Chance Constrained Models

7

Chance constrained programming was developed as a means of describing constraints in mathematical programming models in the form of probability levels of attainment.[1] Consideration of chance constraints allows decision makers to consider mathematical programming objectives in terms of the probability of their attainment. If α is a predetermined confidence level desired by a decision maker, the implication is that a constraint will be violated at most (1−α) of all possible cases.

Chance constraints are thus special types of constraints in mathematical programming models, where there is some objective to be optimized subject to constraints. A typical mathematical programming formulation might be:

$$\begin{array}{l}\text{Maximize } f(X)\\ \text{Subject to : } Ax \leq b\end{array}$$

The objective function *f(X)* can be profit, with the function consisting of *n* variables *X* as the quantities of products produced and *f(X)* including profit contribution rate constants. There can be any number *m* of constraints in *Ax*, each limited by some constant *b*. Chance constraints can be included in *Ax*, leading to a number of possible chance constraint model forms. Charnes and Cooper presented three formulations[2]:

(1) Maximize the expected value of a probabilistic function

$$\begin{array}{l}\text{Maximize E[Y] (where } Y = f(X))\\ \text{Subject to : Pr}\{Ax \leq b\} \geq \alpha\end{array}$$

Any coefficient of this model (*Y*, *A*, *b*) may be probabilistic. The intent of this formulation would be to maximize (or minimize) a function while assuring α probability that a constraint is met. While the expected value of a function usually involves a linear functional form, chance constraints will usually be nonlinear. This

D.L. Olson, D.D. Wu, *Enterprise Risk Management Models*, Springer Texts in Business and Economics, DOI 10.1007/978-3-662-53785-5_7

formulation would be appropriate for many problems seeking maximum profit subject to staying within resource constraints at some specified probability.

(2) Minimize variance
Min Var $[Y]$
Subject to : $\Pr\{Ax \leq b\} \geq \alpha$

The intent is to accomplish some functional performance level while satisfying the chance constraint set. This formulation might be used in identifying portfolio investments with minimum variance, which often is used as a measure of risk.

(3) Maximize probability of satisfying a chance constraint set
$\mathrm{Max}\Pr\{Y \geq \text{target}\}$
Subject to : $\Pr\{Ax \leq b\} \geq \alpha$

This formulation is generally much more difficult to accomplish, especially in the presence of joint chance constraints (where simultaneous satisfaction of chance constraints is required). The only practical means to do this is running a series of models seeking the highest α level yielding a feasible solution.

All three models include a common general chance constraint set, allowing probabilistic attainment of functional levels:

$$\Pr\{Ax \leq b\} \geq \alpha$$

This set is nonlinear, requiring nonlinear programming solution. This inhibits the size of the model to be analyzed, as large values of model parameters m (number of constraints) and especially n (number of variables) make it much harder to obtain a solution.

Most chance constrained applications assume normal distributions for model coefficients. Goicoechea and Duckstein presented deterministic equivalents for non-normal distributions.[3] However, in general, chance constrained models become much more difficult to solve if the variance of parameter estimates increases (the feasible region shrinks drastically when more dispersed distributions are used). The same is true if α is set at too high a value (for the same reason—the feasible region shrinks).

Chance constrained applications also usually assume coefficient independence. This is often appropriate. However, it is not appropriate in many investment analyses. Covariance elements of coefficient estimates can be incorporated within chance constraints, eliminating the need to assume coefficient independence. However, this requires significantly more data, and vastly complicates model data entry.

Chance Constrained Applications

Chance constrained models are not nearly as widespread as linear programming models. A number of applications involve financial planning, to include retirement fund planning models.[4] Chance constraints have also been applied to stress testing value-at-risk (and CVaR).[5] Beyond financial planning, chance constrained models have been applied to supplier selection[6] in operations, as well as in project selection in construction.[7] A multi-attribute model for selection of infrastructure projects in an aerospace firm seeking to maximize company performance subject to probabilistic budget constraints has been presented.[8] There are green chance constrained models seeking efficient climate policies considering available investment streams and renewable energy technologies.[9]

Chance constraints have been incorporated into data envelopment analysis models.[10] Chance constrained programming has been compared with data envelopment analysis and multi-objective programming in a supply chain vendor selection model.[11]

Portfolio Selection

Assume a given sum of money to be invested in n possible securities. We denote by $x = (x_1, \ldots, x_n)$ is an investment proportion vector (also called a portfolio). As for the number of securities n, many large institutions have "approved lists" where n is anywhere from several hundred to a thousand. When attempting to form a portfolio to mimic a large broad based index (like S&P500, EAFE, Wilshire 5000), n can be up to several thousand. Denote by

r_i the percent return of i-th security;Other objectives to characterize the i-th security could be

- s_i is social responsibility of i-th security
- g_i is growth in sales of i-th security
- a_i is amount invested in R&D of i-th security
- d_i is dividends of i-th security
- q_i is liquidity of i-th security

Consideration of such investment objectives will lead to utilization of multi-objective programming models. The investor tries to select several possible securities from the n securities to maximize his profit, which leads to the investor's decision problem as:

$$\begin{aligned} &\text{Max } r_p = \sum\nolimits_{i=1}^{n} r_i x_i \\ &\text{s.t.} Ax \leq b \end{aligned} \tag{1}$$

where

- r_p is percent return on a portfolio over the holding period.
- $Ax \leq b$, the feasible region in decision space

In the investor's decision problem (1), the quantity r_p to be maximized is a random variable because r_p is a function of the individual-security r_i random variables. Therefore, (1) is a *stochastic programming problem*. Stochastic programming models are similar to deterministic optimization problems where the parameters are known only within certain bounds but take advantage of the fact that probability distributions governing the data are known or can be estimated. To solve a stochastic programming problem, we need convert the stochastic programming to an equivalent *deterministic programming problem*. A popular way of doing this is to use utility function $U(\cdot)$], which maps stochastic terms into their deterministic equivalents. For example, by use of the means μ_i, variances σ_{ii} and covariances σ_{ij} of the r_i, a portfolio selection problem is to maximize expected utility.

$$E\left[U\left(r_p\right)\right] = E\left[r_p\right] - \lambda Var\left[r_p\right],$$

where $\lambda \geq 0$ a risk reversion coefficient and may be different from different investors. In other words, a portfolio selection problem can be modeled by a trade-off between the mean and variance of random variable r_p:

$$\text{Max} \;\; E\left[U\left(r_p\right)\right] = E\left[r_p\right] - \lambda Var\left[r_p\right],$$

$$\lambda \geq 0$$

$$Ax \leq b$$

Assuming $[U(r_p)]$ is Taylor series expandable, the validity of $E[U(r_p)]$ and thus the above problem can be guaranteed if $[U(r_p)]$ Taylor series expandable of $\mathbf{r} = (r_1, \ldots, r_n)$ follows the multinormal distribution. Another alternative to Markowitz's mean variance framework, chance constrained programming was employed to model the portfolio selection problem. We will demonstrate the utilization of chance constrained programming to model the portfolio selection problem in the next section.

Demonstration of Chance Constrained Programming

The following example was taken from Lee and Olson (2006).[12] The Hal Chase Investment Planning Agency is in business to help investors optimize their return from investment, to include consideration of risk. Through the use of nonlinear programming models, Hal Chase can control risk.

Hal deals with three investment mediums: a stock fund, a bond fund, and his own Sports and Casino Investment Plan (SCIP). The stock fund is a mutual fund investing in openly traded stocks. The bond fund focuses on the bond market,

Table 7.1 Hal chase investment data

	Stock **S**	Bond **B**	SCIP **G**
Average return	0.148	0.060	0.152
Variance	0.014697	0.000155	0.160791
Covariance with S		0.000468	−0.002222
Covariance with B			−0.000227

which has a much stabler return, although significantly lower expected return. SCIP is a high-risk scheme, often resulting in heavy losses, but occasionally coming through with spectacular gains. In fact, Hal takes a strong interest in SCIP, personally studying investment opportunities and placing investments daily. The return on these mediums, as well as their variance and correlation, are given in Table 7.1:

Note that there is a predictable relationship between the relative performance of the investment opportunities, so the covariance terms report the tendency of investments to do better or worse given that another investment did better or worse. This indicates that variables **S** and **B** tend to go up and down together (although with a fairly weak relationship), while variable **G** tends to move opposite to the other two investment opportunities.

Hal can develop a mathematical programming model to reflect an investor's desire to avoid risk. Hal assumes that return on investments are normally distributed around the average returns reported above. He bases this on painstaking research he has done with these three investment opportunities.

Maximize Expected Value of Probabilistic Function

Using this form, the objective is to maximize return:

$$\text{Expected return} = 0.148\,\mathbf{S} + 0.060\,\mathbf{B} + 0.152\,\mathbf{G}$$

subject to staying within budget:

$$\text{Budget} = 1\,\mathbf{S} + 1\,\mathbf{B} + 1\,\mathbf{G} \leq 1000$$

having a probability of positive return greater than a specified probability:

$$\Pr\{\text{Expected return} \geq 0\} \geq \alpha$$

with all variables greater than or equal to 0:

$$\mathbf{S}, \mathbf{B}, \mathbf{G} \geq 0$$

The solution will depend on the confidence limit α. Using EXCEL, and varying α from 0.5, 0.8, 0.9 and 0.95, we obtain the solutions given in Table 7.2:

Table 7.2 Results for chance constrained formulation (1)

Probability {return ≥ 0}	α	Stock	Bond	Gamble	Expected return
0.50	0	–	–	1000.00	152.00
0.80	0.253	379.91	–	620.09	150.48
0.90	0.842	556.75	–	443.25	149.77
0.95	1.282	622.18	–	377.82	149.51
0.99	2.054	668.92	–	331.08	149.32

The probability determines the penalty function α. At a probability of 0.80, the one-tailed normal z-function is 0.253, and thus the chance constrained is:

$$0.148\,\mathbf{S} + 0.060\,\mathbf{B} + 0.152\,\mathbf{G} - 0.253{*}\mathrm{SQRT}\big(0.014697\mathbf{S}^2 + 0.000936\mathbf{SB} - 0.004444\mathbf{SG} + 0.000155\mathbf{B}^2 - 0.000454\mathbf{BG} + 0.160791\mathbf{G}^2$$

The only difference in the constraint set for the different rows of Table 7.2 is that α is varied. The affect is seen is that investment is shifted from the high risk gamble to a bit safer stock. The stock return has low enough variance to assure the specified probabilities given. Had it been higher, the even safer bond would have entered into the solution at higher specified probability levels.

Minimize Variance

With this chance constrained form, Hal is risk averse. He wants to minimize risk subject to attaining a prescribed level of gain. The variance-covariance matrix measures risk in one form, and Hal wants to minimize this function.

$$\text{Min } 0.014697\mathbf{S}^2 + 0.000936\mathbf{SB} - 0.004444\mathbf{SG} + 0.000155\mathbf{B}^2 - 0.000454\mathbf{BG} + 0.160791\mathbf{G}^2$$

This function can be constrained to reflect other restrictions on the decision. For instance, there typically is some budget of available capital to invest.

$$\mathbf{S} + \mathbf{B} + \mathbf{G} \leq 1000 \quad \text{for a \$1000 budget}$$

Finally, Hal only wants to minimize variance given that he attains a prescribed expected return. Hal wants to explore four expected return levels: \$50/\$1000 invested, \$100/\$1000 invested, \$150/\$1000 invested, and \$200/\$1000 invested. Note that these four levels reflect expected returns of 5, 10, 15, and 20 %.

$$0.148\ \mathbf{S} + 0.06\ \mathbf{B} + 0.152\ \mathbf{G} \geq r \quad \text{where } r = 50,\ 100,\ 150,\ \text{and}\ 200$$

Solution Procedure

The EXCEL input file will start off with the objective, MIN followed by the list of variables. Then we include the constraint set. The constraints can be stated as you want, but the partial derivatives of the variables need to consider each constraint stated in less-than-or-equal-to form. Therefore, the original model is transformed to:

$$\begin{aligned}
&\text{Min}\ .014697\mathbf{S}^2 + .000936\mathbf{SB} - .004444\mathbf{SG} + .000155\mathbf{B}^2 - .000454\mathbf{BG} + .160791\mathbf{G}^2] \\
&\text{st}\ \mathbf{S} + \mathbf{B} + \mathbf{G} \leq 1000 && \text{budget constraint} \\
&\quad 0.148\ \mathbf{S} + 0.06\ \mathbf{B} + 0.152\ \mathbf{G} \geq 50 && \text{gain constraint} \\
&\qquad \mathbf{S},\ \mathbf{B},\ \mathbf{G} \geq 0
\end{aligned}$$

The solution for each of the four gain levels are given in Table 7.3:

The first solution indicates that the lowest variance with an expected return of \$50 per \$1000 invested would be to invest \$825.30 in **B** (the bond fund), \$3.17 in **G** (the risky alternative), and keeping the 171.53 slack. The variance is \$106.002. This will yield an average return of 5 % on the money invested. Increasing specified gain to \$100 yields the designed expected return of \$100 with a variance of \$2928.51. Raising expected gain to 150 yields the prescribed \$150 with a variance of \$42,761.06. Clearly this is a high risk solution. But it also is near the maximum expected return (if all \$1000 was placed on the riskiest alternative, **G**, the expected return would be maximized at \$152 per \$1000 invested). A model specifying a gain of \$200 yields an infeasible solution, and thus by running multiple models, we can identify the maximum gain available (matching the linear programming model without chance constraints). It can easily be seen that lower variance is obtained by investing in bonds, then shifting to stocks, and finally to the high-risk gamble option.

Maximize Probability of Satisfying Chance Constraint

The third chance constrained form is implicitly attained by using the first form example above, stepping up α until the model becomes infeasible. When the probability of satisfying the chance constraint was set too high, a null solution

Table 7.3 Results for chance constrained formulation (2)

Specified Gain	Variance	Stock	Bond	Gamble
≥50	106.00	–	825.30	3.17
≥100	2928.51	406.31	547.55	46.14
≥150	42,761	500.00	–	500.00
≥152	160,791	–	–	1000.00

Table 7.4 Results for chance constrained formulation (3)

α	Stock	Bond	Gamble	Expected return
3	157.84	821.59	20.57	75.78
4	73.21	914.93	11.86	67.53
4.5	406.31	547.55	46.14	64.17
4.8	500.00	–	500.00	61.48
4.9 and up	–	–	–	0

was generated (don't invest anything—keep all the \$1000). Table 7.4 shows solutions obtained, with the highest α yielding a solution being 4.8, associated with a probability very close to 1.0 (0.999999 according to EXCEL).

Real Stock Data

To check the validity of the ideas presented, we took real stock data from the Internet, taking daily stock prices for six dispersed, large firms, as well as the S&P500 index. Data was manipulated to obtain daily rates of return over the period 1999 through 2008 (2639 observations—dividing closing price by closing price of prior day).

$$r = \frac{V_t}{V_{t-1}}$$

where V_t = return for day t and V_{t-1} = return for the prior day. (The arithmetic return yields identical results, only subtracting 1 from each data point.)

$$r_{arith} = \frac{V_t - V_{t-1}}{V_{t-1}}$$

We first looked at possible distributions. Figure 7.1 shows the Crystal Ball best fit for all data (using the Chi-square criterion—same result for Kolmogorov-Smirnov or Anderson criteria), while Fig. 7.2 shows fit with the logistic distribution, and Fig. 7.3 with the normal distribution:

The parameters for the student-t distribution fit was a scale of 0.01, and 2.841 degrees of freedom. For the logistic distribution, the scale parameter was 0.01.

The data had a slight negative skew, with a skewness score of −1.87. It had a high degree of kurtosis (73.65), and thus much more peaked than a normal distribution. This demonstrates "fat tail" distributions that are often associated with financial returns. Figures 7.1, 7.2, 7.3 clearly show how the normal assumption is too spread out for probabilities close to 0.5, and too narrow for the extremes (tails). The logistic distribution gives a better fit, but student-t distribution does better yet.

Table 7.5 shows means standard deviations, and covariances of these investments.

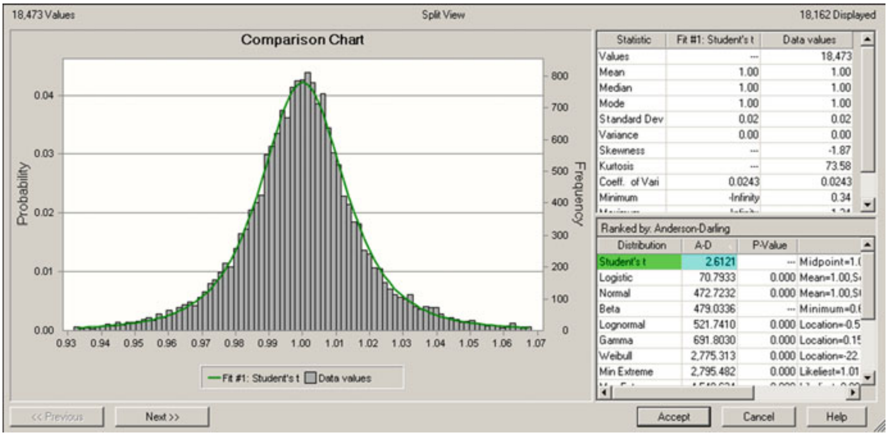

Fig. 7.1 Data distribution fit student-t. ©Oracle. used with permission

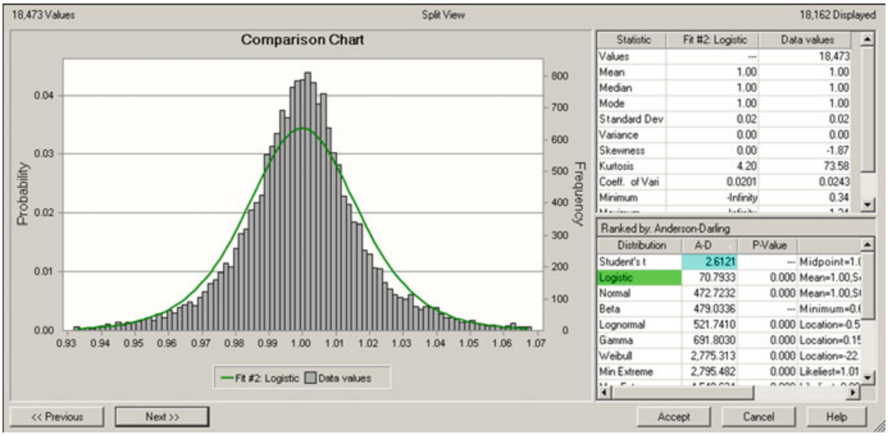

Fig. 7.2 Logistic fit. ©Oracle. used with permission

An alternative statistic for returns is the logarithmic return, or continuously compounded return, using the formula:

$$r_{log} = \ln\left(\frac{V_f}{V_i}\right)$$

The student's t distribution again had the best fit, followed by logistic and normal (see Fig. 7.4):

This data yields slightly different data, as shown in Table 7.6.

Like the arithmetic return, the logarithmic return is centered on 0. There is a difference (slight) between logarithmic return covariances and arithmetic return covariances. The best distribution fit was obtained with the original data (identical

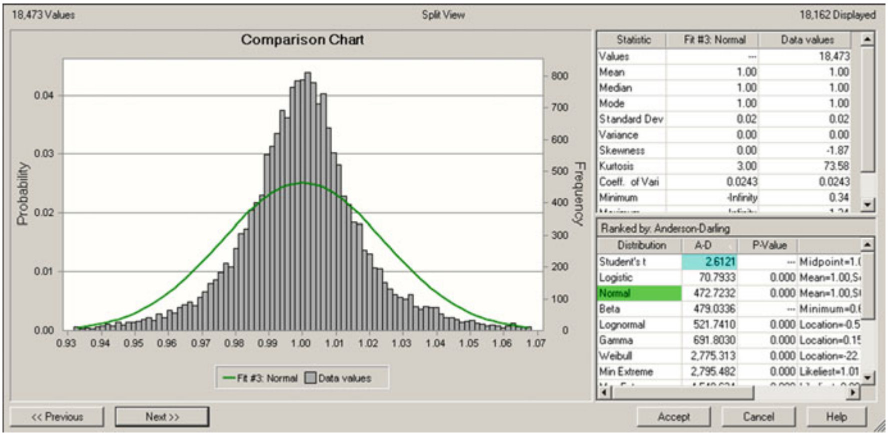

Fig. 7.3 Normal model fit to data. ©Oracle. used with permission

Table 7.5 Daily data

	Ford	IBM	Pfizer	SAP	WalMart	XOM	S&P
Mean	1.00084	1.00033	0.99935	0.99993	1.00021	1.00012	0.99952
Std. Dev	0.03246	0.02257	0.02326	0.03137	0.02102	0.02034	0.01391
Min	0.62822	0.49101	0.34294	0.81797	0.53203	0.51134	0.90965
Max	1.29518	1.13160	1.10172	1.33720	1.11073	1.17191	1.11580
Cov(Ford)	0.00105	0.00019	0.00014	0.00020	0.00016	0.00015	0.00022
Cov(IBM)		0.00051	0.00009	0.00016	0.00013	0.00012	0.00018
Cov(Pfizer)			0.00054	0.00011	0.00014	0.00014	0.00014
Cov(SAP)				0.00098	0.00010	0.00016	0.00016
Cov(WM)					0.00044	0.00011	0.00014
Cov(XOM)						0.00041	0.00015
Cov(S&P)							0.00019

to arithmetic return), so we used that data for our chance constrained calculations. If logarithmic return data was preferred, the data in Table 7.6 could be used in the chance constrained formulations.

Chance Constrained Model Results

We ran the data into chance constrained models assuming a normal distribution for data, using means, variances, and covariances from Table 7.5. The model included a budget limit of $1000, all variables ≥0, (chance constrained to have no loss), obtaining results shown in Table 7.7.

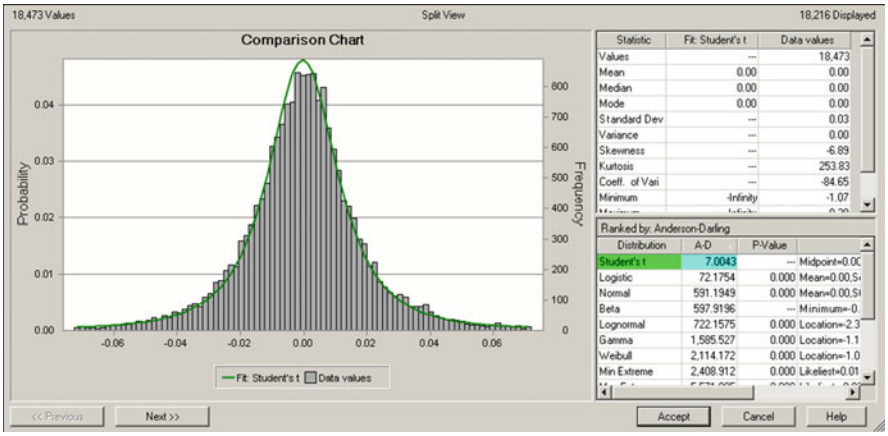

Fig. 7.4 Distribution comparison from Crystal Ball. ©Oracle. used with permission

Maximizing return is a linear programming model, with an obvious solution of investing all available funds in the option with the greatest return (Ford). This has the greatest expected return, but also the highest variance.

Minimizing variance is equivalent to chance constrained form (2). The solution avoided Ford (which had a high variance), and spread the investment out among the other options, but had a small loss.

A series of models using chance constrained form (1) were run. Maximizing expected return subject to investment ≤ 1000 as well as adding the chance constraint $\Pr\{\text{return} \geq 970\}$ was run for both normal and t-distributions.

$$\begin{aligned} &\text{Max expected return} \\ &\text{s.t.} \quad \text{Sum investment} \leq 1000 \\ &\qquad \Pr\{\text{return} \geq 970\} \geq 0.95 \\ &\qquad \text{All investments} \geq 0 \end{aligned}$$

It can be seen in Table 7.6 that the t-distribution was less restrictive, resulting in more investment in the riskier Ford option, but having a slightly higher variance (standard deviation). The chance constraint was binding in both assumptions (normal and Student-t). There was a 0.9 probability return of 979.50, and a 0.8 probability of return of 988.09 by t-distribution. Further chance constraint models were run assuming t-distribution. For the model:

$$\begin{aligned} &\text{Max expected return} \\ &\text{s.t.} \quad \text{Sum investment} \leq 1000 \\ &\qquad \Pr\{\text{return} \geq 970\} \geq 0.95 \\ &\qquad \Pr\{\text{return} \geq 980\} \geq 0.9 \\ &\qquad \text{All investments} \geq 0 \end{aligned}$$

Table 7.6 Daily data for logarithmic return

	Ford	IBM	Pfizer	SAP	WalMart	XOM	S&P
Mean	−0.00029	0.00015	−0.00084	−0.00038	0.00006	−0.00017	−0.00068
Std. Dev	0.03278	0.02455	0.02852	0.03087	0.02254	0.02219	0.01392
Min	−0.46486	−0.71130	−1.07021	−0.20093	−0.63105	−0.67073	−0.09470
Max	0.25865	0.12364	0.09687	0.29058	0.10502	0.15863	0.10957
Cov(Ford)	0.00107	0.00019	0.00013	0.00020	0.00016	0.00015	0.00022
Cov(IBM)		0.00060	0.00009	0.00015	0.00013	0.00012	0.00018
Cov(Pfizer)			0.00081	0.00011	0.00014	0.00013	0.00014
Cov(SAP)				0.00095	0.00010	0.00016	0.00016
Cov(WM)					0.00051	0.00011	0.00014
Cov(XOM)						0.00049	0.00015
Cov(S&P)							0.00019

Table 7.7 Model results

Model	Ford	IBM	Pfizer	SAP	WM	XOM	S&P	Return	Stdev
Max return	1000.000	–	–	–	–	–	–	1000.84	32.404
Min variance	–	45.987	90.869	30.811	127.508	116.004	588.821	999.76	13.156
Normal **Pr{>970}>0.95**	398.381	283.785	–	–	222.557	95.277	–	1000.49	18.534
t **Pr{>970}>0.95**	607.162	296.818	–	–	96.020	–	–	1000.63	23.035
t Pr{>970}>0.95 **Pr{>980}>0.9**	581.627	301.528	–	–	116.845	–	–	1000.61	22.475
t Pr{>970}>0.95 Pr{>980}>0.9 **Pr{>990}>0.8**	438.405	279.287	–	–	220.254	62.054	–	1000.51	19.320
Max Pr{>1000}	16.275	109.867	105.586	38.748	174.570	172.244	382.711	999.91	13.310

The bold emphasis signifies the instance with high variance

The expected return was only slightly less, with the constraint Pr {return ≥ 980} ≥ 0.9 binding. There was a 0.95 probability of return of 970.73, and a 0.8 probability of return of 988.38. A model using three chance constraints was also run:

$$\begin{aligned} &\text{Max expected return} \\ &\text{s.t.} \quad \text{Sum investment} \leq 1000 \\ &\qquad \Pr\{\text{return} \geq 970\} \geq 0.95 \\ &\qquad \Pr\{\text{return} \geq 980\} \geq 0.9 \\ &\qquad \Pr\{\text{return} \geq 990\} \geq 0.8 \\ &\qquad \text{All investments} \geq 0 \end{aligned}$$

This yielded a solution where the 0.95 probability of return was 974.83, the 0.9 probability of return was 982.80, and the 0.8 probability of return was 990 (binding).

Finally, a model was run to maximizing probability of return ≥1000 (chance constrained model type 3).

$$\begin{aligned} &\text{Minimize D} \\ &\text{s.t.} \quad \text{Sum investment} \leq 1000 \\ &\qquad \Pr\{\text{return} \geq 970\} \geq 0.95 \\ &\qquad \Pr\{\text{return} \geq 980\} \geq 0.9 \\ &\qquad D = 1000 - \Pr\{\text{return} \geq 1000\} \geq 0.8 \\ &\qquad \text{All investments} \geq 0 \end{aligned}$$

This was done by setting the deviation from an infeasible target. The solution yielded a negative expected return at a low variance, with the 0.95 probability of return 982.22, the 0.9 probability of return 987.71, and the 0.8 probability of return 992.67.

Conclusions

A number of different types of models can be built using chance constraints. The first form is to maximize the linear expected return subject to attaining specified probabilities of reaching specified targets. The second is to minimize variance. This second form is not that useful, in that the lowest variance is actually to not invest. Here we forced investment of the 1000 capital assumed. The third form is to maximize probability of attaining some target, which in order to be useful, has to be infeasible.

Chance constrained models have been used in many applications. Here we have focused on financial planning, but there have been applications whenever statistical data is available in an optimization problem.

The models presented all were solved with EXCEL SOLVER. In full disclosure, we need to point out that chance constraints create nonlinear optimization models, which are somewhat unstable relative to linear programming models. Solutions are very sensitive to the accuracy of input data. There also are practical limits to model size. The variance-covariance matrix involves a number of parameters to enter into EXCEL functions, which grow rapidly with the number of variables. In the simple example there were three solution variables, with six elements to the variance-covariance matrix. In the real example, there were seven solution variables (investment options). The variance-covariance matrix thus involved 28 nonlinear expressions.

Notes

1. Charnes, A. and Cooper, W.W. (1959). Chance-constrained programming, *Management Science* 6:1, 73–79; Charnes, A. and Cooper, W.W. (1962). Chance-constraints and normal deviates, *Journal of the American Statistical Association* 57, 134–148.
2. Charnes, A. and Cooper, W.W. (1963). Deterministic equivalents for optimizing and satisficing under chance-constraints, *Operations Research* 11:1, 18–39.
3. Goicoechea, A. and Duckstein, L. (1987). Nonnormal deterministic equivalents and a transformation in stochastic mathematical programming, *Applied Mathematics and Computation* 21:1, 51–72.
4. Booth, L. (2004). Formulating retirement targets and the impact of time horizon on asset allocation, *Financial Services Review* 13:1, 1–17.
5. Dupaĉová, J. and Polivka, J. (2007). Stress testing for VaR and CVaR. *Quantitative Finance* 7 (4), 411–421.
6. Bilsel, R.U. and Ravindran, A. (2011). A multiobjective chance constrained programming model for supplier selection under uncertainty. *Transportation Research: Part B* 45(8), 1284–1300.
7. Wibowo, A. and Kochendoerfer, B. (2011). Selecting BOT/PPP infrastructure projects for government guarantee portfolio under conditions of budget and risk in the Indonesian context. *Journal of Construction Engineering & Management* 137(7), 512–522.
8. Gurgur, C.Z. and Morley, C.T. (2008). Lockheed Martin Space Systems Company optimizes infrastructure project-portfolio, *Interfaces* 38:4, 251–262.
9. Held, H., Kriegler, E., Lessmann, K. and Edenhofer, O. (2009). Efficient climate policies under technology and climate uncertainty, *Energy Economics* 31, S50–S61.
10. Cooper, W.W., Deng, H., Huang, Z. and Li, S.X. (2002). Chance constrained programming approaches to technical efficiencies and inefficiencies in stochastic data envelopment analysis, *Journal of the Operational Research Society* 53:12, 1347–1356; Cooper, W.W., Deng, H., Huang, Z. and Li, S.X. (2004). Chance constrained programming approaches to congestion in stochastic data envelopment analysis, *European Journal of Operational Research* 155:2, 487–501.
11. Wu, D. and Olson, D.L. (2008). Supply chain risk, simulation, and vendor selection, *International Journal of Production Economics* 114:2, 646–655.
12. Lee, S.M. and Olson, D.L. (2006). *Introduction to Management Science* 3rd ed. Cincinnati: Thompson.