5 Sequences, Series and Permutations and Combinations

Key Topics

Arithmetic sequence
Arithmetic series
Geometric Sequence
Geometric Series
Simple and compound interest
Annuities
Present Value
Permutations and Combinations
Counting Principle

5.1 Introduction

The goal of this chapter is to provide an introduction to sequences and series, including arithmetic and geometric sequences, and arithmetic and geometric series. We derive formulae for the sum of an arithmetic series and geometric series, and we discuss the convergence of a geometric series when $|r| < 1$, and the limit of its sum as n gets larger and larger.

We discuss the calculation of simple and compound interest, and the concept of the time value of money, and its application to determine the present value of a payment to be made in the future. We then discuss annuities, which are a series of payments made at regular intervals over a period of time, and we determine the present value of an annuity.

G. O'Regan, *Guide to Discrete Mathematics*, Texts in Computer Science,
DOI 10.1007/978-3-319-44561-8_5

We consider the counting principle where one operation has m possible outcomes and a second operation has n possible outcomes. We determine that the total number of outcomes after performing the first operation is followed by the second operation to be $m \times n$. A permutation is an arrangement of a given number of objects, by taking some or all of them at a time. The order of the arrangement is important, as the arrangement ‘abc’ is different from ‘cba’. A combination is a selection of a number of objects in any order, where the order of the selection is unimportant. That is, the selection ‘abc’ is the same as the selection ‘cba’.

5.2 Sequences and Series

A sequence $a_1, a_2, \ldots a_n \ldots$ is any succession of terms (usually numbers), and we discussed the Fibonacci sequence earlier in Chap. 4. Each term in the Fibonacci sequence (apart from the first two terms) is obtained from the sum of the previous two terms in the sequence.

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

A sequence may be finite (with a fixed number of terms) or infinite. The Fibonacci sequence is infinite whereas the sequence 2, 4, 6, 8, 10 is finite. We distinguish between convergent and divergent sequences, where a *convergent* sequence approaches a certain value as n gets larger and larger (technically we say that ${}_{n\to\propto}\lim a_n$ exists (i.e. the limit of a_n exists). Otherwise, the sequence is said to be *divergent*.

Often, there is a mathematical expression for the nth term in a sequence (e.g. for the sequence of even integers 2, 4, 6, 8, … the general expression for a_n is given by $a_n = 2n$). Clearly, the sequence of the even integers is divergent, as it does not approach a particular value and as n gets larger and larger. Consider the following sequence:

$$1, -1, 1, -1, 1, -1$$

Then this sequence is divergent since it does not approach a certain value, as n gets larger and larger and since it continues to alternate between 1 and −1. The formula for the nth term in the sequence may be given by

$$(-1)^{n.+1}$$

The sequence 1, 1/2, 1/3. 1/4, …1/n … is convergent and it converges to 0. The nth term in the sequence is given by 1/n, and as n gets larger and larger it gets closer and closer to 0.

A series is the sum of the terms in a sequence, and the sum of the first n terms of the sequence a_1, a_2, …. a_n … is given by $a_1 + a_2 + \cdots + a_n$ which is denoted by

$$\sum_{k=1}^{n} a_k$$

A series is convergent if its sum approaches a certain value S as n gets larger and larger, and this is written formally as

$$\lim_{n \to \infty} \sum_{k=1}^{n} a_k = S$$

Otherwise, the series is said to be divergent.

5.3 Arithmetic and Geometric Sequences

Consider the sequence 1, 4, 7, 10,… where each term is obtained from the previous term by adding the constant value 3. This is an example of an arithmetic sequence, and there is a difference of 3 between any term and the previous one. The general form of a term in this sequence is $a_n = 3n - 2$.

The general form of an *arithmetic sequence* is given by

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots\ a+(n-1)d, \cdots$$

The value a is the initial term in the sequence, and the value d is the constant difference between a term and its successor. For the sequence, 1, 4, 7, …, we have $a = 1$ and $d = 3$, and the sequence is not convergent. In fact, all arithmetic sequences (apart from the constant sequence a, a, … a which converges to a) are divergent.

Consider, the sequence 1, 3, 9, 27, 81,… where each term is achieved from the previous term by multiplying by the constant value 3. This is an example of a geometric sequence, and the general form of a geometric sequence is given by

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1}$$

The first term in the geometric sequence is a and r is the common ratio. Each term is obtained from the previous one by multiplying by the common ratio r. For the sequence 1, 3, 9, 27 the value of a is 1 and r is 3.

A geometric sequence is convergent if $r < 1$, and for this case it converges to 0. It is also convergent if $r = 1$, as for this case it is simply the constant sequence a, a, a, …, which converges to a. For the case where $r > 1$ the sequence is divergent.

5.4 Arithmetic and Geometric Series

An arithmetic series is the sum of the terms in an arithmetic sequence, and a geometric sequence is the sum of the terms in a geometric sequence. It is possible to derive a simple formula for the sum of the first n terms in an arithmetic and geometric series.

Arithmetic Series
We write the series two ways: first the normal left to right addition, and then the reverse, and then we add both series together.

$$\begin{aligned}
Sn &= a \quad + (a+d) + (a+2d) + (a+3d) + \ldots + (a+(n-1)d) \\
Sn &= a+(n-1)d + a+(n-2)d + \ldots + \quad + (a+d) + a \\
&- \\
2Sn &= [2a+(n-1)d] + [2a+(n-1)d] + \ldots + [2a+(n-1)d] \qquad (n\,\text{times}) \\
2Sn &= n \times [2a+(n-1)d]
\end{aligned}$$

Therefore, we conclude that

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

Example (**Arithmetic Series**) Find the sum of the first n terms in the following arithmetic series 1, 3, 5, 7, 9.

Solution
Clearly, $a = 1$ and $d = 2$. Therefore, applying the formula we get

$$S_n = \frac{n}{2}[2.1 + (n-1)2] = \frac{2n^2}{2} = n^2$$

Geometric Series
For a geometric series we have

$$\begin{aligned}
S_n &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \\
\Rightarrow rS_n &= \quad ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \\
&- \\
\Rightarrow rS_n - S_n &= ar^n - a = a(r^n - 1) \\
\Rightarrow (r-1)S_n &= a(r^n - 1)
\end{aligned}$$

Therefore, we conclude that (where $r \neq 1$) that

$$S_n = a\frac{(r^n - 1)}{r - 1} = a\frac{(1 - r^n)}{1 - r}$$

The case of when $r = 1$ corresponds to the arithmetic series $a + a + \cdots + a$, and the sum of this series is simply na. The geometric series converges when $|r| < 1$ as $r^n \to 0$ as $n \to \propto$, and so

$$S_n \to \frac{a}{1 - r} \qquad as\, n \to \propto$$

Example (**Geometric Series**) Find the sum of the first n terms in the following geometric series 1, 1/2, 1/4, 1/8, … What is the sum of the series?

Solution
Clearly, $a = 1$ and $r = {}^1/_2$. Therefore, applying the formula we get

$$S_n = 1\frac{(1 - 1/2^n)}{1 - 1/2} = \frac{(1 - 1/2^n)}{1 - 1/2} = 2(1 - 1/2^n)$$

The sum of the series is the limit of the sum of the first n terms as n approaches infinity. This is given by

$$\lim_{n \to \propto} S_n = \lim_{n \to \propto} 2(1 - 1/2^n) = 2$$

5.5 Simple and Compound Interest

Savers receive interest on placing deposits at the bank for a period of time, whereas lenders pay interests on their loans to the bank. We distinguish between simple and compound interest, where *simple interest* is always calculated on the original principal, whereas for *compound interest*, the interest is added to the principal sum, so that interest is also earned on the added interest for the next compounding period.

For example, if Euro 1000 is placed on deposit at a bank with an interest rate of 10 % per annum for 2 years, it would earn a total of Euro 200 in simple interest. The interest amount is calculated by

$$\frac{1000 * 10 * 2}{100} \qquad = \text{Euro}\, 200$$

The general formula for calculating simple interest on principal P, at a rate of interest I, and for time T (in years:) is

$$A = \frac{P \times I \times T}{100}$$

The calculation of compound interest is more complicated as may be seen from the following example:

Example (**Compound Interest**) Calculate the interest earned and what the new principal will be on Euro 1000, which is placed on deposit at a bank, with an interest rate of 10 % per annum (compound) for 3 years.

Solution
At the end of year 1, Euro 100 of interest is earned, and this is capitalized making the new principal at the start of year 2 Euro 1100. At the end of year 2, Euro 110 is earned in interest, making the new principal at the start of year 3 Euro 1210. Finally, at the end of year 3 a further Euro 121 is earned in interest, and so the new principal is Euro 1331 and the total interest earned for the 3 years is the sum of the interest earned for each year (i.e. Euro 331). This may be seen from Table 5.1.

The new principal each year is given by the geometric sequence with a = 1000 and r = 10/100 = 0.1.

$$1000,\ 1000(1.1),\ 1000(1.1)^2,\ 1000(1.1)^3, \ldots\ldots$$

In general, if a principal amount P is invested for T years at a rate R of interest (r is expressed as a proportion, i.e. $r = R/100$) then it will amount to

$$\mathrm{A} = P(1+r)^T$$

For our example above, A = 1000, T = 3 and r = 0.1. Therefore,

$$\begin{aligned}\mathrm{A} &= 1000(1.1)^3 \\ &= 1331 (\text{as before})\end{aligned}$$

There are variants of the compound interest formula to cover situations where there are m-compounding periods per year, and so the reader may consult the available texts.

Table 5.1 Calculation of compound interest

Year	Principal	Interest earned
1	1000	100
2	1100	110
3	1210	121

5.6 Time Value of Money and Annuities

The time value of money discusses the concept that the earlier that cash is received the greater value it has to the recipient. Similarly, the later that a cash payment is made, the lower its value to the recipient, and the lower its cost to the payer.

This is clear if we consider the example of a person who receives $1000 now and a person who receives $1000 five years from now. The person who receives $1000 now is able to invest it and to receive annual interest on the principal, whereas the other person who receives $1000 in 5 years earns no interest during the period. Further, the inflation during the period means that the purchasing power of $1000 is less in 5 years time and is less than it is today.

We presented the general formula for what the future value of a principal P invested for n years at a compound rate r of interest as $A = P\ (1 + r)^n$

We can determine the present value of an amount A received in n years time at a discount rate r by

$$P = \frac{A}{(1+r)^n}$$

An annuity is a series of equal cash payments made at regular intervals over a period of time, and so there is a need to calculate the present value of the series of payments made over the period. The actual method of calculation is clear from Table 5.2.

Example (Annuities) Calculate the present value of a series of payments of $1000 (made at the end of each year) with the payments made for 5 years at a discount rate of 10 %.

Solution

The regular payment A is 1000, the rate r is 0.1 and $n = 5$. The present value of the payment received at the end of year of year 1 is 1000/1.1 = 909.91; at the end of year 2 it is $1000/(1.1)^2 = 826.45$; and so on. The total present value of the payments over the 5 years is given by the sum of the individual present values and is $3791 (Table 5.2).

We may easily derive a formula for the present value of a series of payments A over a period of n years at a discount rate of r as follows: Clearly, the present value is given by

Table 5.2 Calculation of present value of annuity

Year	Amount	Present value ($r = 0.1$)
1	1000	909.91
2	1000	826.44
3	1000	751.31
4	1000	683.01
5	1000	620.92

$$\frac{A}{(1+r)} + \frac{A}{(1+r)^2} + \cdots + \frac{A}{(1+r)^n}$$

This is a geometric series where the constant ratio is $\frac{1}{1+r}$ and the present value of the annuity is given by its sum

$$\text{PV} = \frac{A}{r}[1 - \frac{1}{(1+r)^n}]$$
$$\text{PV} = \frac{1000}{0.1}[1 - \frac{1}{(1.1)^5}]$$

For the example above we apply the formula and get

$$= 10{,}000(0.3791)$$
$$= \$3791$$

5.7 Permutations and Combinations

A permutation is an arrangement of a given number of objects, by taking some or all of them at a time. A combination is a selection of a number of objects where the order of the selection is unimportant. Permutations and combinations are defined in terms of the factorial function, which was defined in Chap. 4. Recall that $n! = n(n - 1)\cdots 3.2.1$.

Principles of Counting

(a) Suppose one operation has m possible outcomes and a second operation has n possible outcomes, then the total number of possible outcomes when performing the first operation followed by the second operation is $m \times n$. (**Product Rule**).
(b) Suppose one operation has m possible outcomes and a second operation has n possible outcomes then the possible outcomes of the first operation **or** the second operation is given by $m + n$. (***Sum Rule***)

Example (**Counting Principle (*a*)**) Suppose a dice is thrown and a coin is then tossed. How many different outcomes are there and what are they?

Solution
There are six possible outcomes from a throw of the dice: 1, 2, 3, 4, 5 or 6, and there are two possible outcomes from the toss of a coin: H or T. Therefore, the total number of outcomes is determined from the product rule as $6 \times 2 = 12$. The outcomes are given by

$$(1, H),\ (2, H),\ (3, H),\ (4, H),\ (5, H),\ (6, H),\ (1, T),\ (2, T),\ (3, T),\ (4, T),\ (5, T),\ (6, T)$$

Example (**Counting Principle (*b*)**) Suppose a dice is thrown and if the number is even a coin is tossed and if it is odd then there is a second throw of the dice. How many different outcomes are there?

Solution
There are two experiments involved with the first experiment involving an even number and a toss of a coin. There are three possible outcomes that result in an even number and two outcomes from the toss of a coin. Therefore, there are $3 \times 2 = 6$ outcomes from the first experiment.

The second experiment involves an odd number from the throw of a dice and the further throw of the dice. There are three possible outcomes that result in an odd number and six outcomes from the throw of a dice. Therefore, there are $3 \times 6 = 18$ outcomes from the second experiment.

Finally, there are six outcomes from the first experiment and 18 outcomes from the second experiment, and so from the sum rule there are a total of $6 + 18 = 24$ outcomes.

Pigeonhole Principle
The pigeonhole principle states that if n items are placed into m containers (with $n > m$) then at least one container must contain more than one item.

Examples (**Pigeonhole Principle**)

(a) Suppose there is a group of 367 people then there must be at least two people with the same birthday.
This is clear as there are 365 days in a year (with 366 days in a leap year), and so as there are at most 366 possible birthdays in a year. The group size is 367 people, and so there must be at least two people with the same birthday.
(b) Suppose that a class of 102 students are assessed in an examination (the outcome from the exam is a mark between 0 and 100). Then, there are at least two students who receive the same mark.
This is clear as there are 101 possible outcomes from the test (as the mark that a student may achieve is between is between 0 and 100), and as there are 102 students in the class and 101 possible outcomes from the test, then there must be at least two students who receive the same mark.

Permutations
A permutation is an arrangement of a number of objects in a definite order.

Consider the three letters A, B and C. If these letters are written in a row then there are six possible arrangements:

ABC or ACB or BAC or BCA or CAB or CBA

There is a choice of three letters for the first place, then there is a choice of two letters for the second place, and there is only one choice for the third place. Therefore, there are $3 \times 2 \times 1 = 6$ arrangements.

If there are n different objects to arrange then the total number of arrangements (permutations) of n objects is given by $n! = n(n-1)(n-2) \ldots 3.2.1$.

Consider the four letters A, B, C and D. How many arrangements (taking two letters at a time with no repetition) of these letters can be made?

There are four choices for the first letter and three choices for the second letter, and so there are 12 possible arrangements. These are given by

AB or AC or AD or BA or BC or BD or CA or CB or CD or DA or DB or DC

The total number of arrangements of n different objects taking r at a time ($r \leq n$) is given by ${}^{n}P_{r} = n(n-1)(n-2) \ldots (n-r+1)$. It may also be written as

$$nP_r = \frac{n!}{(n-r)!}$$

Example (**Permutations**) Suppose A, B, C, D, E and F are six students. How many ways can they be seated in a row if

(i) There is no restriction on the seating.
(ii) A and B must sit next to one another
(iii) A and B must not sit next to one another

Solution
For unrestricted seating the number of arrangements is given by $6.5.4.3.2.1 = 6! = 720$.

For the case where A and B must be seated next to one another, then consider A and B as one person, and then the five people may be arranged in $5! = 120$ ways. There are $2! = 2$ ways in which AB may be arranged, and so there are $2! \times 5! = 240$ arrangements.

AB	C	D	E	F

For the case where A and B must not be seated next to one another, then this is given by the difference between the total number of arrangements and the number of arrangements with A and B together: i.e. $720 - 240 = 480$.

Combinations

A combination is a selection of a number of objects in any order, and the order of the selection is unimportant, in that both AB and BA represent the same selection. The total number of arrangements of n different objects taking r at a time is given by ^nP_r, and we can determine that the number of ways that r objects can be selected from n different objects from this, as each selection may be permuted $r!$ times, and so the total number of selections is $r! \times$ total number of combinations. That is, $^nP_r = r! \times {^nC_r}$, and we may also write this as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$$

It is clear from the definition that

$$\binom{n}{r} = \binom{n}{n-r}$$

Example 1 (**Combinations**) How many ways are there to choose a team of 11 players from a panel of 15 players?

Solution

Clearly, the number of ways is given by $\binom{15}{11} = \binom{15}{4}$

That is, 15.14.13.12/4.3.2.1 = 1365.

Example 2 (**Combinations**) How many ways can a committee of four people be chosen from a panel of 10 people where

(i) There is no restriction on membership of the panel.
(ii) A certain person must be a member.
(iii) A certain person must not be a member.

Solution

For (i) with no restrictions on membership the number of selections of a committee of four people from a panel of 10 people is given by $\binom{10}{4} = 210$

For (ii) where one person must be a member of the committee then this involves choosing three people from a panel of nine people and is given by $\binom{9}{3} = 84$

For (iii) where one person must not be a member of the committee then this involves choosing four people from a panel of nine people, and is given by $\binom{9}{4} = 126$

5.8 Review Questions

1. Determine the formula for the general term and the sum of the following arithmetic sequence:

$$1, 4, 7, 10, \ldots.$$

2. Write down the formula for the *n*th term in the following sequence:

$$1/4, 1/12, 1/36, 1/108, \ldots.$$

3. Find the sum of the following geometric sequence:

$$1/3, 1/6, 1/12, 1/24, \ldots.$$

4. How many years will it take a principal of \$5000 to exceed \$10,000 at a constant annual growth rate of 6 % compound interest?
5. What is the present value of \$5000 to be receive in 5 years time at a discount rate of 7 %?
6. Determine the present value of a 20-year annuity of an annual payment of \$5000 per year at a discount rate of 5 %.
7. How many different five-digit numbers can be formed from the digits 1, 2, 3, 4, 5 where

   (i) No restrictions on digits and repetitions allowed.
   (ii) The number is odd and no repetitions are allowed.
   (iii) The number is even and repetitions are allowed.

8. (i) How many ways can a group of five people be selected from nine people?
   (ii) How many ways can a group be selected if two particular people are always included?
   (iii) How many ways can a group be selected if two particular people are always excluded?

5.9 Summary

This chapter provided a brief introduction to sequences and series, including arithmetic and geometric sequences, and arithmetic series and geometric series. We derived formulae for the sum of an arithmetic series and geometric series, and we discussed the convergence of a geometric series when $|r| < 1$.

We discussed the calculation of simple and compound interest, and the concept of the time value of money, and its application to determine the present value of a payment to be made in the future. We discussed annuities, which are a series of payments made at regular intervals over a period of time, and we calculated the present value of an annuity.

We considered counting principles including the product and sum rules. The product rule is concerned with where one operation has m possible outcomes and a second operation has n possible outcomes then the total number of possible outcomes when performing the first operation followed by the second operation is $m \times n$.

We discussed the pigeonhole principle, which states that if n items are placed into m containers (with $n > m$) then at least one container must contain more than one item. We discussed permutations and combinations where permutations are an arrangement of a given number of objects, by taking some or all of them at a time. A combination is a selection of a number of objects in any order, and the order of the selection is unimportant.