.

Aristotle was born in Macedonia and became a student of Plato in Athens (Fig. 1.8). Plato had founded a school (known as Plato's academy) in Athens in the fourth century B.C., and this school remained open until 529 A.D. Aristotle founded his own school (known as the Lyceum) in Athens. He was also the tutor of Alexander the Great. He made contributions to physics, biology, logic, politics, ethics and metaphysics.

Aristotle's starting point to the acquisition of knowledge was the senses, as he believed that these were essential to acquire knowledge. This position is the opposite from Plato who argued that the senses deceive and should not be relied upon. Plato's writings are mainly in dialogues involving his former mentor Socrates.[19]

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[17]The origin of the word 'odometer' is from the Greek words 'οδοζ (meaning journey) and μετρον meaning (measure).

[18]The figures given here are for the distance of one Roman mile. This is given by $\pi 4 * 400 = 12.56 * 400 = 5024$ (which is less than 5280 feet for a standard mile in the Imperial system).

[19]Socrates was a moral philosopher who deeply influenced Plato. His method of enquiry into philosophical problems and ethics was by questioning. Socrates himself maintained that he knew nothing (Socratic ignorance). However, from his questioning it became apparent that those who thought they were clever were not really that clever after all. His approach obviously would not have made him very popular with the citizens of Athens. Socrates had consulted the oracle at Delphi to find out who was the wisest of all men, and he was informed that there was no one wiser

Aristotle made important contributions to formal reasoning with his development of syllogistic logic. His collected works on logic is called the Organon and it was used in his school in Athens. Syllogistic logic (also known as term logic) consists of reasoning with two premises and one conclusion. Each premise consists of two terms and there is a common middle term. The conclusion links the two unrelated terms from the premises. For example

Premise 1	All Greeks are Mortal
Premise 2	Socrates is a Greek
Conclusion	Socrates is Mortal

The common middle term is 'Greek', which appears in the two premises. The two unrelated terms from the premises are 'Socrates' and 'Mortal'. The relationship between the terms in the first premise is that of the universal: i.e. anything or any person that is a Greek is mortal. The relationship between the terms in the second premise is that of the particular: i.e. Socrates is a person that is a Greek. The conclusion from the two premises is that Socrates is mortal: i.e. a particular relationship between the two unrelated terms 'Socrates' and 'Mortal'.

The syllogism above is a valid syllogistic argument. Aristotle studied the various possible syllogistic arguments and determined those that were valid and invalid. Syllogistic logic is described in more detail in Chap. 14. Aristotle's work was highly regarded in classical and medieval times, and Kant believed that there was nothing else to invent in Logic. There was another competing system of logic proposed by the Stoics in Hellenistic times: i.e. an early form of propositional logic that was developed by Chrysippus[20] in the third century B.C. Aristotelian logic is mainly of historical interest today.

Aquinas,[21] a thirteenth century Christian theologian and philosopher, was deeply influenced by Aristotle, and referred to him as the philosopher. Aquinas was an empiricist (i.e. he believed that all knowledge was gained by sense experience), and he used some of Aristotle's arguments to offer five proofs of the existence of God. These arguments included the Cosmological argument and the Design argument. The Cosmological argument used Aristotle's ideas on the scientific method and causation. Aquinas argued that there was a first cause and he deduced that this first cause is God.

1. Every effect has a cause
2. Nothing can cause itself

(Footnote 19 continued)
than him. Socrates was sentenced to death for allegedly corrupting the youth of Athens, and the sentence was carried out by Socrates being forced to take hemlock (a type of poison). The juice of the hemlock plant was prepared for Socrates to drink.

[20]Chrysippus was the head of the Stoics in the third century B.C.

[21]Aquinus's (or St. Thomas's) most famous work is Sumna Theologicae.

Fig. 1.9 Julius Caesar

3. A causal chain cannot be of infinite length
4. Therefore, there must be a first cause.

The Antikythera [5] was an ancient mechanical device that is believed to have been designed to calculate astronomical positions. It was discovered in 1902 in a wreck off the Greek island of Antikythera, and dates from about 80 B.C. It is one of the oldest known geared devices, and it is believed that it was used for calculating the position of the Sun, Moon, Stars and Planets for a particular date entered.

The Romans appear to have been aware of a device similar to the Antikythera that was capable of calculating the position of the planets. The island of Antikythera was well known in the Greek and Roman period for its displays of mechanical engineering.

1.5 The Romans

Rome is said to have been founded[22] by Romulus and Remus about 750 B.C. Early Rome covered a small part of Italy but it gradually expanded in size and importance. It destroyed Carthage[23] in 146 B.C. to become the major power in the Mediterranean. The Romans colonized the Hellenistic world, and they were influenced by Greek culture and mathematics. Julius Caesar conquered the Gauls in 58 B.C. (Fig. 1.9).

[22]The Aenid by Virgil suggests that the Romans were descended from survivors of the Trojan war, and that Aeneas brought surviving Trojans to Rome after the fall of Troy.

[23]Carthage was located in Tunisia, and the wars between Rome and Carthage are known as the Punic wars. Hannibal was one of the great Carthaginan military commanders, and during the second Punic war, he brought his army to Spain, marched through Spain and crossed the Pyrnees. He then marched along southern France and crossed the Alps into Northern Italy. His army also consisted of war elephants. Rome finally defeated Carthage and destroyed the city.

Fig. 1.10 Roman numbers

I = 1
V = 5
X = 10
L = 50
C = 100
D = 500
M = 1000

The Gauls consisted of several disunited Celtic[24] tribes. Vercingetorix succeeded in uniting them, but he was defeated by at the siege of Alesia in 52 B.C.

The Roman number system uses letters to represented numbers and a number consists of a sequence of letters. The evaluation rules specify that if a number follows a smaller number then the smaller number is subtracted from the larger number: e.g. IX represents 9 and XL represents 40. Similarly, if a smaller number followed a larger number they were generally added: e.g. MCC represents 1200. They had no zero in their number system (Fig. 1.10).

The use of Roman numerals was cumbersome in calculation, and an abacus was often employed. An abacus is a device that is usually of wood and has a frame that holds rods with freely sliding beads mounted on them. It is used as a tool to assist calculation, and it is useful for keeping track of the sums and the carries of calculations.

It consists of several columns in which beads or pebbles are placed. Each column represented powers of 10: i.e. 10^0, 10^1, 10^2, 10^3, etc. The column to the far right represents one; the column to the left 10; next column to the left 100; and so on. Pebbles[25] (calculi) were placed in the columns to represent different numbers: e.g. the number represented by an abacus with four pebbles on the far right; two pebbles in the column to the left; and three pebbles in the next column to the left is 324. The calculations were performed by moving pebbles from column to column.

Merchants introduced a set of weights and measures (including the *libra* for weights and the *pes* for lengths). They developed an early banking system to provide loans for business, and commenced minting money about 290 B.C. The Romans also made contributions to calendars, and Julius Caesar introduced the Julian calendar in 45 B.C. It has a regular year of 365 days divided into 12 months and a leap day is added to February every four years. It remained in use up to the

[24]The Celtic period commenced around 1000 B.C. in Hallstaat (near Salzburg in Austria). The Celts were skilled in working with Iron and Bronze, and they gradually expanded into Europe. They eventually reached Britain and Ireland around 600 B.C. The early Celtic period was known as the ‘Hallstaat period’ and the later Celtic period is known as ‘La Téne’. The later La Téne period is characterized by the quality of ornamentation produced. The Celtic museum in Hallein in Austria provides valuable information and artefacts on the Celtic period. The Celtic language would have similarities to the Irish language. However, the Celts did not employ writing, and the Ogham writing used in Ireland was developed in the early Christian period.

[25]The origin of the word ‘Calculus’ is from Latin and means a small stone or pebble used for counting.

Alphabet Symbol	abcde fghij klmno pqrst uvwxyz
Cipher Symbol	dfegh ijklm nopqr stuvw xyzabc

Fig. 1.11 Caesar cipher

twentieth century, but has since been replaced by the Gregorian calendar. The problem with the Julian calendar is that too many leap years are added over time. The Gregorian calendar was first introduced in 1582.

The Romans employed the mathematics that had been developed by the Greeks. Caesar employed a substitution cipher on his military campaigns to enable important messages to be communicated safely. It involves the substitution of each letter in the plaintext (i.e. the original message) by a letter a fixed number of positions down in the alphabet. For example, a shift of three positions causes the letter B to be replaced by E, the letter C by F, and so on. It is easily broken, as the frequency distribution of letters may be employed to determine the mapping. The cipher is defined in Fig. 1.11.

The process of enciphering a message (i.e. plaintext) involves looking up each letter in the plaintext and writing down the corresponding cipher letter. The decryption involves the reverse operation: i.e. for each cipher letter the corresponding plaintext letter is identified from the table.

The encryption may also be represented using modular arithmetic,[26] with the numbers 0–25 representing the alphabet letters, and addition (modulo 26) is used to perform the encryption.

The emperor Augustus[27] employed a similar substitution cipher (with a shift key of 1). The Caesar cipher remained in use up to the early twentieth century. However, by then frequency analysis techniques were available to break the cipher.

1.6 Islamic Influence

Islamic mathematics refers to mathematics developed in the Islamic world from the birth of Islam in the early seventh century up until the seventeenth century. The Islamic world commenced with the prophet Mohammed in Mecca, and spread throughout the Middle East, North Africa and Spain. The Golden Age of Islamic civilization was from 750 A.D. to 1250 A.D., and during this period enlightened

[26]Modular arithmetic is discussed in chapter seven.

[27]Augustus was the first Roman emperor and his reign ushered in a period of peace and stability following the bitter civil wars. He was the adopted son of Julius Caesar and was called Octavion before he became emperor. The earlier civil wars were between Caesar and Pompey, and following Caesar's assassination civil war broke out between Mark Anthony and Octavion. Octavion defeated Anthony and Cleopatra at the battle of Actium, and became the first Roman emperor, Augusus.

Fig. 1.12 Mohammed Al-Khwarizmi

caliphs recognized the value of knowledge, and sponsored scholars to come to Baghdad to gather and translate the existing world knowledge into Arabic.

This led to the preservation of the Greek texts during the Dark ages in Europe. Further, the Islamic cities of Baghdad, Cordoba and Cairo became key intellectual centres, and scholars added to existing knowledge (e.g. in mathematics, astronomy, medicine and philosophy), as well as translating the known knowledge into Arabic.

The Islamic mathematicians and scholars were based in several countries in the Middle East, North Africa and Spain. Early work commenced in Baghdad, and the mathematicians were also influenced by the work of Hindu mathematicians who had introduced the decimal system and decimal numerals. Among the well-known Islamic scholars are Ibn Al Haytham, a tenth century Iraqi scientist; Mohammed Al-Khwarizmi (Fig. 1.12), a ninth Persian mathematician; Abd Al Rahman al Sufi, a Persian astronomer who discovered the Andromeda galaxy; Ibn Al Nafis, a Syrian who did work on circulation in medicine; Averroes, who was an Aristotelian philosopher from Cordoba in Spain; Avicenna who was a Persian philosopher; and Omar Khayyman who was a Persian Mathematician and poet.

Many caliphs (Muslim rulers) were enlightened and encouraged scholarship in mathematics and science. They has setup a centre for translation and research in Baghdad, and existing Greek texts such as the works of Euclid, Archimedes, Apollonius and Diophantus were translated into Arabic. Al-Khwarizmi made contributions to early classical algebra, and the word algebra comes from the Arabic word '*al jabr*' that appears in a textbook by Al-Khwarizmi. The origin of the word *algorithm* is from the name of the Islamic scholar 'Al-Khwarizmi'.

Education was important during the Golden Age, and the Al Azhar University in Cairo (Fig. 1.13) was established in 970 A.D., and the Al-Qarawiyyin University in Fez, Morocco was established in 859 A.D. The Islamic World has created beautiful architecture and art including the ninth century Great Mosque of Samarra in Iraq; the tenth century Great Mosque of Cordoba; and the eleventh century Alhambra in Grenada.

Fig. 1.13 Al Azhar University, Cairo

The Moors[28] invaded Spain in the eighth century A.D., and they ruled large parts of the Peninsula for several centuries. Moorish Spain became a centre of learning, and this led to Islamic and other scholars coming to study at the universities in Spain. Many texts on Islamic mathematics were translated from Arabic into Latin, and these were invaluable in the renaissance in European learning and mathematics from the thirteenth century. The Moorish influence[29] in Spain continued until the time of the Catholic Monarchs[30] in the fifth century. Ferdinand and Isabella united Spain, defeated the Moors in Andalusia, and expelled them from Spain.

The Islamic contribution to algebra was an advance on the achievements of the Greeks. They developed a broader theory that treated rational and irrational numbers as algebraic objects, and moved away from the Greek concept of mathematics as being essentially Geometry. Later Islamic scholars applied algebra to arithmetic and geometry, and studied curves using equations. This included contributions to

[28]The origin of the word 'Moor' is from the Greek work μυοροζ meaning very dark. It referred to the fact that many of the original Moors who came to Spain were from Egypt, Tunisia and other parts of North Africa.

[29]The Moorish influence includes the construction of various castles (*alcazar*), fortresses (*alcalzaba*) and mosques. One of the most striking Islamic sites in Spain is the palace of Alhambra in Granada, and it represents the zenith of Islamic art.

[30]The Catholic Monarchs refer to Ferdinand of Aragon and Isabella of Castille who married in 1469. They captured Granada (the last remaining part of Spain controlled by the Moors) in 1492.

reduce geometric problems such as duplicating the cube to algebraic problems. Eventually this led to the use of symbols in the fifteenth century such as

$$x^n \cdot x^m = x^{m+n}.$$

The poet Omar Khayman was also a mathematician who did work on the classification of cubic equations with geometric solutions. Other scholars made contributions to the theory of numbers: e.g. a theorem that allows pairs of amicable numbers to be found. Amicable numbers are two numbers such that each is the sum of the proper divisors of the other. They were aware of Wilson's theory in number theory: i.e. for p prime then p divides $(p - 1)! +1$.

The Islamic world was tolerant of other religious belief systems during the Golden Age, and there was freedom of expression provided that it did not infringe on the rights of others. It began to come to an end following the Mongol invasion and sack of Baghdad in the late 1250s and the Crusades. It continued to some extent until the conquest by Ferdinand and Isabella of Andalusia in the late fifteenth century.

1.7 Chinese and Indian Mathematics

The development of mathematics commenced in China about 1000 B.C., and was independent of developments in other countries. The emphasis was on problem solving rather than on conducting formal proofs. It was concerned with finding the solution to practical problems such as the calendar, the prediction of the positions of the heavenly bodies, land measurement, conducting trade and the calculation of taxes.

The Chinese employed counting boards as mechanical aids for calculation from the fourth century B.C. These are similar to abaci and are usually made of wood or metal, and contained carved grooves between which beads, pebbles or metal discs were moved.

Early Chinese mathematics was written on bamboo strips and included work on arithmetic and astronomy. The Chinese method of learning and calculation in mathematics was learning by analogy. This involves a person acquiring knowledge from observation of how a problem is solved, and then applying this knowledge for problem solving to similar kinds of problems.

They had their version of Pythagoras's Theorem and applied it to practical problems. They were familiar with the Chinese remainder theorem, the formula for finding the area of a triangle, as well as showing how polynomial equations (up to degree ten) could be solved. They showed how geometric problems could be solved by algebra, how roots of polynomials could be solved, how quadratic and simultaneous equations could be solved, and how the area of various geometric shapes such as rectangles, trapezia and circles could be computed. Chinese mathematicians were familiar with the formula to calculate the volume of a sphere. The best approximation that the Chinese had to π was 3.14159, and this was obtained by approximations from inscribing regular polygons with 3×2^n sides in a circle.

The Chinese made contributions to number theory including the summation of arithmetic series and solving simultaneous congruences. The Chinese remainder theorem deals with finding the solutions to a set of simultaneous congruences in modular arithmetic. Chinese astronomers made accurate observations, which were used to produce a new calendar in the sixth century. This was known as the Taming Calendar and it was based on a cycle of 391 years.

Indian mathematicians have made important contributions such as the development of the decimal notation for numbers that is now used throughout the world. This was developed in India sometime between 400 B.C. and 400 A.D. Indian mathematicians also invented zero and negative numbers, and also did early work on the trigonometric functions of sine and cosine. The knowledge of the decimal numerals reached Europe through Arabic mathematicians, and the resulting system is known as the Hindu–Arabic numeral system.

The Sulva Sutras is a Hindu text that documents Indian mathematics and it dates from about 400 B.C. They were familiar with the statement and proof of Pythagoras's theorem, Rational numbers, quadratic equations, as well as the calculation of the square root of 2 to five decimal places.

1.8 Review Questions

1. Discuss the strengths and weaknesses of the various numbering system.
2. Describe the ciphers used during the Roman civilization and write a program to implement one of these.
3. Discuss the nature of an algorithm and its importance in computing.
4. Discuss the working of an abacus and its application to calculation.
5. What are the differences between syllogistic logic and stoic logic?
6. Describe the main achievements of the Islamic world in mathematics.

1.9 Summary

Software is pervasive in the modern world, and it has transformed the world in which we live in. New technology has led to improvements in all aspects of our lives including medicine, transport, education, and so on. The pace of change of new technology is relentless, with new versions of technology products becoming available several times a year.

This chapter considered some of the contributions of early civilizations to computing. We commenced our journey with an examination of some of the contributions of the Babylonians. We then moved forward to consider some of the achievements of the Egyptians, the Greek and Romans; Islamic scholars; and the Indians and Chinese.

The Babylonians recorded their mathematical knowledge on clay cuneiform tablets. These tablets included tables for multiplication, division, squares and square roots and the calculation of area. They were familiar with techniques that allowed the solution of a linear equation and one root of a quadratic equation to be determined.

The Egyptian civilization developed along the River Nile, and they applied their knowledge of mathematics to solve practical problem such as measuring the annual Nile flooding, and constructing temples and pyramids.

The Greeks and the later Hellenistic period made important contributions to western civilization. Their contributions to mathematics included the Euclidean algorithm, which is used to determine the greatest common divisor of two numbers. Eratosthenes developed an algorithm to determine the prime numbers up to a given number. Archimedes invented the 'Archimedes Screw', the 'Archimedes Claw', and a type of heat ray.

The Islamic civilization helped to preserve western knowledge that was lost during the dark ages in Europe, and they also continued to develop mathematics and algebra. Hindu mathematicians introduced the decimal notation that is familiar today. Islamic mathematicians adopted it and the resulting system is known as the Hindu–Arabic system.

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