Integration of Functions of Two Variables

17

In Sect. 11.3 we have shown how to calculate the volume of solids of revolution. If there is no rotational symmetry, however, one needs an extension of integral calculus to functions of two variables. This arises, for example, if one wants to find the volume of a solid that lies between a domain D in the (x, y)-plane and the graph of a non-negative function $z = f(x, y)$. In this section we will extend the notion of Riemann integrals from Chap. 11 to double integrals of functions of two variables. Important tools for the computation of double integrals are their representation as iterated integrals and the transformation formula (change of coordinates). The integration of functions of several variables occurs in numerous applications, a few of which we will discuss.

17.1 Double Integrals

We start with the integration of a real-valued function $z = f(x, y)$ which is defined on a rectangle $R = [a, b] \times [c, d]$. More general domains of integration $D \subset \mathbb{R}^2$ will be discussed below. Since we know from Sect. 11.1 that Riemann integrable functions are necessarily bounded, we assume in the whole section that f is bounded. If f is non-negative, the integral should be interpretable as the volume of the solid with base R and top surface given by the graph of f (see Fig. 17.2). This motivates the following approach in which the solid is approximated by a sum of cuboids.

We place a rectangular grid G over the domain R by partitioning the intervals $[a, b]$ and $[c, d]$ like in Sect. 11.1:

$$Z_x : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$
$$Z_y : c = y_0 < y_1 < y_2 < \cdots < y_{m-1} < y_m = d.$$

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The rectangular grid is made up of the small rectangles

$$[x_{i-1}, x_i] \times [y_{j-1}, y_j], \quad i = 1, \dots, n, \; j = 1, \dots, m.$$

The *mesh size* $\Phi(G)$ is the length of the largest subinterval involved:

$$\Phi(G) = \max\big(|x_i - x_{i-1}|, |y_j - y_{j-1}| \; ; \; i = 1, \dots, n, \; j = 1, \dots, m\big).$$

Finally we choose an arbitrary intermediate point $\mathbf{p}_{ij} = (\xi_{ij}, \eta_{ij})$ in each of the rectangles of the grid, see Fig. 17.1.

The double sum

$$S = \sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_{ij}, \eta_{ij})(x_i - x_{i-1})(y_j - y_{j-1})$$

is again called a *Riemann sum*. Since the volume of a cuboid with base $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ and height $f(\xi_{ij}, \eta_{ij})$ is

$$f(\xi_{ij}, \eta_{ij})(x_i - x_{i-1})(y_j - y_{j-1}),$$

the above Riemann sum is an approximation to the volume under the graph of f (Fig. 17.2).

Like in Sect. 11.1, the integral is now defined as a limit of Riemann sums. We consider a sequence $G_1, G_2, G_3, \dots$ of grids whose mesh size $\Phi(G_N)$ tends to zero as $N \to \infty$ and the corresponding Riemann sums S_N.

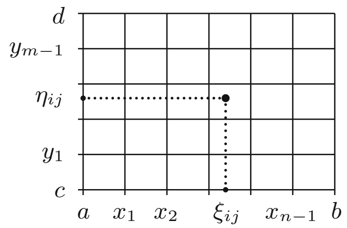


Fig. 17.1 Partitioning the rectangle R

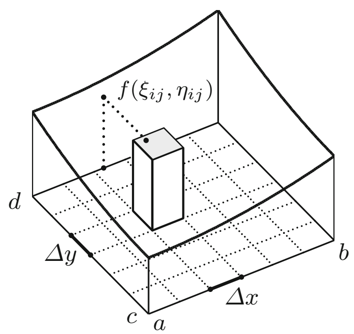


Fig. 17.2 Volume and approximation by cuboids

Definition 17.1 A bounded function $z = f(x, y)$ is called *Riemann integrable* on $R = [a, b] \times [c, d]$ if for arbitrary sequences of grids $(G_N)_{N\geq 1}$ with $\Phi(G_N) \to 0$ the corresponding Riemann sums $(S_N)_{N\geq 1}$ tend to the same limit $I(f)$, independently of the choice of intermediate points. This limit

$$I(f) = \iint_R f(x, y)\,\mathrm{d}(x, y)$$

is called the *double integral* of f on R.

Experiment 17.2 Study the M-file `mat17_1.m` and experiment with different randomly chosen Riemann sums for the function $z = x^2 + y^2$ on the rectangle $[0, 1] \times [0, 1]$. What happens if you choose finer and finer grids?

As in the case of one variable, one may use the definition of the double integral for obtaining a numerical approximation to the integral. However, it is of little use for the analytic evaluation of integrals. In Sect. 11.1 the fundamental theorem of calculus has proven helpful, here the representation as *iterated integral* does. In this way the computation of double integrals is reduced to the integration of functions in one variable.

Proposition 17.3 (The double integral as iterated integral) *If a bounded function f and its partial functions $x \mapsto f(x, y)$, $y \mapsto f(x, y)$ are Riemann integrable on $R = [a, b] \times [c, d]$, then the mappings $x \mapsto \int_c^d f(x, y)\,\mathrm{d}y$ and $y \mapsto \int_a^b f(x, y)\,\mathrm{d}x$ are Riemann integrable as well and*

$$\iint_R f(x, y)\,\mathrm{d}(x, y) = \int_a^b \left(\int_c^d f(x, y)\,\mathrm{d}y\right) \mathrm{d}x = \int_c^d \left(\int_a^b f(x, y)\,\mathrm{d}x\right) \mathrm{d}y.$$

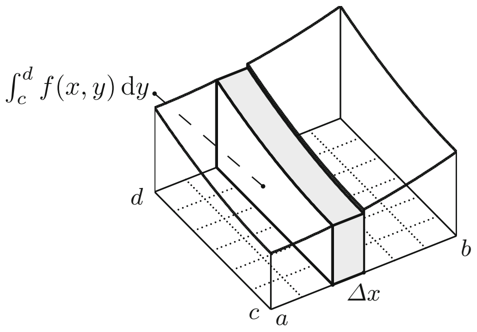
Outline of the proof. If one chooses intermediate points in the Riemann sums of the special form $\mathbf{p}_{ij} = (\xi_i, \eta_j)$ with $\xi_i \in [x_{i-1}, x_i]$, $\eta_j \in [y_{j-1}, y_j]$, then

$$\iint_R f(x, y)\,\mathrm{d}(x, y) \approx \sum_{i=1}^n \left(\sum_{j=1}^m f(\xi_i, \eta_j)(y_j - y_{j-1})\right)(x_i - x_{i-1})$$
$$\approx \sum_{i=1}^n \left(\int_c^d f(\xi_i, y)\,\mathrm{d}y\right)(x_i - x_{i-1}) \approx \int_a^b \left(\int_c^d f(x, y)\,\mathrm{d}y\right) \mathrm{d}x$$

and likewise for the second statement by changing the order. For a rigorous proof of this argument, we refer to the literature, for instance [4, Theorem 8.13 and Corollary]. □

Figure 17.3 serves to illustrate Proposition 17.3. The volume is approximated by summation of thin slices parallel to the axis instead of small cuboids. Proposition 17.3

Fig. 17.3 The double integral as iterated integral



states that the volume of the solid is obtained by integration over the area of the cross sections (perpendicular to the x- or y-axis). In this form Proposition 17.3 is called *Cavalieri's principle*.[1] In general integration theory one also speaks of *Fubini's theorem*.[2] Since in the case of integrability the order of integration does not matter, one often omits the brackets and writes

$$\iint_R f(x,y)\,\mathrm{d}(x,y) = \iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

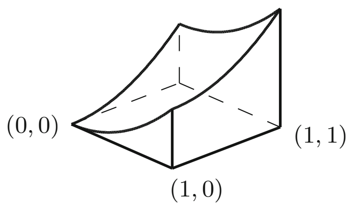
Example 17.4 Let $R = [0,1] \times [0,1]$. The volume of the body

$$B = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in R,\ 0 \le z \le x^2 + y^2\}$$

is obtained using Proposition 17.3 as follows, see also Fig. 17.4:

$$\begin{aligned}\iint_R \left(x^2+y^2\right)\mathrm{d}(x,y) &= \int_0^1 \left(\int_0^1 \left(x^2+y^2\right)\mathrm{d}y\right)\mathrm{d}x \\ &= \int_0^1 \left(x^2 y + \frac{y^3}{3}\right)\Big|_{y=0}^{y=1} \mathrm{d}x = \int_0^1 \left(x^2 + \frac{1}{3}\right)\mathrm{d}x = \left(\frac{x^3}{3} + \frac{x}{3}\right)\Big|_{x=0}^{x=1} = \frac{2}{3}.\end{aligned}$$

Fig. 17.4 The body B



[1] B. Cavalieri, 1598–1647.
[2] G. Fubini, 1879–1943.

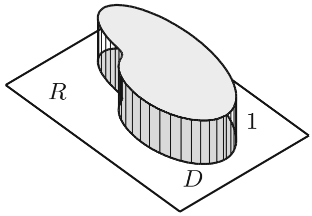


Fig. 17.5 Area as volume of the cylinder of height one

We now turn to the integration over more general (bounded) domains $D \subset \mathbb{R}^2$. The *indicator function* of the domain D is

$$\mathbb{1}_D(x, y) = \begin{cases} 1, & (x, y) \in D, \\ 0, & (x, y) \notin D. \end{cases}$$

We can enclose the bounded domain D in a rectangle R ($D \subset R$). If the Riemann integral of the indicator function of D exists, then it represents the volume of the cylinder of height one and base D and thus the area of D (Fig. 17.5). The result obviously does not depend on the size of the surrounding rectangle since the indicator function assumes the value zero outside the domain D.

Definition 17.5 Let D be a bounded domain and R an enclosing rectangle.

(a) If the indicator function of D is Riemann integrable then the domain D is called *measurable* and one sets

$$\iint_D \mathrm{d}(x, y) = \iint_R \mathbb{1}_D(x, y)\, \mathrm{d}(x, y).$$

(b) A subset $N \subset \mathbb{R}^2$ is called *set of measure zero*, if $\iint_N \mathrm{d}(x, y) = 0$.

(c) For a bounded function $z = f(x, y)$, its integral over a measurable domain D is defined as

$$\iint_D f(x, y)\, \mathrm{d}(x, y) = \iint_R f(x, y) \mathbb{1}_D(x, y)\, \mathrm{d}(x, y),$$

if $f(x, y)\mathbb{1}_D(x, y)$ is Riemann integrable.

Sets of measure zero are, for example, single points, straight line segments or segments of differentiable curves in the plane. Item (c) of the definition states that the integral of a function f over a domain D is determined by continuing f to a larger rectangle R and assigning the value zero outside D.

Remark 17.6 (a) If D is a measurable domain, N a set of measure zero and f is integrable over the respective domains then

$$\iint_D f(x, y)\, \mathrm{d}(x, y) = \iint_{D\setminus N} f(x, y)\, \mathrm{d}(x, y).$$

(b) Let $D = D_1 \cup D_2$. If $D_1 \cap D_2$ is a set of measure zero then

$$\iint_D f(x, y)\, \mathrm{d}(x, y) = \iint_{D_1} f(x, y)\, \mathrm{d}(x, y) + \iint_{D_2} f(x, y)\, \mathrm{d}(x, y).$$

The integral over the entire domain D is thus obtained as sum of the integrals over subdomains. The proof of this statement can easily be obtained by working with Riemann sums.

An important class of domains D on which integration is simple are the so-called *normal domains*.

Definition 17.7 (a) A subset $D \subset \mathbb{R}^2$ is called *normal domain of type I* if

$$D = \{(x, y) \in \mathbb{R}^2 \; ; \; a \le x \le b,\; v(x) \le y \le w(x)\}$$

with certain continuously differentiable lower and upper bounding functions $x \mapsto v(x)$, $x \mapsto w(x)$.

(b) A subset $D \subset \mathbb{R}^2$ is called *normal domain of type II*

$$D = \{(x, y) \in \mathbb{R}^2 \; ; \; c \le y \le d,\; l(y) \le x \le r(y)\}$$

with certain continuously differentiable left and right bounding functions $x \mapsto l(x)$, $x \mapsto r(x)$.

Figure 17.6 shows examples of normal domains.

Proposition 17.8 (Integration over normal domains) *Let D be a normal domain and $f : D \to \mathbb{R}$ continuous. For normal domains of type I, one has*

$$\iint_D f(x, y)\, \mathrm{d}(x, y) = \int_a^b \left(\int_{v(x)}^{w(x)} f(x, y)\, \mathrm{d}y \right) \mathrm{d}x$$

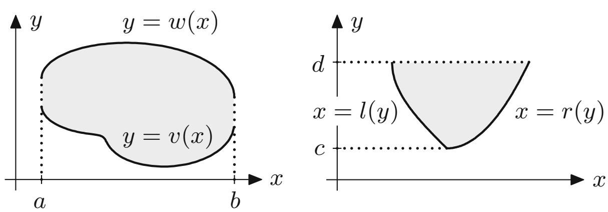


Fig. 17.6 Normal domains of type I and II

and for normal domains of type II

$$\iint_D f(x, y)\,\mathrm{d}(x, y) = \int_c^d \left(\int_{l(y)}^{r(y)} f(x, y)\,\mathrm{d}x \right) \mathrm{d}y.$$

Proof The statements follow from Proposition 17.3. We recall that f is extended by zero outside of D. For details we refer to the remark at the end of [4, Chap. 8.3]. □

Example 17.9 For the calculation of the volume of the body lying between the triangle $D = \{(x, y)\ ;\ 0 \le x \le 1, 0 \le y \le 1 - x\}$ and the graph of $z = x^2 + y^2$, we interpret D as normal domain of type I with the boundaries $v(x) = 0$, $w(x) = 1 - x$. Consequently

$$\begin{aligned}\iint_D \left(x^2 + y^2\right) \mathrm{d}(x, y) &= \int_0^1 \left(\int_0^{1-x} \left(x^2 + y^2\right) \mathrm{d}y \right) \mathrm{d}x \\ &= \int_0^1 \left(x^2 y + \frac{y^3}{3} \right) \Big|_{y=0}^{y=1-x} \mathrm{d}x = \int_0^1 \left(x^2(1-x) + \frac{(1-x)^3}{3} \right) \mathrm{d}x = \frac{1}{6},\end{aligned}$$

as can be seen by multiplying out and integrating term by term.

17.2 Applications of the Double Integral

For modelling purposes it is useful to introduce a simplified notation for Riemann sums. In the case of equidistant partitions Z_x, Z_y where all subintervals have the same lengths, one writes

$$\Delta x = x_i - x_{i-1}, \quad \Delta y = y_j - y_{j-1}$$

and calls

$$\Delta A = \Delta x\, \Delta y$$

the *area element of the grid G*. If one then takes the right upper corner $\mathbf{p}_{ij} = (x_i, y_j)$ of the subrectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ as an intermediate point, the corresponding Riemann sum reads

$$S = \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\, \Delta A = \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\, \Delta x\, \Delta y.$$

Application 17.10 (Mass as integral of the density) A thin plane object D has density $\rho(x, y)$ [mass/unit area] at the point (x, y). If the density ρ is constant everywhere then its total mass is simply the product of density and area. In the case of variable density (e.g. due to a change of the material properties from point to point), we partition D in smaller rectangles with sides Δx, Δy. The mass contained in such a small rectangle around (x, y) is approximately equal to $\rho(x, y)\,\Delta x\,\Delta y$. The total mass is thus approximately equal to

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\rho(x_i, y_j)\,\Delta x\,\Delta y.$$

However, this is just a Riemann sum for

$$M = \iint_D \rho(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

This consideration shows that the integral of the density function is a feasible model for representing the total mass of a two-dimensional object.

Application 17.11 (Centre of gravity) We consider a two-dimensional flat object D as in Application 17.10. The two statical moments of a small rectangle close to (x, y) with respect to a point (x^*, y^*) are

$$(x - x^*)\rho(x, y)\,\Delta x\,\Delta y, \quad (y - y^*)\rho(x, y)\,\Delta x\,\Delta y,$$

see Fig. 17.7.

The relevance of the statical moments can be seen if one considers the object under the influence of gravity. Multiplied by the gravitational acceleration g one obtains the moments of force with respect to the axes through (x^*, y^*) in direction of the coordinates (force times lever arm). The *centre of gravity* of the two-dimensional object D is the point (x_S, y_S) with respect to which the total statical moments vanish:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - x_S)\,\rho(x_i, y_j)\,\Delta x\,\Delta y \approx 0, \quad \sum_{i=1}^{n}\sum_{j=1}^{m}(y_j - y_S)\,\rho(x_i, y_j)\,\Delta x\,\Delta y \approx 0.$$

In the limit, as the mesh size of the grid tends to zero, one obtains

$$\iint_D (x - x_S)\,\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0, \quad \iint_D (y - y_S)\,\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0$$

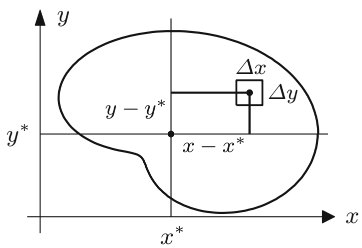


Fig. 17.7 The statical moments

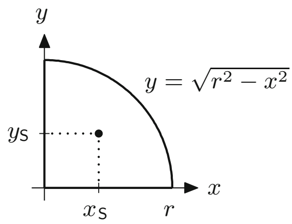


Fig. 17.8 Centre of gravity of the quarter circle

as defining equations for the centre of gravity; i.e.,

$$x_S = \frac{1}{M}\iint_D x\,\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y, \quad y_S = \frac{1}{M}\iint_D y\,\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y,$$

where M denotes the total mass as in Application 17.10.

For the special case of a constant density $\rho(x, y) \equiv 1$ one obtains the *geometric centre of gravity* of the domain D.

Example 17.12 (Geometric centre of gravity of a quarter circle) Let D be the quarter circle of radius r about $(0, 0)$ in the first quadrant; i.e., $D = \{(x, y)\ ;\ 0 \le x \le r,\ 0 \le y \le \sqrt{r^2 - x^2}\}$ (Fig. 17.8). With density $\rho(x, y) \equiv 1$ one obtains the area M as $r^2\pi/4$. The first statical moment is

$$\iint_D x\,\mathrm{d}x\,\mathrm{d}y = \int_0^r \left(\int_0^{\sqrt{r^2-x^2}} x\,\mathrm{d}y\right)\mathrm{d}x = \int_0^r \left(xy\Big|_{y=0}^{y=\sqrt{r^2-x^2}}\right)\mathrm{d}x$$
$$= \int_0^r x\sqrt{r^2 - x^2}\,\mathrm{d}x = -\frac{1}{3}\big(r^2 - x^2\big)^{3/2}\Big|_{x=0}^{x=r} = \frac{1}{3}r^3.$$

The x-coordinate of the centre of gravity is thus given by $x_S = \frac{4}{r^2\pi}\cdot\frac{1}{3}r^3 = \frac{4r}{3\pi}$. For reasons of symmetry, one has $y_S = x_S$.

17.3 The Transformation Formula

Similar to the substitution rule for one-dimensional integrals (Sect. 10.2), the transformation formula for double integrals makes it possible to change coordinates on the domain D of integration. For the purpose of this section it is convenient to assume that D is an open subset of $\mathbb{R}^2$ (see Definition 9.1).

Definition 17.13 A bijective, differentiable mapping $\mathbf{F} : D \to B = \mathbf{F}(D)$ between two open subsets $D, B \subset \mathbb{R}^2$ is called a *diffeomorphism* if the inverse mapping $\mathbf{F}^{-1}$ is also differentiable.

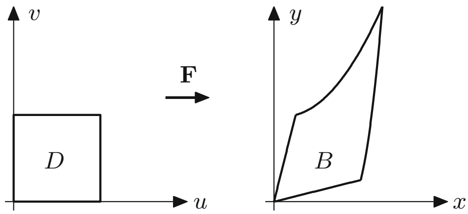


Fig. 17.9 Transformation of a planar domain

We use the following notation for the variables:

$$\mathbf{F} : D \to B : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(u, v) \\ y(u, v) \end{bmatrix}.$$

Figure 17.9 shows the image B of the domain $D = (0, 1) \times (0, 1)$ under the transformation

$$\mathbf{F} : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u + v/4 \\ u/4 + v + u^2 v^2 \end{bmatrix}.$$

The aim is to transform the integral of a real-valued function f over the domain B to one over D.

For this purpose we lay a grid G over the domain D in the (u, v)-plane and select a rectangle, for instance with the left lower corner (u, v) and sides spanned by the vectors

$$\begin{bmatrix} \Delta u \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \Delta v \end{bmatrix}.$$

The image of this rectangle under the transformation **F** will in general have a curvilinear boundary. In a first approximation we replace it by a parallelogram. In linear approximation (see Sect. 15.4) we have the following:

$$\mathbf{F}(u + \Delta u, v) \approx \mathbf{F}(u, v) + \mathbf{F}'(u, v) \begin{bmatrix} \Delta u \\ 0 \end{bmatrix},$$

$$\mathbf{F}(u, v + \Delta v) \approx \mathbf{F}(u, v) + \mathbf{F}'(u, v) \begin{bmatrix} 0 \\ \Delta v \end{bmatrix}.$$

The approximating parallelogram is thus spanned by the vectors

$$\begin{bmatrix} \dfrac{\partial x}{\partial u}(u, v) \\ \dfrac{\partial y}{\partial u}(u, v) \end{bmatrix} \Delta u, \quad \begin{bmatrix} \dfrac{\partial x}{\partial v}(u, v) \\ \dfrac{\partial y}{\partial v}(u, v) \end{bmatrix} \Delta v$$

and has the area (see Appendix A.5)

$$\left| \det \begin{bmatrix} \dfrac{\partial x}{\partial u}(u, v) & \dfrac{\partial x}{\partial v}(u, v) \\ \dfrac{\partial y}{\partial u}(u, v) & \dfrac{\partial y}{\partial v}(u, v) \end{bmatrix} \Delta u \, \Delta v \right| = \left| \det \mathbf{F}'(u, v) \right| \Delta u \, \Delta v.$$

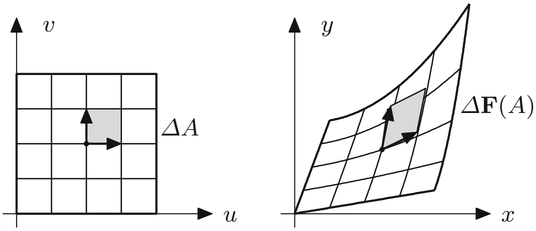


Fig. 17.10 Transformation of an area element

In short, the area element $\varDelta A = \varDelta u\,\varDelta v$ is changed by the transformation $\mathbf{F}$ to the area element $\varDelta \mathbf{F}(A) = \big|\det \mathbf{F}'(u, v)\big|\, \varDelta u\,\varDelta v$ (see Fig. 17.10).

Proposition 17.14 (Transformation formula for double integrals) *Let D, B be open, bounded subsets of $\mathbb{R}^2$, $\mathbf{F} : D \to B$ a diffeomorphism and $f : B \to \mathbb{R}$ a bounded mapping. Then*

$$\iint_B f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_D f\big(\mathbf{F}(u, v)\big)\,\big|\det \mathbf{F}'(u, v)\big|\,\mathrm{d}u\,\mathrm{d}v,$$

as long as the functions f and $f(\mathbf{F})\,\big|\det \mathbf{F}'\big|$ are Riemann integrable.

Outline of the proof. We use Riemann sums on the transformed grid and obtain

$$\begin{aligned}\iint_B f(x, y)\,\mathrm{d}x\,\mathrm{d}y &\approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\,\varDelta \mathbf{F}(A)\\ &\approx \sum_{i=1}^{n}\sum_{j=1}^{m} f\big(x(u_i, v_j), y(u_i, v_j)\big)\,\big|\det \mathbf{F}'(u_i, v_j)\big|\,\varDelta u\,\varDelta v\\ &\approx \iint_D f\big(x(u, v), y(u, v)\big)\,\big|\det \mathbf{F}'(u, v)\big|\,\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

A rigorous proof is tedious and requires a careful study of the boundary of the domain D and the behaviour of the transformation $\mathbf{F}$ near the boundary (see for instance [3, Chap. 19, Theorem 4.7]). □

Example 17.15 The area of the domain B from Fig. 17.9 can be calculated using the transformation formula with $f(x, y) = 1$ as follows. We have

$$\mathbf{F}'(u, v) = \begin{bmatrix} 1 & 1/4 \\ 1/4 + 2uv^2 & 1 + 2u^2 v \end{bmatrix},$$

$$\big|\det \mathbf{F}'(u, v)\big| = \left|\frac{15}{16} + 2u^2 v - \frac{1}{2}uv^2\right|$$

and thus

$$\iint_B \mathrm{d}x\,\mathrm{d}y = \iint_D \left|\det \mathbf{F}'(u,v)\right| \mathrm{d}u\,\mathrm{d}v$$
$$= \int_0^1 \left(\int_0^1 \left(\frac{15}{16} + 2u^2 v - \frac{1}{2}uv^2 \right) \mathrm{d}v \right) \mathrm{d}u$$
$$= \int_0^1 \left(\frac{15}{16} + u^2 - \frac{1}{6}u \right) \mathrm{d}u = \frac{15}{16} + \frac{1}{3} - \frac{1}{12} = \frac{19}{16}.$$

Example 17.16 (Volume of a hemisphere in polar coordinates) We represent a hemisphere of radius R by the three-dimensional domain

$$\{(x, y, z)\ ;\ 0 \le x^2 + y^2 \le R^2, 0 \le z \le \sqrt{R^2 - x^2 - y^2}\}.$$

Its volume is obtained by integration of the function $f(x, y) = \sqrt{R^2 - x^2 - y^2}$ over the base $B = \{(x, y)\ ;\ 0 \le x^2 + y^2 \le R^2\}$. In polar coordinates

$$\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2 : \begin{bmatrix} r \\ \varphi \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix}$$

the area B can be represented as the image $\mathbf{F}(D)$ of the rectangle $D = [0, R] \times [0, 2\pi]$. However, in order to fulfil the assumptions of Proposition 17.14 we have to switch to open domains on which $\mathbf{F}$ is a diffeomorphism. We can obtain this, for instance, by removing the boundary and the half ray $\{(x, y)\ ;\ 0 \le x \le R,\ y = 0\}$ of the circle B and the boundary of the rectangle D. On the smaller domains D', B' obtained in this way, $\mathbf{F}$ is a diffeomorphism. However, since B differs from B' and D differs from D' by sets of measure zero, the value of the integral is not changed if one replaces B by B' and D by D', see Remark 17.6. We have

$$\mathbf{F}'(r, \varphi) = \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix}, \quad \left|\det \mathbf{F}'(r, \varphi)\right| = r.$$

Substituting $x = r\cos\varphi$, $y = r\sin\varphi$ results in $x^2 + y^2 = r^2$ and we obtain the volume from the transformation formula as

$$\iint_B \sqrt{R^2 - x^2 - y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^R \int_0^{2\pi} \sqrt{R^2 - r^2}\; r\,\mathrm{d}\varphi\,\mathrm{d}r$$
$$= \int_0^R 2\pi r\sqrt{R^2 - r^2}\,\mathrm{d}r$$
$$= -\frac{2\pi}{3}\left(R^2 - r^2\right)^{3/2}\Big|_{r=0}^{r=R} = \frac{2\pi}{3}R^3,$$

which coincides with the known result from elementary geometry.

17.4 Exercises

1. Compute the volume of the parabolic dome $z = 2 - x^2 - y^2$ above the quadratic domain $D : -1 \le x \le 1, -1 \le y \le 1$.
2. (From statics) Compute the axial moment of inertia $\iint_D y^2 \,\mathrm{d}x\,\mathrm{d}y$ of a rectangular cross section $D : 0 \le x \le b, -h/2 \le y \le h/2$, where $b > 0, h > 0$.
3. Compute the volume of the body bounded by the plane $z = x + y$ above the domain $D : 0 \le x \le 1, 0 \le y \le \sqrt{1-x^2}$.
4. Compute the volume of the body bounded by the plane $z = 6 - x - y$ above the domain D, which is bounded by the y-axis and the straight lines $x + y = 6$, $x + 3y = 6$ $(x \ge 0, y \ge 0)$.
5. Compute the geometric centre of gravity of the domain $D : 0 \le x \le 1, 0 \le y \le 1 - x^2$.
6. Compute the area and the geometric centre of gravity of the semi-ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1, \quad y \ge 0.$$

   Hint. Introduce elliptic coordinates $x = ar\cos\varphi$, $y = br\sin\varphi$, $0 \le r \le 1$, $0 \le \varphi \le \pi$, compute the Jacobian and use the transformation formula.
7. (From statics) Compute the axial moment of inertia of a ring with inner radius R_1 and outer radius R_2 with respect to the central axis, i.e. the integral $\iint_D (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y$ over the domain $D : R_1 \le \sqrt{x^2+y^2} \le R_2$.
8. Modify the M-file `mat17_1.m` so that it can evaluate Riemann sums over equidistant partitions with $\Delta x \ne \Delta y$.
9. Let the domain D be bounded by the curves

$$y = x \quad \text{and} \quad y = x^2, \qquad 0 \le x \le 1.$$

   (a) Sketch D.
   (b) Compute the area of D by means of the double integral $F = \iint_D \mathrm{d}(x, y)$.
   (c) Compute the statical moments $\iint_D x \; \mathrm{d}(x, y)$ und $\iint_D y \; \mathrm{d}(x, y)$.
10. Compute the statical moment $\iint_D y \; \mathrm{d}(x, y)$ of the half-disk

$$D = \{(x, y) \in \mathbb{R}^2;\ -1 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\}$$

   (a) as a double integral, writing D as a normal domain of type I;
   (b) by transformation to polar coordinates.
11. The following integral is written in terms of a normal domain of type II:

$$\int_0^1 \int_y^{y^2+1} x^2 y \,\mathrm{d}x\mathrm{d}y.$$

(a) Compute the integral.
(b) Sketch the domain and represent it as a normal domain of type I.
(c) Interchange the order of integration and recompute the integral.

Hint. In (c) two summands are needed.