



The notion of a function is the mathematical way of formalising the idea that one or more *independent quantities* are assigned to one or more *dependent quantities*. Functions in general and their investigation are at the core of analysis. They help to model dependencies of variable quantities, from simple planar graphs, curves and surfaces in space to solutions of differential equations or the algorithmic construction of fractals. On the one hand, this chapter serves to introduce the basic concepts. On the other hand, the most important examples of real-valued, elementary functions are discussed in an informal way. These include the power functions, the exponential functions and their inverses. Trigonometric functions will be discussed in Chap. 3, complex-valued functions in Chap. 4.

2.1 Basic Notions

The simplest case of a real-valued function is a double-row list of numbers, consisting of values from an *independent* quantity x and corresponding values of a *dependent* quantity y .

Experiment 2.1 Study the mapping $y = x^2$ with the help of MATLAB. First choose the region D in which the x -values should vary, for instance $D = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$. The command

$$x = -1 : 0.01 : 1;$$

produces a list of x -values, the row vector

$$x = [x_1, x_2, \dots, x_n] = [-1.00, -0.99, -0.98, \dots, 0.99, 1.00].$$

Using

$$y = x.^2;$$

a row vector of the same length of corresponding y -values is generated. Finally `plot(x, y)` plots the points $(x_1, y_1), \dots, (x_n, y_n)$ in the coordinate plane and connects them with line segments. The result can be seen in Fig. 2.1.

In the general mathematical framework we do not just want to assign finite lists of values. In many areas of mathematics functions defined on arbitrary sets are needed. For the general set-theoretic notion of a function we refer to the literature, e.g. [3, Chap.0.2]. This section is dedicated to *real-valued functions*, which are central in analysis.

Definition 2.2 A real-valued function f with domain D and range \mathbb{R} is a rule which assigns to every $x \in D$ a real number $y \in \mathbb{R}$.

In general, D is an arbitrary set. In this section, however, it will be a subset of \mathbb{R} . For the expression *function* we also use the word *mapping* synonymously. A function is denoted by

$$f : D \rightarrow \mathbb{R} : x \mapsto y = f(x).$$

The *graph of the function* f is the set

$$\Gamma(f) = \{(x, y) \in D \times \mathbb{R}; y = f(x)\}.$$

In the case of $D \subset \mathbb{R}$ the graph can also be represented as a subset of the coordinate plane. The set of the actually assumed values is called *image of f* or *proper range*:

$$f(D) = \{f(x); x \in D\}.$$

Example 2.3 A part of the graph of the quadratic function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$ is shown in Fig. 2.2. If one chooses the domain to be $D = \mathbb{R}$, then the image is the interval $f(D) = [0, \infty)$.

An important tool is the concept of *inverse functions*, whether to solve equations or to find new types of functions. If and in which domain a given function has an inverse depends on two main properties, the injectivity and the surjectivity, which we investigate on their own at first.

Definition 2.4 (a) A function $f : D \rightarrow \mathbb{R}$ is called *injective* or *one-to-one*, if different arguments always have different function values:

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

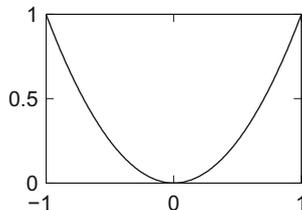


Fig. 2.1 A function

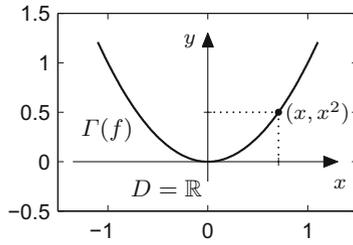


Fig. 2.2 Quadratic function

(b) A function $f : D \rightarrow B \subset \mathbb{R}$ is called *surjective* or *onto* from D to B , if each $y \in B$ appears as a function value:

$$\forall y \in B \exists x \in D : y = f(x).$$

(c) A function $f : D \rightarrow B$ is called *bijective*, if it is injective and surjective.

Figures 2.3 and 2.4 illustrate these notions.

Surjectivity can always be enforced by reducing the range B ; for example, $f : D \rightarrow f(D)$ is always surjective. Likewise, injectivity can be obtained by restricting the domain to a subdomain.

If $f : D \rightarrow B$ is bijective, then for every $y \in B$ there exists *exactly one* $x \in D$ with $y = f(x)$. The mapping $y \mapsto x$ then defines the inverse of the mapping $x \mapsto y$.

Definition 2.5 If the function

$$f : D \rightarrow B : y = f(x),$$

is bijective, then the assignment

$$f^{-1} : B \rightarrow D : x = f^{-1}(y),$$

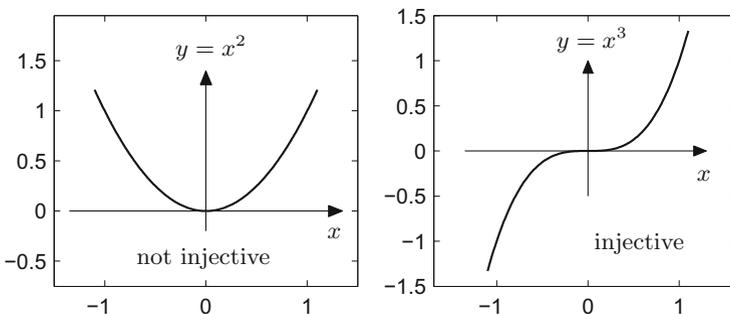


Fig. 2.3 Injectivity

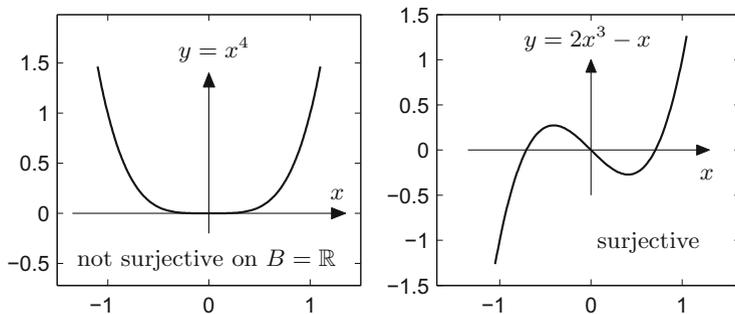
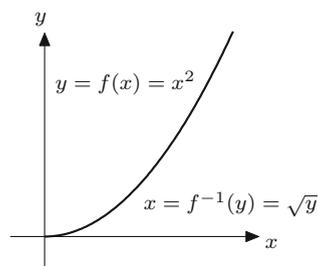


Fig. 2.4 Surjectivity

Fig. 2.5 Bijectivity and inverse function



which maps each $y \in B$ to the unique $x \in D$ with $y = f(x)$ is called the *inverse function* of the function f .

Example 2.6 The quadratic function $f(x) = x^2$ is bijective from $D = [0, \infty)$ to $B = [0, \infty)$. In these intervals ($x \geq 0$, $y \geq 0$) one has

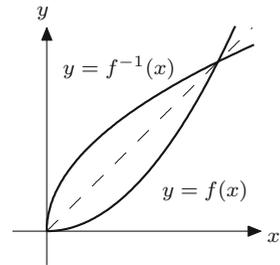
$$y = x^2 \Leftrightarrow x = \sqrt{y}.$$

Here \sqrt{y} denotes the positive square root. Thus the inverse of the quadratic function on the above intervals is given by $f^{-1}(y) = \sqrt{y}$; see Fig. 2.5.

Once one has found the inverse function f^{-1} , it is usually written with variables $y = f^{-1}(x)$. This corresponds to flipping the graph of $y = f(x)$ about the diagonal $y = x$, as is shown in Fig. 2.6.

Experiment 2.7 The term inverse function is clearly illustrated by the MATLAB plot command. The graph of the inverse function can easily be plotted by interchanging the variables, which exactly corresponds to flipping the lists $y \leftrightarrow x$. For example,

Fig. 2.6 Inverse function and reflection in the diagonal



the graphs in Fig. 2.6 are obtained by

```
x = 0:0.01:1;
y = x.^2;
plot(x,y)
hold on
plot(y,x)
```

How the formatting, the dashed diagonal and the labelling are obtained can be learned from the M-file `mat02_1.m`.

2.2 Some Elementary Functions

The elementary functions are the powers and roots, exponential functions and logarithms, trigonometric functions and their inverse functions, as well as all functions which are obtained by combining these. We are going to discuss the most important basic types which have historically proven to be of importance for applications. The trigonometric functions will be dealt with in Chap. 3.

Linear functions (straight lines). A *linear function* $\mathbb{R} \rightarrow \mathbb{R}$ assigns each x -value a fixed multiple as y -value, i.e.,

$$y = kx.$$

Here

$$k = \frac{\text{increase in height}}{\text{increase in length}} = \frac{\Delta y}{\Delta x}$$

is the *slope* of the graph, which is a *straight line* through the origin. The connection between the slope and the angle between the straight line and x -axis is discussed in Sect. 3.1. Adding an *intercept* $d \in \mathbb{R}$ translates the straight line d units in y -direction (Fig. 2.7). The equation is then

$$y = kx + d.$$

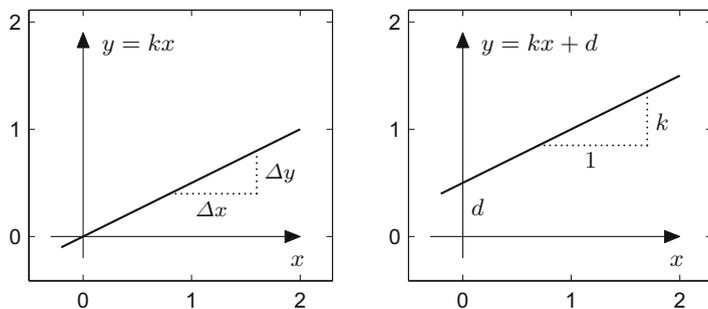


Fig. 2.7 Equation of a straight line

Quadratic parabolas. The quadratic function with domain $D = \mathbb{R}$ in its basic form is given by

$$y = x^2.$$

Compression/stretching, horizontal and vertical translation are obtained via

$$y = \alpha x^2, \quad y = (x - \beta)^2, \quad y = x^2 + \gamma.$$

The effect of these transformations on the graph can be seen in Fig. 2.8.

- $\alpha > 1$... compression in x -direction
- $0 < \alpha < 1$... stretching in x -direction
- $\alpha < 0$... reflection in the x -axis
- $\beta > 0$... translation to the right $\gamma > 0$... translation upwards
- $\beta < 0$... translation to the left $\gamma < 0$... translation downwards

The general quadratic function can be reduced to these cases by *completing the square*:

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\ &= \alpha(x - \beta)^2 + \gamma. \end{aligned}$$

Power functions. In the case of an integer exponent $n \in \mathbb{N}$ the following rules apply

$$\begin{aligned} x^n &= x \cdot x \cdot x \cdot \dots \cdot x \quad (n \text{ factors}), \quad x^1 = x, \\ x^0 &= 1, \quad x^{-n} = \frac{1}{x^n} \quad (x \neq 0). \end{aligned}$$

The behaviour of $y = x^3$ can be seen in the picture on the right-hand side of Fig. 2.3, the one of $y = x^4$ in the picture on the left-hand side of Fig. 2.4. The graphs for odd and even powers behave similarly.

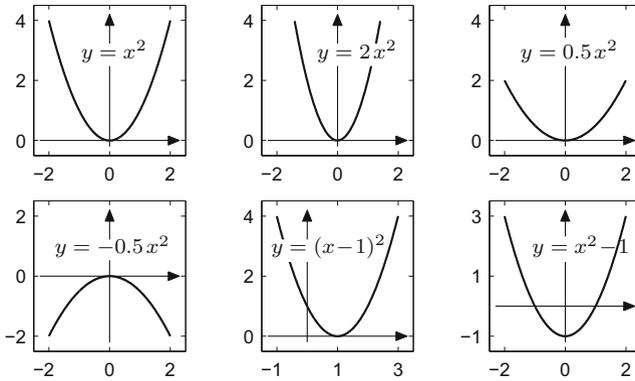


Fig. 2.8 Quadratic parabolas

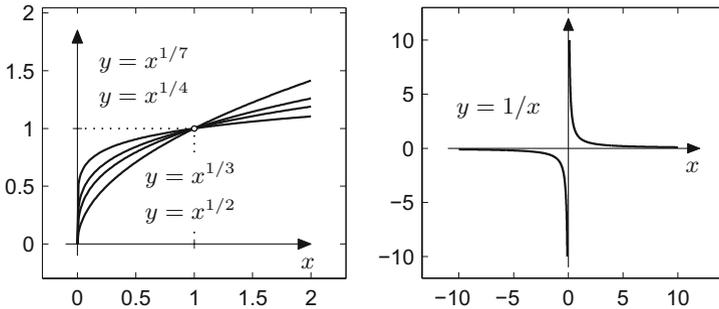


Fig. 2.9 Power functions with fractional and negative exponents

As an example of fractional exponents we consider the *root functions* $y = \sqrt[n]{x} = x^{1/n}$ for $n \in \mathbb{N}$ with domain $D = [0, \infty)$. Here $y = \sqrt[n]{x}$ is defined as the inverse function of the n th power, see Fig. 2.9 left. The graph of $y = x^{-1}$ with domain $D = \mathbb{R} \setminus \{0\}$ is pictured in Fig. 2.9 right.

Absolute value, sign and indicator function. The graph of the *absolute value function*

$$y = |x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0 \end{cases}$$

has a kink at the point $(0, 0)$, see Fig. 2.10 left.

The graph of the *sign function* or *signum function*

$$y = \text{sign } x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0 \end{cases}$$

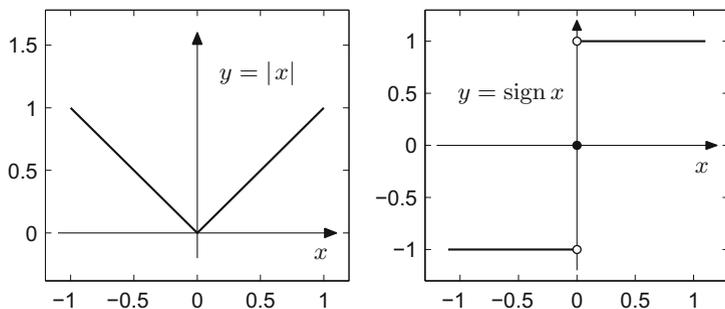


Fig. 2.10 Absolute value and sign

has a jump at $x = 0$ (Fig. 2.10 right). The *indicator function* of a subset $A \subset \mathbb{R}$ is defined as

$$\mathbb{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Exponential functions and logarithms. *Integer powers* of a number $a > 0$ have just been defined. Fractional (rational) powers give

$$a^{1/n} = \sqrt[n]{a}, \quad a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}.$$

If r is an arbitrary real number then a^r is defined by its approximations $a^{m/n}$, where $\frac{m}{n}$ is the rational approximation to r obtained by decimal expansion.

Example 2.8 2^π is defined by the sequence

$$2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, 2^{3.1415}, \dots,$$

where

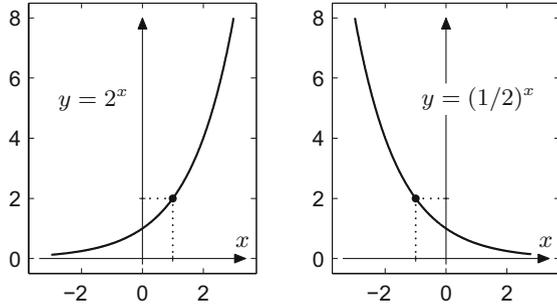
$$2^{3.1} = 2^{31/10} = \sqrt[10]{2^{31}}; \quad 2^{3.14} = 2^{314/100} = \sqrt[100]{2^{314}}; \quad \dots \text{ etc.}$$

This somewhat informal introduction of the exponential function should be sufficient to have some examples at hand for applications in the following sections. With the tools we have developed so far we cannot yet show that this process of approximation actually leads to a well-defined mathematical object. The success of this process is based on the *completeness* of the real numbers. This will be thoroughly discussed in Chap. 5.

From the definition above we obtain that the following rules of calculation are valid for rational exponents:

$$\begin{aligned} a^r a^s &= a^{r+s} \\ (a^r)^s &= a^{rs} = (a^s)^r \\ a^r b^r &= (ab)^r \end{aligned}$$

Fig. 2.11 Exponential functions



for $a, b > 0$ and arbitrary $r, s \in \mathbb{Q}$. The fact that these rules are also true for real-valued exponents $r, s \in \mathbb{R}$ can be shown by employing a limiting argument.

The graph of the exponential function with base a , the function $y = a^x$, increases for $a > 1$ and decreases for $a < 1$, see Fig. 2.11. Its proper range is $B = (0, \infty)$; the exponential function is bijective from \mathbb{R} to $(0, \infty)$. Its inverse function is the logarithm to the base a (with domain $(0, \infty)$ and range \mathbb{R}):

$$y = a^x \iff x = \log_a y.$$

For example, $\log_{10} 2$ is the power by which 10 needs to be raised to obtain 2:

$$2 = 10^{\log_{10} 2}.$$

Other examples are, for instance:

$$2 = \log_{10}(10^2), \quad \log_{10} 10 = 1, \quad \log_{10} 1 = 0, \quad \log_{10} 0.001 = -3.$$

Euler's number¹ e is defined by

$$\begin{aligned} e &= 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots \\ &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots = \sum_{j=0}^{\infty} \frac{1}{j!} \\ &\approx 2.718281828459045235360287471\dots \end{aligned}$$

That this summation of infinitely many numbers can be defined rigorously will be proven in Chap. 5 by invoking the completeness of the real numbers. The logarithm to the base e is called *natural logarithm* and is denoted by \log :

$$\log x = \log_e x$$

¹L. Euler, 1707–1783.

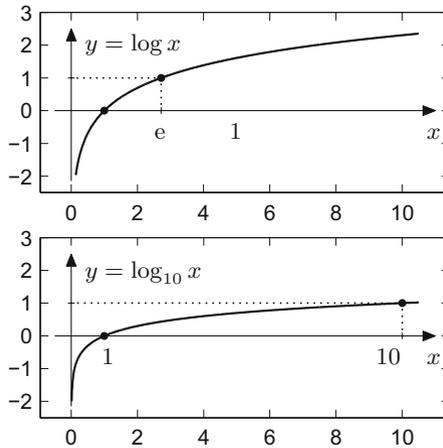


Fig. 2.12 Logarithms to the base e and to the base 10

In some books the natural logarithm is denoted by $\ln x$. We stick to the notation $\log x$ which is used, e.g., in MATLAB. The following rules are obtained directly by rewriting the rules for the exponential function:

$$\begin{aligned} u &= e^{\log u} \\ \log(uv) &= \log u + \log v \\ \log(u^z) &= z \log u \end{aligned}$$

for $u, v > 0$ and arbitrary $z \in \mathbb{R}$. In addition, it holds that

$$u = \log(e^u)$$

for all $u \in \mathbb{R}$, and $\log e = 1$. In particular it follows from the above that

$$\log \frac{1}{u} = -\log u, \quad \log \frac{v}{u} = \log v - \log u.$$

The graphs of $y = \log x$ and $y = \log_{10} x$ are shown in Fig. 2.12.

Hyperbolic functions and their inverses. Hyperbolic functions and their inverses will mainly be needed in Chap. 14 for the parametric representation of hyperbolas, in Chap. 10 for evaluating integrals and in Chap. 19 for explicitly solving some differential equations.

The *hyperbolic sine*, the *hyperbolic cosine* and the *hyperbolic tangent* are defined by

$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \quad \cosh x = \frac{1}{2}(e^x + e^{-x}), \quad \tanh x = \frac{\sinh x}{\cosh x}$$

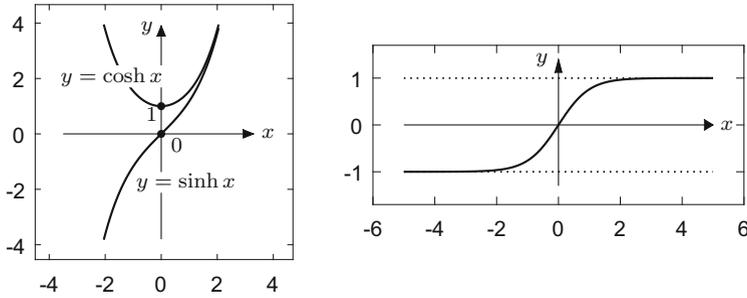


Fig. 2.13 Hyperbolic sine and cosine (left), and hyperbolic tangent (right)

for $x \in \mathbb{R}$. Their graphs are displayed in Fig. 2.13. An important property is the identity

$$\cosh^2 x - \sinh^2 x = 1,$$

which can easily be verified by inserting the defining expressions.

Figure 2.13 shows that the hyperbolic sine is invertible as a function from $\mathbb{R} \rightarrow \mathbb{R}$, the hyperbolic cosine is invertible as a function from $[0, \infty) \rightarrow [1, \infty)$, and the hyperbolic tangent is invertible as a function from $\mathbb{R} \rightarrow (-1, 1)$. The *inverse hyperbolic functions*, also known as *area functions*, are referred to as *inverse hyperbolic sine (cosine, tangent)* or *area hyperbolic sine (cosine, tangent)*. They can be expressed by means of logarithms as follows (see Exercise 15):

$$\begin{aligned} \operatorname{arsinh} x &= \log(x + \sqrt{x^2 + 1}), & \text{for } x \in \mathbb{R}, \\ \operatorname{arcosh} x &= \log(x + \sqrt{x^2 - 1}), & \text{for } x \geq 1, \\ \operatorname{artanh} x &= \frac{1}{2} \log \frac{1+x}{1-x}, & \text{for } |x| < 1. \end{aligned}$$

2.3 Exercises

- How does the graph of an arbitrary function $y = f(x) : \mathbb{R} \rightarrow \mathbb{R}$ change under the transformations

$$y = f(ax), \quad y = f(x - b), \quad y = cf(x), \quad y = f(x) + d$$

with $a, b, c, d \in \mathbb{R}$? Distinguish the following different cases for a :

$$a < -1, \quad -1 \leq a < 0, \quad 0 < a \leq 1, \quad a > 1,$$

and for b, c, d the cases

$$b, c, d > 0, \quad b, c, d < 0.$$

Sketch the resulting graphs.

2. Let the function $f : D \rightarrow \mathbb{R} : x \mapsto 3x^4 - 2x^3 - 3x^2 + 1$ be given. Using MATLAB plot the graphs of f for

$$D = [-1, 1.5], \quad D = [-0.5, 0.5], \quad D = [0.5, 1.5].$$

Explain the behaviour of the function for $D = \mathbb{R}$ and find

$$f([-1, 1.5]), \quad f((-0.5, 0.5)), \quad f((-\infty, 1]).$$

3. Which of the following functions are injective/surjective/bijective?

$$f : \mathbb{N} \rightarrow \mathbb{N} : n \mapsto n^2 - 6n + 10;$$

$$g : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto |x + 1| - 3;$$

$$h : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto x^3.$$

Hint. Illustrative examples for the use of the MATLAB plot command may be found in the M-file `mat02_2.m`.

4. Sketch the graph of the function $y = x^2 - 4x$ and justify why it is bijective as a function from $D = (-\infty, 2]$ to $B = [-4, \infty)$. Compute its inverse function on the given domain.
5. Check that the following functions $D \rightarrow B$ are bijective in the given regions and compute the inverse function in each case:

$$\begin{aligned} y &= -2x + 3, & D &= \mathbb{R}, \quad B = \mathbb{R}; \\ y &= x^2 + 1, & D &= (-\infty, 0], \quad B = [1, \infty); \\ y &= x^2 - 2x - 1, & D &= [1, \infty), \quad B = [-2, \infty). \end{aligned}$$

6. Find the equation of the straight line through the points (1, 1) and (4, 3) as well as the equation of the quadratic parabola through the points (-1, 6), (0, 5) and (2, 21).
7. Let the amount of a radioactive substance at time $t = 0$ be A grams. According to the law of radioactive decay, there remain $A \cdot q^t$ grams after t days. Compute q for radioactive iodine 131 from its half life (8 days) and work out after how many days $\frac{1}{100}$ of the original amount of iodine 131 is remaining.

Hint. The half life is the time span after which only half of the initial amount of radioactive substance is remaining.

8. Let I [Watt/cm²] be the sound intensity of a sound wave that hits a detector surface. According to the Weber–Fechner law, its sound level L [Phon] is computed by

$$L = 10 \log_{10}(I/I_0)$$

where $I_0 = 10^{-16}$ W/cm². If the intensity I of a loudspeaker produces a sound level of 80 Phon, which level is then produced by an intensity of $2I$ by two loudspeakers?

9. For $x \in \mathbb{R}$ the floor function $\lfloor x \rfloor$ denotes the largest integer not greater than x , i.e.,

$$\lfloor x \rfloor = \max \{n \in \mathbb{N}; n \leq x\}.$$

Plot the following functions with domain $D = [0, 10]$ using the MATLAB command `floor`:

$$y = \lfloor x \rfloor, \quad y = x - \lfloor x \rfloor, \quad y = (x - \lfloor x \rfloor)^3, \quad y = (\lfloor x \rfloor)^3.$$

Try to program correct plots in which the vertical connecting lines do not appear.

10. A function $f : D = \{1, 2, \dots, N\} \rightarrow B = \{1, 2, \dots, N\}$ is given by the list of its function values $y = (y_1, \dots, y_N)$, $y_i = f(i)$. Write a MATLAB program which determines whether f is bijective. Test your program by generating random y -values using

$$(a) \quad y = \text{unirnd}(N, 1, N), \quad (b) \quad y = \text{randperm}(N).$$

Hint. See the two M-files `mat02_ex12a.m` and `mat02_ex12b.m` or the Python-file `python02_ex12`.

11. Draw the graph of the function $f : \mathbb{R} \rightarrow \mathbb{R} : y = ax + \text{sign } x$ for different values of a . Distinguish between the cases $a > 0$, $a = 0$, $a < 0$. For which values of a is the function f injective and surjective, respectively?
12. Let $a > 0$, $b > 0$. Verify the *laws of exponents*

$$a^r a^s = a^{r+s}, \quad (a^r)^s = a^{rs}, \quad a^r b^r = (ab)^r$$

for rational $r = k/l$, $s = m/n$.

Hint. Start by verifying the laws for integer r and s (and arbitrary $a, b > 0$). To prove the first law for rational $r = k/l$, $s = m/n$, write

$$(a^{k/l} a^{m/n})^{ln} = (a^{k/l})^{ln} (a^{m/n})^{ln} = a^{kn} a^{lm} = a^{kn+lm}$$

using the third law for integer exponents and inspection; conclude that

$$a^{k/l} a^{m/n} = a^{(kn+lm)/ln} = a^{k/l+m/n}.$$

13. Using the arithmetics of exponentiation, verify the rules $\log(uv) = \log u + \log v$ and $\log u^z = z \log u$ for $u, v > 0$ and $z \in \mathbb{R}$.
Hint. Set $x = \log u$, $y = \log v$, so $uv = e^x e^y$. Use the laws of exponents and take the logarithm.
14. Verify the identity $\cosh^2 x - \sinh^2 x = 1$.
15. Show that $\text{arsinh } x = \log(x + \sqrt{x^2 + 1})$ for $x \in \mathbb{R}$.

Hint. Set $y = \text{arsinh } x$ and solve the identity $x = \sinh y = \frac{1}{2}(e^y - e^{-y})$ for y . Substitute $u = e^y$ to derive the quadratic equation $u^2 - 2xu - 1 = 0$ for u . Observe that $u > 0$ to select the appropriate root of this equation.