19 Differential Equations

In this chapter we discuss the theory of initial value problems for ordinary differential equations. We limit ourselves to scalar equations here; systems will be discussed in the next chapter.

After presenting the general definition of a differential equation and the geometric significance of its direction field, we start with a detailed discussion of first-order linear equations. As important applications we discuss the modelling of growth and decay processes. Subsequently, we investigate questions of existence and (local) uniqueness of the solution of general differential equations and discuss the method of power series. We also study the qualitative behaviour of solutions close to an equilibrium point. Finally, we discuss the solution of second-order linear problems with constant coefficients.

19.1 Initial Value Problems

Differential equations are equations involving a (sought after) function and its derivative(s). They play a decisive role in modelling time-dependent processes.

Definition 19.1 Let $D \subset \mathbb{R}^2$ be open and $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ continuous. The equation

$$y'(x) = f\big(x, y(x)\big)$$

is called (an ordinary) *first-order differential equation*. A *solution* is a differentiable function $y : I \to D$ which satisfies the equation for all $x \in I$.

M. Oberguggenberger and A. Ostermann, *Analysis for Computer Scientists*, Undergraduate Topics in Computer Science,
https://doi.org/10.1007/978-3-319-91155-7_19

One often suppresses the *independent variable* x in the notation and writes the above problem for short as

$$y' = f(x, y).$$

The sought after function y in this equation is also called the *dependent variable* (depending on x).

In modelling time-dependent processes, one usually denotes the independent variable by t (for time) and the dependent variable by $x = x(t)$. In this case one writes the first-order differential equation as

$$\dot{x}(t) = f\big(t, x(t)\big)$$

or for short as $\dot{x} = f(t, x)$.

Example 19.2 (Separation of the variables) We want to find all functions $y = y(x)$ satisfying the equation $y'(x) = x \cdot y(x)^2$. In this example one obtains the solutions by *separating the variables.* For $y \neq 0$ one divides the differential equation by y^2 and gets

$$\frac{1}{y^2} \cdot y' = x.$$

The left-hand side of this equation is of the form $g(y) \cdot y'$. Let $G(y)$ be an antiderivative of $g(y)$. According to the chain rule, and recalling that y is a function of x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x} G(y) = \frac{\mathrm{d}}{\mathrm{d}y} G(y) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = g(y) \cdot y'.$$

In our example we have $g(y) = y^{-2}$ and $G(y) = -y^{-1}$, consequently

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{y}\right) = \frac{1}{y^2} \cdot y' = x.$$

Integration of this equation with respect to x results in

$$-\frac{1}{y} = \frac{x^2}{2} + C,$$

where C denotes an arbitrary integration constant. By elementary manipulations we find

$$y = \frac{1}{-x^2/2 - C} = \frac{2}{K - x^2}$$

with the constant $K = -2C$.

The function $y = 0$ is also a solution of the differential equation. Formally, one obtains it from the above solution by setting $K = \infty$. The example shows that differential equations have infinitely many solutions in general. By requiring an additional condition, a unique solution can be selected. For example, setting $y(0) = 1$ gives $y(x) = 2/(2 - x^2)$.

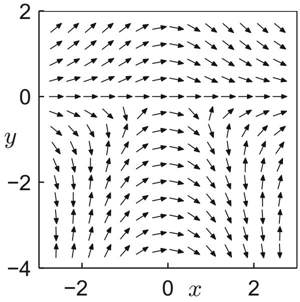


Fig. 19.1 The direction field of $y' = -2xy/(x^2 + 2y)$

Definition 19.3 The differential equation $y'(x) = f\big(x, y(x)\big)$ together with the additional condition $y(x_0) = y_0$, i.e.,

$$y'(x) = f\big(x, y(x)\big), \quad y(x_0) = y_0,$$

is called *initial value problem*. A solution of an initial value problem is a (continuously) differentiable function $y(x)$, which satisfies the differential equation and the *initial condition* $y(x_0) = y_0$.

Geometric interpretation of a differential equation. For a given first-order differential equation

$$y' = f(x, y), \qquad (x, y) \in D \subset \mathbb{R}^2$$

one searches for a differentiable function $y = y(x)$ whose graph lies in D and whose tangents have the slopes $\tan \varphi = y'(x) = f\big(x, y(x)\big)$ for each x. By plotting short arrows with slopes $\tan \varphi = f(x, y)$ at the points $(x, y) \in D$ one obtains the *direction field* of the differential equation. The direction field is *tangential* to the solution curves and offers a good visual impression of their shapes. Figure 19.1 shows the direction field of the differential equation

$$y' = -\frac{2xy}{x^2 + 2y}.$$

The right-hand side has singularities along the curve $y = -x^2/2$ which is reflected by the behaviour of the arrows in the lower part of the figure.

Experiment 19.4 Visualise the direction field of the above differential equation with the applet *Dynamical systems in the plane*.

19.2 First-Order Linear Differential Equations

Let $a(x)$ and $g(x)$ be functions defined on some interval. The equation

$$y' + a(x)\,y = g(x)$$

is called a *first-order linear differential equation.* The function a is the *coefficient*, the right-hand side g is called *inhomogeneity*. The differential equation is called *homogeneous*, if $g = 0$, otherwise *inhomogeneous*. First we state the following important result.

Proposition 19.5 (Superposition principle) *If y and z are solutions of a linear differential equation with possibly different inhomogeneities*

$$y'(x) + a(x)\,y(x) = g(x),$$
$$z'(x) + a(x)\,z(x) = h(x),$$

then their linear combination

$$w(x) = \alpha y(x) + \beta z(x), \qquad \alpha,\ \beta \in \mathbb{R}$$

solves the linear differential equation

$$w'(x) + a(x)\,w(x) = \alpha g(x) + \beta h(x).$$

Proof This so-called *superposition principle* follows from the linearity of the derivative and the linearity of the equation. □

In a first step we compute all solutions of the homogeneous equation. We will use the superposition principle later to find all solutions of the inhomogeneous equation.

Proposition 19.6 *The general solution of the* homogeneous *differential equation*

$$y' + a(x)\,y = 0$$

is

$$y_h(x) = K\mathrm{e}^{-A(x)}$$

with $K \in \mathbb{R}$ and an arbitrary antiderivative $A(x)$ of $a(x)$.

Proof For $y \neq 0$ we separate the variables

$$\frac{1}{y} \cdot y' = -a(x)$$

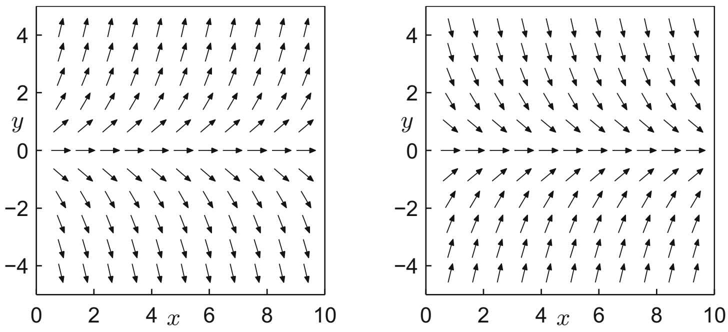


Fig. 19.2 The direction field of $y' = y$ (left) and $y' = -y$ (right)

and use

$$\frac{\mathrm{d}}{\mathrm{d}y} \log|y| = \frac{1}{y}$$

to obtain

$$\log|y| = -A(x) + C$$

by integrating the equation. From that we infer

$$|y(x)| = \mathrm{e}^{-A(x)}\mathrm{e}^{C}.$$

This formula shows that $y(x)$ cannot change sign since the right-hand side is never zero. Thus $K = \mathrm{e}^C \cdot \operatorname{sign} y(x)$ is a constant as well, and the formula

$$y(x) = \operatorname{sign} y(x) \cdot |y(x)| = K\mathrm{e}^{-A(x)}, \qquad K \in \mathbb{R}$$

yields all solutions of the homogeneous equation. □

Example 19.7 The linear differential equation

$$\dot{x} = ax$$

with *constant* coefficient a has the general solution

$$x(t) = K\mathrm{e}^{at}, \qquad K \in \mathbb{R}.$$

The constant K is determined by $x(0)$, for example.

The direction field of the differential equation $y' = ay$ (depending on the sign of the coefficient) is shown in Fig. 19.2.

Interpretation. Let $x(t)$ be a time-dependent function which describes a growth or decay process (population increase/decrease, change of mass, etc.). We consider a time interval $[t, t+h]$ with $h > 0$. For $x(t) \neq 0$ the relative change of x in this time interval is given by

$$\frac{x(t+h)-x(t)}{x(t)} = \frac{x(t+h)}{x(t)} - 1.$$

The relative *rate of change* (change per unit of time) is thus

$$\frac{x(t+h)-x(t)}{t+h-t} \cdot \frac{1}{x(t)} = \frac{x(t+h)-x(t)}{h \cdot x(t)}.$$

For an *ideal* growth process this rate only depends on time t. In the limit $h \to 0$ this leads to the *instantaneous relative rate of change*

$$a(t) = \lim_{h\to 0} \frac{x(t+h)-x(t)}{h \cdot x(t)} = \frac{\dot{x}(t)}{x(t)}.$$

Ideal growth processes thus may be modelled by the linear differential equation

$$\dot{x}(t) = a(t)x(t).$$

Example 19.8 (Radioactive decay) Let $x(t)$ be the concentration of a radioactive substance at time t. In radioactive decay the rate of change does not depend on time and is negative,

$$a(t) \equiv a < 0.$$

The solution of the equation $\dot{x} = ax$ with initial value $x(0) = x_0$ is

$$x(t) = \mathrm{e}^{at} x_0.$$

It is exponentially decreasing and $\lim_{t\to\infty} x(t) = 0$, see Fig. 19.3. The *half life* T, the time in which half of the substance has decayed, is obtained from

$$\frac{x_0}{2} = \mathrm{e}^{aT} x_0 \qquad \text{as} \qquad T = -\frac{\log 2}{a}.$$

The half life for $a = -2$ is indicated in Fig. 19.3 by the dotted lines.

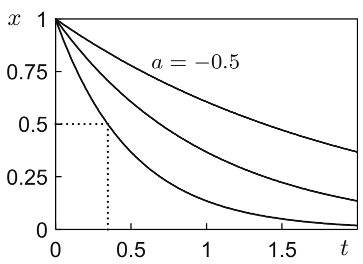


Fig. 19.3 Radioactive decay with constants $a = -0.5, -1, -2$ (top to bottom)

Example 19.9 (Population models) Let $x(t)$ be the size of a population at time t, modelled by $\dot{x} = ax$. If a constant, positive rate of growth $a > 0$ is presumed then the population grows exponentially

$$x(t) = \mathrm{e}^{at} x_0, \qquad \lim_{t\to\infty} |x(t)| = \infty.$$

One calls this behaviour *Malthusian law*.[1] In 1839 Verhulst suggested an improved model which also takes limited resources into account

$$\dot{x}(t) = \big(\alpha - \beta x(t)\big) \cdot x(t) \qquad \text{with} \quad \alpha, \beta > 0.$$

The corresponding discrete model was already discussed in Example 5.3, where L denoted the quotient α/β.

The rate of growth in Verhulst's model is population dependent, namely equal to $\alpha - \beta x(t)$, and decreases *linearly* with increasing population. Verhulst's model can be solved by separating the variables (or with maple). One obtains

$$x(t) = \frac{\alpha}{\beta + C\alpha \mathrm{e}^{-\alpha t}}$$

and thus, independently of the initial value,

$$\lim_{t\to\infty} x(t) = \frac{\alpha}{\beta},$$

see also Fig. 19.4. The *stationary* solution $x(t) = \alpha/\beta$ is an *asymptotically stable equilibrium point* of Verhulst's model, see Sect. 19.5.

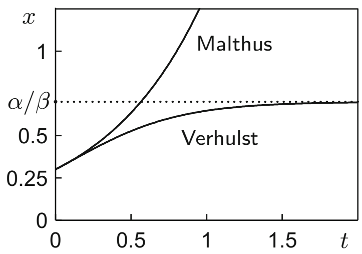
Variation of constants. We now turn to the solution of the *inhomogeneous* equation

$$y' + a(x)y = g(x).$$

We already know the general solution

$$y_{\mathrm{h}}(x) = c \cdot \mathrm{e}^{-A(x)}, \qquad c \in \mathbb{R}$$

Fig. 19.4 Population increase according to Malthus and Verhulst



[1] T.R. Malthus, 1766–1834.

of the homogeneous equation with the antiderivative

$$A(x) = \int_{x_0}^{x} a(\xi)\,\mathrm{d}\xi.$$

We look for a particular solution of the inhomogeneous equation of the form

$$y_\mathrm{p}(x) = c(x) \cdot y_\mathrm{h}(x) = c(x) \cdot \mathrm{e}^{-A(x)},$$

where we allow the constant $c = c(x)$ to be a function of x (variation of constants). Substituting this formula into the inhomogeneous equation and differentiating using the product rule yields

$$\begin{aligned} y_\mathrm{p}'(x) + a(x)\,y_\mathrm{p}(x) &= c'(x)\,y_\mathrm{h}(x) + c(x)\,y_\mathrm{h}'(x) + a(x)\,y_\mathrm{p}(x) \\ &= c'(x)\,y_\mathrm{h}(x) - a(x)\,c(x)\,y_\mathrm{h}(x) + a(x)\,y_\mathrm{p}(x) \\ &= c'(x)\,y_\mathrm{h}(x). \end{aligned}$$

If one equates this expression with the inhomogeneity $g(x)$, one recognises that $c(x)$ fulfils the differential equation

$$c'(x) = \mathrm{e}^{A(x)} g(x)$$

which can be solved by integration

$$c(x) = \int_{x_0}^{x} \mathrm{e}^{A(\xi)} g(\xi)\,\mathrm{d}\xi.$$

We thus obtain the following proposition.

Proposition 19.10 *The differential equation*

$$y' + a(x)y = g(x)$$

has the general solution

$$y(x) = \mathrm{e}^{-A(x)} \left(\int_{x_0}^{x} \mathrm{e}^{A(\xi)} g(\xi)\,\mathrm{d}\xi + K \right)$$

with $A(x) = \int_{x_0}^{x} a(\xi)\,\mathrm{d}\xi$ *and an arbitrary constant* $K \in \mathbb{R}$.

Proof By the above considerations, the function $y(x)$ is a solution of the differential equation $y' + a(x)y = g(x)$. Conversely, let $z(x)$ be any other solution. Then, according to the *superposition principle*, the difference $z(x) - y(x)$ is a solution of the homogeneous equation, so

$$z(x) = y(x) + c\,\mathrm{e}^{-A(x)}.$$

Therefore, $z(x)$ also has the form stated in the proposition. □

Corollary 19.11 *Let y_p be an arbitrary solution of the inhomogeneous linear differential equation*

$$y' + a(x)y = g(x).$$

Then, its general solution can be written as

$$y(x) = y_p(x) + y_h(x) = y_p(x) + K\,\mathrm{e}^{-A(x)}, \qquad K \in \mathbb{R}.$$

Proof This statement follows from the proof of Proposition 19.10 or directly from the superposition principle. □

Example 19.12 We solve the problem $y' + 2y = \mathrm{e}^{4x} + 1$. The solution of the homogeneous equation is $y_\mathrm{h}(x) = c\,\mathrm{e}^{-2x}$. A particular solution can be found by variation of constants. From

$$c(x) = \int_0^x \mathrm{e}^{2\xi}\big(\mathrm{e}^{4\xi} + 1\big)\,\mathrm{d}\xi = \frac{1}{6}\,\mathrm{e}^{6x} + \frac{1}{2}\,\mathrm{e}^{2x} - \frac{2}{3}$$

it follows that

$$y_\mathrm{p}(x) = \frac{1}{6}\,\mathrm{e}^{4x} - \frac{2}{3}\,\mathrm{e}^{-2x} + \frac{1}{2}.$$

The general solution is thus

$$y(x) = y_\mathrm{p}(x) + y_\mathrm{h}(x) = K\,\mathrm{e}^{-2x} + \frac{1}{6}\,\mathrm{e}^{4x} + \frac{1}{2}.$$

Here, we have combined the two terms containing e^{-2x}. The new constant K can be determined from an additional initial condition $y(0) = \alpha$, namely

$$K = \alpha - \frac{2}{3}.$$

19.3 Existence and Uniqueness of the Solution

Finding analytic solutions of differential equations can be a difficult problem and is often impossible. Apart from some types of differential equations (e.g., linear problems or equations with separable variables), there is no general procedure to determine the solution explicitly. Thus numerical methods are used frequently (see Chap. 21). In the following we discuss the existence and uniqueness of solutions of general initial value problems.

Proposition 19.13 (Peano's theorem[2]) *If the function f is continuous in a neighbourhood of (x_0, y_0), then the initial value problem*

$$y' = f(x, y), \qquad y(x_0) = y_0$$

has a solution $y(x)$ for x close to x_0.

Instead of a proof (see [11, Part I, Theorem 7.6]), we discuss the limitations of this proposition. First it only guarantees the existence of a local solution in the neighbourhood of the initial value. The next example shows that one cannot expect more, in general.

Example 19.14 We solve the differential equation $\dot{x} = x^2,\ x(0) = 1$. Separation of the variables yields

$$\int \frac{\mathrm{d}x}{x^2} = \int \mathrm{d}t = t + C,$$

and thus

$$x(t) = \frac{1}{1-t}.$$

This function has a singularity at $t = 1$, where the solution ceases to exist. This behaviour is called *blow up*.

Furthermore, Peano's theorem does not give any information on how many solutions an initial value problem has. In general, solutions need not be unique, as it is shown in the following example.

Example 19.15 The initial value problem $y' = 2\sqrt{|y|},\ y(0) = 0$ has infinitely many solutions

$$y(x) = \begin{cases} (x-b)^2, & b < x, \\ 0, & -a \le x \le b, \\ -(x-a)^2, & x < -a, \end{cases} \qquad a, b \ge 0 \text{ arbitrary}.$$

For example, for $x < -a$, one verifies at once

$$y'(x) = -2(x-a) = 2(a-x) = 2|x-a| = 2\sqrt{(x-a)^2} = 2\sqrt{|y|}.$$

Thus the continuity of f is not sufficient to guarantee the uniqueness of the solution of initial value problems. One needs somewhat more regularity, namely Lipschitz[3] continuity with respect to the second variable (see also Definition C.14).

[2] G. Peano, 1858–1932.
[3] R. Lipschitz, 1832–1903.

Definition 19.16 Let $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$. The function f is said to satisfy a *Lipschitz condition* with *Lipschitz constant* L on D, if the inequality $|f(x, y) - f(x, z)| \le L\,|y - z|$ holds for all points $(x, y), (x, z) \in D$.

According to the mean value theorem (Proposition 8.4)

$$f(x, y) - f(x, z) = \frac{\partial f}{\partial y}(x, \xi)(y - z)$$

for every differentiable function. If the derivative is bounded, then the function satisfies a Lipschitz condition. In this case one can choose

$$L = \sup \left| \frac{\partial f}{\partial y}(x, \xi) \right|.$$

Counterexample 19.17 The function $g(x, y) = \sqrt{|y|}$ does not satisfy a Lipschitz condition in any D that contains a point with $y = 0$ because

$$\frac{|g(x, y) - g(x, 0)|}{|y - 0|} = \frac{\sqrt{|y|}}{|y|} = \frac{1}{\sqrt{|y|}} \to \infty \qquad \text{for} \quad y \to 0.$$

Proposition 19.18 *If the function f satisfies a Lipschitz condition in the neighbourhood of (x_0, y_0), then the initial value problem*

$$y' = f(x, y), \qquad y(x_0) = y_0$$

has a unique *solution $y(x)$ for x close to x_0.*

Proof We only show uniqueness, the existence of a solution $y(x)$ on the interval $[x_0, x_0 + H]$ follows (for small H) from Peano's theorem. Uniqueness is proven indirectly. Assume that z is another solution, *different* from y, on the interval $[x_0, x_0 + H]$ with $z(x_0) = y_0$. The number

$$x_1 = \inf\big\{x \in \mathbb{R}\,;\; x_0 \le x \le x_0 + H \;\text{ and }\; y(x) \ne z(x)\big\}$$

is thus well-defined. We infer from the continuity of y and z that $y(x_1) = z(x_1)$. Now we choose $h > 0$ so small that $x_1 + h \le x_0 + H$ and integrate the differential equation

$$y'(x) = f\big(x, y(x)\big)$$

from x_1 to $x_1 + h$. This gives

$$y(x_1 + h) - y(x_1) = \int_{x_1}^{x_1+h} y'(x)\,\mathrm{d}x = \int_{x_1}^{x_1+h} f\big(x, y(x)\big)\,\mathrm{d}x$$

and

$$z(x_1 + h) - y(x_1) = \int_{x_1}^{x_1+h} f\big(x, z(x)\big)\, \mathrm{d}x.$$

Subtracting the first formula above from the second yields

$$z(x_1 + h) - y(x_1 + h) = \int_{x_1}^{x_1+h} \Big(f\big(x, z(x)\big) - f\big(x, y(x)\big)\Big)\, \mathrm{d}x.$$

The Lipschitz condition on f gives

$$\begin{aligned} |z(x_1 + h) - y(x_1 + h)| &\le \int_{x_1}^{x_1+h} \big| f\big(x, z(x)\big) - f\big(x, y(x)\big)\big|\, \mathrm{d}x \\ &\le L \int_{x_1}^{x_1+h} |z(x) - y(x)|\, \mathrm{d}x. \end{aligned}$$

Let now

$$M = \max\big\{|z(x) - y(x)|\,;\; x_1 \le x \le x_1 + h\big\}.$$

Due to the continuity of y and z, this maximum exists, see the discussion after Proposition 6.15. After possibly decreasing h this maximum is attained at $x_1 + h$ and

$$M = |z(x_1 + h) - y(x_1 + h)| \le L \int_{x_1}^{x_1+h} M\, \mathrm{d}x \le Lh\, M.$$

For a sufficiently small h, namely $Lh < 1$, the inequality

$$M \le Lh\, M$$

implies $M = 0$. Since one can choose h arbitrarily small, $y(x) = z(x)$ holds true for $x_1 \le x \le x_1 + h$ in contradiction to the definition of x_1. Hence the assumed different solution z does not exist. □

19.4 Method of Power Series

We have encountered several examples of functions that can be represented as series, e.g. in Chap. 12. Motivated by this we try to solve the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0$$

by means of a series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

We will use the fact that *convergent power series* can be differentiated and rearranged term by term, see for instance [3, Chap. 9, Corollary 7.4].

Example 19.19 We solve once more the linear initial value problem

$$y' = y, \qquad y(0) = 1.$$

For that we differentiate the ansatz

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

term by term with respect to x

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

and substitute the result into the differential equation to get

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \;=\; a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Since this equation has to hold for all x, the unknowns a_n can be determined by equating the coefficients of same powers of x. This gives

$$\begin{aligned} a_1 &= a_0, & 2a_2 &= a_1, \\ 3a_3 &= a_2, & 4a_4 &= a_3, \end{aligned}$$

and so on. Due to $a_0 = y(0) = 1$ this infinite system of equations can be solved recursively. One obtains

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = \frac{1}{2!}, \quad a_3 = \frac{1}{3!}, \; \ldots \;, \; a_n = \frac{1}{n!}$$

and thus the (expected) solution

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \mathrm{e}^x.$$

Example 19.20 (A particular Riccati differential equation[4]) For the solution of the initial value problem

$$y' = y^2 + x^2, \qquad y(0) = 1,$$

[4] J.F. Riccati, 1676–1754.

we make the ansatz

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

The initial condition $y(0) = 1$ immediately gives $a_0 = 1$. First, we compute the product (see also Proposition C.10)

$$\begin{aligned} y(x)^2 &= (1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)^2 \\ &= 1 + 2a_1 x + (a_1^2 + 2a_2)x^2 + (2a_3 + 2a_2 a_1)x^3 + \cdots \end{aligned}$$

and substitute it into the differential equation

$$\begin{aligned} &a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \\ &\qquad = 1 + 2a_1 x + (1 + a_1^2 + 2a_2)x^2 + (2a_3 + 2a_2 a_1)x^3 + \cdots \end{aligned}$$

Equating coefficients results in

$$\begin{aligned} a_1 &= 1, & & \\ 2a_2 &= 2a_1, & a_2 &= 1 \\ 3a_3 &= 1 + a_1^2 + 2a_2, & a_3 &= 4/3 \\ 4a_4 &= 2a_3 + 2a_2 a_1, & a_4 &= 7/6, \quad \ldots \end{aligned}$$

Thus we obtain a good approximation to the solution for small x

$$y(x) = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \mathcal{O}(x^5).$$

The maple command

```
dsolve({diff(y(x),x)=x^2+y(x)^2, y(0)=1}, y(x), series);
```

carries out the above computations.

19.5 Qualitative Theory

Often one can describe the qualitative behaviour of the solutions of differential equations without solving the equations themselves. As the simplest case we discuss the stability of nonlinear differential equations in the neighbourhood of an equilibrium point. A differential equation is called *autonomous*, if its right-hand side does not explicitly depend on the independent variable.

Definition 19.21 The point $y^\star \in \mathbb{R}$ is called an *equilibrium* of the autonomous differential equation $y' = f(y)$, if $f(y^\star) = 0$.

Equilibrium points are particular solutions of the differential equation, so-called stationary solutions.

In order to investigate solutions in the neighbourhood of an equilibrium point, we *linearise* the differential equation at the equilibrium. Let

$$w(x) = y(x) - y^\star$$

denote the distance of the solution $y(x)$ from the equilibrium. Taylor series expansion of f shows that

$$w' = y' = f(y) = f(y) - f(y^\star) = f'(y^\star)w + \mathcal{O}(w^2),$$

hence

$$w'(x) = \big(a + \mathcal{O}(w)\big)w$$

with $a = f'(y^\star)$. It is decisive how solutions of this problem behave for small w. Obviously the value of the coefficient $a + \mathcal{O}(w)$ is crucial. If $a < 0$, then $a + \mathcal{O}(w) < 0$ for sufficiently small w and the function $|w(x)|$ decreases. If on the other hand $a > 0$, then the function $|w(x)|$ increases for small w. With these considerations one has proven the following proposition.

Proposition 19.22 *Let $y^\star$ be an* equilibrium point *of the differential equation $y' = f(y)$ and assume that $f'(y^\star) < 0$. Then all solutions of the differential equation with initial value $w(0)$ close to $y^\star$ satisfy the estimate*

$$|w(x)| \le C \cdot \mathrm{e}^{bx} \cdot |w(0)|$$

with constants $C > 0$ and $b < 0$.

Under the conditions of the proposition one calls the equilibrium point *asymptotically stable*. An asymptotically stable equilibrium attracts all solutions in a sufficiently small neighbourhood (exponentially fast), since due to $b < 0$

$$|w(x)| \to 0 \qquad \text{as } x \to \infty.$$

Example 19.23 Verhulst's model

$$y' = (\alpha - \beta y)y, \qquad \alpha, \beta > 0$$

has two equilibrium points, namely $y_1^\star = 0$ and $y_2^\star = \alpha/\beta$. Due to

$$f'(y_1^\star) = \alpha - 2\beta y_1^\star = \alpha, \qquad f'(y_2^\star) = \alpha - 2\beta y_2^\star = -\alpha,$$

$y_1^\star = 0$ is *unstable* and $y_2^\star = \alpha/\beta$ is *asymptotically stable*.

19.6 Second-Order Problems

The equation

$$y''(x) + ay'(x) + by(x) = g(x)$$

is called a second-order linear differential equation with *constant* coefficients a, b and inhomogeneity g.

Example 19.24 (Mass–spring–damper model) According to Newton's second law of motion, a mass–spring system is modelled by the second-order differential equation

$$y''(x) + ky(x) = 0,$$

where $y(x)$ denotes the position of the mass and k is the stiffness of the spring. The solution of this equation describes a free vibration without damping and excitation. A more realistic model is obtained by adding a viscous damping force $-cy'(x)$ and an external excitation $g(x)$. This results in the differential equation

$$my''(x) + cy'(x) + ky(x) = g(x),$$

which is of the above form.

By introducing the new variable $z(x) = y'(x)$, the homogeneous problem

$$y'' + ay' + by = 0$$

can be rewritten as a system of first-order equations

$$\begin{aligned} y' &= z \\ z' &= -by - az, \end{aligned}$$

see Chap. 20, where this approach is worked out in detail.

Here, we will follow a different idea. Let α and β denote the roots of the quadratic equation

$$\lambda^2 + a\lambda + b = 0,$$

which is called the *characteristic equation* of the homogeneous problem. Then, the second-order problem

$$y''(x) + ay'(x) + by(x) = g(x)$$

can be factorised in the following way:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + a\frac{\mathrm{d}}{\mathrm{d}x} + b\right) y(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x} - \beta\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\right) y(x) = g(x).$$

Setting

$$w(x) = y'(x) - \alpha y(x),$$

we obtain the following first-order linear differential equation for w

$$w'(x) - \beta w(x) = g(x).$$

This problem has the general solution (see Proposition 19.10)

$$w(x) = K_2 \mathrm{e}^{\beta(x-x_0)} + \int_{x_0}^{x} \mathrm{e}^{\beta(x-\xi)} g(\xi)\,\mathrm{d}\xi$$

with some constant K_2. Inserting this expression into the definition of w shows that y is the solution of the first-order problem

$$y'(x) - \alpha y(x) = K_2 \mathrm{e}^{\beta(x-x_0)} + \int_{x_0}^{x} \mathrm{e}^{\beta(x-\xi)} g(\xi)\,\mathrm{d}\xi.$$

Let us assume for a moment that $\alpha \neq \beta$. Applying once more Proposition 19.10 for the solution of this problem gives

$$\begin{aligned} y(x) &= K_1 \mathrm{e}^{\alpha(x-x_0)} + \int_{x_0}^{x} \mathrm{e}^{\alpha(x-\eta)} w(\eta)\,\mathrm{d}\eta \\ &= K_1 \mathrm{e}^{\alpha(x-x_0)} + K_2 \int_{x_0}^{x} \mathrm{e}^{\alpha(x-\eta)} \mathrm{e}^{\beta(\eta-x_0)}\,\mathrm{d}\eta \\ &\qquad + \int_{x_0}^{x} \mathrm{e}^{\alpha(x-\eta)} \int_{x_0}^{\eta} \mathrm{e}^{\beta(\eta-\xi)} g(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned}$$

Since

$$\begin{aligned} \int_{x_0}^{x} \mathrm{e}^{\alpha(x-\eta)} \mathrm{e}^{\beta(\eta-x_0)}\,\mathrm{d}\eta &= \mathrm{e}^{\alpha x - \beta x_0} \int_{x_0}^{x} \mathrm{e}^{\eta(\beta-\alpha)}\,\mathrm{d}\eta \\ &= \frac{1}{\beta-\alpha}\left(\mathrm{e}^{\beta(x-x_0)} - \mathrm{e}^{\alpha(x-x_0)}\right), \end{aligned}$$

we finally obtain

$$y(x) = c_1 \mathrm{e}^{\alpha(x-x_0)} + c_2 \mathrm{e}^{\beta(x-x_0)} + \int_{x_0}^{x} \mathrm{e}^{\alpha(x-\eta)} \int_{x_0}^{\eta} \mathrm{e}^{\beta(\eta-\xi)} g(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

with

$$c_1 = K_1 - \frac{K_2}{\beta-\alpha}, \qquad c_2 = \frac{K_2}{\beta-\alpha}.$$

By setting $g = 0$, one obtains the general solution of the homogeneous problem

$$y_\mathrm{h}(x) = c_1 \mathrm{e}^{\alpha(x-x_0)} + c_2 \mathrm{e}^{\beta(x-x_0)}.$$

The double integral

$$\int_{x_0}^{x} \mathrm{e}^{\alpha(x-\eta)} \int_{x_0}^{\eta} \mathrm{e}^{\beta(\eta-\xi)} g(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

is a particular solution of the inhomogeneous problem. Note that, due to the linearity of the problem, the superposition principle (see Proposition 19.5) is again valid.

Summarising the above calculations gives the following two propositions.

Proposition 19.25 *Consider the homogeneous differential equation*

$$y''(x) + ay'(x) + by(x) = 0$$

and let α and β denote the roots of its characteristic equation

$$\lambda^2 + a\lambda + b = 0.$$

The general (real) solution of this problem is given by

$$y_h(x) = \begin{cases} c_1 \mathrm{e}^{\alpha x} + c_2 \mathrm{e}^{\beta x} & \text{for } \alpha \neq \beta \in \mathbb{R}, \\ (c_1 + c_2 x)\mathrm{e}^{\alpha x} & \text{for } \alpha = \beta \in \mathbb{R}, \\ \mathrm{e}^{\rho x}\big(c_1 \cos(\theta x) + c_2 \sin(\theta x)\big) & \text{for } \alpha = \rho + \mathrm{i}\theta, \quad \rho, \theta \in \mathbb{R}, \end{cases}$$

for arbitrary real constants c_1 and c_2.

Proof Since the characteristic equation has real coefficients, the roots are either both real or conjugate complex, i.e. $\alpha = \overline{\beta}$. The case $\alpha \neq \beta$ was already considered above. In the complex case where $\alpha = \rho + \mathrm{i}\theta$, we use Euler's formula

$$\mathrm{e}^{\alpha x} = \mathrm{e}^{\rho x}\big(\cos(\theta x) + \mathrm{i}\sin(\theta x)\big).$$

This shows that $c_1 \mathrm{e}^{\rho x}\cos(\theta x)$ and $c_2 \mathrm{e}^{\rho x}\sin(\theta x)$ are the searched for real solutions. Finally, in the case $\alpha = \beta$, the above calculations show

$$\begin{aligned} y_\mathrm{h}(x) &= K_1 \mathrm{e}^{\alpha(x-x_0)} + K_2 \int_{x_0}^{x} \mathrm{e}^{\alpha(x-\eta)} \mathrm{e}^{\alpha(\eta-x_0)}\,\mathrm{d}\eta \\ &= (c_1 + c_2 x)\mathrm{e}^{\alpha x} \end{aligned}$$

with $c_1 = (K_1 - K_2 x_0)\mathrm{e}^{-\alpha x_0}$ and $c_2 = K_2 \mathrm{e}^{-\alpha x_0}$. □

Proposition 19.26 *Let y_p be an arbitrary solution of the inhomogeneous differential equation*

$$y''(x) + ay'(x) + by(x) = g(x).$$

Then its general solution can be written as

$$y(x) = y_h(x) + y_p(x)$$

where y_h is the general solution of the homogeneous problem.

Proof Superposition principle. □

Example 19.27 In order to find the general solution of the inhomogeneous differential equation

$$y''(x) - 4y(x) = \mathrm{e}^x$$

we first consider the homogeneous part. Its characteristic equation $\lambda^2 - 4 = 0$ has the roots $\lambda_1 = 2$ and $\lambda_2 = -2$. Therefore,

$$y_\mathrm{h}(x) = c_1\mathrm{e}^{2x} + c_2\mathrm{e}^{-2x}.$$

A particular solution of the inhomogeneous problem is found by the general formula

$$\begin{aligned} y_\mathrm{p}(x) &= \int_0^x \mathrm{e}^{2(x-\eta)} \int_0^\eta \mathrm{e}^{-2(\eta-\xi)}\mathrm{e}^\xi \,\mathrm{d}\xi \,\mathrm{d}\eta \\ &= \mathrm{e}^{2x} \int_0^x \mathrm{e}^{-4\eta}\frac{1}{3}\left(\mathrm{e}^{3\eta} - 1\right) \mathrm{d}\eta \\ &= \frac{1}{3}\mathrm{e}^{2x}\left(\left(1 - \mathrm{e}^{-x}\right) + \frac{1}{4}\left(\mathrm{e}^{-4x} - 1\right)\right). \end{aligned}$$

Comparing this with y_h shows that the choice $y_\mathrm{p}(x) = -\frac{1}{3}\mathrm{e}^x$ is possible as well, since the other terms solve the homogeneous equation.

In general, however, it is simpler to use as ansatz for y_p a linear combination of the inhomogeneity and its derivatives. In our case, the ansatz would be $y_\mathrm{p}(x) = a\mathrm{e}^x$. Inserting this ansatz into the inhomogeneous problem gives $a - 4a = 1$, which results again in $y_\mathrm{p}(x) = -\frac{1}{3}\mathrm{e}^x$.

Example 19.28 The characteristic equation of the homogeneous problem

$$y''(x) - 10y'(x) + 25y(x) = 0$$

has the double root $\lambda_1 = \lambda_2 = 5$. Therefore, its general solution is

$$y(x) = c_1\mathrm{e}^{5x} + c_2x\mathrm{e}^{5x}.$$

Example 19.29 The characteristic equation of the homogeneous problem

$$y''(x) + 2y'(x) + 2y(x) = 0$$

has the complex conjugate roots $\lambda_1 = -1 + \mathrm{i}$ and $\lambda_2 = -1 - \mathrm{i}$. The complex form of its general solution is

$$y(x) = c_1 \mathrm{e}^{-(1+\mathrm{i})x} + c_2 \mathrm{e}^{-(1-\mathrm{i})x}$$

with complex coefficients c_1 and c_2.

The real form is

$$y(x) = \mathrm{e}^{-x}\big(d_1 \cos x + d_2 \sin x\big)$$

with real coefficients d_1 and d_2.

19.7 Exercises

1. Find the general solution of the following differential equations and sketch some solution curves

$$\text{(a)}\quad \dot{x} = \frac{x}{t}, \qquad \text{(b)}\quad \dot{x} = \frac{t}{x}, \qquad \text{(c)}\quad \dot{x} = \frac{-t}{x}.$$

The direction field is most easily plotted with maple, e.g. with `DEplot`.

2. Using the applet *Dynamical systems in the plane*, solve Exercise 1 by rewriting the respective differential equation as an equivalent autonomous system by adding the equation $\dot{t} = 1$.
Hint. The variables are denoted by x and y in the applet. For example, Exercise 1(a) would have to be written as $x' = x/y$ and $y' = 1$.
3. According to Newton's law of cooling, the rate of change of the temperature x of an object is proportional to the difference of its temperature and the ambient temperature a. This is modelled by the differential equation

$$\dot{x} = k(a - x),$$

where k is a proportionality constant. Find the general solution of this differential equation.
How long does it take to cool down an object from $x(0) = 100°$ to $40°$ at an ambient temperature of $20°$, if it cooled down from $100°$ to $80°$ in 5 minutes?
4. Solve Verhulst's differential equation from Example 19.9 and compute the limit $t \to \infty$ of the solution.

5. A tank contains $100 l$ of liquid A. Liquid B is added at a rate of $5 l/s$, while at the same time the mixture is pumped out with a rate of $10 l/s$. We are interested in the amount $x(t)$ of the liquid B in the tank at time t. From the balance equation $\dot{x}(t) = \text{rate(in)} - \text{rate(out)} = \text{rate(in)} - 10 \cdot x(t)/\text{total amount}(t)$ one obtains the differential equation

$$\dot{x} = 5 - \frac{10x}{100 - 5t}, \quad x(0) = 0.$$

Explain the derivation of this equation in detail and use maple (with `dsolve`) to solve the initial value problem. When is the tank empty?

6. Solve the differential equations

$$\text{(a)} \quad y' = ay, \qquad \text{(b)} \quad y' = ay + 2$$

with the method of power series.

7. Find the solution of the initial value problem

$$\dot{x}(t) = 1 + x(t)^2$$

with initial value $x(0) = 0$. In which interval does the solution exist?

8. Find the solution of the initial value problem

$$\dot{x}(t) + 2x(t) = \mathrm{e}^{4t} + 1$$

with initial value $x(0) = 0$.

9. Find the general solutions of the differential equations

$$\text{(a)} \quad \ddot{x} + 4\dot{x} - 5x = 0, \qquad \text{(b)} \quad \ddot{x} + 4\dot{x} + 5x = 0, \qquad \text{(c)} \quad \ddot{x} + 4\dot{x} = 0.$$

10. Find a particular solution of the problem

$$\ddot{x}(t) + \dot{x}(t) - 6x(t) = t^2 + 2t - 1.$$

Hint. Use the ansatz $y_\mathrm{p}(t) = at^2 + bt + c$.

11. Find the general solution of the differential equation

$$y''(x) + 4y(x) = \cos x$$

and specify the solution for the initial data $y(0) = 1$, $y'(0) = 0$.
Hint. Consider the ansatz $y_\mathrm{p}(x) = k_1 \cos x + k_2 \sin x$.

12. Find the general solution of the differential equation

$$y''(x) + 4y'(x) + 5y(x) = \cos 2x$$

and specify the solution for the initial data $y(0) = 1$, $y'(0) = 0$.
Hint. Consider the ansatz $y_\mathrm{p}(x) = k_1 \cos 2x + k_2 \sin 2x$.

13. Find the general solution of the homogeneous equation

$$y''(x) + 2y'(x) + y(x) = 0.$$