

The graph of a function $y = f(x)$ represents a curve in the plane. This concept, however, is too tight to represent more intricate curves, like loops, self-intersections, or even curves of fractal dimension. The aim of this chapter is to introduce the concept of parametrised curves and to study, in particular, the case of differentiable curves. For the visualisation of the trajectory of a curve, the notions of velocity vector, moving frame, and curvature are important. The chapter contains a collection of geometrically interesting examples of curves and several of their construction principles. Further, the computation of the arc length of differentiable curves is discussed, and an example of a continuous, bounded curve of infinite length is given. The chapter ends with a short outlook on spatial curves. For the vector algebra used in this chapter, we refer to Appendix A.

14.1 Parametrised Curves in the Plane

Definition 14.1 A *parametrised plane curve* is a continuous mapping

$$t \mapsto \mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

of an interval $[a, b]$ to \mathbb{R}^2 ; i.e., both the components $t \mapsto x(t)$ and $t \mapsto y(t)$ are continuous functions.¹ The variable $t \in [a, b]$ is called *parameter of the curve*.

¹Concerning the vector notation we remark that $x(t)$, $y(t)$ actually represent the coordinates of a point in \mathbb{R}^2 . It is, however, common practise and useful to write this point as a position vector, thus the column notation.

Example 14.2 An object that is thrown at height h with horizontal velocity v_H and vertical velocity v_V has the trajectory

$$\begin{aligned} x(t) &= v_H t, \\ y(t) &= h + v_V t - \frac{g}{2} t^2, \quad 0 \leq t \leq t_0, \end{aligned}$$

where t_0 is the positive solution of the equation $h + v_V t_0 - \frac{g}{2} t_0^2 = 0$ (time of impact, see Fig. 14.1). In this example, we can eliminate t and represent the trajectory as the graph of a function (ballistic curve). We have $t = x/v_H$, and thus

$$y = h + \frac{v_V}{v_H} x - \frac{g}{2v_H^2} x^2.$$

Example 14.3 A circle of radius R with centre at the origin has the parametric representation

$$\begin{aligned} x(t) &= R \cos t, \\ y(t) &= R \sin t, \quad 0 \leq t \leq 2\pi. \end{aligned}$$

In this case, t can be interpreted as the angle between the position vector and the positive x -axis (Fig. 14.1). The components $x = x(t)$, $y = y(t)$ satisfy the quadratic equation

$$x^2 + y^2 = R^2;$$

however, one cannot represent the circle in its entirety as the graph of a function.

Experiment 14.4 Open the M-file `mat14_1.m` and discuss which curve is being represented. Compare with the M-files `mat14_2.m` to `mat14_4.m`. Are these the same curves?

Experiment 14.4 suggests that one can view curves statically as a set of points in the plane or dynamically as the trajectory of a moving point. Both perspectives are of importance in applications.

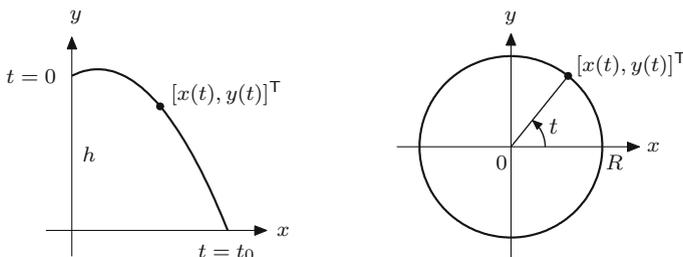


Fig. 14.1 Parabolic trajectory and circle

The kinematic point of view. In the kinematic interpretation, one considers the parameter t of the curve as time and the curve as path. Different parametrisations of the same geometric object are viewed as different curves.

The geometric point of view. In the geometric interpretation, the location, the moving sense and the number of cycles are considered as the defining properties of a curve. The particular parametrisation, however, is irrelevant.

A strictly monotonically increasing, continuous mapping of an interval $[\alpha, \beta]$ to $[a, b]$,

$$\varphi : [\alpha, \beta] \rightarrow [a, b]$$

is called a *change of parameter*. The curve

$$\tau \mapsto \xi(\tau), \quad \alpha \leq \tau \leq \beta$$

is called a *reparametrisation* of the curve

$$t \mapsto \mathbf{x}(t), \quad a \leq t \leq b,$$

if it is obtained through a change of parameter $t = \varphi(\tau)$; i.e.,

$$\xi(\tau) = \mathbf{x}(\varphi(\tau)).$$

From the geometric point of view, the parametrised curves $\tau \mapsto \xi(\tau)$ and $t \mapsto \mathbf{x}(t)$ are identified. A *plane curve* Γ is an *equivalence class of parametrised curves* which can be transformed to one another by reparametrisation.

Example 14.5 We consider the segment of a parabola, parametrised by

$$\Gamma : \mathbf{x}(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}, \quad -1 \leq t \leq 1.$$

Reparametrisations are for instance

$$\begin{aligned} \varphi : \left[-\frac{1}{2}, \frac{1}{2}\right] &\rightarrow [-1, 1], & \varphi(\tau) &= 2\tau, \\ \tilde{\varphi} : [-1, 1] &\rightarrow [-1, 1], & \tilde{\varphi}(t) &= \tau^3. \end{aligned}$$

Consequently,

$$\xi(\tau) = \begin{bmatrix} 2\tau \\ 4\tau^2 \end{bmatrix}, \quad -\frac{1}{2} \leq \tau \leq \frac{1}{2}$$

and

$$\boldsymbol{\eta}(\tau) = \begin{bmatrix} \tau^3 \\ \tau^6 \end{bmatrix}, \quad -1 \leq \tau \leq 1$$

geometrically represent the same curve. However,

$$\begin{aligned} \psi : [-1, 1] &\rightarrow [-1, 1], & \psi(\tau) &= -\tau, \\ \tilde{\psi} : [0, 1] &\rightarrow [-1, 1], & \tilde{\psi}(\tau) &= -1 + 8\tau(1 - \tau) \end{aligned}$$

are not reparametrisations and yield other curves, namely

$$\begin{aligned} \mathbf{y}(\tau) &= \begin{bmatrix} -\tau \\ \tau^2 \end{bmatrix}, & -1 \leq \tau \leq 1, \\ \mathbf{z}(\tau) &= \begin{bmatrix} -1 + 8\tau(1 - \tau) \\ (-1 + 8\tau(1 - \tau))^2 \end{bmatrix}, & 0 \leq \tau \leq 1. \end{aligned}$$

In the first case, the moving sense of Γ is reversed, and in the second case, the curve is traversed twice.

Experiment 14.6 Modify the M-files from Experiment 14.4 so that the curves from Example 14.5 are represented.

Algebraic curves. These are obtained as the set of zeros of polynomials in two variables. As examples we had already parabola and circle

$$y - x^2 = 0, \quad x^2 + y^2 - R^2 = 0.$$

One can also create cusps and loops in this way.

Example 14.7 Neil's² parabola

$$y^2 - x^3 = 0$$

has a cusp at $x = y = 0$ (Fig. 14.2). Generally, one obtains algebraic curves from

$$y^2 - (x + p)x^2 = 0, \quad p \in \mathbb{R}.$$

For $p > 0$ they have a loop. A parametric representation of this curve is, for instance,

$$\begin{aligned} x(t) &= t^2 - p, & -\infty < t < \infty. \\ y(t) &= t(t^2 - p), \end{aligned}$$

²W. Neil, 1637–1670.

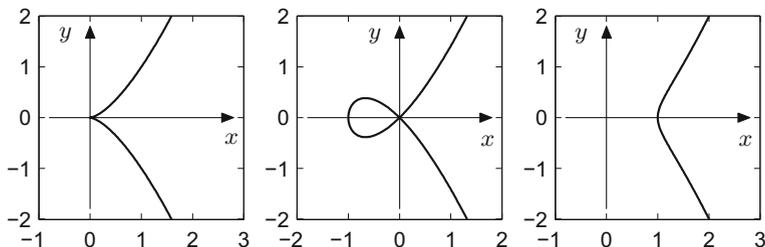


Fig. 14.2 Neil's parabola, the α -curve and an elliptic curve

In the following we will primarily deal with curves which are given by differentiable parametrisations.

Definition 14.8 If a plane curve $\Gamma : t \mapsto \mathbf{x}(t)$ has a parametrisation whose components $t \mapsto x(t)$, $t \mapsto y(t)$ are differentiable, then Γ is called a *differentiable curve*. If the components are k -times differentiable, then Γ is called a k -times differentiable curve.

The graphical representation of a differentiable curve does not have to be smooth but may have cusps and corners, as Example 14.7 shows.

Example 14.9 (Straight line and half ray) The parametric representation

$$t \mapsto \mathbf{x}(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \quad -\infty < t < \infty$$

describes a straight line through the point $\mathbf{x}_0 = [x_0, y_0]^T$ in the direction $\mathbf{r} = [r_1, r_2]^T$. If one restricts the parameter t to $0 \leq t < \infty$ one obtains a half ray. The parametrisation

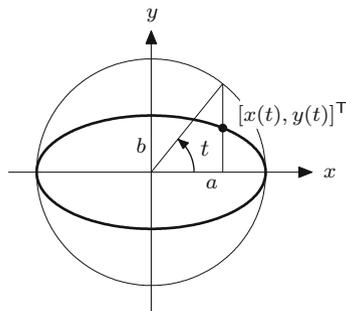
$$\mathbf{x}_H(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t^2 \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \quad -\infty < t < \infty$$

leads to a double passage through the half ray.

Example 14.10 (Parametric representation of an ellipse) The equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Fig. 14.3 Parametric representation of the ellipse



A parametric representation (single passage in counterclockwise sense) is obtained by

$$\begin{aligned} x(t) &= a \cos t, \\ y(t) &= b \sin t, \end{aligned} \quad 0 \leq t \leq 2\pi.$$

This can be seen by substituting these expressions into the equation of the ellipse. The meaning of the parameter t can be seen from Fig. 14.3.

Example 14.11 (Parametric representation of a hyperbola) The hyperbolic sine and the hyperbolic cosine have been introduced in Sect. 2.2. The important identity

$$\cosh^2 t - \sinh^2 t = 1$$

has been noted there. It shows that

$$\begin{aligned} x(t) &= a \cosh t, \\ y(t) &= b \sinh t, \end{aligned} \quad -\infty < t < \infty$$

is a parametric representation of the right branch of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

which is highlighted in Fig. 14.4.

Example 14.12 (Cycloids) A circle with radius R rolls (without sliding) along the x -axis. If the starting position of the centre M is initially $M = (0, R)$, its position will be $M_t = (Rt, R)$ after a turn of angle t . A point P with starting position $P = (0, R - A)$ thus moves to $P_t = M_t - (A \sin t, A \cos t)$.

The trajectory of the point P is called a *cycloid*. It is parametrised by

$$\begin{aligned} x(t) &= Rt - A \sin t, & -\infty < t < \infty. \\ y(t) &= R - A \cos t, \end{aligned}$$

Compare Fig. 14.5 for the derivation and Fig. 14.6 for some possible shapes of cycloids.

Definition 14.13 Let $\Gamma : t \mapsto \mathbf{x}(t)$ be a differentiable curve. The rate of change of the position vector with regard to the parameter of the curve

$$\dot{\mathbf{x}}(t) = \lim_{h \rightarrow 0} \frac{1}{h} (\mathbf{x}(t+h) - \mathbf{x}(t)) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix}$$

is called the *velocity vector* at the point $\mathbf{x}(t)$ of the curve. If $\dot{\mathbf{x}}(t) \neq \mathbf{0}$ one defines the *tangent vector*

$$\mathbf{T}(t) = \frac{\dot{\mathbf{x}}(t)}{\|\dot{\mathbf{x}}(t)\|} = \frac{1}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix}$$

and the *normal vector*

$$\mathbf{N}(t) = \frac{1}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \begin{bmatrix} -\dot{y}(t) \\ \dot{x}(t) \end{bmatrix}$$

Fig. 14.4 Parametric representation of the right branch of a hyperbola

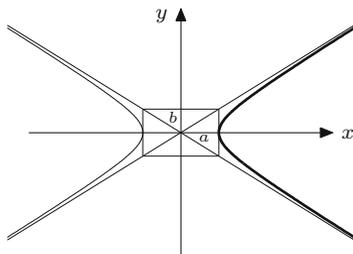


Fig. 14.5 Parametrisation of a cycloid

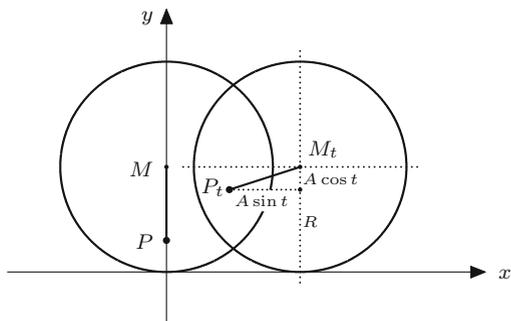


Fig. 14.6 Cycloids for
 $A = R/2, R, 3R/2$

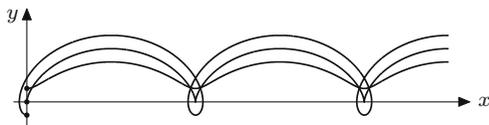
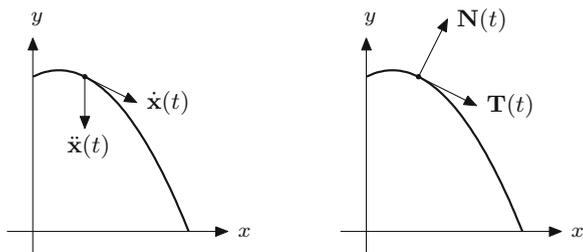


Fig. 14.7 Velocity vector,
 acceleration vector, tangent
 vector, normal vector



of the curve. The pair $(\mathbf{T}(t), \mathbf{N}(t))$ is called *moving frame*. If the curve Γ is twice differentiable then the *acceleration vector* is given by

$$\ddot{\mathbf{x}}(t) = \begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{bmatrix}.$$

In the kinematic interpretation the parameter t is the time and $\dot{\mathbf{x}}(t)$ the velocity vector in the physical sense. If it is different from zero, it points in the direction of the tangent (as limit of secant vectors). The tangent vector is just the unit vector of the same direction. By rotation of 90° in the counterclockwise sense we obtain the normal vector of the curve, see Fig. 14.7.

Experiment 14.14 Open the Java applet *Parametric curves in the plane*. Plot the curves from Example 14.5 and the corresponding velocity and acceleration vectors. Use the moving frame to visualise the kinematic curve progression.

Example 14.15 For the parabola from Example 14.2 we get

$$\begin{aligned} \dot{x}(t) &= v_H, & \ddot{x}(t) &= 0, \\ \dot{y}(t) &= v_V - gt, & \ddot{y}(t) &= -g, \end{aligned}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{v_H^2 + (v_V - gt)^2}} \begin{bmatrix} v_H \\ v_V - gt \end{bmatrix},$$

$$\mathbf{N}(t) = \frac{1}{\sqrt{v_H^2 + (v_V - gt)^2}} \begin{bmatrix} gt - v_V \\ v_H \end{bmatrix}.$$

14.2 Arc Length and Curvature

We start with the question whether and how a length can be assigned to a curve segment. Let a continuous curve

$$\Gamma : t \mapsto \mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad a \leq t \leq b$$

be given. For a partition $Z: a = t_0 < t_1 < \dots < t_n = b$ of the parameter interval we consider the (inscribed) polygonal chain through the points

$$\mathbf{x}(t_0), \mathbf{x}(t_1), \dots, \mathbf{x}(t_n).$$

The length of the largest subinterval is again denoted by $\Phi(Z)$. The length of the polygonal chain is

$$L_n = \sum_{i=1}^n \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}.$$

Definition 14.16 (Curves of finite length) A plane curve Γ is called *rectifiable* or of *finite length* if the lengths L_n of all inscribed polygonal chains Z_n converge towards one (and the same) limit provided that $\Phi(Z_n) \rightarrow 0$.

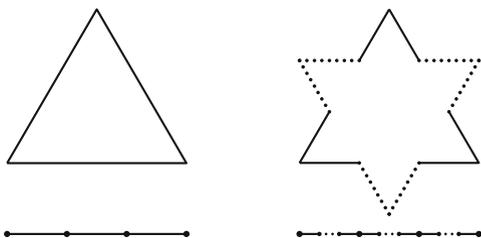
Example 14.17 (Koch's snowflake) Koch's snowflake was introduced in Sect. 9.1 as an example of a finite region whose boundary has the fractal dimension $d = \log 4 / \log 3$ and infinite length. This was proven by the fact that the boundary can be constructed as the limit of polygonal chains whose lengths tend to infinity. It remains to verify that the boundary of Koch's snowflake is indeed a continuous, parametrised curve. This can be seen as follows. The snowflake of depth 0 is an equilateral triangle, for instance with the vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{R}^2$. Using the unit interval $[0, 1]$ we obtain a continuous parametrisation

$$\mathbf{x}_0(t) = \begin{cases} \mathbf{p}_1 + 3t(\mathbf{p}_2 - \mathbf{p}_1), & 0 \leq t \leq \frac{1}{3}, \\ \mathbf{p}_2 + (3t - 1)(\mathbf{p}_3 - \mathbf{p}_2), & \frac{1}{3} \leq t \leq \frac{2}{3}, \\ \mathbf{p}_3 + (3t - 2)(\mathbf{p}_1 - \mathbf{p}_3), & \frac{2}{3} \leq t \leq 1. \end{cases}$$

We parametrise the snowflake of depth 1 by splitting the three intervals $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$, $[\frac{2}{3}, 1]$ into three parts each and using the middle parts for the parametrisation of the inserted next smaller angle (Fig. 14.8). Continuing in this way we obtain a sequence of parametrisations

$$t \mapsto \mathbf{x}_0(t), \quad t \mapsto \mathbf{x}_1(t), \quad \dots, \quad t \mapsto \mathbf{x}_n(t), \quad \dots$$

Fig. 14.8 Parametrisation of the boundary of Koch's snowflake



This is a sequence of continuous functions $[0, 1] \rightarrow \mathbb{R}^2$ which, due to its construction, converges uniformly (see Definition C.5). According to Proposition C.6 the limit function

$$\mathbf{x}(t) = \lim_{n \rightarrow \infty} \mathbf{x}_n(t), \quad t \in [0, 1]$$

is continuous (and obviously parametrises the boundary of Koch's snowflake).

This example shows that continuous curves can be infinitely long even if the parameter of the curve only varies in a bounded interval $[a, b]$. That such a behaviour does not appear for differentiable curves is shown by the next proposition.

Proposition 14.18 (Length of differentiable curves) *Every continuously differentiable curve $t \mapsto \mathbf{x}(t)$, $t \in [a, b]$ is rectifiable. Its length is*

$$L = \int_a^b \|\dot{\mathbf{x}}(t)\| dt = \int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt.$$

Proof We only give the proof for the somewhat simpler case that the components of the velocity vector $\dot{\mathbf{x}}(t)$ are Lipschitz continuous (see Appendix C.4), for instance with a Lipschitz constant C . We start with a partition $Z: a = t_0 < t_1 < \dots < t_n = b$ of $[a, b]$ with corresponding $\Phi(Z)$. The integral defining L is the limit of Riemann sums

$$\int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt = \lim_{n \rightarrow \infty, \Phi(Z) \rightarrow 0} \sum_{i=1}^n \sqrt{\dot{x}(\tau_i)^2 + \dot{y}(\tau_i)^2} (t_i - t_{i-1}),$$

where $\tau_i \in [t_{i-1}, t_i]$. On the other hand, according to the mean value theorem, Proposition 8.4, the length of the inscribed polygonal chain through $\mathbf{x}(t_0)$, $\mathbf{x}(t_1)$, \dots , $\mathbf{x}(t_n)$ is equal to

$$\begin{aligned} \sum_{i=1}^n \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2} \\ = \sum_{i=1}^n \sqrt{\dot{x}(\rho_i)^2 + \dot{y}(\sigma_i)^2} (t_i - t_{i-1}) \end{aligned}$$

for certain $\rho_i, \sigma_i \in [t_{i-1}, t_i]$. In order to be able to estimate the difference between the Riemann sums and the lengths of the inscribed polygonal chains, we use the inequality (triangle inequality for vectors in the plane)

$$\left| \sqrt{a^2 + b^2} - \sqrt{c^2 + d^2} \right| \leq \sqrt{(a - c)^2 + (b - d)^2},$$

which can be checked directly by squaring. Applying this inequality shows that

$$\begin{aligned} \left| \sqrt{\dot{x}(\tau_i)^2 + \dot{y}(\tau_i)^2} - \sqrt{\dot{x}(\rho_i)^2 + \dot{y}(\sigma_i)^2} \right| \\ \leq \sqrt{(\dot{x}(\tau_i) - \dot{x}(\rho_i))^2 + (\dot{y}(\tau_i) - \dot{y}(\sigma_i))^2} \\ \leq \sqrt{C^2(\tau_i - \rho_i)^2 + C^2(\tau_i - \sigma_i)^2} \\ \leq \sqrt{2}C\Phi(Z). \end{aligned}$$

For the difference between the Riemann sums and the lengths of the polygonal chains one obtains the estimate

$$\begin{aligned} \left| \sum_{i=1}^n \left(\sqrt{\dot{x}(\tau_i)^2 + \dot{y}(\tau_i)^2} - \sqrt{\dot{x}(\rho_i)^2 + \dot{y}(\sigma_i)^2} \right) (t_i - t_{i-1}) \right| \\ \leq \sqrt{2}C\Phi(Z) \sum_{i=1}^n (t_i - t_{i-1}) = \sqrt{2}C\Phi(Z)(b - a). \end{aligned}$$

For $\Phi(Z) \rightarrow 0$, this difference tends to zero. Thus the Riemann sums and the lengths of the inscribed polygonal chains have the same limit, namely L .

The proof of the general case, where the components of the velocity vector are not Lipschitz continuous, is similar. However, one additionally needs the fact that continuous functions on bounded, closed intervals are uniformly continuous. This is briefly addressed near the end of Appendix C.4. \square

Example 14.19 (Length of a circular arc) The parametric representation of a circle of radius R and its derivative is

$$\begin{aligned} x(t) &= R \cos t, & \dot{x}(t) &= -R \sin t, \\ y(t) &= R \sin t, & \dot{y}(t) &= R \cos t, \end{aligned} \quad 0 \leq t \leq 2\pi.$$

The circumference of the circle is thus

$$L = \int_0^{2\pi} \sqrt{(-R \sin t)^2 + (R \cos t)^2} dt = \int_0^{2\pi} R dt = 2R\pi.$$

Experiment 14.20 Use the MATLAB program `mat14_5.m` to approximate the circumference of the unit circle using inscribed polygonal chains. Modify the program so that it approximates the lengths of arbitrary differentiable curves.

Definition 14.21 (Arc length) Let $t \mapsto \mathbf{x}(t)$ be a differentiable curve. The length of the curve segment from the initial parameter value a to the current parameter value t is called the *arc length*,

$$s = L(t) = \int_a^t \sqrt{\dot{x}(\tau)^2 + \dot{y}(\tau)^2} d\tau.$$

The arc length s is a strictly monotonically increasing, continuous (even continuously differentiable) function. It is thus suitable for a reparametrisation $t = L^{-1}(s)$. The curve

$$s \mapsto \boldsymbol{\xi}(s) = \mathbf{x}(L^{-1}(s))$$

is called *parametrised by arc length*.

In the following let $t \mapsto \mathbf{x}(t)$ be a differentiable curve (in the plane). The angle of the tangent vector with the positive x -axis is denoted by $\varphi(t)$; that is,

$$\tan \varphi(t) = \frac{\dot{y}(t)}{\dot{x}(t)}.$$

Definition 14.22 (Curvature of a plane curve) The *curvature* of a differentiable curve in the plane is the rate of change of the angle φ with respect to the arc length,

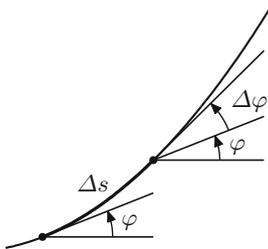
$$\kappa = \frac{d\varphi}{ds} = \frac{d}{ds} \varphi(L^{-1}(s)).$$

Figure 14.9 illustrates this definition. If φ is the angle at the length s of the arc and $\varphi + \Delta\varphi$ the angle at the length $s + \Delta s$, then $\kappa = \lim_{\Delta s \rightarrow 0} \frac{\Delta\varphi}{\Delta s}$. This shows that the value of κ actually corresponds to the intuitive meaning of curvature. Note that the curvature of a plane curve comes with a sign; when reversing the moving sense, the sign changes.

Proposition 14.23 The curvature of a twice continuously differentiable curve at the point $(x(t), y(t))$ of the curve is

$$\kappa(t) = \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{(\dot{x}(t)^2 + \dot{y}(t)^2)^{3/2}}.$$

Fig. 14.9 Curvature



Proof According to the chain rule and the inverse function rule, one gets

$$\kappa = \frac{d}{ds} \varphi(L^{-1}(s)) = \dot{\varphi}(L^{-1}(s)) \cdot \frac{d}{ds} L^{-1}(s) = \dot{\varphi}(L^{-1}(s)) \cdot \frac{1}{\dot{L}(L^{-1}(s))}.$$

Differentiating the arc length

$$s = L(t) = \int_a^t \sqrt{\dot{x}(\tau)^2 + \dot{y}(\tau)^2} d\tau$$

with respect to t gives

$$\frac{ds}{dt} = \dot{L}(t) = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}.$$

Differentiating the relationship $\tan \varphi(t) = \dot{y}(t)/\dot{x}(t)$ leads to

$$\dot{\varphi}(t)(1 + \tan^2 \varphi(t)) = \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{\dot{x}(t)^2},$$

which gives, after substituting the above expression for $\tan \varphi(t)$ and simplifying,

$$\dot{\varphi}(t) = \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{\dot{x}(t)^2 + \dot{y}(t)^2}.$$

If one takes into account the relation $t = L^{-1}(s)$ and substitutes the derived expressions for $\dot{\varphi}(t)$ and $\dot{L}(t)$ into the formula for κ at the beginning of the proof, one obtains

$$\kappa(t) = \frac{\dot{\varphi}(t)}{\dot{L}(t)} = \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{(\dot{x}(t)^2 + \dot{y}(t)^2)^{3/2}},$$

which is the desired assertion. □

Remark 14.24 As a special case, the curvature of the graph of a twice differentiable function $y = f(x)$ can be obtained as

$$\kappa(x) = \frac{f''(x)}{(1 + f'(x)^2)^{3/2}}.$$

This follows easily from the above proposition by using the parametrisation $x = t$, $y = f(t)$.

Example 14.25 The curvature of a circle of radius R , traversed in the positive direction, is constant and equal to $\kappa = \frac{1}{R}$. Indeed

$$\begin{aligned} x(t) &= R \cos t, & \dot{x}(t) &= -R \sin t, & \ddot{x}(t) &= -R \cos t, \\ y(t) &= R \sin t, & \dot{y}(t) &= R \cos t, & \ddot{y}(t) &= -R \sin t, \end{aligned}$$

and thus

$$\kappa = \frac{R^2 \sin^2 t + R^2 \cos^2 t}{(R^2 \sin^2 t + R^2 \cos^2 t)^{3/2}} = \frac{1}{R}.$$

One obtains the same result from the following geometric consideration. At the point $(x, y) = (R \cos t, R \sin t)$ the angle φ of the tangent vector with the positive x -axis is equal to $t + \pi/2$, and the arc length is $s = Rt$. Therefore $\varphi = s/R + \pi/2$ which differentiated with respect to s gives $\kappa = 1/R$.

Definition 14.26 The *osculating circle* at a point of a differentiable curve is the circle which has the same tangent and the same curvature as the curve.

According to Example 14.25 it follows that the osculating circle has the radius $\frac{1}{|\kappa(t)|}$ and its centre $\mathbf{x}_c(t)$ lies on the normal of the curve. It is given by

$$\mathbf{x}_c(t) = \mathbf{x}(t) + \frac{1}{\kappa(t)} \mathbf{N}(t).$$

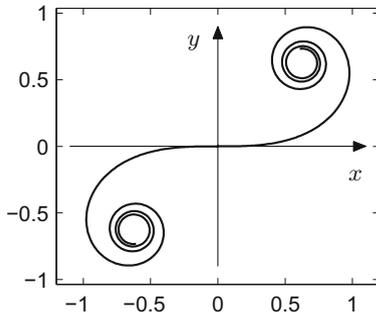
Example 14.27 (Clothoid) The *clothoid* is a curve whose curvature is proportional to its arc length. In applications it serves as a connecting link from a straight line (with curvature 0) to a circular arc (with curvature $\frac{1}{R}$). It is used in railway engineering and road design. Its defining property is

$$\kappa(s) = \frac{d\varphi}{ds} = c \cdot s$$

for a certain $c \in \mathbb{R}$. If one starts with curvature 0 at $s = 0$ then the angle is equal to

$$\varphi(s) = \frac{c}{2} s^2.$$

Fig. 14.10 Clothoid



We use s as the curve parameter.

Differentiating the relation

$$s = \int_0^s \sqrt{\dot{x}(\sigma)^2 + \dot{y}(\sigma)^2} \, d\sigma$$

shows that

$$1 = \sqrt{\dot{x}(s)^2 + \dot{y}(s)^2};$$

thus, the velocity vector of a curve parametrised by arc length has length one. This implies in particular

$$\frac{dx}{ds} = \cos \varphi(s), \quad \frac{dy}{ds} = \sin \varphi(s).$$

From there we can compute the parametrisation of the curve:

$$x(s) = \int_0^s \frac{dx}{ds}(\sigma) \, d\sigma = \int_0^s \cos \varphi(\sigma) \, d\sigma = \int_0^s \cos\left(\frac{c}{2}\sigma^2\right) \, d\sigma,$$

$$y(s) = \int_0^s \frac{dy}{ds}(\sigma) \, d\sigma = \int_0^s \sin \varphi(\sigma) \, d\sigma = \int_0^s \sin\left(\frac{c}{2}\sigma^2\right) \, d\sigma.$$

The components of the curve are thus given by Fresnel's integrals. The shape of the curve is displayed in Fig. 14.10, its numerical calculation can be seen in the MATLAB program `mat14_6.m`.

14.3 Plane Curves in Polar Coordinates

By writing the parametric representation in the form

$$\begin{aligned}x(t) &= r(t) \cos t, \\y(t) &= r(t) \sin t\end{aligned}$$

in polar coordinates with t as angle and $r(t)$ as radius, one obtains a simple way of representing many curves. By convention negative radii are plotted in opposite direction of the ray with angle t .

Example 14.28 (Spirals) The Archimedean³ spiral is defined by

$$r(t) = t, \quad 0 \leq t < \infty,$$

the logarithmic spiral by

$$r(t) = e^t, \quad -\infty < t < \infty,$$

the hyperbolic spiral by

$$r(t) = \frac{1}{t}, \quad 0 < t < \infty.$$

Typical parts of these spirals are displayed in Fig. 14.11.

Experiment 14.29 Study the behaviour of the logarithmic spiral near the origin using the zoom tool (use the M-file `mat14_7.m`).

Example 14.30 (Loops) Loops are obtained by choosing $r(t) = \cos nt$, $n \in \mathbb{N}$. In Cartesian coordinates the parametric representation thus reads

$$\begin{aligned}x(t) &= \cos nt \cos t, \\y(t) &= \cos nt \sin t.\end{aligned}$$

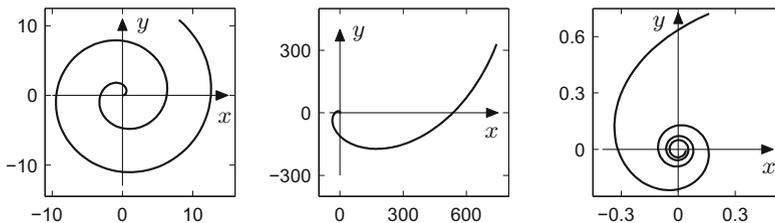


Fig. 14.11 Archimedean, logarithmic and hyperbolic spirals

³Archimedes of Syracuse, 287–212 B.C.

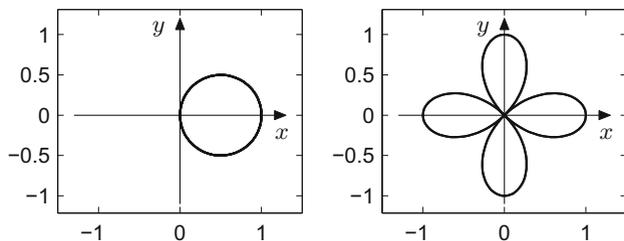


Fig. 14.12 Loops with $r = \cos t$ and $r = \cos 2t$

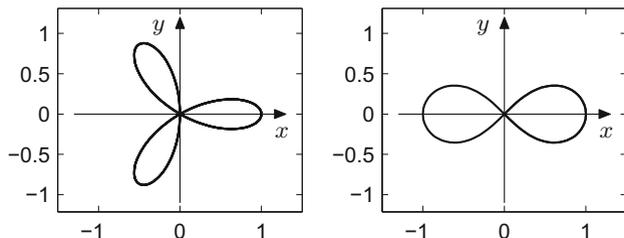


Fig. 14.13 Loops with $r = \cos 3t$ and $r = \pm\sqrt{\cos 2t}$

The choice $n = 1$ results in a circle of radius $\frac{1}{2}$ about $(\frac{1}{2}, 0)$, for odd n one obtains n leaves, for even n one obtains $2n$ leaves, see Figs. 14.12 and 14.13.

The *figure eight* from Fig. 14.13 is obtained by $r(t) = \sqrt{\cos 2t}$ and $r(t) = -\sqrt{\cos 2t}$, respectively, for $-\frac{\pi}{4} < t < \frac{\pi}{4}$, where the positive root gives the right leaf and the negative root the left leaf. This curve is called *lemniscate*.

Example 14.31 (Cardioid) The *cardioid* is a special epicycloid, where one circle is rolling around another circle with the same radius A . Its parametric representation is

$$\begin{aligned} x(t) &= 2A \cos t + A \cos 2t, \\ y(t) &= 2A \sin t + A \sin 2t \end{aligned}$$

for $0 \leq t \leq 2\pi$. The cardioid with radius $A = 1$ is shown in Fig. 14.14.

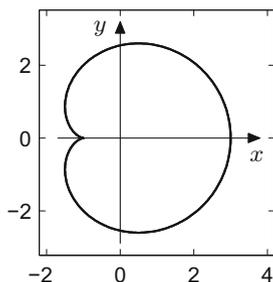


Fig. 14.14 Cardioid with $A = 1$

14.4 Parametrised Space Curves

In the same way as for plane curves, a *parametrised curve in space* is defined as a continuous mapping of an interval $[a, b]$ to \mathbb{R}^3 ,

$$t \mapsto \mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, \quad a \leq t \leq b.$$

The curve is called *differentiable*, if all three components $t \mapsto x(t)$, $t \mapsto y(t)$, $t \mapsto z(t)$ are differentiable real-valued functions.

Velocity and tangent vector of a differentiable curve in space are defined as in the planar case by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix}, \quad \mathbf{T}(t) = \frac{\dot{\mathbf{x}}(t)}{\|\dot{\mathbf{x}}(t)\|} = \frac{1}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2}} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix}.$$

The second derivative $\ddot{\mathbf{x}}(t)$ is the acceleration vector. In the spatial case there is a *normal plane* to the curve which is spanned by the *normal vector*

$$\mathbf{N}(t) = \frac{1}{\|\dot{\mathbf{T}}(t)\|} \dot{\mathbf{T}}(t)$$

and the *binormal vector*

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t),$$

provided that $\dot{\mathbf{x}}(t) \neq \mathbf{0}$, $\dot{\mathbf{T}}(t) \neq \mathbf{0}$. The formula

$$0 = \frac{d}{dt} 1 = \frac{d}{dt} \|\mathbf{T}(t)\|^2 = 2\langle \mathbf{T}(t), \dot{\mathbf{T}}(t) \rangle$$

(which is verified by a straightforward computation) implies that $\dot{\mathbf{T}}(t)$ is perpendicular to $\mathbf{T}(t)$. Therefore, the three vectors $(\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t))$ form an orthogonal basis in \mathbb{R}^3 , called the *moving frame* of the curve.

Rectifiability of a curve in space is defined in analogy to Definition 14.16 for plane curves. The *length* of a differentiable curve in space can be computed by

$$L = \int_a^b \|\dot{\mathbf{x}}(t)\| dt = \int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2} dt.$$

Also, the *arc length* can be defined similarly to the planar case (Definition 14.21).

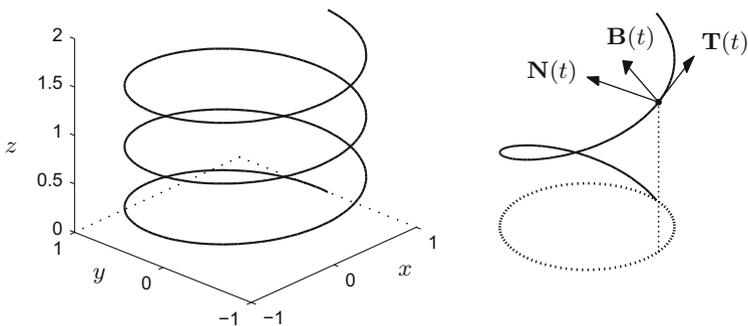


Fig. 14.15 Helix with tangent, normal and binormal vector

Example 14.32 (Helix) The parametric representation of the helix is

$$\mathbf{x}(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}, \quad -\infty < t < \infty.$$

We obtain

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}, & \mathbf{T}(t) &= \frac{1}{\sqrt{2}} \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}, \\ \dot{\mathbf{T}}(t) &= \frac{1}{\sqrt{2}} \begin{bmatrix} -\cos t \\ -\sin t \\ 0 \end{bmatrix}, & \mathbf{N}(t) &= \begin{bmatrix} -\cos t \\ -\sin t \\ 0 \end{bmatrix} \end{aligned}$$

with binormal vector

$$\mathbf{B}(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix} \times \begin{bmatrix} -\cos t \\ -\sin t \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sin t \\ -\cos t \\ 1 \end{bmatrix}.$$

The formula for the arc length of the helix, counting from the origin, is particularly simple:

$$L(t) = \int_0^t \|\dot{\mathbf{x}}(\tau)\| \, d\tau = \int_0^t \sqrt{2} \, d\tau = \sqrt{2}t.$$

Figure 14.15 was drawn using the MATLAB commands

```
t=0 : pi/100 : 6 * pi;
plot3(cos(t), sin(t), t/10).
```

The Java applet *Parametric curves in space* offers dynamic visualising possibilities of those and other curves in space and of their moving frames.

14.5 Exercises

1. Find out which geometric formation is represented by the set of zeros of the polynomial $y^2 - x(x^2 - 1) = 0$. Visualise the curve in maple using the command `implicitplot`. Can you parametrise it as a continuous curve?
2. Verify that the algebraic curves $y^2 - (x + p)x^2 = 0$, $p \in \mathbb{R}$ (Example 14.7) can be parametrised by

$$\begin{aligned} x(t) &= t^2 - p, \\ y(t) &= t(t^2 - p), \quad -\infty < t < \infty. \end{aligned}$$

Visualise the curves for $p = -1, 0, 1$ in maple using the command `implicitplot`.

3. Using MATLAB or maple, investigate the shape of Lissajous figures⁴

$$x(t) = \sin(w_1 t), \quad y(t) = \cos(w_2 t)$$

and

$$x(t) = \sin(w_1 t), \quad y(t) = \cos\left(w_2 t + \frac{\pi}{2}\right).$$

Consider the cases $w_2 = w_1$, $w_2 = 2w_1$, $w_2 = \frac{3}{2}w_1$ and explain the results.

The following exercises use the Java applets *Parametric curves in the plane* and *Parametric curves in space*.

4. (a) Using the Java applet analyse where the cycloid

$$\begin{aligned} x(t) &= t - 2 \sin t, \\ y(t) &= 1 - 2 \cos t, \quad -2\pi \leq t \leq 2\pi. \end{aligned}$$

has its maximal speed ($\|\dot{\mathbf{x}}(t)\| \rightarrow \max$) and check your result by hand.

⁴J.A. Lissajous, 1822–1880.

(b) Discuss and explain the shape of the loops

$$\begin{aligned}x(t) &= \cos nt \cos t, \\y(t) &= \cos nt \sin t,\end{aligned} \quad 0 \leq t \leq 2\pi.$$

for $n = 1, 2, 3, 4, 5$ using the Java applets (plot the moving frame).

5. Study the velocity and the acceleration of the following curves by using the Java applet. Verify your results by computing the points where the curve has either a horizontal tangent ($\dot{x}(t) \neq 0, \dot{y}(t) = 0$) or a vertical tangent ($\dot{x}(t) = 0, \dot{y}(t) \neq 0$), or is singular ($\dot{x}(t) = 0, \dot{y}(t) = 0$).

(a) Cycloid:

$$\begin{aligned}x(t) &= t - \sin t, \\y(t) &= 1 - \cos t,\end{aligned} \quad -2\pi \leq t \leq 2\pi.$$

(b) Cardioid:

$$\begin{aligned}x(t) &= 2 \cos t + \cos 2t, \\y(t) &= 2 \sin t + \sin 2t,\end{aligned} \quad 0 \leq t \leq 2\pi.$$

6. Analyse and explain the trajectories of the curves

$$\begin{aligned}\mathbf{x}(t) &= \begin{bmatrix} 1 - 2t^2 \\ (1 - 2t^2)^2 \end{bmatrix}, \quad -1 \leq t \leq 1, \\ \mathbf{y}(t) &= \begin{bmatrix} \cos t \\ \cos^2 t \end{bmatrix}, \quad 0 \leq t \leq 2\pi, \\ \mathbf{z}(t) &= \begin{bmatrix} t \cos t \\ t^2 \cos^2 t \end{bmatrix}, \quad -2 \leq t \leq 2.\end{aligned}$$

Are these curves (geometrically) equivalent?

7. (a) Compute the curvature $\kappa(t)$ of the branch of the hyperbola

$$\begin{aligned}x(t) &= \cosh t, \\y(t) &= \sinh t,\end{aligned} \quad -\infty < t < \infty.$$

(b) Determine its osculating circle (centre and radius) at $t = 0$.

8. Consider the ellipse

$$\mathbf{x}(t) = \begin{bmatrix} 2 \cos t \\ \sin t \end{bmatrix}, \quad -\pi \leq t \leq \pi.$$

- (a) Compute its velocity vector $\dot{\mathbf{x}}(t)$, its acceleration vector $\ddot{\mathbf{x}}(t)$ as well as the moving frame $(\mathbf{T}(t), \mathbf{N}(t))$.
 (b) Compute its curvature $\kappa(t)$ and determine the osculating circle (centre and radius) at $t = 0$.

9. (a) Analyse the trajectory of the astroid

$$\mathbf{x}(t) = \begin{bmatrix} \cos^3 t \\ \sin^3 t \end{bmatrix}, \quad 0 \leq t \leq 2\pi.$$

- (b) Compute the length of the part of the astroid which lies in the first quadrant.
 10. (a) Compute the velocity vector $\dot{\mathbf{x}}(t)$ and the moving frame $(\mathbf{T}(t), \mathbf{N}(t))$ for the segment

$$\mathbf{x}(t) = \begin{bmatrix} e^t \cos t \\ e^t \sin t \end{bmatrix}, \quad 0 \leq t \leq \pi/2$$

of the logarithmic spiral. At what point in the interval $[0, \pi/2]$ does it have a vertical tangent?

- (b) Compute the length of the segment. Deduce a formula for its arc length $s = L(t)$.
 (c) Reparametrise the spiral by its arc length, i.e., compute $\boldsymbol{\xi}(s) = \mathbf{x}(L^{-1}(s))$ and verify that $\|\dot{\boldsymbol{\xi}}(s)\| = 1$.
 11. (Application of the secant and cosecant functions) Analyse what plane curves are determined in polar coordinates by

$$r(t) = \sec t, \quad -\pi/2 < t < \pi/2 \quad \text{and} \quad r(t) = \csc t, \quad 0 < t < \pi.$$

12. (a) Determine the tangent and the normal to the graph of the function $y = 1/x$ at $(x_0, y_0) = (1, 1)$ and compute its curvature at that point.
 (b) Suppose the graph of the function $y = 1/x$ is to be replaced by a circular arc at x_0 , i.e., for $x \geq 1$. Find the centre and the radius of a circle which admits a smooth transition (same tangent, same curvature).
 13. (a) Analyse the space curve

$$\mathbf{x}(t) = \begin{bmatrix} \cos t \\ \sin t \\ 2 \sin \frac{t}{2} \end{bmatrix}, \quad 0 \leq t \leq 4\pi$$

using the applet.

- (b) Check that the curve is the intersection of the cylinder $x^2 + y^2 = 1$ with the sphere $(x + 1)^2 + y^2 + z^2 = 4$.

Hint. Use $\sin^2 \frac{t}{2} = \frac{1}{2}(1 - \cos t)$.

14. Using MATLAB, maple or the applet, sketch and discuss the space curves

$$\mathbf{x}(t) = \begin{bmatrix} t \cos t \\ t \sin t \\ 2t \end{bmatrix}, \quad 0 \leq t < \infty,$$

and

$$\mathbf{y}(t) = \begin{bmatrix} \cos t \\ \sin t \\ 0 \end{bmatrix}, \quad 0 \leq t \leq 4\pi.$$

15. Sketch and discuss the space curves

$$\mathbf{x}(t) = \begin{bmatrix} t \\ t \\ t^3 \end{bmatrix}, \quad \mathbf{y}(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}, \quad 0 \leq t < 1.$$

Compute their velocity vectors $\dot{\mathbf{x}}(t)$, $\dot{\mathbf{y}}(t)$ and their acceleration vectors $\ddot{\mathbf{x}}(t)$, $\ddot{\mathbf{y}}(t)$.

16. Sketch the space curve

$$\mathbf{x}(t) = \begin{bmatrix} \sqrt{2}t \\ \cosh t \\ \cosh t \end{bmatrix}, \quad 0 \leq t < 1.$$

Compute its moving frame $(\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t))$ as well as its length.

17. Sketch the space curve

$$\mathbf{x}(t) = \begin{bmatrix} \cos t \\ \sin t \\ t^{3/2} \end{bmatrix}, \quad 0 \leq t < 2\pi,$$

and compute its length.