

Key Topics

- Matrix
- Matrix Operations
- Inverse of a Matrix
- Determinant
- Eigen Vectors and Values
- Cayley–Hamilton Theorem
- Cramer’s Rule

8.1 Introduction

A *matrix* is a rectangular array of numbers that consists of horizontal rows and vertical columns. A matrix with m rows and n columns is termed an $m \times n$ matrix, where m and n are its dimensions. A matrix with an equal number of rows and columns (e.g. n rows and n columns) is termed a *square* matrix. Figure 8.1 is an example of a square matrix with four rows and four columns.

The entry in the i th row and the j th column of a matrix A is denoted by $A[i, j]$, $A_{i, j}$, or a_{ij} , and the matrix A may be denoted by the formula for its (i, j) th entry: i.e. (a_{ij}) where i ranges from 1 to m and j ranges from 1 to n .

An $m \times 1$ matrix is termed a *column vector*, and a $1 \times n$ matrix is termed a *row vector*. Any row or column of a $m \times n$ matrix determines a row or column vector which is obtained by removing the other rows (respectively, columns) from the

$$\begin{pmatrix} 6 & 0 & -2 & 3 \\ 4 & 2 & 3 & 7 \\ 11 & -5 & 5 & 3 \\ 3 & -5 & -8 & 1 \end{pmatrix}$$

Fig. 8.1 Example of a 4×4 square matrix

matrix. For example, the row vector $(11, -5, 5, 3)$ is obtained from the matrix example by removing rows 1, 2, and 4 of the matrix.

Two matrices A and B are equal if they are both of the same dimensions, and if $a_{ij} = b_{ij}$ for each $i = 1, 2, \dots, m$ and each $j = 1, 2, \dots, n$.

Matrices be added or multiplied (provided certain conditions are satisfied). There are identity matrices under the addition and multiplication binary operations such that the addition of the (additive) identity matrix to any matrix A yields A and similarly for the multiplicative identity. Square matrices have inverses (provided that their determinant is non-zero), and every square matrix satisfies its characteristic polynomial.

It is possible to consider matrices with infinite rows and columns, and although it is not possible to write down such matrices explicitly it is still possible to add, subtract and multiply by a scalar provided there is a well-defined entry in each (i, j) th element of the matrix.

Matrices are an example of an algebraic structure known as *algebra*. Chapter 6 discussed several algebraic structures such as groups, rings, fields and vector spaces. The matrix algebra for $m \times n$ matrices A, B, C and scalars λ, μ satisfies the following properties (there are additional multiplicative properties for square matrices).

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $A + 0 = 0 + A = A$
4. $A + (-A) = (-A) + A = 0$
5. $\lambda(A + B) = \lambda A + \lambda B$
6. $(\lambda + \mu)A = \lambda A + \mu B$
7. $\lambda(\mu A) = (\lambda\mu) A$
8. $1A = A$

Matrices have many applications including their use in graph theory to keep track of the distance between pairs of vertices in the graph; a rotation matrix may be employed to represent the rotation of a vector in three-dimensional space. The product of two matrices represents the composition of two linear transformations, and matrices may be employed to determine the solution to a set of linear equations.

They also arise in computer graphics and may be employed to project a three-dimensional image onto a two-dimensional screen. It is essential to employ efficient algorithms for matrix computation, and this is an active area of research in the field of numerical analysis.

8.2 Two \times Two Matrices

Matrices arose in practice as a means of solving a set of linear equations. One of the earliest examples of their use is in a Chinese text dating from between 300 B.C. and 200 A.D. The Chinese text showed how matrices could be employed to solve simultaneous equations. Consider the set of equations:

$$\begin{aligned}ax + by &= r \\cx + dy &= s\end{aligned}$$

Then the coefficients of the linear equations in x and y above may be represented by the matrix A , where A is given by:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The linear equations may be represented as the multiplication of the matrix A and a vector \underline{x} resulting in a vector \underline{v} :

$$A\underline{x} = \underline{v}.$$

The matrix representation of the linear equations and its solution are as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix}$$

The vector \underline{x} may be calculated by determining the inverse of the matrix A (provided that its inverse exists). The vector \underline{x} is then given by:

$$\underline{x} = A^{-1}\underline{v}$$

The solution to the set of linear equations is then given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} r \\ s \end{pmatrix}$$

The inverse of a matrix A exists if and only if its *determinant* is non-zero, and if this is the case the vector \underline{x} is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix}$$

The determinant of a 2×2 matrix A is given by:

$$\det A = ad - cb.$$

The determinant of a 2×2 matrix is denoted by:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

A key property of determinants is that

$$\det(AB) = \det(A) \cdot \det(B)$$

The transpose of a 2×2 matrix A (denoted by A^T) involves exchanging rows and columns, and is given by:

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

The inverse of the matrix A (denoted by A^{-1}) is given by:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Further, $A \cdot A^{-1} = A^{-1} \cdot A = I$ where I is the identity matrix of the algebra of 2×2 matrices under multiplication. That is:

$$AA^{-1} = A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The addition of two 2×2 matrices A and B is given by a matrix whose entries are the addition of the individual components of A and B . The addition of two matrices is commutative and we have:

$$A + B = B + A = \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix}$$

where A , B are given as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

The identity matrix under addition is given by the matrix whose entries are all 0, and it has the property that $A + 0 = 0 + A = A$.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The multiplication of two 2×2 matrices is given as

$$AB = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$$

The multiplication of matrices is not commutative: i.e. $AB \neq BA$. The multiplicative identity matrix I has the property that $A \cdot I = I \cdot A = A$, and it is given as

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A matrix A may be multiplied by a scalar λ , and this yields the matrix λA where each entry in A is multiplied by the scalar λ . That is the entries in the matrix λA are λa_{ij} .

8.3 Matrix Operations

More general sets of linear equations may be solved with $m \times n$ matrices (i.e. a matrix with m rows and n columns) or square $n \times n$ matrices. In this section we consider several matrix operations including addition, subtraction, multiplication of matrices, scalar multiplication and the transpose of a matrix.

The addition and subtraction of two matrices A , B is meaningful if and only if A and B have the same dimensions: i.e. they are both $m \times n$ matrices. In this case, $A + B$ is defined by adding the corresponding entries:

$$(A + B)_{ij} = A_{ij} + B_{ij}$$

The additive identity matrix for the square $n \times n$ matrices is denoted by 0 , where 0 is a $n \times n$ matrix whose entries are zero: i.e. $r_{ij} = 0$ for all i, j where $1 \leq i \leq n$ and $1 \leq j \leq n$.

The scalar multiplication of a matrix A by a scalar k is meaningful and the resulting matrix kA is given by:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3p} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ c_{31} & c_{32} & \cdots & c_{3p} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$

m rows, n columns n rows, p columns m rows, p columns

Fig. 8.2 Multiplication of two matrices

$$(kA)_{ij} = kA_{ij}$$

The multiplication of two matrices A and B is meaningful if and only if the number of columns of A is equal to the number of rows of B (Fig. 8.2): i.e. A is an $m \times n$ matrix and B is a $n \times p$ matrix and the resulting matrix AB is a $m \times p$ matrix.

Let $A = (a_{ij})$ where i ranges from 1 to m and j ranges from 1 to n , and let $B = (b_{jl})$ where j ranges from 1 to n and l ranges from 1 to p . Then AB is given by (c_{il}) where i ranges from 1 to m and l ranges from 1 to p with c_{il} given as

$$c_{il} = \sum_{k=1}^n a_{ik}b_{kl}.$$

That is, the entry (c_{il}) is given by multiplying the i th row in A by the l th column in B followed by a summation. Matrix multiplication is not commutative: i.e. $AB \neq BA$.

The identity matrix I is a $n \times n$ matrix and the entries are given by r_{ij} where $r_{ii} = 1$ and $r_{ij} = 0$ where $i \neq j$ (Fig. 8.3). A matrix that has non-zero entries only on the diagonal is termed a *diagonal matrix*. A triangular matrix is a square matrix in which all the entries above or below the main diagonal are zero. A matrix is an *upper triangular* matrix if all entries below the main diagonal are zero, and *lower triangular* if all of the entries above the main diagonal are zero. Upper triangular and lower triangular matrices form a sub algebra of the algebra of square matrices.

A key property of the identity matrix is that for all $n \times n$ matrices A we have:

$$AI = IA = A$$

The inverse of a $n \times n$ matrix A is a matrix A^{-1} such that:

$$AA^{-1} = A^{-1}A = I$$

The inverse A^{-1} exists if and only if the determinant of A is non-zero.

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Fig. 8.3 Identity matrix I_n

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \dots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \dots & a_{m3} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{mn} \end{pmatrix}$$

m rows, n columns *n rows, m columns*

Fig. 8.4 Transpose of a matrix

The *transpose* of a matrix $A = (a_{ij})$ involves changing the rows to columns and vice versa to form the transpose matrix A^T . The result of the operation is that the $m \times n$ matrix A is converted to the $n \times m$ matrix A^T (Fig. 8.4). It is defined by:

$$(A^T)_{ij} = (A_{ji}) \quad 1 \leq j \leq n, \text{ and } 1 \leq i \leq m$$

A matrix is *symmetric* if it is equal to its transpose: i.e. $A = A^T$.

8.4 Determinants

The determinant is a function defined on square matrices and its value is a scalar. A key property of determinants is that a matrix is invertible if and only if its determinant is non-zero. The determinant of a 2×2 matrix is given by:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Fig. 8.5 Determining the (i, j) minor of A

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3j} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{pmatrix} =$$

i, j minor of A

The determinant of a 3×3 matrix is given by:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

Cofactors

Let A be an $n \times n$ matrix. For $1 \leq i, j \leq n$, the (i, j) minor of A is defined to be the $(n - 1) \times (n - 1)$ matrix obtained by deleting the i th row and j th column of A (Fig. 8.5).

The shaded row is the i th row and the shaded column is the j th column. These are both deleted from A to form the (i, j) minor of A , and this is a $(n - 1) \times (n - 1)$ matrix.

The (i, j) cofactor of A is defined to be $(-1)^{i+j}$ times the determinant of the (i, j) minor. The (i, j) cofactor of A is denoted by $K_{ij}(A)$.

The cofactor matrix $Cof A$ is formed in this way where the (i, j) th element in the cofactor matrix is the (i, j) cofactor of A .

Definition of Determinant

The determinant of a matrix is defined as

$$\det A = \sum_{j=1}^n A_{ij}K_{ij}$$

In other words, the determinant of A is determined by taking any row of A and multiplying each element by the corresponding cofactor and adding the results. The determinant of the product of two matrices is the product of their determinants.

$$\det(AB) = \det A \times \det B$$

Definition

The *adjugate* of A is the $n \times n$ matrix $Adj(A)$ whose (i, j) entry is the (j, i) cofactor $K_{ji}(A)$ of A . That is, the adjugate of A is the transpose of the cofactor matrix of A .

Inverse of A

The inverse of A is determined from the determinant of A and the adjugate of A . That is,

$$A^{-1} = \frac{1}{\det A} Adj A = \frac{1}{\det A} (Cof A)^T$$

A matrix is invertible if and only if its determinant is non-zero: i.e. A is invertible if and only if $\det(A) \neq 0$.

Cramer's Rule

Cramer's rule is a theorem that expresses the solution to a system of linear equations with several unknowns using the determinant of a matrix. There is a unique solution if the determinant of the matrix is non-zero.

For a system of linear equations of the $A\underline{x} = \underline{y}$ where \underline{x} and \underline{y} are n -dimensional column vectors, then if $\det A \neq 0$ then the unique solution for each x_i is

$$x_i = \frac{\det U_i}{\det A}$$

where U_i is the matrix obtained from A by replacing the i th column in A by the \underline{y} -column.

Characteristic Equation

For every $n \times n$ matrix A there is a polynomial equation of degree n satisfied by A . The *characteristic polynomial* of A is a polynomial in x of degree n . It is given as

$$cA(x) = \det(xI - A).$$

Cayley-Hamilton Theorem

Every matrix A satisfies its characteristic polynomial: i.e. $p(A) = 0$ where $p(x)$ is the characteristic polynomial of A .

8.5 Eigen Vectors and Values

A number λ is an eigenvalue of a $n \times n$ matrix A if there is a non-zero vector \underline{v} such that the following equation holds:

$$A\underline{v} = \lambda \underline{v}$$

This method for solving a set of linear equations is best illustrated by an example, and we consider an example taken from [1]. Then, the solution to a set of linear equations with four unknowns may be determined as follows:

$$\begin{aligned} E_1 : & x_1 + x_2 + 3x_4 = 4 \\ E_2 : & 2x_1 + x_2 - x_3 + x_4 = 1 \\ E_3 : & 3x_1 - x_2 - x_3 + 2x_4 = -3 \\ E_4 : & -x_1 + 2x_2 + 3x_3 - x_4 = 4 \end{aligned}$$

First, the unknown x_1 is eliminated from E_2 , E_3 , and E_4 and this is done by replacing E_2 with $E_2 - 2E_1$; replacing E_3 with $E_3 - 3E_1$; and replacing E_4 with $E_4 + E_1$. The resulting system is

$$\begin{aligned} E_1 : & x_1 + x_2 + 3x_4 = 4 \\ E_2 : & -x_2 - x_3 - 5x_4 = -7 \\ E_3 : & -4x_2 - x_3 - 7x_4 = -15 \\ E_4 : & 3x_2 + 3x_3 + 2x_4 = 8 \end{aligned}$$

The next step is then to eliminate x_2 from E_3 and E_4 . This is done by replacing E_3 with $E_3 - 4E_2$ and replacing E_4 with $E_4 + 3E_2$. The resulting system is now in triangular form and the unknown variable may be solved easily by backward substitution. That is, we first use equation E_4 to find the solution to x_4 and then we use equation E_3 to find the solution to x_3 . We then use equations E_2 and E_1 to find the solutions to x_2 and x_1 .

$$\begin{aligned} E_1 : & x_1 + x_2 + 3x_4 = 4 \\ E_2 : & -x_2 - x_3 - 5x_4 = -7 \\ E_3 : & 3x_3 + 13x_4 = 13 \\ E_4 : & -13x_4 = -13 \end{aligned}$$

The usual approach to Gaussian elimination is to do it with an augmented matrix. That is, the set of equations is a $n \times n$ matrix and it is augmented by the column vector to form the augmented $n \times n + 1$ matrix. Gaussian elimination is then applied to the matrix to put it into triangular form, and it is then easy to solve the unknowns.

The other common approach to solving a set of linear equation is to employ Cramer's rule, which was discussed in Sect. 13.4. Finally, another possible (but computationally expensive) approach to solving the set of linear equations $A\underline{x} = \underline{v}$ is to compute the determinant and inverse of A , and to then compute $\underline{x} = A^{-1}\underline{v}$.

8.7 Review Questions

1. Show how 2×2 matrices may be added and multiplied.
2. What is the additive identity for 2×2 matrices? The multiplicative identity?
3. What is the determinant of a 2×2 matrix?
4. Show that a 2×2 matrix is invertible if its determinant is non-zero.
5. Describe general matrix algebra including addition and multiplication, determining the determinant and inverse of a matrix.
6. What is Cramer's rule?
7. Show how Gaussian elimination may be used to solve a set of linear equations.
8. Write a program to find the inverse of a 3×3 and then a $(n \times n)$ matrix.

8.8 Summary

A matrix is a rectangular array of numbers that consists of horizontal rows and vertical columns. A matrix with m rows and n columns is termed an $m \times n$ matrix, where m and n are its dimensions. A matrix with an equal number of rows and columns (e.g. n rows and n columns) is termed a square matrix.

Matrices arose in practice as a means of solving a set of linear equations, and one of the earliest examples of their use is from a Chinese text dating from between 300 B.C. and 200 A.D.

Matrices of the same dimensions may be added, subtracted, and multiplied by a scalar. Two matrices A and B may be multiplied provided that the number of columns of A equals the number of rows in B .

Matrices have an identity matrix under addition and multiplication, and a square matrix has an inverse provided that its determinant is non-zero. The inverse of a matrix involves determining its determinant, constructing the cofactor matrix, and transposing the cofactor matrix.

The solution to a set of linear equations may be determined by Gaussian elimination to convert the matrix to upper triangular form, and then employing backward substitution. Another approach is to use Cramer's rule.

Eigenvalues and eigenvectors lead to the characteristic polynomial and every matrix satisfies its characteristic polynomial. The characteristic polynomial is of degree n , and a square $n \times n$ matrix has at most n eigenvalues.

Reference

1. Numerical Analysis. 4th Edition. Richard L. Burden and J. Douglas Faires. PWS Kent. 1989.