Chapter 10

Dynamics

In this chapter, we turn our attention to continuous dynamical systems, which are governed by first and second order linear systems of ordinary differential equations. Such systems, whose unvarying equilibria were the subject of Chapter 6, include the dynamical motions of mass–spring chains and structures, and the time-varying voltages and currents in simple electrical circuits. Dynamics of continuous media, including fluids, solids, and gases, are modeled by infinite-dimensional dynamical systems described by partial differential equations, [**61**, **79**], and will not be treated in this text, nor will we venture into the vastly more complicated realm of nonlinear dynamics, [**34**, **41**].

Chapter 8 developed the basic mathematical tools — eigenvalues and eigenvectors — used in the analysis of linear systems of ordinary differential equations. For a first order system, the resulting *eigensolutions* describe the basic modes of exponential growth, decay, or periodic behavior. In particular, the stability properties of an equilibrium solution are (mostly) determined by the eigenvalues. Most of the phenomenology inherent in linear dynamics can already be observed in the two-dimensional situation, and we devote Section 10.3 to a complete description of first order planar linear systems. In Section 10.4, we re-interpret the solution to a first order system in terms of the matrix exponential, which is defined by analogy with the usual scalar exponential function. Matrix exponentials are particularly effective for solving inhomogeneous or forced linear systems, and also appear in applications to geometry, computer graphics and animation, theoretical physics, and mechanics.

As a consequence of Newton's laws of motion, mechanical vibrations are modeled by second order dynamical systems. For stable configurations with no frictional damping, the eigensolutions constitute the system's normal modes, each periodically vibrating with its associated natural frequency. The full dynamics is obtained by linear superposition of the periodic normal modes, but the resulting solution is, typically, no longer periodic. Such quasi-periodic motion may seem quite chaotic — even though mathematically it is merely a combination of finitely many simple periodic solutions. When subjected to an external periodic forcing, the system usually remains in a quasi-periodic motion that superimposes a periodic response onto its own internal vibrations. However, attempting to force the system at one of its natural frequencies, as prescribed by its eigenvalues, may induce a resonant vibration, of progressively unbounded amplitude, resulting in a catastrophic breakdown of the physical apparatus. In contrast, frictional effects, depending on first order derivatives/velocities, serve to damp out the quasi-periodic vibrations and similarly help mitigate the dangers of resonance.

10.1 Basic Solution Techniques

Our initial focus will be on systems

$$\frac{d\mathbf{u}}{dt} = A\,\mathbf{u} \tag{10.1}$$

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consisting of n first order linear ordinary differential equations in the n unknowns $\mathbf{u}(t) = (\,u_1(t), \ldots, u_n(t)\,)^T \in \mathbb{R}^n$. In an *autonomous* system, the time variable does not appear explicitly, and so the coefficient matrix A, of size $n \times n$, is a constant real† matrix. Nonautonomous systems, in which $A(t)$ is time-dependent, are considerably more difficult to analyze, and we refer the reader to a more advanced text such as [**36**].

As we saw in Section 8.1, a vector-valued exponential function

$$\mathbf{u}(t) = e^{\lambda t}\,\mathbf{v},$$

in which λ is a constant scalar and $\mathbf{v}$ a constant vector, describes a solution to (10.1) if and only if

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Hence, assuming $\mathbf{v} \neq \mathbf{0}$, the scalar λ must be an eigenvalue of A, and $\mathbf{v}$ the corresponding eigenvector. The resulting exponential function will be called an *eigensolution* of the linear system. Since the system is linear and homogeneous, linear superposition allows us to combine the basic eigensolutions to form more general solutions.

If the coefficient matrix A is complete (diagonalizable), then, by definition, its eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis. The corresponding eigensolutions

$$\mathbf{u}_1(t) = e^{\lambda_1 t}\,\mathbf{v}_1, \qquad \ldots \qquad \mathbf{u}_n(t) = e^{\lambda_n t}\,\mathbf{v}_n,$$

will form a basis for the solution space to the system. Hence, the general solution to a first order linear system with complete coefficient matrix has the form

$$\mathbf{u}(t) = c_1\mathbf{u}_1(t) + \;\cdots\; + c_n\mathbf{u}_n(t) = c_1\,e^{\lambda_1 t}\,\mathbf{v}_1 + \;\cdots\; + c_n\,e^{\lambda_n t}\,\mathbf{v}_n, \tag{10.2}$$

where $c_1, \ldots, c_n$ are constants, which are uniquely prescribed by the initial conditions

$$\mathbf{u}(t_0) = \mathbf{u}_0. \tag{10.3}$$

This all follows from the basic existence and uniqueness theorem for ordinary differential equations, which will be discussed shortly.

Example 10.1. Let us solve the coupled pair of ordinary differential equations

$$\frac{du}{dt} = 3\,u + v, \qquad\qquad \frac{dv}{dt} = u + 3\,v.$$

We first write the system in matrix form (10.1) with unknown $\mathbf{u}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ and coefficient matrix $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. According to Example 8.5, the eigenvalues and eigenvectors of A are

$$\lambda_1 = 4, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad\qquad \lambda_2 = 2, \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Both eigenvalues are simple, and so A is a complete matrix. The resulting eigensolutions

$$\mathbf{u}_1(t) = e^{4\,t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{4\,t} \\ e^{4t} \end{pmatrix}, \qquad\qquad \mathbf{u}_2(t) = e^{2\,t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\,e^{2\,t} \\ e^{2t} \end{pmatrix},$$

† Extending the solution techniques to complex systems with complex coefficient matrices is straightforward, but will not be treated here.

form a basis of the solution space, and so the general solution is a linear combination

$$\mathbf{u}(t) = c_1\, e^{4\,t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\, e^{2\,t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1\, e^{4\,t} - c_2\, e^{2\,t} \\ c_1\, e^{4t} + c_2\, e^{2t} \end{pmatrix}, \qquad \text{hence} \qquad \begin{aligned} u(t) &= c_1\, e^{4\,t} - c_2\, e^{2\,t}, \\ v(t) &= c_1\, e^{4t} + c_2\, e^{2t}, \end{aligned}$$

in which c_1, c_2 are arbitrary constants.

The Phase Plane

As noted above, a wide variety of physical systems are modeled by second order ordinary differential equations. Your first course on ordinary differential equations, e.g., [**7**, **22**], covered the basic solution technique for constant coefficient scalar equations, which we quickly review in the context of an example.

Example 10.2. To solve the homogeneous ordinary differential equation

$$\frac{d^2u}{dt^2} + \frac{du}{dt} - 6\,u = 0, \tag{10.4}$$

we begin with the exponential ansatz[†]

$$u(t) = e^{\lambda\,t},$$

where the constant factor λ is to be determined. Substituting into the differential equation leads immediately to the *characteristic equation*

$$\lambda^2 + \lambda - 6 = 0, \qquad \text{with roots} \qquad \lambda_1 = 2, \qquad \lambda_2 = -3.$$

Therefore, $e^{2\,t}$ and $e^{-3\,t}$ are individual solutions. Since the equation is of second order, Theorem 7.34 implies that they form a basis for the two-dimensional solution space, and hence the general solution can be written as a linear combination

$$u(t) = c_1\, e^{2\,t} + c_2\, e^{-3\,t}, \tag{10.5}$$

where c_1, c_2 are arbitrary constants.

There is a standard trick to convert a second order equation

$$\frac{d^2u}{dt^2} + \alpha\,\frac{du}{dt} + \beta\,u = 0 \tag{10.6}$$

into a first order system. One introduces the so-called *phase plane variables*[‡]

$$u_1 = u, \qquad u_2 = \dot{u} = \frac{du}{dt}\,. \tag{10.7}$$

Assuming α, β are constants, the phase plane variables satisfy

$$\frac{du_1}{dt} = \frac{du}{dt} = u_2, \qquad \frac{du_2}{dt} = \frac{d^2u}{dt^2} = -\,\beta\,u - \alpha\,\frac{du}{dt} = -\,\beta\,u_1 - \alpha\,u_2.$$

[†] See the footnote on p. 379 for an explanation of the term "ansatz", a.k.a. "inspired guess".

[‡] We will often use dots as a shorthand notation for time derivatives.

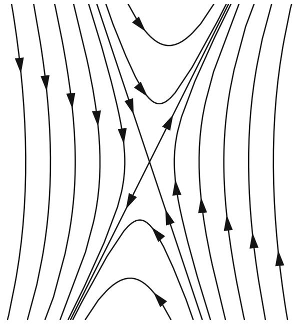

Figure 10.1. Phase Portrait of $\dot{u}_1 = u_2$, $\dot{u}_2 = 6\,u_1 - u_2$.

In this manner, the second order equation (10.6) is converted into a first order system

$$\dot{\mathbf{u}} = A\,\mathbf{u}, \qquad \text{where} \qquad \mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix}. \tag{10.8}$$

Every solution $u(t)$ to the second order equation yields a solution $\mathbf{u}(t) = (u(t), \dot{u}(t))^T$ to the first order system (10.8), whose second component is merely its time derivative. Conversely, if $\mathbf{u}(t) = (\,u_1(t), u_2(t)\,)^T$ is any solution to (10.8), then its first component $u(t) = u_1(t)$ defines a solution to the original scalar equation (10.6). We conclude that the two are completely equivalent, in the sense that solving one will immediately resolve the other.

The variables $(u_1, u_2)^T = (u, \dot{u})^T$ serve as coordinates in the *phase plane* $\mathbb{R}^2$. The solutions $\mathbf{u}(t)$ parameterize curves in the phase plane, known as the solution *trajectories* or *orbits*. In particular, the equilibrium solution $\mathbf{u}(t) \equiv \mathbf{0}$ remains fixed at the origin, and so its trajectory is a single point. Assuming $\beta \neq 0$, every other solution describes a genuine curve, whose tangent direction $\dot{\mathbf{u}} = d\mathbf{u}/dt$ at a point $\mathbf{u}$ is prescribed by the right-hand side of the differential equation, namely $\dot{\mathbf{u}} = A\,\mathbf{u}$. The collection of all possible solution trajectories is called the *phase portrait* of the system. An important fact is that, in an autonomous first order system, *the phase plane trajectories never cross*. This striking property, which is also valid for nonlinear systems, is a consequence of the uniqueness properties of solutions, [**7**, **36**]. Thus, the phase portrait consists of a family of non-intersecting curves that, when combined with the equilibrium points, fill out the entire phase plane. The direction of motion along a trajectory will be indicated graphically by a small arrow; nearby trajectories are all traversed in the same direction. The one feature that is not so easily pictured in the phase portrait is the continuously varying speed at which the solution moves along its trajectory. Plotting this requires a more complicated three-dimensional diagram using time as the third coordinate.

Example 10.2 (continued). For the second order equation (10.4), the equivalent phase plane system is

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix} \mathbf{u}, \qquad \text{or, in full detail,} \qquad \begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= 6\,u_1 - u_2. \end{aligned} \tag{10.9}$$

Our previous solution formula (10.5) implies that the solution to the phase plane system (10.9) is given by

$$u_1(t) = u(t) = c_1\,e^{2\,t} + c_2\,e^{-3\,t}, \qquad u_2(t) = \frac{du}{dt} = 2\,c_1\,e^{2\,t} - 3\,c_2\,e^{-3\,t},$$

and hence
$$\mathbf{u}(t) = \begin{pmatrix} c_1\, e^{2t} + c_2\, e^{-3t} \\ 2\, c_1\, e^{2t} - 3\, c_2\, e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} e^{2t} \\ 2\, e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-3t} \\ -3\, e^{-3t} \end{pmatrix}. \tag{10.10}$$
A sketch of the phase portrait, indicating several representative trajectories, appears in Figure 10.1. The solutions with $c_2 = 0$ go out to ∞ along the two rays in the directions $(1, 2)^T$ and $(-1, -2)^T$, whereas those with $c_1 = 0$ come in to the origin along the rays in the directions $(1, -3)^T$ and $(-1, 3)^T$. All other non-equilibrium solutions move along hyperbolic trajectories whose asymptotes, in forward and backward time, are one of these four rays.

With some practice, one learns to understand the temporal behavior of the solution by studying its phase plane trajectory. We will investigate the qualitative and quantitative behavior of phase plane systems in depth in Section 10.3.

Exercises

10.1.1. Choose one or more of the following differential equations, and then: (*a*) Solve the equation directly. (*b*) Write down its phase plane equivalent, and the general solution to the phase plane system. (*c*) Plot at least four representative trajectories to illustrate the phase portrait. (*d*) Choose two trajectories in your phase portrait and graph the corresponding solution curves $u(t)$. Explain in your own words how the orbit and the solution graph are related. (*i*) $\ddot{u} + 4u = 0$, (*ii*) $\ddot{u} - 4u = 0$, (*iii*) $\ddot{u} + 2\dot{u} + u = 0$, (*iv*) $\ddot{u} + 4\dot{u} + 3u = 0$, (*v*) $\ddot{u} - 2\dot{u} + 10u = 0$.

10.1.2. (a) Convert the third order equation $\dfrac{d^3u}{dt^3} + 3\dfrac{d^2u}{dt^2} + 4\dfrac{du}{dt} + 12u = 0$ into a first order system in three variables by setting $u_1 = u$, $u_2 = \dot{u}$, $u_3 = \ddot{u}$. (*b*) Solve the equation directly, and then use this to write down the general solution to your first order system. (*c*) What is the dimension of the solution space?

10.1.3. Convert the second order coupled system of ordinary differential equations
$$\ddot{u} = a\,\dot{u} + b\,\dot{v} + c\,u + d\,v, \qquad \ddot{v} = p\,\dot{u} + q\,\dot{v} + r\,u + s\,v,$$
into a first order system involving four variables.

♢ 10.1.4. (a) Show that if $\mathbf{u}(t)$ solves $\dot{\mathbf{u}} = A\,\mathbf{u}$, then its *time reversal*, defined as $\mathbf{v}(t) = \mathbf{u}(-t)$, solves $\dot{\mathbf{v}} = B\,\mathbf{v}$, where $B = -A$. (*b*) Explain why the two systems have the same phase portraits, but the direction of motion along the trajectories is reversed. (*c*) Apply time reversal to the system(s) you derived in Exercise 10.1.1. (*d*) What is the effect of time reversal on the original second order equation?

♡ 10.1.5. A first order linear system $\dot{u} = a\,u + b\,v$, $\dot{v} = c\,u + d\,v$, can be converted into a single second order differential equation by the following device. Assuming that $b \neq 0$, solve the system for v and $\dot{v}$ in terms of u and $\dot{u}$. Then differentiate your equation for v with respect to t, and eliminate $\dot{v}$ from the resulting pair of equations. The result is a second order ordinary differential equation for $u(t)$. (a) Write out the second order equation in terms of the coefficients a, b, c, d of the first order system. (*b*) Show that there is a one-to-one correspondence between solutions of the system and solutions of the scalar differential equation. (*c*) Use this method to solve the following linear systems, and sketch the resulting phase portraits. (*i*) $\dot{u} = v$, $\dot{v} = -u$, (*ii*) $\dot{u} = 2u + 5v$, $\dot{v} = -u$, (*iii*) $\dot{u} = 4u - v$, $\dot{v} = 6u - 3v$, (*iv*) $\dot{u} = u + v$, $\dot{v} = u - v$, (*v*) $\dot{u} = v$, $\dot{v} = 0$. (*d*) Show how to obtain a second order equation satisfied by $v(t)$ by an analogous device. Are the second order equations for u and for v the same? (*e*) Discuss how you might proceed if $b = 0$.

10.1.6. (a) Show that if $\mathbf{u}(t)$ solves $\dot{\mathbf{u}} = A\,\mathbf{u}$, then $\mathbf{v}(t) = \mathbf{u}(2t)$ solves $\dot{\mathbf{v}} = B\,\mathbf{v}$, where $B = 2A$.
(b) How are the solution trajectories of the two systems related?

♢ 10.1.7. Let A be a constant $n \times n$ matrix. Let $\mathbf{u}(t)$ be a solution to the system $\dfrac{d\mathbf{u}}{dt} = A\,\mathbf{u}$.
(a) Show that its derivatives $\dfrac{d^k\mathbf{u}}{dt^k}$ for $k = 1, 2, \ldots$, are also solutions.
(b) Show that $\dfrac{d^k\mathbf{u}}{dt^k} = A^k\mathbf{u}$.

10.1.8. *True or false*: Each solution to a phase plane system moves at a constant speed along its trajectory.

10.1.9. *True or false*: The phase plane trajectories (10.10) for $(\,c_1, c_2\,)^T \neq \mathbf{0}$ are hyperbolas.

♠ 10.1.10. Use a three-dimensional graphics package to plot solution curves $(\,t, u_1(t), u_2(t)\,)^T$ of the phase plane systems in Exercise 10.1.1. Discuss their shape and explain how they are related to the phase plane trajectories.

Existence and Uniqueness

Before delving further into our subject, it will help to briefly summarize the basic existence and uniqueness theorems as they apply to linear systems of ordinary differential equations. Even though we will study only the constant coefficient case in detail, these results are equally applicable to non-autonomous systems, and so — but only in this subsection — we allow the coefficient matrix to depend continuously on t. A key fact is that a system of n first order ordinary differential equations requires n initial conditions — one for each variable — in order to uniquely specify its solution. More precisely:

Theorem 10.3. Let $A(t)$ be an $n \times n$ matrix and $\mathbf{f}(t)$ an n-component column vector each of whose entries is a continuous functions on the interval† $a < t < b$. Set an initial time $a < t_0 < b$ and an initial vector $\mathbf{b} \in \mathbb{R}^n$. Then the *initial value problem*

$$\frac{d\mathbf{u}}{dt} = A(t)\,\mathbf{u} + \mathbf{f}(t), \qquad\qquad \mathbf{u}(t_0) = \mathbf{b}, \tag{10.11}$$

admits a unique solution $\mathbf{u}(t)$ that is defined for all $a < t < b$.

For completeness, we have included an inhomogeneous forcing term $\mathbf{f}(t)$ in the system. We will not prove Theorem 10.3, which is a direct consequences of the more general existence and uniqueness theorem for nonlinear systems of ordinary differential equations. Full details can be found in most texts on ordinary differential equations, including [**7**, **22**, **36**]. In the homogeneous case, when $\mathbf{f}(t) \equiv \mathbf{0}$, uniqueness of solutions implies that the solution with zero initial conditions, $\mathbf{u}(t_0) = \mathbf{0}$, is the trivial zero solution: $\mathbf{u}(t) \equiv \mathbf{0}$ for *all* t. In other words, if you start at an equilibrium, you remain there for all time. Moreover, you can never arrive at equilibrium in a finite amount of time, since if $\mathbf{u}(t_1) = \mathbf{0}$, then, again by uniqueness, $\mathbf{u}(t) \equiv \mathbf{0}$ for all $t < t_1$ (and $\geq t_1$, too).

Uniqueness has another important consequence: linear independence of solutions needs be checked only at a single point.

† We allow a and b to be infinite.

Lemma 10.4. The solutions $\mathbf{u}_1(t), \ldots, \mathbf{u}_k(t)$ to a first order homogeneous linear system $\dot{\mathbf{u}} = A(t)\,\mathbf{u}$ are linearly independent if and only if their initial values $\mathbf{u}_1(t_0), \ldots, \mathbf{u}_k(t_0)$ are linearly independent vectors in $\mathbb{R}^n$.

Proof: If the solutions were linearly dependent, one could find (constant) scalars $c_1, \ldots, c_k$, not all zero, such that

$$\mathbf{u}(t) = c_1\,\mathbf{u}_1(t) + \cdots + c_k\,\mathbf{u}_k(t) \equiv \mathbf{0}. \tag{10.12}$$

The equation holds, in particular, at $t = t_0$,

$$\mathbf{u}(t_0) = c_1\,\mathbf{u}_1(t_0) + \cdots + c_k\,\mathbf{u}_k(t_0) = \mathbf{0}. \tag{10.13}$$

This immediately proves linear dependence of the initial vectors.

Conversely, if the initial values $\mathbf{u}_1(t_0), \ldots, \mathbf{u}_k(t_0)$ are linearly dependent, then (10.13) holds for some $c_1, \ldots, c_k$, not all zero. Linear superposition implies that the self-same linear combination $\mathbf{u}(t) = c_1\mathbf{u}_1(t) + \cdots + c_k\mathbf{u}_k(t)$ is a solution to the system, with zero initial condition. By uniqueness, $\mathbf{u}(t) \equiv \mathbf{0}$ for all t, and so (10.12) holds, proving linear dependence of the solutions. *Q.E.D.*

Warning. This result is *not* true if the functions are not solutions to a *first order* linear system. For example, $\mathbf{u}_1(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$, $\mathbf{u}_2(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$, are linearly independent vector-valued functions, but, at time $t = 0$, the vectors $\mathbf{u}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{u}_2(0)$ are linearly dependent. Even worse, $\mathbf{u}_1(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$, $\mathbf{u}_2(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$, define linearly dependent vectors at every specified value of t. Nevertheless, as vector-valued functions, they are linearly independent. (Why?) In view of Lemma 10.4, neither pair of vector-valued functions can be solutions to a common first order homogeneous linear system.

The next result tells us how many different solutions are required in order to construct the general solution by linear superposition.

Theorem 10.5. Let $\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)$ be n linearly independent solutions to the homogeneous system of n first order linear ordinary differential equations $\dot{\mathbf{u}} = A(t)\,\mathbf{u}$. Then the general solution is a linear combination $\mathbf{u}(t) = c_1\mathbf{u}_1(t) + \cdots + c_n\mathbf{u}_n(t)$ depending on n arbitrary constants $c_1, \ldots, c_n$.

Proof: If we have n linearly independent solutions $\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)$, then Lemma 10.4 implies that, at the initial time t_0, the vectors $\mathbf{u}_1(t_0), \ldots, \mathbf{u}_n(t_0)$ are linearly independent, and hence form a basis for $\mathbb{R}^n$. This means that we can express an arbitrary initial condition

$$\mathbf{u}(t_0) = \mathbf{b} = c_1\mathbf{u}_1(t_0) + \cdots + c_n\mathbf{u}_n(t_0)$$

as a linear combination of the initial vectors. Superposition and uniqueness of solutions implies that the corresponding solution to the initial value problem (10.11) is given by the same linear combination

$$\mathbf{u}(t) = c_1\mathbf{u}_1(t) + \cdots + c_n\mathbf{u}_n(t).$$

We conclude that every solution to the ordinary differential equation can be written in the prescribed form, which thus forms the general solution. *Q.E.D.*

Complete Systems

Thus, given a system of n homogeneous linear differential equations $\dot{\mathbf{u}} = A\mathbf{u}$, the immediate goal is to find n linearly independent solutions. Each eigenvalue λ and eigenvector $\mathbf{v}$ of its (constant) coefficient matrix A leads to an exponential eigensolution $\mathbf{u}(t) = e^{\lambda t}\mathbf{v}$. The eigensolutions will be linearly independent if and only if the eigenvectors are — this follows directly from Lemma 10.4. Thus, if the $n \times n$ matrix admits an eigenvector basis, i.e., it is complete, then we have the requisite number of solutions, and hence have solved the differential equation.

Theorem 10.6. If the $n \times n$ matrix A is complete, then the general (complex) solution to the autonomous linear system $\dot{\mathbf{u}} = A\mathbf{u}$ is given by

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n, \tag{10.14}$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the eigenvector basis, $\lambda_1, \ldots, \lambda_n$ the corresponding eigenvalues. The constants $c_1, \ldots, c_n$ are uniquely specified by the initial conditions $\mathbf{u}(t_0) = \mathbf{b}$.

Proof: Since the eigenvectors are linearly independent, the eigensolutions define linearly independent vectors $\mathbf{u}_1(0) = \mathbf{v}_1, \ldots, \mathbf{u}_n(0) = \mathbf{v}_n$ at the initial time $t = 0$. Lemma 10.4 implies that the eigensolutions $\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)$ are, indeed, linearly independent. Hence, we know n linearly independent solutions, and the result is an immediate consequence of Theorem 10.5. *Q.E.D.*

Example 10.7. Let us solve the initial value problem

$$\dot{u}_1 = -2u_1 + u_2, \qquad u_1(0) = 3,$$
$$\dot{u}_2 = 2u_1 - 3u_2, \qquad u_2(0) = 0.$$

The coefficient matrix is $A = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}$. A straightforward computation produces its eigenvalues and eigenvectors:

$$\lambda_1 = -4, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \lambda_2 = -1, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Theorem 10.6 assures us that the corresponding eigensolutions

$$\mathbf{u}_1(t) = e^{-4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \mathbf{u}_2(t) = e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

form a basis for the two-dimensional solution space. The general solution is an arbitrary linear combination

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = c_1 e^{-4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 e^{-4t} + c_2 e^{-t} \\ -2c_1 e^{-4t} + c_2 e^{-t} \end{pmatrix},$$

where c_1, c_2 are constant scalars. Once we have the general solution in hand, the final step is to determine the values of c_1, c_2 in order to satisfy the initial conditions. Evaluating the solution at $t = 0$, we find that we need to solve the linear system

$$u_1(0) = c_1 + c_2 = 3, \qquad u_2(0) = -2c_1 + c_2 = 0,$$

for $c_1 = 1$, $c_2 = 2$. Thus, the (unique) solution to the initial value problem is

$$u_1(t) = e^{-4t} + 2e^{-t}, \qquad u_2(t) = -2e^{-4t} + 2e^{-t}. \tag{10.15}$$

Note that both components of the solution decay exponentially fast to 0 as $t \to \infty$.

Example 10.8. Consider the linear initial value problem

$$\begin{aligned} \dot{u}_1 &= u_1 + 2u_2, & u_1(0) &= 2,\\ \dot{u}_2 &= u_2 - 2u_3, & u_2(0) &= -1,\\ \dot{u}_3 &= 2u_1 + 2u_2 - u_3. & u_3(0) &= -2. \end{aligned}$$

The coefficient matrix is $A = \begin{pmatrix} 1 & 2 & 0\\ 0 & 1 & -2\\ 2 & 2 & -1 \end{pmatrix}$. In Example 8.9, we computed its eigenvalues and eigenvectors:

$$\lambda_1 = -1, \qquad \lambda_2 = 1 + 2\,\mathrm{i}\,, \qquad \lambda_3 = 1 - 2\,\mathrm{i}\,,$$
$$\mathbf{v}_1 = \begin{pmatrix} -1\\ 1\\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1\\ \mathrm{i}\\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 1\\ -\mathrm{i}\\ 1 \end{pmatrix}.$$

The corresponding eigensolutions to the system are

$$\mathbf{u}_1(t) = e^{-t}\begin{pmatrix} -1\\ 1\\ 1 \end{pmatrix}, \qquad \widehat{\mathbf{u}}_2(t) = e^{(1+2\,\mathrm{i})\,t}\begin{pmatrix} 1\\ \mathrm{i}\\ 1 \end{pmatrix}, \qquad \widehat{\mathbf{u}}_3(t) = e^{(1-2i)\,t}\begin{pmatrix} 1\\ -\mathrm{i}\\ 1 \end{pmatrix}.$$

The first solution is real, but the second and third, while perfectly valid solutions, are complex-valued, and hence not as convenient to work with if, as in most applications, we are ultimately after real functions. But, since the underlying linear system is real, the general reality principle of Theorem 7.48 tells us that a complex solution can be broken up into its real and imaginary parts, each of which is a *real* solution. Here, applying Euler's formula (3.92) to the complex exponential, we obtain

$$\widehat{\mathbf{u}}_2(t) = e^{(1+2i)\,t}\begin{pmatrix} 1\\ \mathrm{i}\\ 1 \end{pmatrix} = \left(e^t\cos 2t + \mathrm{i}\,e^t\sin 2t\right)\begin{pmatrix} 1\\ \mathrm{i}\\ 1 \end{pmatrix} = \begin{pmatrix} e^t\cos 2t\\ -e^t\sin 2t\\ e^t\cos 2t \end{pmatrix} + \mathrm{i}\begin{pmatrix} e^t\sin 2t\\ e^t\cos 2t\\ e^t\sin 2t \end{pmatrix}.$$

The final two vector-valued functions are independent real solutions, as you can readily check. In this manner, we have produced three linearly independent real solutions

$$\mathbf{u}_1(t) = \begin{pmatrix} -e^{-t}\\ e^{-t}\\ e^{-t} \end{pmatrix}, \qquad \mathbf{u}_2(t) = \begin{pmatrix} e^t\cos 2t\\ -e^t\sin 2t\\ e^t\cos 2t \end{pmatrix}, \qquad \mathbf{u}_3(t) = \begin{pmatrix} e^t\sin 2t\\ e^t\cos 2t\\ e^t\sin 2t \end{pmatrix},$$

which, by Theorem 10.5, form a basis for the three-dimensional solution space to our system. The general solution can be written as a linear combination:

$$\mathbf{u}(t) = c_1\,\mathbf{u}_1(t) + c_2\,\mathbf{u}_2(t) + c_3\,\mathbf{u}_3(t) = \begin{pmatrix} -c_1 e^{-t} + c_2 e^t\cos 2t + c_3 e^t\sin 2t\\ c_1 e^{-t} - c_2 e^t\sin 2t + c_3 e^t\cos 2t\\ c_1 e^{-t} + c_2 e^t\cos 2t + c_3 e^t\sin 2t \end{pmatrix}.$$

The constants c_1, c_2, c_3 are uniquely prescribed by imposing initial conditions. In our case, the solution satisfying

$$\mathbf{u}(0) = \begin{pmatrix} -c_1 + c_2\\ c_1 + c_3\\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 2\\ -1\\ -2 \end{pmatrix} \qquad \text{results in} \qquad \begin{aligned} c_1 &= -2,\\ c_2 &= 0,\\ c_3 &= 1. \end{aligned}$$

Thus, the solution to the original initial value problem is

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} = \begin{pmatrix} 2e^{-t} + e^t \sin 2t \\ -2e^{-t} + e^t \cos 2t \\ -2e^{-t} + e^t \sin 2t \end{pmatrix}.$$

Incidentally, the third complex eigensolution also produces two real solutions, but these reproduce the ones we have already listed, since it is the complex conjugate of the second eigensolution, and so $\widehat{\mathbf{u}}_3(t) = \mathbf{u}_2(t) - \mathrm{i}\,\mathbf{u}_3(t)$. In general, when solving real systems, you need to deal with only one eigenvalue from each complex conjugate pair to construct a complete system of real solutions.

Exercises

10.1.11. Find the solution to the system of differential equations $\dfrac{du}{dt} = 3u + 4v$, $\dfrac{dv}{dt} = 4u - 3v$, with initial conditions $u(0) = 3$ and $v(0) = -2$.

10.1.12. Find the general real solution to the following systems of differential equations:

(a) $\begin{aligned}\dot u_1 &= u_1 + 9u_2,\\ \dot u_2 &= u_1 + 3u_2;\end{aligned}$ (b) $\begin{aligned}\dot x_1 &= 4x_1 + 3x_2,\\ \dot x_2 &= 3x_1 - 4x_2;\end{aligned}$ (c) $\begin{aligned}\dot y_1 &= y_1 - y_2,\\ \dot y_2 &= 2y_1 + 3y_2;\end{aligned}$

(d) $\begin{aligned}\dot y_1 &= y_2,\\ \dot y_2 &= 3y_1 + 2y_3,\\ \dot y_3 &= -y_2;\end{aligned}$ (e) $\begin{aligned}\dot x_1 &= 3x_1 - 8x_2 + 2x_3,\\ \dot x_2 &= -x_1 + 2x_2 + 2x_3,\\ \dot x_3 &= x_1 - 4x_2 + 2x_3;\end{aligned}$ (f) $\begin{aligned}\dot u_1 &= u_1 - 3u_2 + 11u_3,\\ \dot u_2 &= 2u_1 - 6u_2 + 16u_3,\\ \dot u_3 &= u_1 - 3u_2 + 7u_3.\end{aligned}$

10.1.13. Solve the following initial value problems: (a) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$;

(b) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$; (c) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$;

(d) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} -1 & 3 & -3 \\ 2 & 2 & -7 \\ 0 & 3 & -4 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; (e) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 2 & 1 & -6 \\ -1 & 0 & 4 \\ 0 & -1 & -2 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(\pi) = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$;

(f) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(2) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$; (g) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 2 & 1 & -1 & 0 \\ -3 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}$.

10.1.14. (a) Find the solution to the system $\dfrac{dx}{dt} = -x + y$, $\dfrac{dy}{dt} = -x - y$, that has initial conditions $x(0) = 1$, $y(0) = 0$. (b) Sketch a phase portrait of the system that shows several typical solution trajectories, including the solution you found in part (a). Clearly indicate the direction of increasing t on your curves.

10.1.15. A planar steady-state fluid flow has velocity vector field $\mathbf{v} = (2x - 3y, x - y)^T$ at position $\mathbf{x} = (x, y)^T$. The corresponding fluid motion is described by the differential equation $\dfrac{d\mathbf{x}}{dt} = \mathbf{v}$. A floating object starts out at the point $(1, 1)^T$. Find its position after one time unit.

10.1.16. A steady-state fluid flow has velocity vector field $\mathbf{v} = (-2y, 2x, z)^T$ at position $\mathbf{x} = (x, y, z)^T$. Describe the motion of the fluid particles as governed by the differential equation $\dfrac{d\mathbf{x}}{dt} = \mathbf{v}$.

10.1.17. Solve the initial value problem $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} -6 & 1 \\ 1 & -6 \end{pmatrix} \mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Explain how orthogonality can help.

10.1.18. (*a*) Find the eigenvalues and eigenvectors of $K = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.

(*b*) Verify that the eigenvectors are mutually orthogonal. (*c*) Based on part (*a*), is K positive definite, positive semi-definite, or indefinite? (*d*) Solve the initial value problem $\dfrac{d\mathbf{u}}{dt} = K\,\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, using orthogonality to simplify the computations.

10.1.19. Demonstrate that one can also solve the initial value problem in Example 10.8 by writing the solution as a complex linear combination of the complex eigensolutions, and then using the initial conditions to specify the coefficients.

10.1.20. Determine whether the following vector-valued functions are linearly dependent or linearly independent:

(a) $\begin{pmatrix} 1 \\ t \end{pmatrix}, \begin{pmatrix} -t \\ 1 \end{pmatrix}$, (*b*) $\begin{pmatrix} 1+t \\ t \end{pmatrix}, \begin{pmatrix} 1-t^2 \\ t-t^2 \end{pmatrix}$, (*c*) $\begin{pmatrix} 1 \\ t \end{pmatrix}, \begin{pmatrix} t \\ 2 \end{pmatrix}, \begin{pmatrix} -t \\ t \end{pmatrix}$, (*d*) $\begin{pmatrix} e^{-t} \\ -e^{t} \end{pmatrix}, \begin{pmatrix} -e^{-t} \\ e^{t} \end{pmatrix}$,

(*e*) $\begin{pmatrix} e^{2t}\cos 3t \\ -e^{2t}\sin 3t \end{pmatrix}, \begin{pmatrix} e^{2t}\sin 3t \\ e^{2t}\cos 3t \end{pmatrix}$, (*f*) $\begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}, \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$, (*g*) $\begin{pmatrix} 1 \\ t \\ 1-t \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1+3t \\ 2-3t \end{pmatrix}$,

(*h*) $\begin{pmatrix} e^{t} \\ -e^{t} \\ e^{t} \end{pmatrix}, \begin{pmatrix} e^{t} \\ e^{t} \\ -e^{t} \end{pmatrix}, \begin{pmatrix} -e^{t} \\ e^{t} \\ e^{t} \end{pmatrix}$, (*i*) $\begin{pmatrix} e^{t} \\ t e^{t} \\ t^2 e^{t} \end{pmatrix}, \begin{pmatrix} t^2 e^{t} \\ e^{t} \\ t e^{t} \end{pmatrix}, \begin{pmatrix} t e^{t} \\ t^2 e^{t} \\ e^{t} \end{pmatrix}, \begin{pmatrix} e^{t} \\ e^{t} \\ e^{t} \end{pmatrix}$.

♢ 10.1.21. Let A be a constant matrix. Suppose $\mathbf{u}(t)$ solves the initial value problem $\dot{\mathbf{u}} = A\,\mathbf{u}$, $\mathbf{u}(0) = \mathbf{b}$. Prove that the solution to the initial value problem $\dot{\mathbf{u}} = A\,\mathbf{u}$, $\mathbf{u}(t_0) = \mathbf{b}$, is equal to $\tilde{\mathbf{u}}(t) = \mathbf{u}(t - t_0)$. How are the solution trajectories related?

10.1.22. Suppose $\mathbf{u}(t)$ and $\tilde{\mathbf{u}}(t)$ both solve the linear system $\dot{\mathbf{u}} = A\,\mathbf{u}$. (*a*) Suppose they have the same value $\mathbf{u}(t_1) = \tilde{\mathbf{u}}(t_1)$ at any one time t_1. Show that they are, in fact, the same solution: $\mathbf{u}(t) = \tilde{\mathbf{u}}(t)$ for all t. (*b*) What happens if $\mathbf{u}(t_1) = \tilde{\mathbf{u}}(t_2)$ for some $t_1 \neq t_2$? *Hint*: See Exercise 10.1.21.

10.1.23. Prove that the general solution to a linear system $\dot{\mathbf{u}} = \Lambda\,\mathbf{u}$ with diagonal coefficient matrix $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is given by $\mathbf{u}(t) = (c_1 e^{\lambda_1 t}, \ldots, c_n e^{\lambda_n t})^T$.

10.1.24. Show that if $\mathbf{u}(t)$ is a solution to $\dot{\mathbf{u}} = A\,\mathbf{u}$, and S is a constant, nonsingular matrix of the same size as A, then $\mathbf{v}(t) = S\,\mathbf{u}(t)$ solves the linear system $\dot{\mathbf{v}} = B\,\mathbf{v}$, where $B = S\,A\,S^{-1}$ is similar to A.

♢ 10.1.25. (*i*) Combine Exercises 10.1.23–24 to show that if $A = S\,\Lambda\,S^{-1}$ is diagonalizable, then the solution to $\dot{\mathbf{u}} = A\,\mathbf{u}$ can be written as $\mathbf{u}(t) = S\,(c_1 e^{\lambda_1 t}, \ldots, c_n e^{\lambda_n t})^T$, where $\lambda_1, \ldots, \lambda_n$ are its eigenvalues and $S = (\, \mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n \,)$ is the corresponding matrix of eigenvectors. (*ii*) Write the general solution to the systems in Exercise 10.1.13 in this form.

The General Case

Summarizing the preceding subsection, if the coefficient matrix of a homogeneous, autonomous first order linear system is complete, then the eigensolutions form a (complex) basis for the solution space. Assuming the coefficient matrix is real, one obtains a real basis by taking the real and imaginary parts of each complex conjugate pair of solutions. In the incomplete cases, the formulas for the basis solutions are a little more intricate, and involve polynomials as well as (complex) exponentials. Readers who did not cover Section 8.6 are advised to skip ahead to Section 10.2; only Theorem 10.13, which summarizes the key features, will be used in the sequel.

Example 10.9. The simplest incomplete case arises as the phase plane equivalent of a scalar ordinary differential equation whose characteristic equation has a repeated root. For example, to directly solve the second order equation

$$\frac{d^2u}{dt^2} - 2\,\frac{du}{dt} + u = 0, \tag{10.16}$$

we substitute the usual exponential ansatz $u = e^{\lambda t}$, leading to the characteristicequation

$$\lambda^2 - 2\lambda + 1 = 0.$$

There is only one double root, $\lambda = 1$, and hence, up to scalar multiple, only one exponential solution $u_1(t) = e^t$. For a scalar ordinary differential equation, the second, "missing",solution is obtained by simply multiplying the first by t, so that $u_2(t) = t\,e^t$. As a result, the general solution to (10.16) is

$$u(t) = c_1\,u_1(t) + c_2\,u_2(t) = c_1\,e^t + c_2\,t\,e^t.$$

As in (10.8), the equivalent phase plane system is

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\mathbf{u}, \qquad \text{where} \qquad \mathbf{u}(t) = \begin{pmatrix} u(t) \\ \dot u(t) \end{pmatrix}.$$

Note that the coefficient matrix is incomplete — it has $\lambda = 1$ as a double eigenvalue, but only one independent eigenvector, namely $\mathbf{v} = (1,1)^T$. The two linearly independent solutions to the phase plane system can be constructed from the two solutions to the scalar equation. Thus,

$$\mathbf{u}_1(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}, \qquad \mathbf{u}_2(t) = \begin{pmatrix} t\,e^t \\ t\,e^t + e^t \end{pmatrix}$$

form a basis for the two-dimensional solution space. The first is an eigensolution, while the second includes an additional polynomial factor. Observe that, in contrast to the scalar case, the second solution $\mathbf{u}_2$ is *not* obtained from the first by merely multiplying by t.

In general, the eigenvectors of an incomplete matrix fail to form a basis, and, as noted in Section 8.6, can be extended to a Jordan basis. Thus, the key step is to describe the solutions associated with a Jordan chain, cf. Definition 8.47.

Lemma 10.10. Suppose $\mathbf{w}_1, \ldots, \mathbf{w}_k$ form a Jordan chain of length k for the eigenvalue λ of the matrix A. Then there are k linearly independent solutions to the corresponding first order system $\dot{\mathbf{u}} = A\,\mathbf{u}$ having the form

$$\mathbf{u}_1(t) = e^{\lambda t}\mathbf{w}_1, \qquad \mathbf{u}_2(t) = e^{\lambda t}(t\,\mathbf{w}_1 + \mathbf{w}_2), \qquad \mathbf{u}_3(t) = e^{\lambda t}\big(\tfrac12 t^2\,\mathbf{w}_1 + t\,\mathbf{w}_2 + \mathbf{w}_3\big),$$

$$\text{and, in general,} \qquad \mathbf{u}_j(t) = e^{\lambda t}\sum_{i=1}^{j}\frac{t^{j-i}}{(j-i)!}\,\mathbf{w}_i, \qquad 1 \le j \le k. \tag{10.17}$$

The proof is by direct substitution of the formulas (10.17) into the differential equation, invoking the defining relations (8.46) of the Jordan chain as needed; details are left to the reader. If λ is a complex eigenvalue, then the Jordan chain solutions (10.17) will involve complex exponentials. As usual, if A is a real matrix, they can be split into their real and imaginary parts, which are independent real solutions.

Example 10.11. The coefficient matrix of the system

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ -2 & 2 & -4 & 1 & 1 \\ -1 & 0 & -3 & 0 & 0 \\ -4 & -1 & 3 & 1 & 0 \\ 4 & 0 & 2 & -1 & 0 \end{pmatrix} \mathbf{u}$$

is incomplete; it has only 2 linearly independent eigenvectors associated with the eigenvalues 1 and -2. Using the Jordan basis computed in Example 8.52, we produce the following 5 linearly independent solutions:

$$\mathbf{u}_1(t) = e^t\,\mathbf{v}_1, \qquad \mathbf{u}_2(t) = e^t\,(t\,\mathbf{v}_1 + \mathbf{v}_2), \qquad \mathbf{u}_3(t) = e^t\left(\tfrac{1}{2}t^2\,\mathbf{v}_1 + t\,\mathbf{v}_2 + \mathbf{v}_3\right),$$
$$\mathbf{u}_4(t) = e^{-2t}\,\mathbf{v}_4, \qquad \mathbf{u}_5(t) = e^{-2t}\,(t\,\mathbf{v}_4 + \mathbf{v}_5),$$

or, explicitly,

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ -e^t \\ e^t \end{pmatrix}, \quad \begin{pmatrix} 0 \\ e^t \\ 0 \\ -t\,e^t \\ (-1+t)\,e^t \end{pmatrix}, \quad \begin{pmatrix} 0 \\ t\,e^t \\ 0 \\ \left(1-\frac{1}{2}t^2\right)e^t \\ \left(-t+\frac{1}{2}t^2\right)e^t \end{pmatrix}, \quad \begin{pmatrix} -e^{-2t} \\ e^{-2t} \\ e^{-2t} \\ -2\,e^{-2t} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -(1+t)\,e^{-2t} \\ t\,e^{-2t} \\ t\,e^{-2t} \\ -2\,(1+t)\,e^{-2t} \\ e^{-2t} \end{pmatrix}.$$

The first three solutions are associated with the $\lambda_1 = 1$ Jordan chain, the last two with the $\lambda_2 = -2$ chain. The eigensolutions are the pure exponentials $\mathbf{u}_1(t), \mathbf{u}_4(t)$. The general solution to the system is an arbitrary linear combination of these five basis solutions.

Proposition 10.12. Let A be an $n \times n$ matrix. Then the Jordan chain solutions (10.17) constructed from a Jordan basis of A form a basis for the n-dimensional solution space for the corresponding linear system $\dot{\mathbf{u}} = A\,\mathbf{u}$.

The proof of linear independence of the Jordan chain solutions follows, via Lemma 10.4, from the linear independence of the Jordan basis vectors, which are their initial values.

Important qualitative features can be readily gleaned from the algebraic structure of the solution formulas (10.17). The following result describes the principal classes of solutions of homogeneous autonomous linear systems of ordinary differential equations.

Theorem 10.13. Let A be a real $n \times n$ matrix. Every real solution to the linear system $\dot{\mathbf{u}} = A\,\mathbf{u}$ is a linear combination of n linearly independent solutions appearing in the following four classes:

(1) If λ is a complete real eigenvalue of multiplicity m, then there exist m linearly independent solutions of the form

$$\mathbf{u}_k(t) = e^{\lambda t}\,\mathbf{v}_k, \qquad k = 1, \ldots, m,$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly independent eigenvectors.

(2) If $\lambda_\pm = \mu \pm \mathrm{i}\,\nu$ form a pair of complete complex conjugate eigenvalues of multiplicity m, then there exist $2m$ linearly independent real solutions of the forms

$$\begin{aligned} \mathbf{u}_k(t) &= e^{\mu t}\big[\cos(\nu t)\,\mathbf{x}_k - \sin(\nu t)\,\mathbf{y}_k\big], \\ \widehat{\mathbf{u}}_k(t) &= e^{\mu t}\big[\sin(\nu t)\,\mathbf{x}_k + \cos(\nu t)\,\mathbf{y}_k\big], \end{aligned} \qquad k = 1, \ldots, m,$$

where $\mathbf{v}_k = \mathbf{x}_k \pm \mathrm{i}\,\mathbf{y}_k$ are the associated complex conjugate eigenvectors.

(3) If λ is an incomplete real eigenvalue of multiplicity m and r is the dimension of the eigenspace V_λ, then there exist m linearly independent solutions of the form

$$\mathbf{u}_k(t) = e^{\lambda t}\,\mathbf{p}_k(t), \qquad k = 1, \ldots, m,$$

where $\mathbf{p}_k(t)$ is a vector of polynomials of degree $\le m - r$.

(4) If $\lambda_\pm = \mu \pm \mathrm{i}\,\nu$ form a pair of incomplete complex conjugate eigenvalues of multiplicity m, and r is the common dimension of the two eigenspaces, then there exist $2m$ linearly independent real solutions

$$\begin{aligned} \mathbf{u}_k(t) &= e^{\mu t}\,\big[\cos(\nu t)\,\mathbf{p}_k(t) - \sin(\nu t)\,\mathbf{q}_k(t)\big], \\ \widehat{\mathbf{u}}_k(t) &= e^{\mu t}\,\big[\sin(\nu t)\,\mathbf{p}_k(t) + \cos(\nu t)\,\mathbf{q}_k(t)\big], \end{aligned} \qquad k = 1, \ldots, m,$$

where $\mathbf{p}_k(t), \mathbf{q}_k(t)$ are vectors of polynomials of degree $\le m - r$, whose detailed structure can be gleaned from Lemma 10.10.

As a result, every real solution to a homogeneous linear system of ordinary differential equations is a vector-valued function whose entries are linear combinations of functions of the particular form $t^k\,e^{\mu t}\cos \nu t$ and $t^k\,e^{\mu t}\sin \nu t$, i.e., sums of products of exponentials, trigonometric functions, and polynomials. The exponents μ are the real parts of the eigenvalues of the coefficient matrix; the trigonometric frequencies ν are the imaginary parts of the eigenvalues; nonconstant polynomials appear only if the matrix is incomplete.

Exercises

10.1.26. Find the general solution to the linear system $\dfrac{d\mathbf{u}}{dt} = A\,\mathbf{u}$ for the following incomplete coefficient matrices: (a) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & -1 \\ 9 & -4 \end{pmatrix}$, (c) $\begin{pmatrix} -1 & -1 \\ 4 & -5 \end{pmatrix}$, (d) $\begin{pmatrix} 4 & -1 & -3 \\ -2 & 1 & 2 \\ 5 & -1 & -4 \end{pmatrix}$, (e) $\begin{pmatrix} -3 & 1 & 0 \\ 1 & -3 & -1 \\ 0 & 1 & -3 \end{pmatrix}$, (f) $\begin{pmatrix} 3 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, (g) $\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$.

10.1.27. Find a first order system of ordinary differential equations that has the indicated vector-valued function as a solution: (a) $\begin{pmatrix} e^t + e^{2t} \\ 2\,e^t \end{pmatrix}$, (b) $\begin{pmatrix} e^{-t}\cos 3t \\ -3\,e^{-t}\sin 3t \end{pmatrix}$, (c) $\begin{pmatrix} 1 \\ t-1 \end{pmatrix}$, (d) $\begin{pmatrix} \sin 2t - \cos 2t \\ \sin 2t + 3\cos 2t \end{pmatrix}$, (e) $\begin{pmatrix} e^{2t} \\ e^{-3t} \\ e^{2t} - e^{-3t} \end{pmatrix}$, (f) $\begin{pmatrix} \sin t \\ \cos t \\ 1 \end{pmatrix}$, (g) $\begin{pmatrix} t \\ 1-t^2 \\ 1+t \end{pmatrix}$, (h) $\begin{pmatrix} e^t \sin t \\ 2\,e^t\cos t \\ e^t \sin t \end{pmatrix}$.

10.1.28. Which sets of functions in Exercise 10.1.20 can be solutions to a common first order, homogeneous, constant coefficient linear system of ordinary differential equations? If so, find a system they satisfy; if not, explain why not.

10.1.29. Solve the third order equation $\dfrac{d^3u}{dt^3} + 3\,\dfrac{d^2u}{dt^2} + 4\,\dfrac{du}{dt} + 12\,u = 0$ by converting it into a first order system. Compare your answer with what you found in Exercise 10.1.2.

10.1.30. Solve the second order coupled system of ordinary differential equations $\ddot{u} = \dot{u} + u - v$, $\ddot{v} = \dot{v} - u + v$, by converting it into a first order system involving four variables.

10.1.31. Suppose that $\mathbf{u}(t) \in \mathbb{R}^n$ is a polynomial solution to the constant coefficient linear system $\dot{\mathbf{u}} = A\,\mathbf{u}$. What is the maximal possible degree of $\mathbf{u}(t)$? What can you say about A when $\mathbf{u}(t)$ has maximal degree?

♢ 10.1.32. (*a*) Under the assumption that $\mathbf{u}_1, \ldots, \mathbf{u}_k$ form a Jordan chain for the coefficient matrix A, prove that the functions (10.17) are solutions to the system $\dot{\mathbf{u}} = A\,\mathbf{u}$.
(*b*) Prove that they are linearly independent.

10.2 Stability of Linear Systems

With the general solution formulas in hand, we are now ready to study the qualitative features of first order linear dynamical systems. Our primary focus will be on stability properties of the equilibrium solution(s). A solution to an autonomous system of first order ordinary differential equations $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$ is called an *equilibrium solution* if it remains constant for all t, so $\mathbf{u}(t) \equiv \mathbf{u}^\star$. Since its derivative vanishes, this implies that the *equilibrium point* $\mathbf{u}^\star$ satisfies $\mathbf{f}(\mathbf{u}^\star) = \mathbf{0}$. In particular, for a homogeneous linear system $\dot{\mathbf{u}} = A\,\mathbf{u}$, the origin $\mathbf{u}^\star \equiv \mathbf{0}$ is always an equilibrium point, meaning that a solution that starts out at $\mathbf{0}$ remains there. The complete set of equilibrium solutions consists of all points $\mathbf{u}^\star \in \ker A$ in the kernel of the coefficient matrix, and so the set of equilibrium solutions forms a subspace — indeed, an invariant subspace — of the configuration space.

In physical applications, the stability properties of equilibrium solutions is of crucial importance; see the discussion at the beginning of Chapter 5. In general, an equilibrium point is *stable* if *every* solution that starts out nearby stays nearby. An equilibrium is called *asymptotically stable* if the nearby solutions converge to it as time increases. The formal mathematical definitions are as follows.

Definition 10.14. An equilibrium solution $\mathbf{u}^\star$ to an autonomous system of first order ordinary differential equations $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$ is called

- *stable* if for every sufficiently small $\varepsilon > 0$, there exists a $\delta > 0$ such that every solution $\mathbf{u}(t)$ having initial conditions within distance $\delta > \|\,\mathbf{u}(t_0) - \mathbf{u}^\star\,\|$ of the equilibrium remains within distance $\varepsilon > \|\,\mathbf{u}(t) - \mathbf{u}^\star\,\|$ for all $t \geq t_0$.
- *asymptotically stable* if it is stable and, in addition, there exists $\varepsilon_0 > 0$ such that whenever $\|\,\mathbf{u}(t_0) - \mathbf{u}^\star\,\| < \varepsilon_0$, then $\mathbf{u}(t) \to \mathbf{u}^\star$ as $t \to \infty$.

Thus, although solutions nearby a stable equilibrium point may drift slightly farther away, they must remain relatively close. In the case of asymptotic stability, they will eventually return to equilibrium. An equilibrium point is called *globally stable* if the stability condition holds for *all* $\varepsilon > 0$. It is called *globally asymptotically stable* if *every* solution converges to the equilibrium point: $\mathbf{u}(t) \to \mathbf{u}^\star$ as $t \to \infty$.

In the case of a linear system, local (asymptotic) stability implies global (asymptotic) stability. This is because, by linearity, if $\mathbf{u}(t)$ is a solution, then so is the scalar multiple $c\,\mathbf{u}(t)$ for all $c \in \mathbb{R}$, and hence every solution can be scaled to one that remains nearby the

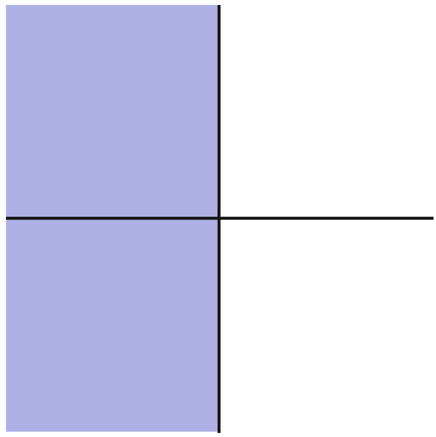

Figure 10.2. The Left Half–Plane.

equilibrium point. We will henceforth omit the redundant term "global" when discussing the stability of a linear system. We will also focus our attention on the particular equilibrium solution $\mathbf{u}^\star = \mathbf{0}$.

Remark. The stability and asymptotic stability of an equilibrium solution are independent of the choice of norm in the definition (although this will affect the dependence of δ on ε). This follows from the equivalence of norms described in Theorem 3.17.

The starting point is a simple calculus lemma, whose proof is left to the reader.

Lemma 10.15. Let μ, ν be real and $k \geq 0$. A function of the form

$$f(t) = t^k\, e^{\mu\, t} \cos \nu\, t \qquad \text{or} \qquad t^k\, e^{\mu\, t} \sin \nu\, t \tag{10.18}$$

will decay to zero for large t, so $\lim\limits_{t \to \infty} f(t) = 0$, if and only if $\mu < 0$. The function remains bounded, so $|\, f(t)\,| \leq C$ for some constant C, for all $t \geq 0$ if and only if either $\mu < 0$, or $\mu = 0$ and $k = 0$.

Loosely put, exponential decay will always overwhelm polynomial growth, while the trigonometric sine and cosine functions remain neutrally bounded. Now, in the solution to our linear system, the functions (10.18) come from the eigenvalues $\lambda = \mu + \mathrm{i}\,\nu$ of the coefficient matrix. The lemma implies that the asymptotic behavior of the solutions, and hence the stability of the system, depends on the sign of $\mu = \operatorname{Re} \lambda$. If $\mu < 0$, then the solutions decay to zero at an exponential rate as $t \to \infty$. If $\mu > 0$, then the solutions become unbounded as $t \to \infty$. In the borderline case $\mu = 0$, the solutions remain bounded, provided that they don't involve any powers of t.

Thus, in order that the equilibrium zero solution be *asymptotically stable*, all the eigenvalues must satisfy $\mu = \operatorname{Re} \lambda < 0$. Or, stated another way, all eigenvalues must lie in the *left half-plane* — the subset of the complex plane $\mathbb{C}$ to the left of the imaginary axis sketched in Figure 10.2. In this manner, we have demonstrated the fundamental asymptotic stability criterion† for linear systems.

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† This is *not* the same as the stability criterion for linear iterative systems, which requires that the eigenvalues of the coefficient matrix lie in the inside the unit circle, cf. Theorem 9.12.

Theorem 10.16. A first order autonomous homogeneous linear system of ordinary differential equations $\dot{\mathbf{u}} = A\mathbf{u}$ has an asymptotically stable zero solution if and only if all the eigenvalues λ of its coefficient matrix A lie in the left half-plane: $\operatorname{Re}\lambda < 0$. If A has one or more eigenvalues with positive real part, $\operatorname{Re}\lambda > 0$, then the zero solution is unstable.

Example 10.17. Consider the system

$$\frac{du}{dt} = 2u - 6v + w, \qquad \frac{dv}{dt} = 3u - 3v - w, \qquad \frac{dw}{dt} = 3u - v - 3w.$$

The coefficient matrix $A = \begin{pmatrix} 2 & -6 & 1 \\ 3 & -3 & -1 \\ 3 & -1 & -3 \end{pmatrix}$ is found to have eigenvalues

$$\lambda_1 = -2, \qquad \lambda_2 = -1 + \mathrm{i}\sqrt{6}\,, \qquad \lambda_3 = -1 - \mathrm{i}\sqrt{6}\,,$$

with respective real parts $-2, -1, -1$. The Stability Theorem 10.16 implies that the equilibrium solution $u_\star \equiv v_\star \equiv w_\star \equiv 0$ is asymptotically stable. Indeed, every solution involves the functions e^{-2t}, $e^{-t}\cos\sqrt{6}\,t$, and $e^{-t}\sin\sqrt{6}\,t$, all of which decay to 0 at an exponential rate. The latter two have the slowest decay rate, and so most solutions to the linear system go to $\mathbf{0}$ in proportion to e^{-t}, i.e., at an exponential rate determined by the least negative real part.

The final statement is a special case of the following general result, whose proof is left to the reader.

Proposition 10.18. If $\mathbf{u}(t)$ is any solution to $\dot{\mathbf{u}} = A\mathbf{u}$, then $\|\,\mathbf{u}(t)\,\| \le C\,e^{a\,t}$ for all $t \ge t_0$ and for all $a > a^\star = \max\{\operatorname{Re}\lambda \mid \lambda \text{ is an eigenvalue of } A\}$, where the constant $C > 0$ depends on the solution and choice of norm. If the eigenvalue(s) λ achieving the maximum, $\operatorname{Re}\lambda = a^\star$, are complete, then one can set $a = a^\star$.

Asymptotic stability implies that the solutions return to equilibrium; *stability* only requires them to stay nearby. The appropriate eigenvalue criterion is readily established.

Theorem 10.19. A first order linear, homogeneous, constant-coefficient system of ordinary differential equations (10.1) has a stable zero solution if and only if all its eigenvalues satisfy $\operatorname{Re}\lambda \le 0$, and, moreover, any eigenvalue lying on the imaginary axis, so $\operatorname{Re}\lambda = 0$, is complete, meaning that it has as many independent eigenvectors as its multiplicity.

Proof: The proof is the same as before, based on Theorem 10.13 and the decay properties in Lemma 10.15. All the eigenvalues with negative real part lead to exponentially decaying solutions — even if they are incomplete. If the coefficient matrix has a complete zero eigenvalue, then the corresponding eigensolutions are all constant, and hence trivially bounded. On the other hand, if 0 is an incomplete eigenvalue, then the associated Jordan chain solutions involve non-constant polynomials, and become unbounded as $t \to \pm\infty$. Similarly, if a purely imaginary eigenvalue is complete, then the associated solutions only involve trigonometric functions, and hence remain bounded, whereas the solutions associated with an incomplete purely imaginary eigenvalue contain polynomials in t multiplying sines and cosines, and hence cannot remain bounded. *Q.E.D.*

A particularly important class of systems consists of the linear *gradient flows*

$$\frac{d\mathbf{u}}{dt} = -\,K\mathbf{u}, \tag{10.19}$$

in which K is a symmetric, positive definite matrix. According to Theorem 8.35, all the eigenvalues of K are real and positive, and so the eigenvalues of the negative definite coefficient matrix $-K$ for the gradient flow system (10.19) are real and negative. Applying Theorem 10.16, we conclude that the zero solution to any gradient flow system (10.19) with negative definite coefficient matrix $-K$ is asymptotically stable. If the coefficient matrix is negative semi-definite, the the equilibrium solutions are stable, since the eigenvalues are necessarily complete.

Example 10.20. On applying the test we learned in Chapter 3, the matrix $K = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$ is seen to be positive definite. The associated gradient flow is

$$\frac{du}{dt} = -u - v, \qquad \frac{dv}{dt} = -u - 5v. \tag{10.20}$$

The eigenvalues and eigenvectors of $-K = \begin{pmatrix} -1 & -1 \\ -1 & -5 \end{pmatrix}$ are

$$\lambda_1 = -3 + \sqrt{5}\,, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 - \sqrt{5} \end{pmatrix}, \qquad \lambda_2 = -3 - \sqrt{5}\,, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 + \sqrt{5} \end{pmatrix}.$$

Therefore, the general solution to the system is

$$\mathbf{u}(t) = c_1\, e^{(-3+\sqrt{5}\,)t} \begin{pmatrix} 1 \\ 2 - \sqrt{5} \end{pmatrix} + c_2\, e^{(-3-\sqrt{5}\,)t} \begin{pmatrix} 1 \\ 2 + \sqrt{5} \end{pmatrix},$$

or, in components,

$$\begin{aligned} u(t) &= c_1\, e^{(-3+\sqrt{5}\,)t} + c_2\, e^{(-3-\sqrt{5}\,)t}, \\ v(t) &= c_1\,(2 - \sqrt{5})\, e^{(-3+\sqrt{5}\,)t} + c_2\,(2 + \sqrt{5})\, e^{(-3-\sqrt{5}\,)t}. \end{aligned}$$

All solutions tend to zero as $t \to \infty$ at the exponential rate prescribed by the least negative eigenvalue, which is $-3 + \sqrt{5} \simeq -.7639$. This confirms the asymptotic stability of the gradient flow.

The reason for the term "gradient flow" is that the vector field $-K\mathbf{u}$ appearing on the right-hand side of (10.19) is, in fact, the negative of the gradient of the quadratic function

$$q(\mathbf{u}) = \tfrac{1}{2}\,\mathbf{u}^T K\,\mathbf{u} = \frac{1}{2}\sum_{i,j=1}^{n} k_{ij}\,u_i\,u_j, \qquad \text{so that} \qquad \nabla q(\mathbf{u}) = K\,\mathbf{u}. \tag{10.21}$$

Thus, we can write (10.19) as

$$\frac{d\mathbf{u}}{dt} = -\,\nabla q(\mathbf{u}). \tag{10.22}$$

For the particular system (10.20),

$$q(u,v) = \tfrac{1}{2}\,(\,u \quad v\,)\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \tfrac{1}{2}\,u^2 + u\,v + \tfrac{5}{2}\,v^2,$$

and so the gradient flow is given by

$$\frac{du}{dt} = -\,\frac{\partial q}{\partial u} = -u - v, \qquad \frac{dv}{dt} = -\,\frac{\partial q}{\partial v} = -u - 5v.$$

As you learn in multivariable calculus, [**2**, **78**], the gradient ∇q of a function q points in the direction of its steepest increase, while its negative $-\nabla q$ points in the direction of steepest decrease. Thus, the solutions to the gradient flow system (10.22) will decrease $q(\mathbf{u})$ as rapidly as possible, tending to its minimum at $\mathbf{u}^\star = \mathbf{0}$. For instance, if $q(u,v)$ represents the height of a hill at position (u,v), then the solutions to (10.22) are the paths of steepest descent followed by, say, water flowing down the hill (provided we ignore inertial effects). In physical applications, the quadratic function (10.21) often represents the potential energy in the system, and the gradient flow models the natural behavior of systems that seek to minimize their energy as rapidly as possible.

Example 10.21. Another extremely important class of dynamical systems comprises the Hamiltonian systems, first developed by the nineteenth-century Irish mathematician William Rowan Hamilton, who also discovered quaternions, developed in Exercise 7.2.23. In particular, a planar *Hamiltonian system* takes the form

$$\frac{du}{dt} = \frac{\partial H}{\partial v}, \qquad\qquad \frac{dv}{dt} = -\,\frac{\partial H}{\partial u}, \tag{10.23}$$

where $H(u,v)$ is known as the *Hamiltonian function*. If

$$H(u,v) = \tfrac{1}{2}\,a\,u^2 + b\,u\,v + \tfrac{1}{2}\,c\,v^2 \tag{10.24}$$

is a quadratic form, then the corresponding Hamiltonian system

$$\dot u = b\,u + c\,v, \qquad\qquad \dot v = -\,a\,u - b\,v, \tag{10.25}$$

is homogeneous linear, with coefficient matrix $A = \begin{pmatrix} b & c \\ -a & -b \end{pmatrix}$. The associated characteristic equation is

$$\det(A - \lambda\,\mathrm{I}\,) = \lambda^2 + (a\,c - b^2) = 0.$$

If H is positive or negative definite, then $a\,c - b^2 > 0$, and so the eigenvalues are purely imaginary: $\lambda = \pm\,\mathrm{i}\sqrt{a\,c - b^2}$ and complete, since they are simple. Thus, the stability criterion of Theorem 10.19 holds, and we conclude that planar Hamiltonian systems with a definite Hamiltonian function are stable. On the other hand, if H is indefinite, then the coefficient matrix has one positive and one negative eigenvalue, and hence the Hamiltonian system is unstable.

In physical applications, the Hamiltonian function $H(u,v)$ represents the energy of the system. According to Exercise 10.2.22, the Hamiltonian energy function is automatically conserved, meaning that it is constant on every solution: $H(u(t),v(t)) =$ constant. This means that the solutions move along its level sets; in the stable cases these are bounded ellipses, whereas in the unstable cases they are unbounded hyperbolas.

Remark. The equations of classical mechanics, such as motion of masses (sun, planets, comets, etc.) under gravitational attraction, can all be formulated as Hamiltonian systems, [**31**]. Moreover, the Hamiltonian formulation is a crucial first step in the physical process of quantizing the classical mechanical equations to determine the quantum mechanical equations of motion, [**54**].

Exercises

10.2.1. Classify the following systems according to whether the origin is (*i*) asymptotically stable, (*ii*) stable, or (*iii*) unstable: (a) $\dfrac{du}{dt} = -\,2\,u - v$, $\dfrac{dv}{dt} = u - 2\,v$; (b) $\dfrac{du}{dt} = 2\,u - 5\,v$, $\dfrac{dv}{dt} = u - v$; (c) $\dfrac{du}{dt} = -\,u - 2\,v$, $\dfrac{dv}{dt} = 2\,u - 5\,v$; (d) $\dfrac{du}{dt} = -\,2\,v$, $\dfrac{dv}{dt} = 8\,u$;

(e) $\dfrac{du}{dt} = -2u - v + w, \quad \dfrac{dv}{dt} = -u - 2v + w, \quad \dfrac{dw}{dt} = -3u - 3v + 2w;$

(f) $\dfrac{du}{dt} = -u - 2v, \quad \dfrac{dv}{dt} = 6u + 3v - 4w, \quad \dfrac{dw}{dt} = 4u - 3w;$

(g) $\dfrac{du}{dt} = 2u - v + 3w, \quad \dfrac{dv}{dt} = u - v + w, \quad \dfrac{dw}{dt} = -4u + v - 5w;$

(h) $\dfrac{du}{dt} = u + v - w, \quad \dfrac{dv}{dt} = -2u - 3v + 3w, \quad \dfrac{dw}{dt} = -v + w.$

10.2.2. Write out the formula for the general real solution to the system in Example 10.17 and verify its stability.

10.2.3. Write out and solve the gradient flow system corresponding to the following quadratic forms: (a) $u^2 + v^2$, (b) uv, (c) $4u^2 - 2uv + v^2$, (d) $2u^2 - uv - 2uw + 2v^2 - vw + 2w^2$.

10.2.4. Write out and solve the Hamiltonian systems corresponding to the first three quadratic forms in Exercise 10.2.3. Which of them are stable?

10.2.5. Which of the following 2×2 systems are gradient flows? Which are Hamiltonian systems? In each case, discuss the stability of the zero solution.

(a) $\begin{array}{l}\dot u = -2u + v,\\ \dot v = u - 2v,\end{array}$ (b) $\begin{array}{l}\dot u = u - 2v,\\ \dot v = -2u + v,\end{array}$ (c) $\begin{array}{l}\dot u = v,\\ \dot v = u,\end{array}$ (d) $\begin{array}{l}\dot u = -v,\\ \dot v = u,\end{array}$ (e) $\begin{array}{l}\dot u = -u - 2v,\\ \dot v = -2u - v.\end{array}$

10.2.6. (a) Show that the matrix $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ has $\lambda = \pm\,\mathrm{i}$ as incomplete complex conjugate eigenvalues. (b) Find the general real solution to $\dot{\mathbf{u}} = A\mathbf{u}$.
(c) Explain the behavior of a typical solution. Why is the zero solution not stable?

10.2.7. Let A be a real 3×3 matrix, and assume that the linear system $\dot{\mathbf{u}} = A\mathbf{u}$ has a periodic solution of period P. Prove that every periodic solution of the system has period P. What other types of solutions can there be? Is the zero solution necessarily stable?

10.2.8. Are the conclusions of Exercise 10.2.7 valid when A is a 4×4 matrix?

10.2.9. Let A be a real 5×5 matrix, and assume that A has eigenvalues $\mathrm{i}, -\mathrm{i}, -2, -1$ (and no others). Is the zero solution to the linear system $\dot{\mathbf{u}} = A\mathbf{u}$ necessarily stable? Explain. Does your answer change if A is 6×6?

10.2.10. Prove that if A is strictly diagonally dominant and each diagonal entry is negative, then the zero equilibrium solution to the linear system of ordinary differential equations $\dot{\mathbf{u}} = A\mathbf{u}$ is asymptotically stable.

10.2.11. *True or false*: The system $\dot{\mathbf{u}} = -H_n\mathbf{u}$, where H_n is the $n \times n$ Hilbert matrix (1.72), is asymptotically stable.

10.2.12. *True or false*: If the zero solution of the linear system of differential equations $\dot{\mathbf{u}} = A\mathbf{u}$ is asymptotically stable, so is the zero solution of the linear iterative system $\mathbf{u}^{(k+1)} = A\mathbf{u}^{(k)}$ with the same coefficient matrix.

10.2.13. Let $\mathbf{u}(t)$ solve $\dot{\mathbf{u}} = A\mathbf{u}$. Let $\mathbf{v}(t) = \mathbf{u}(-t)$ be its time reversal.
(a) Write down the linear system $\dot{\mathbf{v}} = B\mathbf{v}$ satisfied by $\mathbf{v}(t)$. Then classify the following statements as *true* or *false*. As always, explain your answers. (b) If $\dot{\mathbf{u}} = A\mathbf{u}$ is asymptotically stable, then $\dot{\mathbf{v}} = B\mathbf{v}$ is unstable. (c) If $\dot{\mathbf{u}} = A\mathbf{u}$ is unstable, then $\dot{\mathbf{v}} = B\mathbf{v}$ is asymptotically stable. (d) If $\dot{\mathbf{u}} = A\mathbf{u}$ is stable, then $\dot{\mathbf{v}} = B\mathbf{v}$ is stable.

10.2.14. *True or false*: (a) If $\operatorname{tr} A > 0$, then the system $\dot{\mathbf{u}} = A\mathbf{u}$ is unstable.
(b) If $\det A > 0$, then the system $\dot{\mathbf{u}} = A\mathbf{u}$ is unstable.

10.2.15. *True or false*: If K is positive semi-definite, then the zero solution to $\dot{\mathbf{u}} = -K\mathbf{u}$ is stable.

10.2.16. *True or false*: If A is a symmetric matrix, then the system $\dot{\mathbf{u}} = -A^2\mathbf{u}$ has an asymptotically stable equilibrium solution.

10.2.17. Consider the differential equation $\dot{\mathbf{u}} = -K\mathbf{u}$, where K is positive semi-definite. (a) Find all equilibrium solutions. (*b*) Prove that all non-constant solutions decay exponentially fast to some equilibrium. What is the decay rate? (*c*) Is the origin stable, asymptotically stable, or unstable? (*d*) Prove that, as $t \to \infty$, the solution $\mathbf{u}(t)$ converges to the orthogonal projection of its initial vector $\mathbf{a} = \mathbf{u}(0)$ onto $\ker K$.

10.2.18. Suppose that $\mathbf{u}(t)$ satisfies the gradient flow system (10.22).

(a) Prove that $\dfrac{d}{dt}\, q(\mathbf{u}) = -\,\|\, K\mathbf{u}\,\|^2$.

(*b*) Explain why if $\mathbf{u}(t)$ is any nonconstant solution to the gradient flow, then $q(\mathbf{u}(t))$ is a strictly decreasing function of t, thus quantifying how fast a gradient flow decreases energy.

10.2.19. Let $H(u,v) = a u^2 + b\, uv + cv^2$ be a quadratic function. (*a*) Prove that the non-equilibrium trajectories of the associated Hamiltonian system and those of the gradient flow are mutually orthogonal, i.e., they always intersect at right angles. (*b*) Verify this result for the particular quadratic functions (*i*) $u^2 + 3v^2$, (*ii*) uv, by drawing representative trajectories of both systems on the same graph.

10.2.20. *True or false*: If the Hamiltonian system for $H(u,v)$ is stable, then the corresponding gradient flow $\dot{\mathbf{u}} = -\nabla H$ is stable.

10.2.21. *True or false*: A nonzero linear 2×2 gradient flow cannot be a Hamiltonian flow.

♡ 10.2.22. The law of *conservation of energy* states that the energy in a Hamiltonian system is constant on solutions. (*a*) Prove that if $\mathbf{u}(t)$ satisfies the Hamiltonian system (10.23), then $H(\mathbf{u}(t)) = c$ is a constant, and hence solutions $\mathbf{u}(t)$ move along the level sets of the Hamiltonian or energy function. Explain how the value of c is determined by the initial conditions. (*b*) Plot the level curves of the particular Hamiltonian function $H(u,v) = u^2 - 2uv + 2v^2$ and verify that they coincide with the solution trajectories.

10.2.23. *True or false*: A nonzero linear 2×2 gradient flow cannot be a Hamiltonian system.

10.2.24. (a) Explain how to solve the inhomogeneous system $\dfrac{d\mathbf{u}}{dt} = A\mathbf{u} + \mathbf{b}$ when $\mathbf{b}$ is a constant vector belonging to $\operatorname{img} A$. *Hint*: Look at $\mathbf{v}(t) = \mathbf{u}(t) - \mathbf{u}^\star$ where $\mathbf{u}^\star$ is an equilibrium solution. (*b*) Use your method to solve

$$(i)\ \ \frac{du}{dt} = u - 3v + 1,\quad \frac{dv}{dt} = -u - v, \qquad (ii)\ \ \frac{du}{dt} = 4v + 2,\quad \frac{dv}{dt} = -u - 3.$$

♢ 10.2.25. Prove Lemma 10.15.

♢ 10.2.26. Prove Proposition 10.18.

10.3 Two-Dimensional Systems

The two-dimensional case is particularly instructive, since it is relatively easy to analyze, but already manifests most of the key phenomena to be found in higher dimensions. Moreover, the solutions can be easily pictured and their behavior understood through their phase portraits. In this section, we will present a complete classification of the possible qualitative behaviors of real, planar linear dynamical systems.

Setting $\mathbf{u}(t) = (u(t), v(t))^T$, a first order planar homogeneous linear system has the explicit form

$$\frac{du}{dt} = a u + b v, \qquad \frac{dv}{dt} = c u + d v, \tag{10.26}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the (constant) coefficient matrix. As in Section 10.1, we will refer to the uv-plane as the *phase plane*. In particular, the phase plane equivalents (10.8) of second order scalar equations form a subclass thereof.

According to (8.21), the characteristic equation for the given 2×2 matrix is

$$\det(A - \lambda\,\mathrm{I}\,) = \lambda^2 - \tau\,\lambda + \delta = 0, \tag{10.27}$$

where

$$\tau = \operatorname{tr} A = a + d, \qquad \delta = \det A = a\,d - b\,c, \tag{10.28}$$

are, respectively, the trace and the determinant of A. The eigenvalues, and hence the nature of the solutions, are almost entirely determined by these two quantities. The sign of the *discriminant*

$$\Delta = \tau^2 - 4\,\delta = (\operatorname{tr} A)^2 - 4\,\det A = (a - d)^2 + 4\,b\,c \tag{10.29}$$

determines whether the eigenvalues

$$\lambda_\pm = \frac{\tau \pm \sqrt{\Delta}}{2} \tag{10.30}$$

are real or complex, and thereby plays a key role in the classification.

Let us summarize the different possibilities as distinguished by their qualitative behavior. Each category will be illustrated by a representative phase portrait, which displays several typical solution trajectories in the phase plane. A complete portrait gallery of planar systems can be found in Figure 10.3.

Distinct Real Eigenvalues

The coefficient matrix A has two distinct real eigenvalues $\lambda_1 < \lambda_2$ if and only if the discriminant is positive: $\Delta > 0$. In this case, the solutions take the exponential form

$$\mathbf{u}(t) = c_1\, e^{\lambda_1 t}\, \mathbf{v}_1 + c_2\, e^{\lambda_2 t}\, \mathbf{v}_2, \tag{10.31}$$

where $\mathbf{v}_1, \mathbf{v}_2$ are the eigenvectors and c_1, c_2 are arbitrary constants, to be determined by the initial conditions. Let $V_k = \{\, c\,\mathbf{v}_k \mid c \in \mathbb{R}\,\}$ for $k = 1, 2$, denote the two "eigenlines", i.e., the one-dimensional eigenspaces.

The asymptotic behavior of the solutions is governed by the eigenvalues. There are five qualitatively different cases, depending upon their signs. These are listed by their descriptive name, followed by the required conditions on the discriminant, trace, and determinant of the coefficient matrix that serve to prescribe the form of the eigenvalues.

I a. *Stable Node*: $\quad \Delta > 0, \qquad \operatorname{tr} A < 0, \qquad \det A > 0.$

If $\lambda_1 < \lambda_2 < 0$ are both negative, then $\mathbf{0}$ is an asymptotically *stable node*. The solutions all tend to $\mathbf{0}$ as $t \to \infty$. Since the first exponential $e^{\lambda_1 t}$ decreases much faster than the second $e^{\lambda_2 t}$, the first term in the solution (10.31) will soon become negligible, and hence $\mathbf{u}(t) \approx c_2\, e^{\lambda_2 t}\, \mathbf{v}_2$ when t is large, provided $c_2 \neq 0$. Such solutions will arrive at the origin along curves tangent to the eigenline V_2, including those with $c_1 = 0$, which move directly along the eigenline. On the other hand, the solutions with $c_2 = 0$ come in to the origin along the eigenline V_1, at a faster rate. Conversely, as $t \to -\infty$, all solutions become unbounded: $\|\,\mathbf{u}(t)\,\| \to \infty$. In this case, the first exponential grows faster than the second, and so $\mathbf{u}(t) \approx c_1\, e^{\lambda_1 t}\, \mathbf{v}_1$ for $t \ll 0$. In other words, as they escape to ∞, the solution trajectories become more and more parallel to the eigenline V_1 — except for those with $c_1 = 0$, which remain on the eigenline V_2.

I b. *Saddle Point*: $\Delta > 0, \qquad \det A < 0.$

If $\lambda_1 < 0 < \lambda_2$, then $\mathbf{0}$ is an unstable *saddle point*. Solutions (10.31) with $c_2 = 0$ start out on the eigenline V_1 and go in to $\mathbf{0}$ as $t \to \infty$, while solutions with $c_1 = 0$ start on V_2 and go to $\mathbf{0}$ as $t \to -\infty$. All other solutions become unbounded at both large positive and large negative times. As $t \to +\infty$, they asymptotically approach the *unstable eigenline* V_2, while as $t \to -\infty$, they approach the *stable eigenline* V_1.

I c. *Unstable Node*: $\Delta > 0, \qquad \operatorname{tr} A > 0, \qquad \det A > 0.$

If the eigenvalues $0 < \lambda_1 < \lambda_2$ are both positive, then $\mathbf{0}$ is an *unstable node*. The phase portrait is the same as that of a stable node, but the solution trajectories are traversed in the opposite direction. Time reversal $t \to -t$ will convert an unstable node into a stable node and vice versa. Thus, in the unstable case, the solutions all tend to the origin as $t \to -\infty$ and become unbounded as $t \to \infty$. Except for the eigensolutions, they asymptotically approach V_1 as $t \to -\infty$, and become parallel to V_2 as $t \to \infty$.

I d. *Stable Line*: $\Delta > 0, \qquad \operatorname{tr} A < 0, \qquad \det A = 0.$

If $\lambda_1 < \lambda_2 = 0$, then every point on the eigenline V_2 associated with the zero eigenvalue is a stable equilibrium point. The other solutions move along straight lines parallel to V_1, asymptotically approaching one of the equilibrium points on V_2 as $t \to \infty$. On the other hand, as $t \to -\infty$, all solutions except those sitting still on the eigenline move off to ∞.

I e. *Unstable Line*: $\Delta > 0, \qquad \operatorname{tr} A > 0, \qquad \det A = 0.$

This is merely the time reversal of a stable line. If $0 = \lambda_1 < \lambda_2$, then every point on the eigenline V_1 is an equilibrium. The other solutions moves off to ∞ along straight lines parallel to V_2 as $t \to \infty$, and tend to an equilibrium on V_1 as $t \to -\infty$.

Complex Conjugate Eigenvalues

The coefficient matrix A has two complex conjugate eigenvalues

$$\lambda_{\pm} = \mu \pm \mathrm{i}\,\nu, \qquad \text{where} \qquad \mu = \tfrac{1}{2}\,\tau = \tfrac{1}{2}\,\operatorname{tr} A, \qquad \nu = \sqrt{-\Delta},$$

if and only if its discriminant is negative: $\Delta < 0$. In this case, the real solutions can be written in the phase–amplitude form

$$\mathbf{u}(t) = r\,e^{\mu\,t}\,[\,\cos(\nu\,t - \sigma)\,\mathbf{w} - \sin(\nu\,t - \sigma)\,\mathbf{z}\,], \tag{10.32}$$

where $\mathbf{w} \pm \mathrm{i}\,\mathbf{z}$ are the complex eigenvectors. As noted in Exercise 8.3.12, the real vectors $\mathbf{w}, \mathbf{z}$ are always linearly independent. The amplitude r and phase shift σ are uniquely prescribed by the initial conditions. There are three subcases, depending upon the sign of the real part μ, or, equivalently, the sign of the trace of A.

II a. *Stable Focus*: $\Delta < 0, \qquad \operatorname{tr} A < 0.$

If $\mu < 0$, then $\mathbf{0}$ is an asymptotically *stable focus*. As $t \to \infty$, the solutions all spiral in to the origin at an exponential rate $e^{\mu t}$ with a common "frequency" ν — meaning it takes time $2\pi/\nu$ for the solution to spiral once around the origin†. On the other hand, as

† But keep in mind that these solutions are not periodic. Thus, $2\pi/\nu$ is the time interval between successive intersections of the solution and a fixed ray emanating from the origin, e.g., the positive x-axis.

$t \to -\infty$, the solutions spiral off to ∞ at the same exponential rate whilst maintaining their overall frequency.

IIb. *Center*: $\Delta < 0, \qquad \operatorname{tr} A = 0.$

If $\mu = 0$, meaning that the eigenvalues $\lambda_\pm = \pm\,\mathrm{i}\,\nu$ are purely imaginary, then $\mathbf{0}$ is a *center*. The solutions all move periodically around elliptical orbits, with common frequency ν and hence period $2\pi/\nu$. In particular, solutions that start out near $\mathbf{0}$ stay nearby, and hence a center is a stable, but not asymptotically stable, equilibrium.

IIc. *Unstable Focus*: $\Delta < 0, \qquad \operatorname{tr} A > 0.$

If $\mu > 0$, then $\mathbf{0}$ is an *unstable focus*. The phase portrait is the time reversal of a stable focus, with solutions having an unbounded spiral motion as $t \to \infty$, and spiraling in to the origin as $t \to -\infty$, again at an exponential rate $e^{\mu t}$ with a common "frequency" ν.

Incomplete Double Real Eigenvalue

The coefficient matrix has a double real eigenvalue $\lambda = \frac{1}{2}\tau = \frac{1}{2}\operatorname{tr} A$ if and only if the discriminant vanishes: $\Delta = 0$. The formula for the solutions depends on whether the eigenvalue λ is complete. If λ is an incomplete eigenvalue, admitting only one independent eigenvector $\mathbf{v}$, then the solutions are no longer given by simple exponentials. The general solution formula is

$$\mathbf{u}(t) = (c_1 + c_2 t) e^{\lambda t}\,\mathbf{v} + c_2\, e^{\lambda t}\,\mathbf{w}, \tag{10.33}$$

where $(A - \lambda\,\mathrm{I}\,)\mathbf{w} = \mathbf{v}$, and so $\mathbf{v}, \mathbf{w}$ form a Jordan chain for the coefficient matrix. We let $V = \{c\,\mathbf{v}\}$ denote the eigenline associated with the genuine eigenvector $\mathbf{v}$.

IIIa. *Stable Improper Node*: $\Delta = 0, \qquad \operatorname{tr} A < 0, \qquad A \neq \lambda\,\mathrm{I}\,.$

If $\lambda < 0$ then $\mathbf{0}$ is an asymptotically *stable improper node*. Since $t\,e^{\lambda t}$ is larger than $e^{\lambda t}$ for $t > 1$, when $c_2 \neq 0$, the solutions $\mathbf{u}(t) \approx c_2\, t\, e^{\lambda t}$ tend to $\mathbf{0}$ as $t \to \infty$ along a curve that is tangent to the eigenline V, while the eigensolutions with $c_2 = 0$ move in to the origin along the eigenline. Similarly, as $t \to -\infty$, the solutions go off to ∞ in the opposite direction from their approach, becoming more and more parallel to the same eigenline.

IIIb. *Linear Motion*: $\Delta = 0, \qquad \operatorname{tr} A = 0, \qquad A \neq \lambda\,\mathrm{I}\,.$

If $\lambda = 0$, then every point on the eigenline V is an unstable equilibrium point. Every other solution is a linear polynomial in t, and so moves along a straight line parallel to V, going off to ∞ in either direction.

IIIc. *Unstable Improper Node*: $\Delta = 0, \qquad \operatorname{tr} A > 0, \qquad A \neq \lambda\,\mathrm{I}\,.$

If $\lambda > 0$, then $\mathbf{0}$ is an *unstable improper node*. The phase portrait is the time reversal of the stable improper node. Solutions go off to ∞ as t increases, becoming progressively more parallel to the eigenline, and tend to the origin tangent to the eigenline as $t \to -\infty$.

Complete Double Real Eigenvalue

In this case, *every* vector in $\mathbb{R}^2$ is an eigenvector, and so the real solutions take the form $\mathbf{u}(t) = e^{\lambda t}\,\mathbf{v}$, where $\mathbf{v}$ is an *arbitrary* constant vector. In fact, this case occurs if and only if $A = \lambda\,\mathrm{I}$ is a scalar multiple of the identity matrix.

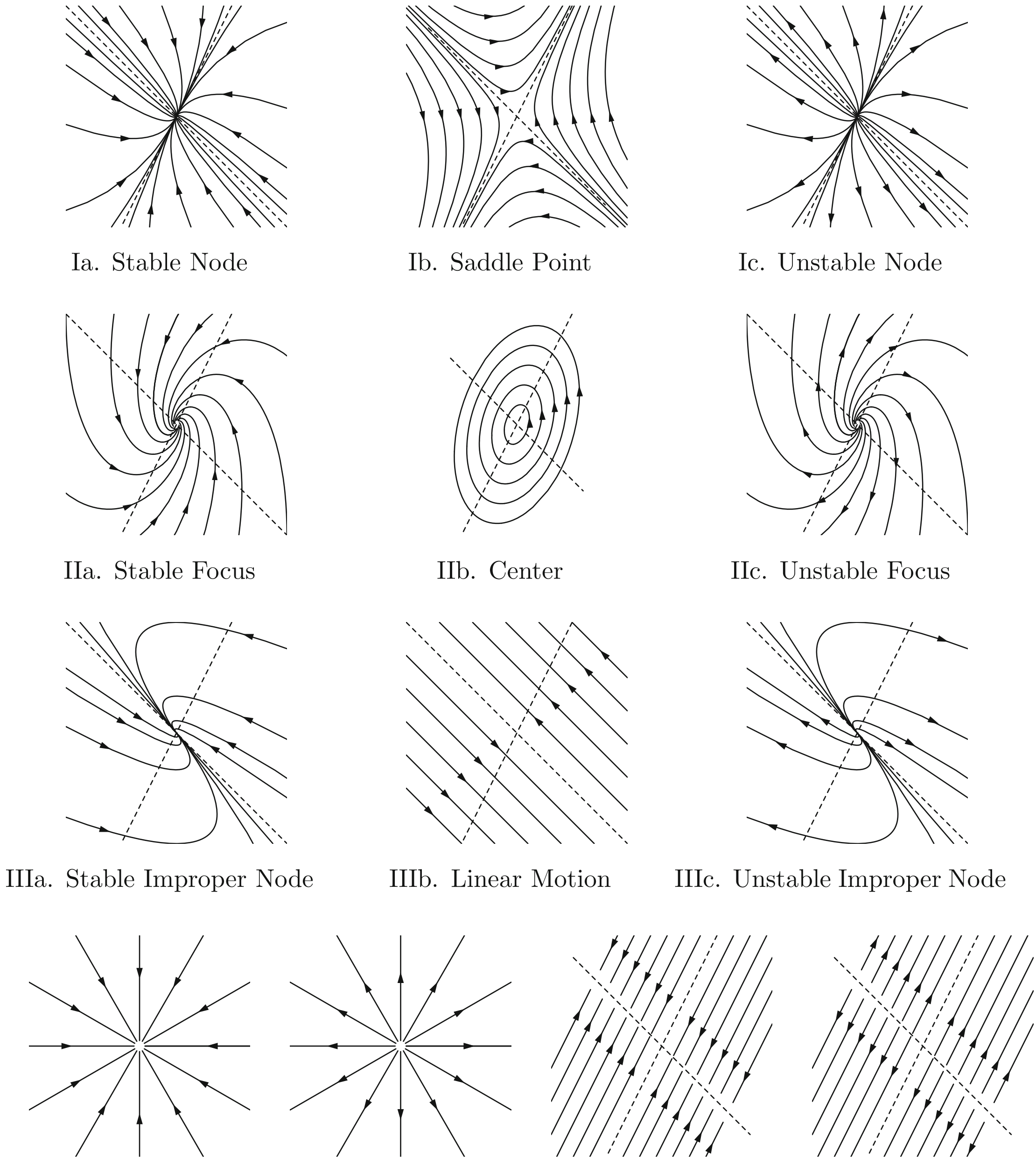


Figure 10.3. Phase Portraits.

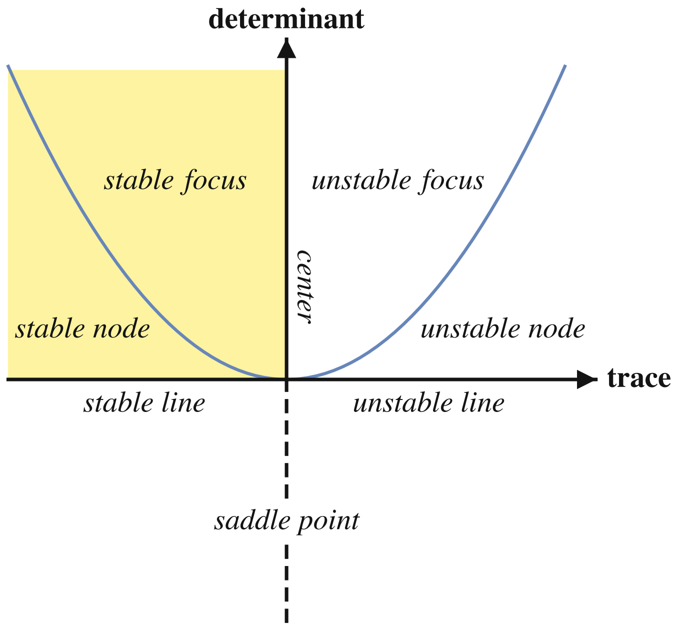


Figure 10.4. Stability Regions for Two–Dimensional Linear Systems.

IV a. *Stable Star*: $A = \lambda\,\mathrm{I}$, $\lambda < 0$.

If $\lambda < 0$, then $\mathbf{0}$ is an asymptotically *stable star*. The solution trajectories are the rays coming in to the origin, and the solutions go to $\mathbf{0}$ at a common exponential rate $e^{\lambda t}$ as $t \to \infty$.

IV b. *Trivial*: $A = \mathrm{O}$.

If $\lambda = 0$, then the only possibility is $A = \mathrm{O}$. Now every solution is constant and every point is a (stable) equilibrium point. Nothing happens! This is the only case not pictured in Figure 10.3.

IV c. *Unstable Star*: $A = \lambda\,\mathrm{I}$, $\lambda > 0$.

If $\lambda > 0$, then $\mathbf{0}$ is an *unstable star*. The phase portrait is the time reversal of the stable star, and so the solutions move out along rays as $t \to \infty$ at an exponential rate $e^{\lambda t}$, while tending to $\mathbf{0}$ as $t \to -\infty$.

Figure 10.4 summarizes the different possibilities, as prescribed by the trace and determinant of the coefficient matrix. The horizontal axis indicates the value of $\tau = \operatorname{tr} A$, while the vertical axis refers to $\delta = \det A$. Points on the parabola $\tau^2 = 4\,\delta$ represent the cases with vanishing discriminant $\Delta = 0$, and correspond to either stars or improper nodes — except for the origin, which is either linear motion or trivial. All the asymptotically stable cases lie in the shaded upper left quadrant where $\operatorname{tr} A < 0$ and $\det A > 0$. The borderline points are either stable centers, when $\operatorname{tr} A = 0$, $\det A > 0$, or stable lines, when $\operatorname{tr} A < 0$, $\det A = 0$, or the origin, which may or may not be stable depending upon whether A is the zero matrix or not. All other values for the trace and determinant result in unstable equilibria. Summarizing:

Proposition 10.22. Let τ, δ denote, respectively, the trace and determinant of the coefficient matrix A of a homogeneous, linear, autonomous planar system of first order ordinary differential equations. Then the system is

(i) *asymptotically stable* if and only if $\delta > 0$ and $\tau < 0$;

(ii) *stable* if and only if $\delta \geq 0$, $\tau \leq 0$, and, if $\delta = \tau = 0$, also $A = \mathrm{O}$.

Remark. Time reversal $t \to -t$ changes the sign of the coefficient matrix $A \to -A$, and hence the sign of its trace, $\tau \to -\tau$, while the determinant $\delta = \det A = \det(-A)$ is unchanged. Thus, the effect is to reflect Figure 10.4 through the vertical axis, interchanging the stable nodes and spirals with their unstable counterparts, while taking saddle points to saddle points.

In physical applications, the parameters occurring in the dynamical system are usually not known exactly, and so the real dynamics may, in fact, be governed by a slight perturbation of the mathematical model. Thus, it is important to know which systems are *structurally stable*, meaning that their basic qualitative features are preserved under sufficiently small changes in the coefficients. Now, a small perturbation will alter the coefficient matrix slightly, and hence shift its trace and determinant by a comparably small amount. The net effect is to slightly perturb its eigenvalues. Therefore, the question of structural stability reduces to whether the eigenvalues have moved sufficiently far to send the system into a different stability regime. Asymptotically stable systems remain asymptotically stable since a sufficiently small perturbation will not alter the signs of the real parts of its eigenvalues. For a similar reason, unstable systems remain unstable under small perturbations. On the other hand, a borderline stable system — either a center or the trivial system — might become either asymptotically stable or unstable, even under a minuscule perturbation. Such results continue to hold, at least locally, even under suitably small nonlinear perturbations, and thereby lie at the foundations of nonlinear dynamics.

Structural stability requires a bit more, since the overall phase portrait should not significantly change. A system in any of the open regions in the Stability Figure 10.4, i.e., a stable or unstable focus, a stable or unstable node, or a saddle point, is structurally stable, whereas a system that lies on the parabola $\tau^2 = 4\,\delta$, or the horizontal axis, or the positive vertical axis, e.g., an improper node, a stable line, etc., is not, since a small perturbation can easily kick it into either of the adjoining regions. Thus, structural stability requires that the eigenvalues be distinct, $\lambda_i \neq \lambda_j$, and have non-zero real part: $\operatorname{Re} \lambda \neq 0$. This final result remains valid for linear systems in higher dimensions, [**36**, **41**]. See also [**69**, **90**] and the brief remarks on page 525 concerning the perturbation theory of eigenvalues, in which Wilkinson's spectral condition number quantifies to what extent the eigenvalues are affected by a perturbation of the coefficient matrix.

Exercises

10.3.1. For each the following: (a) Write the system as $\dot{\mathbf{u}} = A\mathbf{u}$. (b) Find the eigenvalues and eigenvectors of A. (c) Find the general real solution of the system. (d) Draw the phase portrait, indicating its type and stability properties: (i) $\dot{u}_1 = -u_2$, $\dot{u}_2 = 9u_1$, (ii) $\dot{u}_1 = 2u_1 - 3u_2$, $\dot{u}_2 = u_1 - u_2$, (iii) $\dot{u}_1 = 3u_1 - 2u_2$, $\dot{u}_2 = 2u_1 - 2u_2$.

10.3.2. For each of the following systems

$$(i)\ \dot{\mathbf{u}} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{u}, \qquad (ii)\ \dot{\mathbf{u}} = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}\mathbf{u}, \qquad (iii)\ \dot{\mathbf{u}} = \begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix}\mathbf{u}:$$

(a) Find the general real solution. (b) Using the solution formulas obtained in part (a), plot several trajectories of each system. On your graphs, identify the eigenlines (if relevant), and the direction of increasing t on the trajectories. (c) Write down the type and stability properties of the system.

10.3.3. Classify the following systems, and sketch their phase portraits.
(a) $\dfrac{du}{dt} = -u + 4v, \quad \dfrac{dv}{dt} = u - 2v.$ (b) $\dfrac{du}{dt} = -2u + v, \quad \dfrac{dv}{dt} = u - 4v.$ (c) $\dfrac{du}{dt} = 5u + 4v, \quad \dfrac{dv}{dt} = u + 2v.$ (d) $\dfrac{du}{dt} = -3u - 2v, \quad \dfrac{dv}{dt} = 3u + 2v.$

♢ 10.3.4. Justify the solution formulas (10.32) and (10.33).

10.3.5. Sketch the phase portrait for the following systems: (a) $\dot u_1 = u_1 - 3u_2, \quad \dot u_2 = -3u_1 + u_2.$
(b) $\dot u_1 = u_1 - 4u_2, \quad \dot u_2 = u_1 - u_2.$ (c) $\dot u_1 = u_1 + u_2, \quad \dot u_2 = 4u_1 - 2u_2.$ (d) $\dot u_1 = u_1 + u_2, \quad \dot u_2 = u_2.$ (e) $\dot u_1 = \frac{3}{2}u_1 + \frac{5}{2}u_2, \quad \dot u_2 = -\frac{5}{2}u_1 + \frac{3}{2}u_2.$

10.3.6. Which of the 14 possible two-dimensional phase portraits can occur for the phase plane equivalent (10.8) of a second order scalar ordinary differential equation?

10.3.7. Which of the 14 possible two-dimensional phase portraits can occur
(a) for a linear gradient flow (10.19)? (b) for a linear Hamiltonian system (10.25)?

10.3.8. (a) Solve the initial value problem $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix}\mathbf{u}, \quad \mathbf{u}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$
(b) Sketch a picture of your solution curve $\mathbf{u}(t)$, indicating the direction of motion.
(c) Is the origin (i) stable? (ii) asymptotically stable? (iii) unstable? (iv) none of these? Justify your answer.

10.4 Matrix Exponentials

So far, our focus has been on vector-valued solutions $\mathbf{u}(t)$ to homogeneous linear systems of ordinary differential equations

$$\frac{d\mathbf{u}}{dt} = A\,\mathbf{u}. \tag{10.34}$$

An evident, and, in fact, useful, generalization is to look for *matrix solutions*. Specifically, we seek a matrix-valued function $U(t)$ that satisfies the corresponding *matrix differential equation*

$$\frac{dU}{dt} = A\,U(t). \tag{10.35}$$

As with vectors, we compute the derivative of $U(t)$ by differentiating its individual entries. If A is an $n \times n$ matrix, compatibility of matrix multiplication requires that $U(t)$ be of size $n \times k$ for some k. Since matrix multiplication acts column-wise, the individual columns of the matrix solution $U(t) = (\,\mathbf{u}_1(t)\ \ldots\ \mathbf{u}_k(t)\,)$ must solve the original vector system (10.34). Thus, a matrix solution is merely a convenient way of collecting together several different vector solutions. The most important case is that in which $U(t)$ is a square matrix, of size $n \times n$, and so consists of n vector solutions to the system.

Example 10.23. According to Example 10.7, the vector-valued functions

$$\mathbf{u}_1(t) = \begin{pmatrix} e^{-4t} \\ -2e^{-4t} \end{pmatrix}, \qquad \mathbf{u}_2(t) = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix},$$

are both solutions to the linear system

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix} \mathbf{u}.$$

They can be combined to form the matrix solution

$$U(t) = \begin{pmatrix} e^{-4t} & e^{-t} \\ -2e^{-4t} & e^{-t} \end{pmatrix} \qquad \text{satisfying} \qquad \frac{dU}{dt} = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix} U.$$

Indeed, by direct calculation

$$\frac{dU}{dt} = \begin{pmatrix} -4e^{-4t} & -e^{-t} \\ 8e^{-4t} & -e^{-t} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} e^{-4t} & e^{-t} \\ -2e^{-4t} & e^{-t} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix} U.$$

The existence and uniqueness theorems are readily adapted to matrix differential equations, and imply that there is a unique matrix solution to the system (10.35) that has initial conditions

$$U(t_0) = B, \tag{10.36}$$

where B is an $n \times k$ matrix. Note that the j^{th} column $\mathbf{u}_j(t)$ of the matrix solution $U(t)$ satisfies the initial value problem

$$\frac{d\mathbf{u}_j}{dt} = A\,\mathbf{u}_j, \qquad \mathbf{u}_j(t_0) = \mathbf{b}_j,$$

where $\mathbf{b}_j$ denotes the j^{th} column of B.

In the scalar case, the solution to the particular initial value problem

$$\frac{du}{dt} = a\,u, \qquad u(0) = 1,$$

is the ordinary exponential function $u(t) = e^{t\,a}$. Knowing this, we can write down the solution for a more general initial condition

$$u(t_0) = b \qquad \text{as} \qquad u(t) = b\,e^{(t-t_0)\,a}.$$

Let us formulate an analogous initial value problem for a linear system. Recall that, for matrices, the role of the multiplicative unit 1 is played by the identity matrix I. This inspires the following definition.

Definition 10.24. Let A be a square $n \times n$ matrix. The *matrix exponential*

$$U(t) = e^{t\,A} = \exp(t\,A) \tag{10.37}$$

is the unique $n \times n$ matrix solution to the initial value problem

$$\frac{dU}{dt} = A\,U, \qquad U(0) = \mathrm{I}. \tag{10.38}$$

In particular, one computes e^{A} by setting $t = 1$ in the matrix exponential $e^{t\,A}$. The matrix exponential turns out to enjoy almost all the properties you might expect from its scalar counterpart. First, it is defined for all $t \in \mathbb{R}$, and all $n \times n$ matrices, both real and complex. We can rewrite the defining properties (10.38) in the more suggestive form

$$\frac{d}{dt}\,e^{t\,A} = A\,e^{t\,A}, \qquad e^{0\,A} = \mathrm{I}. \tag{10.39}$$

As in the scalar case, once we know the matrix exponential, we are in a position to solve the general initial value problem.

Lemma 10.25. Let A be an $n \times n$ matrix. For any $n \times k$ matrix B, the solution to the initial value problem

$$\frac{dU}{dt} = A\,U, \qquad U(t_0) = B, \qquad \text{is} \qquad U(t) = e^{(t-t_0)\,A}\,B. \tag{10.40}$$

Proof: Since B is a constant matrix,

$$\frac{dU}{dt} = \frac{d}{dt}\left[\, e^{(t-t_0)\,A}\,B \,\right] = A\,e^{(t-t_0)\,A}\,B = A\,U,$$

where we applied the chain rule for differentiation and the first property (10.39). Thus, $U(t)$ is indeed a matrix solution to the system. Moreover, by the second property in (10.39),

$$U(0) = e^{0\,A}\,B = \mathrm{I}\,B = B$$

has the correct initial conditions. *Q.E.D.*

Remark. The computation used in the proof is a special instance of the general *Leibniz rule*

$$\frac{d}{dt}\left[\, M(t)\,N(t) \,\right] = \frac{dM(t)}{dt}\,N(t) + M(t)\,\frac{dN(t)}{dt} \tag{10.41}$$

for the derivative of the product of (compatible) matrix-valued functions $M(t)$ and $N(t)$. The reader is asked to prove this formula in Exercise 10.4.21.

In particular, the solution to the original vector initial value problem

$$\frac{d\mathbf{u}}{dt} = A\,\mathbf{u}, \qquad\qquad \mathbf{u}(t_0) = \mathbf{b},$$

can be written in terms of the matrix exponential:

$$\mathbf{u}(t) = e^{(t-t_0)\,A}\,\mathbf{b}. \tag{10.42}$$

Thus, the matrix exponential provides us with an alternative formula for the solution of autonomous homogeneous first order linear systems, providing us with valuable new insight.

The next step is to find an algorithm for computing the matrix exponential. The solution formula (10.40) gives a hint. Suppose $U(t)$ is any $n \times n$ matrix solution. Then, by uniqueness, $U(t) = e^{t\,A}\,U(0)$, and hence, provided that $U(0)$ is a nonsingular matrix,

$$e^{t\,A} = U(t)\,U(0)^{-1}, \tag{10.43}$$

since $e^{0\,A} = U(0)\,U(0)^{-1} = \mathrm{I}$, as required. Thus, to construct the exponential of an $n \times n$ matrix A, you first need to find a basis of n linearly independent solutions $\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)$ to the linear system $\dot{\mathbf{u}} = A\,\mathbf{u}$ using the eigenvalues and eigenvectors, or, in the incomplete case, the Jordan chains. The resulting $n \times n$ matrix solution $U(t) = \begin{pmatrix} \mathbf{u}_1(t) & \ldots & \mathbf{u}_n(t) \end{pmatrix}$ is then used to produce $e^{t\,A}$ via formula (10.43).

Example 10.26. For the matrix $A = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}$ considered in Example 10.23, we already constructed the nonsingular matrix solution $U(t) = \begin{pmatrix} e^{-4t} & e^{-t} \\ -2\,e^{-4t} & e^{-t} \end{pmatrix}$. Therefore, by (10.43), its matrix exponential is

$$\begin{aligned} e^{tA} &= U(t)\,U(0)^{-1} \\ &= \begin{pmatrix} e^{-4t} & e^{-t} \\ -2e^{-4t} & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac13 e^{-4t} + \frac23 e^{-t} & -\frac13 e^{-4t} + \frac13 e^{-t} \\ -\frac23 e^{-4t} + \frac23 e^{-t} & \frac23 e^{-4t} + \frac13 e^{-t} \end{pmatrix}. \end{aligned}$$

In particular, we obtain $e^A = \exp A$ by setting $t = 1$ in this formula:

$$\exp \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} \frac13 e^{-4} + \frac23 e^{-1} & -\frac13 e^{-4} + \frac13 e^{-1} \\ -\frac23 e^{-4} + \frac23 e^{-1} & \frac23 e^{-4} + \frac13 e^{-1} \end{pmatrix}.$$

Observe that the matrix exponential is *not* obtained by exponentiating the individual matrix entries.

To solve the initial value problem

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix} \mathbf{u}, \qquad \mathbf{u}(0) = \mathbf{b} = \begin{pmatrix} 3 \\ 0 \end{pmatrix},$$

we appeal to formula (10.40), whence

$$\mathbf{u}(t) = e^{tA}\,\mathbf{b} = \begin{pmatrix} \frac13 e^{-4t} + \frac23 e^{-t} & -\frac13 e^{-4t} + \frac13 e^{-t} \\ -\frac23 e^{-4t} + \frac23 e^{-t} & \frac23 e^{-4t} + \frac13 e^{-t} \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-4t} + 2e^{-t} \\ -2e^{-4t} + 2e^{-t} \end{pmatrix}.$$

This reproduces our earlier solution (10.15).

Example 10.27. Suppose $A = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$. Its characteristic equation

$$\det(A - \lambda\,\mathrm{I}) = \lambda^2 + 2\lambda + 5 = 0 \qquad \text{has roots} \qquad \lambda = -1 \pm 2\,\mathrm{i},$$

which are thus the eigenvalues. The corresponding eigenvectors are $\mathbf{v} = \begin{pmatrix} \pm\mathrm{i} \\ 1 \end{pmatrix}$, leading to the complex conjugate solutions

$$\mathbf{u}_1(t) = \begin{pmatrix} \mathrm{i}\,e^{(-1+2\mathrm{i})t} \\ e^{(-1+2\mathrm{i})t} \end{pmatrix}, \qquad \mathbf{u}_2(t) = \begin{pmatrix} -\mathrm{i}\,e^{(-1-2\mathrm{i})t} \\ e^{(-1-2\mathrm{i})t} \end{pmatrix}.$$

We assemble them to form the (complex) matrix solution

$$U(t) = \begin{pmatrix} \mathrm{i}\,e^{(-1+2\mathrm{i})t} & -\mathrm{i}\,e^{(-1-2\mathrm{i})t} \\ e^{(-1+2\mathrm{i})t} & e^{(-1-2\mathrm{i})t} \end{pmatrix}.$$

The corresponding matrix exponential is, therefore,

$$\begin{aligned} e^{tA} &= U(t)\,U(0)^{-1} = \begin{pmatrix} \mathrm{i}\,e^{(-1+2\mathrm{i})t} & -\mathrm{i}\,e^{(-1-2\mathrm{i})t} \\ e^{(-1+2\mathrm{i})t} & e^{(-1-2\mathrm{i})t} \end{pmatrix} \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \dfrac{e^{(-1+2\mathrm{i})t} + e^{(-1-2\mathrm{i})t}}{2} & \dfrac{-e^{(-1+2\mathrm{i})t} + e^{(-1-2\mathrm{i})t}}{2\,\mathrm{i}} \\ \dfrac{e^{(-1+2\mathrm{i})t} - e^{(-1-2\mathrm{i})t}}{2\,\mathrm{i}} & \dfrac{e^{(-1+2\mathrm{i})t} + e^{(-1-2\mathrm{i})t}}{2} \end{pmatrix} = \begin{pmatrix} e^{-t}\cos 2t & -e^{-t}\sin 2t \\ e^{-t}\sin 2t & e^{-t}\cos 2t \end{pmatrix}. \end{aligned}$$

Note that the final expression for the matrix exponential is real, as it must be, since A is a real matrix. (See Exercise 10.4.19.) Also note that it wasn't necessary to find the real solutions to construct the matrix exponential — although this would also have worked and yielded the same result. Indeed, the two columns of e^{tA} form a basis for the space of (real) solutions to the linear system $\dot{\mathbf{u}} = A\mathbf{u}$.

Let us finish by listing some further important properties of the matrix exponential, all of which are direct analogues of the usual scalar exponential function. Proofs are relegated to the exercises. First, the *multiplicative property* says that

$$e^{(s+t)A} = e^{sA}\, e^{tA}, \qquad \text{for all} \qquad s,t \in \mathbb{R}. \tag{10.44}$$

In particular, if we set $s = -t$, the left hand side of (10.44) reduces to the identity matrix, in accordance with the second identity in (10.39), and hence

$$e^{-tA} e^{tA} = \mathrm{I}, \qquad \text{and hence} \qquad e^{-tA} = \left(e^{tA} \right)^{-1}. \tag{10.45}$$

As a consequence, for any A and any $t \in \mathbb{R}$, the exponential e^{tA} is a nonsingular matrix.

Warning. In general,

$$e^{t(A+B)} \neq e^{tA}\, e^{tB}. \tag{10.46}$$

Indeed, according to Proposition 10.30, the left- and right-hand sides of (10.46) are equal for all t if and only if $AB = BA$ — that is, A and B are commuting matrices.

While the matrix exponential can be painful to compute, there is a simple formula for its determinant in terms of the trace of the generating matrix.

Lemma 10.28. Let A be a square matrix. Then $\det e^{tA} = e^{t\, \mathrm{tr}\, A}$.

Proof: According to Exercise 10.4.26, if A has eigenvalues $\lambda_1, \ldots, \lambda_n$, then e^{tA} has eigenvalues $e^{t\lambda_1}, \ldots, e^{t\lambda_n}$. Moreover, using (8.26), its determinant, $\det e^{tA}$, is the product of its eigenvalues, and so

$$\det e^{tA} = e^{t\lambda_1}\, e^{t\lambda_2} \;\cdots\; e^{t\lambda_n} = e^{t(\lambda_1+\lambda_2+\,\cdots\,+\lambda_n)} = e^{t\,\mathrm{tr}\,A},$$

where, by (8.25), we identify the sum of the eigenvalues as the trace of A. *Q.E.D.*

For instance, the matrix $A = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}$ considered above in Example 10.26 has $\mathrm{tr}\, A = (-2) + (-3) = -5$, and hence $\det e^{tA} = e^{-5t}$, as you can easily check.

Finally, we note that the standard exponential series is also valid for matrices:

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \mathrm{I} + tA + \frac{t^2}{2} A^2 + \frac{t^3}{6} A^3 + \;\cdots\; . \tag{10.47}$$

To prove that the series converges, we use the matrix norm convergence criterion in Exercise 9.2.44(*c*). Indeed, the corresponding series of matrix norms is bounded by the scalar exponential series,

$$\| e^{tA} \| \leq \sum_{n=0}^{\infty} \left\| \frac{t^n}{n!} A^n \right\| = \sum_{n=0}^{\infty} \frac{|\,t\,|^n}{n!} \| A^n \| \;\leq\; \sum_{n=0}^{\infty} \frac{|\,t\,|^n}{n!} \| A \|^n = e^{|\,t\,|\,\|A\|},$$

which converges for all t, [**2**, **78**], thereby proving convergence. With this in hand, proving that the exponential series satisfies the defining initial value problem (10.39) is straightforward:

$$\frac{d}{dt} \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} A^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^{n+1} = A \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.$$

Moreover, at $t = 0$, the sum collapses to the identity matrix: $\mathrm{I} = e^{0\,A}$. Thus, formula (10.47) follows from the uniqueness of solutions to the matrix initial value problem.

Exercises

10.4.1. Find the exponentials $e^{t\,A}$ of the following 2×2 matrices:
(a) $\begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, (c) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, (e) $\begin{pmatrix} -1 & 2 \\ -5 & 5 \end{pmatrix}$, (f) $\begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$.

10.4.2. Determine the matrix exponential $e^{t\,A}$ for the following matrices:
(a) $\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$, (c) $\begin{pmatrix} -1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & -1 & -1 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

10.4.3. Verify the determinant formula of Lemma 10.28 for the matrices in Exercises 10.4.1 and 10.4.2.

10.4.4. Solve the indicated initial value problems by first exponentiating the coefficient matrix and then applying formula (10.42): (a) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$,
(b) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 3 & -6 \\ 4 & -7 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, (c) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} -9 & -6 & 6 \\ 8 & 5 & -6 \\ -2 & 1 & 3 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

10.4.5. Find e^{A} when $A =$
(a) $\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$, (c) $\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{pmatrix}$, (e) $\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$.

10.4.6. Let $A = \begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix}$. Show that $e^{A} = \mathrm{I}$.

10.4.7. What is $e^{t\,\mathrm{O}}$ where O is the $n\times n$ zero matrix?

10.4.8. Find all matrices A such that $e^{t\,A} = \mathrm{O}$.

♢ 10.4.9. Explain in detail why the columns of $e^{t\,A}$ form a basis for the solution space to the system $\dot{\mathbf{u}} = A\,\mathbf{u}$.

10.4.10. Let A be a 2×2 matrix such that $\operatorname{tr} A = 0$ and $\delta = \sqrt{\det A} > 0$.
(a) Prove that $e^{A} = (\cos\delta)\,\mathrm{I} + \dfrac{\sin\delta}{\delta}\,A$. *Hint*: Use Exercise 8.2.52.
(b) Establish a similar formula when $\det A < 0$. (c) What if $\det A = 0$?

10.4.11. Show that the origin is an asymptotically stable equilibrium solution to $\dot{\mathbf{u}} = A\,\mathbf{u}$ if and only if $\lim_{t\to\infty} e^{t\,A} = \mathrm{O}$.

10.4.12. Let A be a real square matrix and e^{A} its exponential. Under what conditions does the linear system $\dot{\mathbf{u}} = e^{A}\,\mathbf{u}$ have an asymptotically stable equilibrium solution?

10.4.13. *True or false*: (a) $e^{A^{-1}} = (e^{A})^{-1}$; (b) $e^{A+A^{-1}} = e^{A}\,e^{A^{-1}}$.

♢ 10.4.14. Prove formula (10.44). *Hint*: Fix s and prove that, as functions of t, both sides of the equation define matrix solutions with the same initial conditions. Then use uniqueness.

10.4.15. Prove that A commutes with its exponential: $A\,e^{t\,A} = e^{t\,A}A$.

♢ 10.4.16. (a) Prove that the exponential of the transpose of a matrix is the transpose of its exponential: $e^{t\,A^T} = (e^{t\,A})^T$. (b) What does this imply about the solutions to the linear systems $\dot{\mathbf{u}} = A\,\mathbf{u}$ and $\dot{\mathbf{v}} = A^T\mathbf{v}$?

♢ 10.4.17. Prove that if $A = S\,B\,S^{-1}$ are similar matrices, then so are $e^{t\,A} = S\,e^{t\,B}\,S^{-1}$.

10.4.18. Prove that $e^{t\,(A-\lambda\,\mathrm{I}\,)} = e^{-t\,\lambda}\,e^{t\,A}$ by showing that both sides are matrix solutions to the same initial value problem.

♢ 10.4.19. Let A be a real matrix. (*a*) Explain why e^A is a real matrix. (*b*) Prove that $\det e^A > 0$.

10.4.20. Show that $\operatorname{tr} A = 0$ if and only if $\det e^{t\,A} = 1$ for all t.

♢ 10.4.21. Justify the matrix Leibniz rule (10.41) using the formula for matrix multiplication.

10.4.22. Prove that if $U(t)$ is any matrix solution to $\dfrac{dU}{dt} = A\,U$, then so is $\widetilde{U}(t) = U(t)\,C$, where C is any constant matrix (of compatible size).

♢ 10.4.23. Prove that if $A = \begin{pmatrix} B & \mathrm{O} \\ \mathrm{O} & C \end{pmatrix}$ is a block diagonal matrix, then so is $e^{t\,A} = \begin{pmatrix} e^{t\,B} & \mathrm{O} \\ \mathrm{O} & e^{t\,C} \end{pmatrix}$.

♢ 10.4.24. (*a*) Prove that if $J_{0,n}$ is an $n \times n$ Jordan block matrix with 0 diagonal entries, cf. (8.49), then

$$e^{t\,J_{0,n}} = \begin{pmatrix} 1 & t & \dfrac{t^2}{2} & \dfrac{t^3}{6} & \cdots & \dfrac{t^n}{n!} \\ 0 & 1 & t & \dfrac{t^2}{2} & \cdots & \dfrac{t^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & t & \cdots & \dfrac{t^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & t \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

(*b*) Determine the exponential of a general Jordan block matrix $J_{\lambda,n}$. *Hint*: Use Exercise 10.4.18. (*c*) Explain how you can use the Jordan canonical form to compute the exponential of a matrix. *Hint*: Use Exercise 10.4.23.

♢ 10.4.25. Diagonalization provides an alternative method for computing the exponential of a complete matrix. (*a*) First show that if $D = \operatorname{diag}(d_1, \ldots, d_n)$ is a diagonal matrix, so is $e^{t\,D} = \operatorname{diag}(e^{t\,d_1}, \ldots, e^{t\,d_n})$. (*b*) Second, using Exercise 10.4.17, prove that if $A = S\,D\,S^{-1}$ is diagonalizable, so is $e^{t\,A} = S\,e^{t\,D}\,S^{-1}$. (*c*) When possible, use diagonalization to compute the exponentials of the matrices in Exercises 10.4.1–2.

♢ 10.4.26. (*a*) Prove that if λ is an eigenvalue of A, then $e^{t\,\lambda}$ is an eigenvalue of $e^{t\,A}$. What is the eigenvector? (*b*) Show that the eigenvalues have the same multiplicities.
Hint: Combine the Jordan canonical form (8.51) with Exercises 10.4.24 and 10.4.25.

♢ 10.4.27. Let A be a symmetric matrix with Spectral Decomposition

$$A = \lambda_1\,P_1 + \lambda_2\,P_2 + \cdots + \lambda_k\,P_k,$$

as in (8.37). Prove that

$$e^{t\,A} = e^{t\,\lambda_1}\,P_1 + e^{t\,\lambda_2}\,P_2 + \cdots + e^{t\,\lambda_k}\,P_k.$$

♢ 10.4.28. (*a*) Show that $U(t)$ satisfies the matrix differential equation $\dot{U} = U\,B$ if and only if $U(t) = C\,e^{t\,B}$, where $C = U(0)$. (*b*) If $U(0)$ is nonsingular, then $U(t)$ also satisfies a matrix differential equation of the form $\dot{U} = A\,U$. Is $A = B$? *Hint*: Use Exercise 10.4.17.

10.4.29. *True or false*: The solution to the non-autonomous initial value problem

$$\dot{\mathbf{u}} = A(t)\,\mathbf{u}, \quad \mathbf{u}(0) = \mathbf{b}, \qquad \text{is} \qquad \mathbf{u}(t) = \exp\left(\int_0^t A(s)\,ds\right)\mathbf{b}.$$

♡ 10.4.30. (*a*) Suppose $\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)$ are vector-valued functions whose values at each point t are linearly independent vectors in $\mathbb{R}^n$. Show that they form a basis for the solution space of a homogeneous constant coefficient linear system $\dot{\mathbf{u}} = A\mathbf{u}$ if and only if each $d\mathbf{u}_j/dt$ is a linear combination of $\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)$. *Hint*: Use Exercise 10.4.28. (*b*) Show that a function $\mathbf{u}(t)$ belongs to the solution space of a homogeneous constant coefficient linear system $\dot{\mathbf{u}} = A\mathbf{u}$ if and only if $\dfrac{d^n\mathbf{u}}{dt^n}$ is a linear combination of $\mathbf{u}, \dfrac{d\mathbf{u}}{dt}, \ldots, \dfrac{d^{n-1}\mathbf{u}}{dt^{n-1}}$. *Hint*: Use Exercise 10.1.7.

♡ 10.4.31. By a (natural) *logarithm* of a matrix B we mean a matrix A such that $e^A = B$.
(*a*) Explain why only nonsingular matrices can have a logarithm.
(*b*) Comparing Exercises 10.4.6–7, explain why the matrix logarithm is not unique.
(*c*) Find all real logarithms of the 2×2 identity matrix $\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *Hint*: Use Exercise 10.4.26.

Applications in Geometry

Matrix exponentials are an effective tool for understanding the linear transformations that appear in geometry and group theory, [**93**], quantum mechanics, [**54**], computer graphics and animation, [**5**, **12**, **72**], computer vision, [**73**], and the symmetry analysis of differential equations, [**13**, **60**]. We will only be able to scratch the surface of this important and active area of contemporary mathematical research.

Let A be an $n \times n$ matrix. For each $t \in \mathbb{R}$, the corresponding exponential e^{tA} is itself an $n \times n$ matrix and thus defines a linear transformation on the vector space $\mathbb{R}^n$:

$$L_t[\mathbf{x}] = e^{tA}\mathbf{x} \qquad \text{for} \qquad \mathbf{x} \in \mathbb{R}^n.$$

In this manner, each square matrix A generates a family of invertible linear transformations, parameterized by $t \in \mathbb{R}$. The resulting linear transformations are not arbitrary, but are subject the following three rules:

$$L_t \circ L_s = L_{t+s} = L_s \circ L_t, \qquad L_0 = \mathrm{I}, \qquad L_{-t} = L_t^{-1}. \tag{10.48}$$

These are merely restatements of three of the basic matrix exponential properties listed in (10.39, 44, 45). In particular, every transformation in the family commutes with every other one.

In geometry, the family of transformations $L_t = e^{tA}$ is said to form a *one-parameter group*[†], [**60**], with t the parameter, and the matrix A is referred to as its *infinitesimal generator*. Indeed, by the series formula (10.39) for the matrix exponential,

$$L_t[\mathbf{x}] = e^{tA}\mathbf{x} = \left(\mathrm{I} + tA + \tfrac{1}{2}t^2A^2 + \cdots\right)\mathbf{x} = \mathbf{x} + tA\mathbf{x} + \tfrac{1}{2}t^2A^2\mathbf{x} + \cdots. \tag{10.49}$$

When t is small, we can truncate the exponential series and approximate the transformation by the linear function

$$F_t[\mathbf{x}] = (\mathrm{I} + tA)\mathbf{x} = \mathbf{x} + tA\mathbf{x} \tag{10.50}$$

defined by the infinitesimal generator. We already made use of such approximations when we discussed the rigid motions and mechanisms of structures in Chapter 6. As t varies, the

[†] See also Exercise 4.3.24 for the general definition of a group.

group transformations (10.49) typically move a point $\mathbf{x}$ along a curved trajectory. Under the first order approximation (10.50), the point $\mathbf{x}$ moves along a straight line in the direction $\mathbf{b} = A\mathbf{x}$ — the tangent line to the curved trajectory. Thus, *the infinitesimal generator of a one-parameter group prescribes the tangent line approximation to the nonlinear motion prescribed by the group transformations.*

Most of the linear transformations of interest in the above-mentioned applications arise in this fashion. Let's look briefly at a few basic examples.

(a) When $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ represents a shearing transformation. The group laws (10.48) imply that the composition of a shear of magnitude s and a shear of magnitude t in the same direction is another shear of magnitude $s+t$.

(b) When $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $e^{tA} = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}$ represents a uniform scaling transformation. Composition and inverses of such scaling transformations are also scalings.

(c) When $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $e^{tA} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, which, for $t > 0$, represents a stretch in the x direction and a contraction in the y direction.

(d) When $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ is the matrix for a plane rotation, around the origin, by angle t. The group laws (10.48) say that the composition of a rotation through angle s followed by a rotation through angle t is a rotation through angle $s+t$, as previously noted in Example 7.12. Also, the inverse of a rotation through angle t is a rotation through angle $-t$.

Observe that the infinitesimal generator of this one-parameter group of plane rotations is a 2×2 skew-symmetric matrix. This turns out to be a general fact: rotations in higher dimensions are also generated by skew-symmetric matrices.

Lemma 10.29. If $A^T = -A$ is a skew-symmetric matrix, then, for all $t \in \mathbb{R}$, its matrix exponential $Q(t) = e^{tA}$ is a proper orthogonal matrix.

Proof: According to equation (10.45) and Exercise 10.4.16,

$$Q(t)^{-1} = e^{-tA} = e^{tA^T} = \left(e^{tA}\right)^T = Q(t)^T,$$

which proves orthogonality. Properness, $\det Q = +1$, follows from Lemma 10.28 using the fact that $\operatorname{tr} A = 0$, since all the diagonal entries of a skew-symmetric matrix are 0. *Q.E.D.*

With some more work, it can be shown that every proper orthogonal matrix is the exponential of some skew-symmetric matrix, albeit not a unique one. Thus, the $\frac{1}{2}n(n-1)$–dimensional vector space of $n \times n$ skew-symmetric matrices generates the group of rotations in n-dimensional Euclidean space. In the three-dimensional case, the three matrices A_x, A_y, A_z listed below form a basis and serve to generate, respectively, the one-parameter groups of counterclockwise rotations around the x-, y-, and z-axes:

$$A_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad e^{tA_x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix},$$
$$A_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad e^{tA_y} = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}, \tag{10.51}$$
$$A_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e^{tA_z} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since every other skew-symmetric matrix can be expressed as a linear combination of A_x, A_y, and A_z, every rotation can, in a sense, be generated by these three basic types. This reconfirms our earlier observations concerning the number of rigid motions (rotations and translations) experienced by an unattached structure; see Section 6.3 for details.

In the three-dimensional case, it can be shown that every non-zero skew-symmetric 3×3 matrix A is singular, with one-dimensional kernel. Let $\mathbf{0} \neq \mathbf{v} \in \ker A$ be the null eigenvector. Then the matrix exponentials e^{tA} form the one-parameter group of rotations around the axis defined by $\mathbf{v}$. For instance, referring to (10.51), $\ker A_x$ is spanned by $\mathbf{e}_1 = (1,0,0)^T$, reconfirming that it generates the rotations around the x-axis. Details can be found in Exercise 10.4.38.

Noncommutativity of linear transformations is reflected in the noncommutativity of their infinitesimal generators. Recall, (1.12), that the *commutator* of two $n \times n$ matrices A, B is

$$[A,B] = AB - BA. \tag{10.52}$$

Thus, A and B commute if and only if $[A,B] = \mathrm{O}$. We use the exponential series (10.47) to evaluate the commutator of the corresponding matrix exponentials:

$$\begin{aligned} \left[e^{tA}, e^{tB} \right] &= e^{tA} e^{tB} - e^{tB} e^{tA} \\ &= \left(\mathrm{I} + tA + \tfrac{1}{2} t^2 A^2 + \cdots \right)\left(\mathrm{I} + tB + \tfrac{1}{2} t^2 B^2 + \cdots \right) - \\ &\qquad - \left(\mathrm{I} + tB + \tfrac{1}{2} t^2 B^2 + \cdots \right)\left(\mathrm{I} + tA + \tfrac{1}{2} t^2 A^2 + \cdots \right) \\ &= t^2 (AB - BA) + \cdots = t^2 [A,B] + \cdots . \end{aligned} \tag{10.53}$$

In particular, if the groups commute, then $[A,B] = \mathrm{O}$. The converse is also true, since if $AB = BA$ then all terms in the two series commute, and hence the matrix exponentials also commute.

Proposition 10.30. The matrix exponentials e^{tA} and e^{tB} commute for all t if and only if the matrices A and B commute:

$$e^{tA} e^{tB} = e^{tB} e^{tA} = e^{t(A+B)} \qquad \text{provided} \qquad AB = BA. \tag{10.54}$$

In particular, the non-commutativity of three-dimensional rotations follows from the non-commutativity of their infinitesimal skew-symmetric generators. For instance, the commutator of the generators of rotations around the x- and y-axes is the generator of rotations around the z-axis: $\left[A_x, A_y \right] = A_z$, since

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, to a first order (or, more correctly, second order) approximation, the difference between x and y rotations is, interestingly, a z rotation.

Exercises

10.4.32. Find the one-parameter groups generated by the following matrices and interpret geometrically: What are the trajectories? What are the fixed points?

(a) $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, (c) $\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}$, (e) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

10.4.33. Write down the one-parameter groups generated by the following matrices and interpret. What are the trajectories? What are the fixed points?

(a) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (c) $\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, (e) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

10.4.34. (a) Find the one-parameter group of rotations generated by the skew-symmetric matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$. (b) As noted above, e^{tA} represents a family of rotations around a fixed axis in $\mathbb{R}^3$. What is the axis?

10.4.35. Choose two of the groups in Exercise 10.4.32 or 10.4.33, and determine whether or not they commute by looking at their infinitesimal generators. Then verify your conclusion by directly computing the commutator of the corresponding matrix exponentials.

10.4.36. (a) Prove that the commutator of two upper triangular matrices is upper triangular. (b) Prove that the commutator of two skew-symmetric matrices is skew symmetric. (c) Is the commutator of two symmetric-matrices symmetric?

♢ 10.4.37. Prove that the *Jacobi identity*

$$[\,[\,A,B\,],C\,]+[\,[\,C,A\,],B\,]+[\,[\,B,C\,],A\,]=\mathrm{O} \tag{10.55}$$

is valid for three $n\times n$ matrices A,B,C.

♡ 10.4.38. Let $\mathbf{0}\neq\mathbf{v}\in\mathbb{R}^3$. (a) Show that the cross product $L_{\mathbf{v}}[\mathbf{x}]=\mathbf{v}\times\mathbf{x}$ defines a linear transformation on $\mathbb{R}^3$. (b) Find the 3×3 matrix representative $A_{\mathbf{v}}$ of $L_{\mathbf{v}}$ and show that it is skew-symmetric. (c) Show that every non-zero skew-symmetric 3×3 matrix defines such a cross product map. (d) Show that $\ker A_{\mathbf{v}}$ is spanned by $\mathbf{v}$. (e) Justify the fact that the matrix exponentials $e^{tA_{\mathbf{v}}}$ are rotations around the axis $\mathbf{v}$. Thus, the cross product with a vector serves as the infinitesimal generator of the one-parameter group of rotations around $\mathbf{v}$.

♡ 10.4.39. Given a unit vector $\|\mathbf{u}\|=1$ in $\mathbb{R}^3$, let $A=A_{\mathbf{u}}$ be the corresponding skew-symmetric 3×3 matrix that satisfies $A\mathbf{x}=\mathbf{u}\times\mathbf{x}$, as in Exercise 10.4.38. (a) Prove the *Euler–Rodrigues formula* $e^{tA}=\mathrm{I}+(\sin t)A+(1-\cos t)\,A^2$. *Hint*: Use the matrix exponential series (10.47). (b) Show that $e^{tA}=\mathrm{I}$ if and only if t is an integer multiple of 2π. (c) Generalize parts (a) and (b) to a non-unit vector $\mathbf{v}\neq\mathbf{0}$.

♡ 10.4.40. Let $A=\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{b}=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. (a) Show that the solution to the linear system $\dot{\mathbf{x}}=A\mathbf{x}$ represents a rotation of $\mathbb{R}^3$ around the z-axis. What is the trajectory of a point $\mathbf{x}_0$? (b) Show that the solution to the inhomogeneous system $\dot{\mathbf{x}}=A\mathbf{x}+\mathbf{b}$ represents a screw motion of $\mathbb{R}^3$ around the z-axis. What is the trajectory of a point $\mathbf{x}_0$? (c) More generally, given $\mathbf{0}\neq\mathbf{a}\in\mathbb{R}^3$, show that the solution to $\dot{\mathbf{x}}=\mathbf{a}\times\mathbf{x}+\mathbf{a}$ represents a family of screw motions along the axis $\mathbf{a}$.

10.4.41. Let A be an $n \times n$ matrix whose last row has all zero entries. Prove that the last row of e^{tA} is $\mathbf{e}_n^T = (0, \ldots, 0, 1)$.

10.4.42. Let $A = \begin{pmatrix} B & \mathbf{c} \\ \mathbf{0} & 0 \end{pmatrix}$ be in block form, where B is an $n \times n$ matrix, $\mathbf{c} \in \mathbb{R}^n$, while $\mathbf{0}$ denotes the zero row vector with n entries. Show that its matrix exponential is also in block form $e^{tA} = \begin{pmatrix} e^{tB} & \mathbf{f}(t) \\ \mathbf{0} & 1 \end{pmatrix}$. Can you find a formula for $\mathbf{f}(t)$?

♢ 10.4.43. According to Exercise 7.3.10, an $(n+1) \times (n+1)$ matrix of the block form $\begin{pmatrix} A & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix}$ in which A is an $n \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$ can be identified with the affine transformation $F[\mathbf{x}] = A\mathbf{x} + \mathbf{b}$ on $\mathbb{R}^n$. Exercise 10.4.42 shows that every matrix in the one-parameter group e^{tB} generated by $B = \begin{pmatrix} A & \mathbf{b} \\ \mathbf{0} & 0 \end{pmatrix}$ has such a form, and hence we can identify e^{tB} as a family of affine maps on $\mathbb{R}^n$. Describe the affine transformations of $\mathbb{R}^2$ generated by the following matrices:

(a) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (c) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$.

Invariant Subspaces and Linear Dynamical Systems

Invariant subspaces, as per Definition 8.27, play an important role in the study of homogeneous linear dynamical systems. In general, a subset $S \subset \mathbb{R}^n$ is called *invariant* for the homogeneous linear dynamical system $\dot{\mathbf{u}} = A\mathbf{u}$ if, whenever the initial condition $\mathbf{u}(t_0) = \mathbf{b} \in S$, then the solution $\mathbf{u}(t) \in S$ for all $t \in \mathbb{R}$.

Proposition 10.31. If $V \subset \mathbb{R}^n$ is an invariant subspace of the matrix A, then it is invariant under the corresponding homogeneous linear dynamical system.

Proof: Given that $\mathbf{b} \in V$, we have $A\mathbf{b} \in V$, $A^2\mathbf{b} \in V$, and, in general, $A^n\mathbf{b} \in V$ for each $n \geq 0$. Thus every term in the matrix exponential series for the solution (10.42), namely

$$\mathbf{u}(t) = e^{(t-t_0)A}\,\mathbf{b} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, A^n \mathbf{b},$$

belongs to V and hence, because V is closed, so does their sum: $\mathbf{u}(t) \in V$. *Q.E.D.*

As we know, the (complex) invariant subspaces of a complete matrix are spanned by its (complex) eigenvectors. According to the general Stability Theorem 10.13, these come in three flavors, depending upon whether the real part of the corresponding eigenvalue is less than, equal to, or greater than 0. The first kind, with $\operatorname{Re}\lambda < 0$, correspond to the asymptotically stable eigensolutions $\mathbf{u}(t) = e^{\lambda t}\mathbf{v} \to \mathbf{0}$ as $t \to \infty$. The second kind, with zero real part, correspond to stable eigensolutions that remain bounded for all t, by completeness. The third kind, with $\operatorname{Re}\lambda > 0$, correspond to unstable eigensolutions that become unbounded at an exponential rate as $t \to \infty$. A similar result holds for the corresponding real solutions of a complete real matrix. If the matrix is incomplete, then the solutions corresponding to Jordan chains with eigenvalues having negative real part are also asymptotically stable; those corresponding to Jordan chains with eigenvalues having positive real part remain exponentially unstable. If any purely imaginary eigenvalue is incomplete, then the polynomial factor in front of the corresponding Jordan chain solution makes it unstable, becoming unbounded at a polynomial rate. An example of the latter behavior is provided by a planar system that has 0 as its incomplete eigenvalue, producing unstable linear motion. The minimum dimension of a (real) system possessing a non-zero, incomplete purely imaginary eigenvalue is 4.

This motivates dissecting the underlying vector space into three invariant subspaces, having only the zero vector in common, that capture the three possible modes of behavior. We state the definition in the real case, leaving the simpler complex version to the reader.

Definition 10.32. Let A be a real $n \times n$ matrix. We define the following invariant subspaces spanned by the real and imaginary parts of the eigenvectors and Jordan chains corresponding to the eigenvalues with the following properties:

(i) negative real part: the *stable subspace* $S \subset \mathbb{R}^n$;

(ii) zero real part: the *center subspace* $C \subset \mathbb{R}^n$;

(iii) positive real part: the *unstable subspace* $U \subset \mathbb{R}^n$.

If there are no eigenvalues of the specified type, the corresponding invariant subspace is trivial. For example, if the associated linear system has asymptotically stable zero solution, then $S = \mathbb{R}^n$ while $C = U = \{\mathbf{0}\}$. The stable, unstable, and center subspaces are complementary, as in Exercise 2.2.24, in the sense that their pairwise intersections are trivial: $S \cap C = S \cap U = C \cap U = \{\mathbf{0}\}$, and their sum $S + C + U = \mathbb{R}^n$, in the sense that every $\mathbf{v} \in \mathbb{R}^n$ can be, in fact uniquely, written as a sum $\mathbf{v} = \mathbf{s} + \mathbf{c} + \mathbf{u}$ of vectors in each subspace: $\mathbf{s} \in S$, $\mathbf{c} \in C$, $\mathbf{u} \in U$.

Since each of these subspaces is invariant, if the initial condition belongs to one of them, so does the corresponding solution. In view of the solution formulas in Theorem 10.13, we deduce the following more intrinsic characterizations, in terms of the asymptotic behavior of the solutions to the homogeneous linear dynamical system.

Theorem 10.33. Let A be an $n \times n$ matrix. Let $\mathbf{0} \neq \mathbf{b} \in \mathbb{R}^n$, and let $\mathbf{u}(t)$ be a solution to the associated initial value problem $\dot{\mathbf{u}} = A\mathbf{u}$, $\mathbf{u}(t_0) = \mathbf{b}$. Then $\mathbf{b}$ and hence $\mathbf{u}(t)$ are in:

(i) the stable subspace S if and only if $\mathbf{u}(t) \to \mathbf{0}$ as $t \to \infty$, or, alternatively, $\|\mathbf{u}(t)\| \to \infty$ at an exponential rate as $t \to -\infty$;

(ii) the center subspace C if and only if $\mathbf{u}(t)$ is bounded or $\|\mathbf{u}(t)\| \to \infty$ at a polynomial rate as $t \to \pm\infty$;

(iii) the unstable subspace U if and only if $\|\mathbf{u}(t)\| \to \infty$ at an exponential rate as $t \to \infty$, or, alternatively, $\mathbf{u}(t) \to \mathbf{0}$ as $t \to -\infty$.

Example 10.34. For example, the matrix $A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$ has eigenvalues and eigenvectors

$$\lambda_1 = 0, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad \lambda_2 = -2, \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \qquad \lambda_3 = -3, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Thus, the stable subspace is the plane spanned by $\mathbf{v}_2$ and $\mathbf{v}_3$, whose nonzero solutions tend to the origin as $t \to \infty$ at an exponential rate; the center subspace is the line spanned by $\mathbf{v}_1$, all of whose solutions are constant; the unstable subspace is trivial: $U = \{\mathbf{0}\}$. So the origin is a stable, but not asymptotically stable, equilibrium point.

The Center Manifold Theorem, a celebrated result in nonlinear dynamics, [**34**], states that the above formulated linear splitting into stable, center, and unstable regimes carries over to nonlinear systems in a neighborhood of an equilibrium point. Roughly speaking, suppose that $\mathbf{u}_0$ is an equilibrium point of the nonlinear systems of ordinary differential equations $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$, so that $\mathbf{f}(\mathbf{u}_0) = \mathbf{0}$. Let A be the *linearization* of $\mathbf{f}(\mathbf{u})$ at $\mathbf{u}_0$, meaning its Jacobian matrix. so $A = \mathbf{f}'(\mathbf{u}_0) = (\partial f_i / \partial u_j)$, evaluated at the equilibrium point. Then, in a neighborhood of $\mathbf{u}_0$, the dynamical system admits three invariant curved manifolds, meaning curves, surfaces, and their higher-dimensional counterparts, called the *stable*, *center*, and *unstable manifolds* that are tangent to (or equivalently, approximated by) the corresponding invariant subspaces of its linearization matrix A. Solutions evolving on the stable and unstable manifolds exhibit behaviors similar to those of the linear system. In particular, solutions on the stable manifold converge to the equilibrium, $\mathbf{u}(t_0) \to \mathbf{u}_0$ as $t \to \infty$, at an exponential rate governed by the corresponding eigenvalues of A, while those on the unstable manifold move away from the equilibrium point $\mathbf{u}_0$ — although one cannot say what happens to them once they exit the neighborhood, once the nonlinear effects take over. Solutions on the center manifold have more subtle dynamical behavior, that depends on the detailed structure of the nonlinear terms. In this manner, one can effectively argue that, near a fixed point, all the interesting dynamics takes place on the center manifold.

Exercises

10.4.44. (*a*) Given a homogeneous linear dynamical system with invariant stable, unstable, and center subspaces S, U, C, explain why the origin is asymptotically stable if and only if $C = U = \{\mathbf{0}\}$. (*b*) Is the origin stable if $U = \{\mathbf{0}\}$ but $C \neq \{\mathbf{0}\}$?

10.4.45. Find the (real) stable, unstable, and center subspaces of the following linear systems:

(*a*) $\begin{aligned}\dot u_1 &= 9u_2,\\ \dot u_2 &= -u_1;\end{aligned}$ (*b*) $\begin{aligned}\dot x_1 &= 4x_1 + x_2,\\ \dot x_2 &= 3x_1;\end{aligned}$ (*c*) $\begin{aligned}\dot y_1 &= y_1 - y_2,\\ \dot y_2 &= 2y_1 + 3y_2;\end{aligned}$ (*d*) $\begin{aligned}\dot z_1 &= z_2,\\ \dot z_2 &= 3z_1 + 2z_3,\\ \dot y_3 &= -z_2;\end{aligned}$

(*e*) $\begin{aligned}\dot u_1 &= u_1 - 3u_2 + 11u_3,\\ \dot u_2 &= 2u_1 - 6u_2 + 16u_3,\\ \dot u_3 &= u_1 - 3u_2 + 7u_3,\end{aligned}$ (*f*) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} -1 & 3 & -3 \\ 2 & 2 & -7 \\ 0 & 3 & -4 \end{pmatrix}\mathbf{u}$, (*g*) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}\mathbf{u}$.

♢ 10.4.46. State and prove a counterpart to Definition 10.32 and Theorem 10.33 for a homogeneous linear iterative system.

Inhomogeneous Linear Systems

We now direct our attention to inhomogeneous linear systems of ordinary differential equations. For simplicity, we consider only first order† systems of the form

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} + \mathbf{f}(t), \tag{10.56}$$

in which A is a constant $n \times n$ matrix and $\mathbf{f}(t)$ is a vector-valued function of t that can be interpreted as a collection of time-varying external forces acting on the system. According to Theorem 7.38, the solution to the inhomogeneous system will have the general form

$$\mathbf{u}(t) = \mathbf{u}^\star(t) + \mathbf{z}(t)$$

† Higher order systems can, as in the phase plane construction, (10.8), always be converted into first order systems involving additional variables.

where $\mathbf{u}^\star(t)$ is a particular solution, representing a response to the forcing, while $\mathbf{z}(t)$ is a solution to the corresponding homogeneous system $\dot{\mathbf{z}} = A\,\mathbf{z}$, representing the system's internal motion. Since we now know how to find the solution $\mathbf{z}(t)$ to the homogeneous system, the only task is to find one particular solution to the inhomogeneous system.

In your first course on ordinary differential equations, you probably encountered a method known as *variation of parameters* for constructing particular solutions of inhomogeneous scalar ordinary differential equations, [**7**]. The method can be readily adapted to first order systems. Recall that, in the scalar case, to solve the inhomogeneous equation

$$\frac{du}{dt} = a\,u + f(t), \qquad \text{we set} \qquad u(t) = e^{t\,a}\,v(t), \tag{10.57}$$

where the function $v(t)$ is to be determined. Differentiating, we obtain

$$\frac{du}{dt} = a\,e^{t\,a}\,v(t) + e^{t\,a}\,\frac{dv}{dt} = a\,u + e^{t\,a}\,\frac{dv}{dt}\,.$$

Therefore, $u(t)$ satisfies the differential equation (10.57) if and only if

$$\frac{dv}{dt} = e^{-t\,a}\,f(t).$$

Since the right-hand side of the latter is known, $v(t)$ can be immediately found by a direct integration.

The method can be extended to the vector-valued situation as follows. We replace the scalar exponential by the exponential of the coefficient matrix, setting

$$\mathbf{u}(t) = e^{t\,A}\,\mathbf{v}(t), \tag{10.58}$$

where $\mathbf{v}(t)$ is a vector-valued function that is to be determined. Combining the product rule for matrix multiplication (10.41) with (10.39) yields

$$\frac{d\mathbf{u}}{dt} = \frac{d}{dt}\,(e^{t\,A}\,\mathbf{v}) = \frac{de^{t\,A}}{dt}\,\mathbf{v} + e^{t\,A}\,\frac{d\mathbf{v}}{dt} = A\,e^{t\,A}\,\mathbf{v} + e^{t\,A}\,\frac{d\mathbf{v}}{dt} = A\,\mathbf{u} + e^{t\,A}\,\frac{d\mathbf{v}}{dt}\,.$$

Comparing with the differential equation (10.56), we conclude that

$$\frac{d\mathbf{v}}{dt} = e^{-\,t\,A}\,\mathbf{f}(t).$$

Integrating† both sides from the initial time t_0 to time t produces, by the Fundamental Theorem of Calculus,

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} e^{-\,s\,A}\,\mathbf{f}(s)\,ds, \qquad \text{where} \qquad \mathbf{v}(t_0) = e^{-\,t_0 A}\,\mathbf{u}(t_0). \tag{10.59}$$

Substituting back into (10.58) leads to a general formula for the solution to the inhomogeneous linear system.

Theorem 10.35. The solution to the initial value problem

$$\frac{d\mathbf{u}}{dt} = A\,\mathbf{u} + \mathbf{f}(t), \quad \mathbf{u}(t_0) = \mathbf{b}, \qquad \text{is} \qquad \mathbf{u}(t) = e^{(t-t_0)\,A}\,\mathbf{b} \,+ \int_{t_0}^{t} e^{(t-s)\,A}\,\mathbf{f}(s)\,ds. \tag{10.60}$$

† As with differentiation, vector-valued and matrix-valued functions are integrated entry-wise.

In the solution formula, the integral term can be viewed as a particular solution $\mathbf{u}^\star(t)$, namely the one satisfying the initial condition $\mathbf{u}^\star(t_0) = \mathbf{0}$, while the first summand, $\mathbf{z}(t) = e^{(t-t_0)\,A}\,\mathbf{b}$ for $\mathbf{b} \in \mathbb{R}^n$, constitutes the general solution to the homogeneous system.

Example 10.36. Our goal is to solve the initial value problem

$$\begin{aligned} \dot u_1 &= 2u_1 - u_2, & u_1(0) &= 1,\\ \dot u_2 &= 4u_1 - 3u_2 + e^t, & u_2(0) &= 0. \end{aligned} \tag{10.61}$$

The first step is to determine the eigenvalues and eigenvectors of the coefficient matrix:

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix} \qquad \text{so} \qquad \lambda_1 = 1, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \lambda_2 = -2, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

The resulting eigensolutions form the columns of the nonsingular matrix solution

$$U(t) = \begin{pmatrix} e^t & e^{-2t} \\ e^t & 4e^{-2t} \end{pmatrix}, \qquad \text{hence} \qquad e^{tA} = U(t)\,U(0)^{-1} = \begin{pmatrix} \frac43 e^t - \frac13 e^{-2t} & -\frac13 e^t + \frac13 e^{-2t} \\ \frac43 e^t - \frac43 e^{-2t} & -\frac13 e^t + \frac43 e^{-2t} \end{pmatrix}.$$

Since $t_0 = 0$, the two constituents of the solution formula (10.60) are

$$e^{tA}\,\mathbf{b} = \begin{pmatrix} \frac43 e^t - \frac13 e^{-2t} & -\frac13 e^t + \frac13 e^{-2t} \\ \frac43 e^t - \frac43 e^{-2t} & -\frac13 e^t + \frac43 e^{-2t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac43 e^t - \frac13 e^{-2t} \\ \frac43 e^t - \frac43 e^{-2t} \end{pmatrix},$$

which forms the solution to the homogeneous system for the given nonzero initial conditions, and

$$\begin{aligned} \int_0^t e^{(t-s)\,A}\,\mathbf{f}(s)\,ds &= \int_0^t \begin{pmatrix} \frac43 e^{t-s} - \frac13 e^{-2(t-s)} & -\frac13 e^{t-s} + \frac13 e^{-2(t-s)} \\ \frac43 e^{t-s} - \frac43 e^{-2(t-s)} & -\frac13 e^{t-s} + \frac43 e^{-2(t-s)} \end{pmatrix} \begin{pmatrix} 0 \\ e^s \end{pmatrix} ds \\ &= \int_0^t \begin{pmatrix} -\frac13 e^t + \frac13 e^{-2t+3s} \\ -\frac13 e^t + \frac43 e^{-2t+3s} \end{pmatrix} ds = \begin{pmatrix} -\frac13 t e^t + \frac19 (e^t - e^{-2t}) \\ -\frac13 t e^t + \frac49 (e^t - e^{-2t}) \end{pmatrix}, \end{aligned}$$

which is the particular solution to the inhomogeneous system that satisfies the homogeneous initial conditions $\mathbf{u}(0) = \mathbf{0}$. The solution to our initial value problem is their sum:

$$\mathbf{u}(t) = \begin{pmatrix} -\frac13 t e^t + \frac{13}{9} e^t - \frac49 e^{-2t} \\ -\frac13 t e^t + \frac{16}{9} e^t - \frac{16}{9} e^{-2t} \end{pmatrix}.$$

Exercises

10.4.47. Solve the following initial value problems: (a) $\begin{aligned} \dot u_1 &= 2u_1 - u_2, & u_1(0) &= 0,\\ \dot u_2 &= 4u_1 - 3u_2 + e^{2t}, & u_2(0) &= 0. \end{aligned}$

(b) $\begin{aligned} \dot u_1 &= -u_1 + 2u_2 + e^t, & u_1(1) &= 1,\\ \dot u_2 &= 2u_1 - u_2 + e^t, & u_2(1) &= 1. \end{aligned}$ (c) $\begin{aligned} \dot u_1 &= -u_2, & u_1(0) &= 0,\\ \dot u_2 &= 4u_1 + \cos t, & u_2(0) &= 1. \end{aligned}$

(d) $\begin{aligned} \dot u &= 3u + v + 1, & u(1) &= 1,\\ \dot v &= 4u + t, & v(1) &= -1. \end{aligned}$ (e) $\begin{aligned} \dot p &= p + q + t, & p(0) &= 0,\\ \dot q &= -p - q + t, & q(0) &= 0. \end{aligned}$

10.4.48. Solve the following initial value problems:

(a) $\begin{aligned} \dot u_1 &= -2u_2 + 2u_3, & u_1(0) &= 1,\\ \dot u_2 &= -u_1 + u_2 - 2u_3 + t, & u_2(0) &= 0,\\ \dot u_3 &= -3u_1 + u_2 - 2u_3 + 1, & u_3(0) &= 0. \end{aligned}$ (b) $\begin{aligned} \dot u_1 &= u_1 - 2u_2, & u_1(0) &= -1,\\ \dot u_2 &= -u_2 + e^{-t}, & u_2(0) &= 0,\\ \dot u_3 &= 4u_1 - 4u_2 - u_3, & u_3(0) &= -1. \end{aligned}$

10.4.49. Suppose that λ is *not* an eigenvalue of A. Show that the inhomogeneous system $\dot{\mathbf{u}} = A\mathbf{u} + e^{\lambda t}\mathbf{v}$ has a solution of the form $\mathbf{u}^\star(t) = e^{\lambda t}\mathbf{w}$, where $\mathbf{w}$ is a constant vector. What is the general solution?

10.4.50. (*a*) Write down an integral formula for the solution to the initial value problem
$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} + \mathbf{b}, \quad \mathbf{u}(0) = \mathbf{0}, \quad \text{where } \mathbf{b} \text{ is a } \textit{constant} \text{ vector.}$$
(*b*) Suppose $\mathbf{b} \in \operatorname{img} A$. Do you recover the solution you found in Exercise 10.2.24?

10.5 Dynamics of Structures

Chapter 6 was concerned with the equilibrium configurations of mass–spring chains and, more generally, structures constructed out of elastic bars. We are now able to undertake an analysis of their dynamical motions, which are governed by second order systems of ordinary differential equations. The same systems also serve to model the vibrations of molecules, of fundamental importance in chemistry and spectroscopy, [**91**]. As in the first order case, the eigenvalues of the coefficient matrix play an essential role in both the explicit solution formula and the system's qualitative behavior(s).

Let us begin with a mass–spring chain consisting of n masses $m_1, \ldots, m_n$ connected together in a row and, possibly, to top and bottom supports by springs. As in Section 6.1, that is, we restrict our attention to purely one-dimensional motion of the masses in the direction of the chain. Thus the collective motion of the chain is prescribed by the displacement vector $\mathbf{u}(t) = (u_1(t), \ldots, u_n(t))^T$ whose i^{th} entry represents the displacement from equilibrium of the i^{th} mass. Since we are now interested in dynamics, the displacements are allowed to depend on time, t.

The motion of each mass is subject to Newton's Second Law:
$$\text{Force} = \text{Mass} \times \text{Acceleration}. \tag{10.62}$$
The acceleration of the i^{th} mass is the second derivative $\ddot{u}_i = d^2u_i/dt^2$ of its displacement $u_i(t)$, so the right-hand sides of Newton's Law is $m_i\ddot{u}_i$. These form the entries of the vector $M\ddot{\mathbf{u}}$ obtained by multiplying the acceleration vector by the diagonal, positive definite mass matrix $M = \operatorname{diag}(m_1, \ldots, m_n)$. Keep in mind that the masses of the springs are assumed to be negligible in this model.

If, to begin with, we assume that there are no frictional effects, then the force exerted on each mass is the difference between the external force, if any, and the internal force due to the elongations of its two connecting springs. According to (6.11), the internal forces are the entries of the product $K\mathbf{u}$, where $K = A^T C A$ is the stiffness matrix, constructed from the chain's (reduced) incidence matrix A, and the diagonal matrix of spring constants C. Thus, Newton's law immediately leads to the linear system of second order differential equations
$$M\frac{d^2\mathbf{u}}{dt^2} = \mathbf{f}(t) - K\mathbf{u}, \tag{10.63}$$
governing the dynamical motions of the masses under a possibly time-dependent external force $\mathbf{f}(t)$. Such systems are also used to model the undamped dynamical motion of structures and molecules as well as resistanceless (superconducting) electrical circuits.

As always, the first order of business is to analyze the corresponding homogeneous system
$$M\frac{d^2\mathbf{u}}{dt^2} + K\mathbf{u} = \mathbf{0}, \tag{10.64}$$
modeling the unforced motions of the physical apparatus.

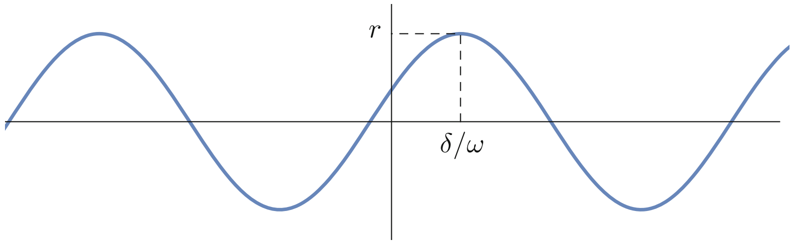


Figure 10.5. Vibration of a Mass.

Example 10.37. The simplest case is that of a single mass connected to a fixed support by a spring. Assuming no external force, the dynamical system (10.64) reduces to a homogeneous second order scalar equation

$$m \frac{d^2 u}{dt^2} + k\,u = 0, \tag{10.65}$$

in which $m > 0$ is the mass, while $k > 0$ is the spring's stiffness. The general solution to (10.65) is

$$u(t) = c_1 \cos \omega\, t + c_2 \sin \omega\, t = r \cos(\omega\, t - \delta), \qquad \text{where} \qquad \omega = \sqrt{\frac{k}{m}} \tag{10.66}$$

is the natural frequency of vibration. In the second expression, we have used the phase-amplitude equation (2.7) to rewrite the solution as a single cosine with

$$\text{amplitude} \quad r = \sqrt{c_1^2 + c_2^2} \qquad \text{and phase shift} \quad \delta = \tan^{-1} \frac{c_2}{c_1}\,. \tag{10.67}$$

Thus, the mass' motion is periodic, with period $P = 2\pi/\omega$. The stiffer the spring or the lighter the mass, the faster the vibrations. Take note of the square root in the frequency formula; quadrupling the mass slows down the vibrations by only a factor of two.

The constants c_1, c_2 — or their phase–amplitude counterparts r, δ — are determined by the initial conditions. Physically, we need to specify both an initial position and an initial velocity

$$u(t_0) = a, \qquad \dot u(t_0) = b, \tag{10.68}$$

in order to uniquely prescribe the subsequent motion of the system. The resulting solution is most conveniently written in the form

$$\begin{gathered} u(t) = a \cos \omega\,(t - t_0) + \frac{b}{\omega}\, \sin \omega\,(t - t_0) = r \cos\bigl[\,\omega\,(t - t_0) - \delta\,\bigr], \\ \text{with amplitude} \quad r = \sqrt{a^2 + \frac{b^2}{\omega^2}} \quad \text{and phase shift} \quad \delta = \tan^{-1} \frac{b}{a\,\omega}\,. \end{gathered} \tag{10.69}$$

A typical solution is plotted in Figure 10.5.

Let us turn to a more general second order system. To begin with, let us assume that the masses are all the same and equal to 1 (in some appropriate units), so that (10.64) reduces to

$$\frac{d^2 \mathbf{u}}{dt^2} + K \mathbf{u} = \mathbf{0}. \tag{10.70}$$

Inspired by the form of the solution of the scalar equation, let us try a trigonometric ansatz for the solution, setting

$$\mathbf{u}(t) = \cos(\omega\, t)\, \mathbf{v}, \tag{10.71}$$

in which the vibrational frequency ω is a constant scalar and $\mathbf{v} \neq \mathbf{0}$ a constant vector. Differentiation produces

$$\frac{d\mathbf{u}}{dt} = -\,\omega\,\sin(\omega\,t)\,\mathbf{v}, \qquad \frac{d^2\mathbf{u}}{dt^2} = -\,\omega^2\,\cos(\omega\,t)\,\mathbf{v}, \qquad \text{whereas} \qquad K\,\mathbf{u} = \cos(\omega\,t)\,K\,\mathbf{v},$$

since the cosine factor is a scalar. Therefore, (10.71) will solve the second order system (10.70) if and only if

$$K\,\mathbf{v} = \omega^2\,\mathbf{v}. \tag{10.72}$$

The result is in the form of the eigenvalue equation $K\mathbf{v} = \lambda\,\mathbf{v}$ for the stiffness matrix K, with eigenvector $\mathbf{v} \neq \mathbf{0}$ and eigenvalue

$$\lambda = \omega^2. \tag{10.73}$$

Now, the scalar equation has both cosine and sine solutions. By the same token, the ansatz $\mathbf{u}(t) = \sin(\omega\,t)\,\mathbf{v}$ leads to the *same* eigenvector equation (10.72). We conclude that each positive eigenvalue leads to two different periodic trigonometric solutions.

Summarizing, we have established:

Lemma 10.38. If $\mathbf{v}$ is an eigenvector of the positive definite matrix K with eigenvalue $\lambda = \omega^2 > 0$, then $\mathbf{u}(t) = \cos(\omega\,t)\,\mathbf{v}$ and $\mathbf{u}(t) = \sin(\omega\,t)\,\mathbf{v}$ are both solutions to the homogeneous second order system $\ddot{\mathbf{u}} + K\,\mathbf{u} = \mathbf{0}$.

Stable Structures

Let us begin with the motion of a stable mass–spring chain or structure, of the type introduced in Section 6.3. According to Theorem 6.8, stability requires that the reduced stiffness matrix be positive definite: $K > 0$. Theorem 8.35 then says that all the eigenvalues of K are strictly positive, $\lambda_i > 0$, which is good, since it implies that the vibrational frequencies $\omega_i = \sqrt{\lambda_i}$ are all real. Moreover, positive definite matrices are always complete, and so K possesses an orthogonal eigenvector basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbb{R}^n$ corresponding to its eigenvalues $\lambda_1, \ldots, \lambda_n$, listed in accordance with their multiplicities. This yields a total of $2\,n$ linearly independent trigonometric eigensolutions, namely

$$\begin{aligned} \mathbf{u}_i(t) &= \cos(\omega_i\,t)\,\mathbf{v}_i = \cos\big(\sqrt{\lambda_i}\,t\big)\,\mathbf{v}_i, \\ \widetilde{\mathbf{u}}_i(t) &= \sin(\omega_i\,t)\,\mathbf{v}_i = \sin\big(\sqrt{\lambda_i}\,t\big)\,\mathbf{v}_i, \end{aligned} \qquad i = 1, \ldots, n, \tag{10.74}$$

which is precisely the number required by the general existence and uniqueness theorems for linear ordinary differential equations. The general solution to (10.70) is an arbitrary linear combination of the eigensolutions:

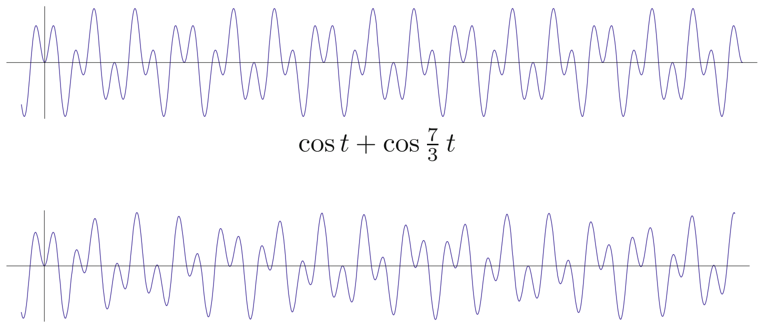
$$\mathbf{u}(t) = \sum_{i=1}^{n} \big[\, c_i \cos(\omega_i\,t) + d_i \sin(\omega_i\,t)\,\big]\,\mathbf{v}_i = \sum_{i=1}^{n} r_i \cos(\omega_i\,t - \delta_i)\,\mathbf{v}_i. \tag{10.75}$$

The $2\,n$ coefficients c_i, d_i — or their phase–amplitude counterparts $r_i \geq 0$ and $0 \leq \delta_i < 2\,\pi$ — are uniquely determined by the initial conditions. As in (10.68), we need to specify both the initial positions and initial velocities of all the masses; this requires a total of $2\,n$ initial conditions

$$\mathbf{u}(t_0) = \mathbf{a}, \qquad \dot{\mathbf{u}}(t_0) = \mathbf{b}. \tag{10.76}$$

Suppose $t_0 = 0$; then substituting the solution formula (10.75) into the initial conditions, we obtain

$$\mathbf{u}(0) = \sum_{i=1}^{n} c_i\,\mathbf{v}_i = \mathbf{a}, \qquad \dot{\mathbf{u}}(0) = \sum_{i=1}^{n} \omega_i\,d_i\,\mathbf{v}_i = \mathbf{b}.$$



$\cos t + \cos \frac{7}{3}\, t$

$\cos t + \cos \sqrt{5}\, t$

Figure 10.6. Periodic and Quasi–Periodic Functions.

Since the eigenvectors are orthogonal, the coefficients are immediately found by our orthogonal basis formula (4.7), whence

$$c_i = \frac{\langle\, \mathbf{a}\,, \mathbf{v}_i \,\rangle}{\|\, \mathbf{v}_i \,\|^2}\,, \qquad\qquad d_i = \frac{\langle\, \mathbf{b}\,, \mathbf{v}_i \,\rangle}{\omega_i \,\|\, \mathbf{v}_i \,\|^2}\,. \tag{10.77}$$

The eigensolutions (10.74) are also known as the *normal modes of vibration* of the system, and the $\omega_i = \sqrt{\lambda_i}$ its *natural frequencies*, which are the *square roots of the eigenvalues of the stiffness matrix* K. Each eigensolution is a periodic, vector-valued function of period $P_i = 2\pi/\omega_i$. Linear combinations of such periodic functions are called *quasi-periodic*, because they are *not*, typically, periodic!

A simple example is provided by the family of functions

$$f(t) = \cos t + \cos \omega\, t.$$

If $\omega = p/q \in \mathbb{Q}$ is a rational number, so $p, q, \in \mathbb{Z}$ with $q > 0$, then $f(t)$ is a periodic function, since $f(t + 2\pi q) = f(t)$, where $2\pi q$ is the minimal period, provided that p and q have no common factors. However, if ω is an irrational number, then $f(t)$ is not periodic. You are encouraged to carefully inspect the graphs in Figure 10.6. The first is periodic — can you spot where it begins to repeat? — whereas the second is only quasi-periodic and never quite succeeds in repeating its behavior. The general solution (10.75) to a vibrational system is similarly quasi-periodic, and is periodic only when *all* the frequency ratios ω_i/ω_j are rational numbers. To the uninitiated, such quasi-periodic motions may appear to be rather chaotic,† even though they are built from a few simple periodic constituents. Most structures and circuits exhibit quasi-periodic vibrational motions. Let us analyze a couple of simple examples.

Example 10.39. Consider a chain consisting of two equal unit masses connected to top and bottom supports by three springs, as in Figure 10.7, with incidence matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$. If the spring constants are c_1, c_2, c_3 (labeled from top to bottom),

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† This is *not* true chaos, which is is an inherently nonlinear phenomenon, [**56**].

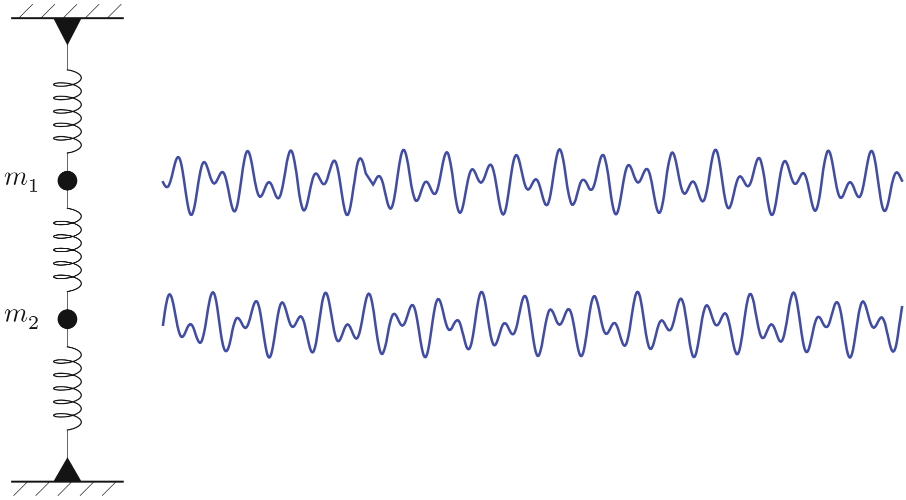


Figure 10.7. Motion of a Double Mass–Spring Chain with Fixed Supports.

then the stiffness matrix is

$$K = A^T C A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{pmatrix}.$$

The eigenvalues and eigenvectors of K will prescribe the normal modes and vibrational frequencies of our two–mass chain.

Let us look in detail when the springs are identical, and choose our units so that $c_1 = c_2 = c_3 = 1$. The resulting stiffness matrix $K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ has eigenvalues and eigenvectors

$$\lambda_1 = 1, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad\qquad \lambda_2 = 3, \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The general solution to the system is then

$$\mathbf{u}(t) = r_1 \cos(t - \delta_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + r_2 \cos(\sqrt{3}\, t - \delta_2) \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The first summand is the normal mode vibrating at the relatively slow frequency $\omega_1 = 1$, with the two masses moving in tandem. The second normal mode vibrate faster, with frequency $\omega_2 = \sqrt{3} \simeq 1.73205$, in which the two masses move in opposing directions. The general motion is a linear combination of these two normal modes. Since the frequency ratio $\omega_2/\omega_1 = \sqrt{3}$ is irrational, the motion is quasi-periodic. The system never quite returns to its initial configuration — unless it happens to be vibrating in one of the normal modes. A graph of some typical displacements of the masses is plotted in Figure 10.7.

If we eliminate the bottom spring, so the masses are just hanging from the top support as in Figure 10.8, then the reduced incidence matrix $A^\star = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ loses its last row.

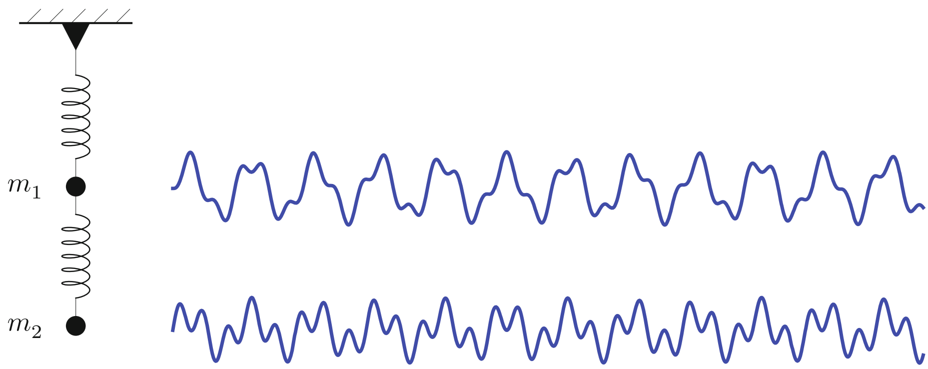


Figure 10.8. Motion of a Double Mass–Spring Chain with One Free End.

Assuming that the springs have unit stiffnesses $c_1 = c_2 = 1$, the corresponding reduced stiffness matrix is

$$K^\star = (A^\star)^T A^\star = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$\lambda_1 = \frac{3-\sqrt{5}}{2}, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ \dfrac{\sqrt{5}+1}{2} \end{pmatrix}, \qquad\qquad \lambda_2 = \frac{3+\sqrt{5}}{2}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -\dfrac{\sqrt{5}-1}{2} \end{pmatrix}.$$

The general solution to the system is the quasi-periodic linear combination

$$\mathbf{u}(t) = r_1 \cos\left(\frac{\sqrt{5}-1}{2}\, t - \delta_1\right) \begin{pmatrix} 1 \\ \dfrac{\sqrt{5}+1}{2} \end{pmatrix} + r_2 \cos\left(\frac{\sqrt{5}+1}{2}\, t - \delta_2\right) \begin{pmatrix} 1 \\ -\dfrac{\sqrt{5}-1}{2} \end{pmatrix}.$$

The slower normal mode, with frequency $\omega_1 = \sqrt{\dfrac{3-\sqrt{5}}{2}} = \dfrac{\sqrt{5}-1}{2} \simeq .61803$, has the masses moving in tandem, with the bottom mass moving proportionally $\dfrac{\sqrt{5}+1}{2} \simeq 1.61803$ farther. The faster normal mode, with frequency $\omega_2 = \sqrt{\dfrac{3+\sqrt{5}}{2}} = \dfrac{\sqrt{5}+1}{2} \simeq 1.61803$, has the masses moving in opposing directions, with the top mass experiencing the larger displacement. Thus, removing the bottom support has caused both modes to vibrate slower. A typical solution is plotted in Figure 10.8.

Example 10.40. Consider a three mass–spring chain, with unit springs and masses, and both ends attached to fixed supports. The stiffness matrix $K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ has eigenvalues and eigenvectors

$$\lambda_1 = 2 - \sqrt{2}, \qquad \lambda_2 = 2, \qquad \lambda_3 = 2 + \sqrt{2},$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.$$

The three normal modes, from slowest to fastest, have frequencies

(a) $\omega_1 = \sqrt{2-\sqrt{2}}$: all three masses move in tandem, with the middle one moving $\sqrt{2}$ times as far.

(b) $\omega_2 = \sqrt{2}$: the two outer masses move in opposing directions, while the middle mass does not move.

(c) $\omega_3 = \sqrt{2+\sqrt{2}}$: the two outer masses move in tandem, while the inner mass moves $\sqrt{2}$ times as far in the opposing direction.

The general motion is a quasi-periodic combination of these three normal modes. As such, to the naked eye it can look very complicated. Our mathematical analysis unmasks the innate simplicity, where the complex dynamics are, in fact, entirely governed by just three fundamental modes of vibration.

Exercises

10.5.1. A 6 kilogram mass is connected to a spring with stiffness 21 kg/sec^2. Determine the frequency of vibration in hertz (cycles per second).

10.5.2. The lowest audible frequency is about 20 hertz = 20 cycles per second. How small a mass would need to be connected to a unit spring to produce a fast enough vibration to be audible? (As always, we assume the spring has negligible mass, which is probably not so reasonable in this situation.)

10.5.3. Graph the following functions. Which are periodic? quasi-periodic? If periodic, what is the (minimal) period? (a) $\sin 4t + \cos 6t$, (b) $1 + \sin \pi t$, (c) $\cos \frac{1}{2}\pi t + \cos \frac{1}{3}\pi t$, (d) $\cos t + \cos \pi t$, (e) $\sin \frac{1}{4}t + \sin \frac{1}{5}t + \sin \frac{1}{6}t$, (f) $\cos t + \cos \sqrt{2}\,t + \cos 2t$, (g) $\sin t \,\sin 3t$.

10.5.4. What is the minimal period of a function of the form $\cos \dfrac{p}{q}\, t + \cos \dfrac{r}{s}\, t$, assuming that each fraction is in lowest terms, i.e., its numerator and denominator have no common factors?

10.5.5. (a) Determine the natural frequencies of the Newtonian system $\dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix} \mathbf{u} = \mathbf{0}$.
(b) What is the dimension of the space of solutions? Explain your answer.
(c) Write out the general solution. (d) For which initial conditions is the resulting motion (i) periodic? (ii) quasi-periodic? (iii) both? (iv) neither? Justify your answer.

10.5.6. Answer Exercise 10.5.5 for the system $\dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 73 & 36 \\ 36 & 52 \end{pmatrix} \mathbf{u} = \mathbf{0}$.

10.5.7. Find the general solution to the following second order systems:
(a) $\dfrac{d^2u}{dt^2} = -3u + 2v,\quad \dfrac{d^2v}{dt^2} = 2u - 3v.$ (b) $\dfrac{d^2u}{dt^2} = -11u - 2v,\quad \dfrac{d^2v}{dt^2} = -2u - 14v.$
(c) $\dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} \mathbf{u} = \mathbf{0}$, (d) $\dfrac{d^2\mathbf{u}}{dt^2} = \begin{pmatrix} -6 & 4 & -1 \\ 4 & -6 & 1 \\ -1 & 1 & -11 \end{pmatrix} \mathbf{u}$.

10.5.8. Two masses are connected by three springs to top and bottom supports. Can you find a collection of spring constants c_1, c_2, c_3 such that all vibrations are periodic?

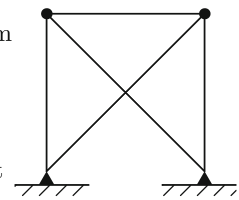
♠ 10.5.9. Suppose the bottom support in the mass–spring chain in Example 10.40 is removed.
(a) Do you predict that the vibration rate will (i) speed up, (ii) slow down, or (iii) stay the same? (b) Verify your prediction by computing the new vibrational frequencies.
(c) Suppose the middle mass is displaced by a unit amount and then let go. Compute and graph the solutions in both situations. Discuss what you observe.

10.5.10. Show that a single mass that is connected to both the top and bottom supports by two springs of stiffnesses c_1, c_2 will vibrate in the same manner as if it were connected to only one support by a spring with the combined stiffness $c = c_1 + c_2$.

♠ 10.5.11. (*a*) Describe, quantitatively and qualitatively, the normal modes of vibration for a mass–spring chain consisting of 3 unit masses, connected to top and bottom supports by unit springs. (*b*) Answer the same question when the bottom support is removed.

♡ 10.5.12. Find the vibrational frequencies for a mass–spring chain with n identical masses, connected by $n+1$ identical springs to both top and bottom supports. Is there any sort of limiting behavior as $n \to \infty$? *Hint*: See Exercise 8.2.48.

♣ 10.5.13. Suppose the illustrated planar structure has unit masses at the nodes and the bars are all of unit stiffness. (*a*) Write down the system of differential equations that describes the dynamical vibrations of the structure. (*b*) How many independent modes of vibration are there? (*c*) Find numerical values for the vibrational frequencies. (*d*) Describe what happens when the structure vibrates in each of the normal modes. (*e*) Suppose the left-hand mass is displaced a unit horizontal distance. Determine the subsequent motion.



10.5.14. When does a homogeneous real first order linear system $\dot{\mathbf{u}} = A\mathbf{u}$ have a quasi-periodic solution? What is the smallest dimension in which this can occur?

♣ 10.5.15. Suppose you are given n different springs. In which order should you connect them to unit masses so that the mass–spring chain vibrates the fastest? Does your answer depend upon the relative sizes of the spring constants? Does it depend upon whether the bottom mass is attached to a support or left hanging free? First try the case of three springs with spring stiffnesses $c_1 = 1, \;\; c_2 = 2, \;\; c_3 = 3$. Then try varying the stiffnesses. Finally, predict what will happen with 4 or 5 springs, and see whether you can make a general conjecture.

Unstable Structures

So far, we have just dealt with the stable case, in which the stiffness matrix K is positive definite. Unstable configurations, which can admit rigid motions and/or mechanisms, will provide additional complications. The simplest is a single mass that is not attached to any spring. Since the mass experiences no restraining force, its motion is governed by the elementary second order ordinary differential equation

$$m\,\frac{d^2u}{dt^2} = 0. \tag{10.78}$$

The general solution is

$$u(t) = c\,t + d. \tag{10.79}$$

If $c = 0$, the mass sits at a fixed position, while when $c \neq 0$, it moves along a straight line with constant velocity.

More generally, suppose that the stiffness matrix K for our structure is only positive semi-definite. Each vector $\mathbf{0} \neq \mathbf{v} \in \ker K$ represents a mode of instability of the system. Since $K\mathbf{v} = \mathbf{0}$, the vector $\mathbf{v}$ is a *null eigenvector* with associated eigenvalue $\lambda = 0$. Lemma 10.38 provides us with two solutions to the dynamical equations (10.70) of "frequency" $\omega = \sqrt{\lambda} = 0$. The first, $\mathbf{u}(t) = \cos(\omega\,t)\,\mathbf{v} \equiv \mathbf{v}$ is a constant solution, i.e., an equilibrium configuration of the system. Thus, an unstable system does not have a unique equilibrium position, since every null eigenvector $\mathbf{v} \in \ker K$ is a constant solution. On the other hand, the second solution, $\mathbf{u}(t) = \sin(\omega\,t)\,\mathbf{v} \equiv \mathbf{0}$, is trivial, and so doesn't help in constructing the requisite $2n$ linearly independent basis solutions. To find the missing

Figure 10.9. A Triatomic Molecule.

solution(s), let us again argue in analogy with the scalar case (10.79), and try $\mathbf{u}(t) = t\,\mathbf{v}$. Fortunately, this works, since $\dot{\mathbf{u}} = \mathbf{v}$, so $\ddot{\mathbf{u}} = \mathbf{0}$. Also, $K\mathbf{u} = t\,K\mathbf{v} = \mathbf{0}$, and hence $\mathbf{u}(t) = t\,\mathbf{v}$ solves the system $\ddot{\mathbf{u}} + K\mathbf{u} = \mathbf{0}$. Therefore, to each element of the kernel of the stiffness matrix — i.e., each rigid motion and mechanism — there is a two-dimensional family of solutions

$$\mathbf{u}(t) = (c\,t + d)\,\mathbf{v}. \tag{10.80}$$

When $c = 0$, the solution $\mathbf{u}(t) = d\,\mathbf{v}$ reduces to a constant equilibrium; when $c \neq 0$, it is moving off to ∞ with constant velocity in the null direction $\mathbf{v}$, and so represents an unstable mode of the system. The general solution will be a linear superposition of the vibrational modes corresponding to the positive eigenvalues and the unstable linear motions corresponding to the independent null eigenvectors.

Remark. If the null direction $\mathbf{v} \in \ker K$ represents a rigid translation, then the entire structure will move in that direction. If $\mathbf{v}$ represents an infinitesimal rotation, then, because our model is based on a linear approximation to the true nonlinear motions, the individual masses will move along straight lines, which are the tangent approximations to the circular motion that occurs in the true physical, nonlinear regime. We refer to the earlier discussion in Chapter 6 for details. Finally, if we excite a mechanism, then the masses will again follow straight lines, moving in different directions, whereas in the nonlinear real world the masses may move along much more complicated curved trajectories. For small motions, the distinction is not so important, while larger displacements, such as occur in the design of robots, platforms, and autonomous vehicles, [**57**, **75**], will require dealing with the vastly more complicated nonlinear dynamical equations.

Example 10.41. Consider a system of three unit masses connected in a line by two unit springs, but not attached to any fixed supports, as illustrated in Figure 10.9. This chain could be viewed as a simplified model of an (unbent) triatomic molecule that is allowed to move only in the vertical direction. The incidence matrix is $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$, and, since we are dealing with unit springs, the stiffness matrix is

$$K = A^T A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

The eigenvalues and eigenvectors of K are easily found:

$$\lambda_1 = 0, \qquad \lambda_2 = 1, \qquad \lambda_3 = 3,$$
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Each positive eigenvalue provides two trigonometric solutions, while the zero eigenvalue leads to solutions that are constant or depend linearly on t. This yields the required six basis solutions:

$$\mathbf{u}_1(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{u}_3(t) = \begin{pmatrix} \cos t \\ 0 \\ -\cos t \end{pmatrix}, \qquad \mathbf{u}_5(t) = \begin{pmatrix} \cos\sqrt{3}\,t \\ -2\cos\sqrt{3}\,t \\ \cos\sqrt{3}\,t \end{pmatrix},$$
$$\mathbf{u}_2(t) = \begin{pmatrix} t \\ t \\ t \end{pmatrix}, \qquad \mathbf{u}_4(t) = \begin{pmatrix} \sin t \\ 0 \\ -\sin t \end{pmatrix}, \qquad \mathbf{u}_6(t) = \begin{pmatrix} \sin\sqrt{3}\,t \\ -2\sin\sqrt{3}\,t \\ \sin\sqrt{3}\,t \end{pmatrix}.$$

The first solution $\mathbf{u}_1(t)$ is a constant, equilibrium mode, where the masses rest at a fixed common distance from their reference positions. The second solution $\mathbf{u}_2(t)$ is the unstable mode, corresponding to a uniform rigid translation of the molecule that does not stretch the interconnecting springs. The final four solutions represent vibrational modes. In the first pair, $\mathbf{u}_3(t), \mathbf{u}_4(t)$, the two outer masses move in opposing directions, while the middle mass remains fixed, while the final pair, $\mathbf{u}_5(t), \mathbf{u}_6(t)$ has the two outer masses moving in tandem, while the inner mass moves twice as far in the opposite direction. The general solution is a linear combination of the six normal modes,

$$\mathbf{u}(t) = c_1\,\mathbf{u}_1(t) + \cdots + c_6\,\mathbf{u}_6(t), \tag{10.81}$$

and corresponds to the entire molecule moving at a fixed velocity while the individual masses perform a quasi-periodic vibration.

Let us see whether we can predict the motion of the molecule from its initial conditions

$$\mathbf{u}(0) = \mathbf{a}, \qquad \dot{\mathbf{u}}(0) = \mathbf{b},$$

where $\mathbf{a} = (\,a_1, a_2, a_3\,)^T$ indicates the initial displacements of the three atoms, while $\mathbf{b} = (\,b_1, b_2, b_3\,)^T$ are their initial velocities. Substituting the solution formula (10.81) leads to the two linear systems

$$c_1\,\mathbf{v}_1 + c_3\,\mathbf{v}_2 + c_5\,\mathbf{v}_3 = \mathbf{a}, \qquad c_2\,\mathbf{v}_1 + c_4\,\mathbf{v}_2 + \sqrt{3}\,c_6\,\mathbf{v}_3 = \mathbf{b},$$

for the coefficients $c_1, \ldots, c_6$. As in (10.77), we can use the orthogonality of the eigenvectors to immediately compute the coefficients:

$$c_1 = \frac{\mathbf{a}\cdot\mathbf{v}_1}{\|\mathbf{v}_1\|^2} = \frac{a_1+a_2+a_3}{3}, \quad c_3 = \frac{\mathbf{a}\cdot\mathbf{v}_2}{\|\mathbf{v}_2\|^2} = \frac{a_1-a_3}{2}, \quad c_5 = \frac{\mathbf{a}\cdot\mathbf{v}_3}{\|\mathbf{v}_3\|^2} = \frac{a_1-2a_2+a_3}{6},$$
$$c_2 = \frac{\mathbf{b}\cdot\mathbf{v}_1}{\|\mathbf{v}_1\|^2} = \frac{b_1+b_2+b_3}{3}, \quad c_4 = \frac{\mathbf{b}\cdot\mathbf{v}_2}{\|\mathbf{v}_2\|^2} = \frac{b_1-b_3}{2}, \quad c_6 = \frac{\mathbf{b}\cdot\mathbf{v}_3}{\sqrt{3}\,\|\mathbf{v}_3\|^2} = \frac{b_1-2b_2+b_3}{6\sqrt{3}}.$$

In particular, the unstable translational mode is excited if and only if $c_2 \neq 0$, and this occurs if and only if there is a nonzero net initial velocity of the molecule: $b_1 + b_2 + b_3 \neq 0$. In this case, the vibrating molecule will run off to ∞ at a uniform velocity $c = c_2 = \frac{1}{3}(b_1 + b_2 + b_3)$ equal to the average of the individual initial velocities. On the other hand, if $b_1 + b_2 + b_3 = 0$, then the atoms will vibrate quasi-periodically, with frequencies 1 and $\sqrt{3}$, around its fixed center of mass.

The observations established in this example hold, in fact, in complete generality. Let us state the result, leaving the details of the proof as an exercise for the reader.

Theorem 10.42. The general solution to an unstable second order linear system $\ddot{\mathbf{u}} + K\mathbf{u} = \mathbf{0}$ with positive semi-definite coefficient matrix $K \geq 0$ is a linear combination of a quasi-periodic or periodic vibrations and a uniform linear motion at a fixed velocity in the direction of a null eigenvector $\mathbf{v} \in \ker K$. In particular, the system will just vibrate around a fixed position if and only if the initial velocity $\dot{\mathbf{u}}(t_0) \in (\ker K)^\perp = \operatorname{img} K$ lies in the image of the coefficient matrix.

As in Chapter 6, the unstable modes $\mathbf{v} \in \ker K$ correspond to either rigid motions or to mechanisms of the structure. Thus, to prevent a structure from exhibiting an unstable motion, one has to ensure that the initial velocity is orthogonal to all of the unstable modes. (The value of the initial position is not an issue.) This is the dynamical counterpart of the requirement that an external force be orthogonal to all unstable modes in order to maintain equilibrium in the structure, as in Theorem 6.8.

Systems with Differing Masses

When a chain or structure has different masses at the nodes, the (unforced) Newtonian equations of motion take the more general form

$$M\,\frac{d^2\mathbf{u}}{dt^2} + K\mathbf{u} = \mathbf{0}, \qquad \text{or, equivalently,} \qquad \frac{d^2\mathbf{u}}{dt^2} = -\,M^{-1}K\mathbf{u} = -\,P\,\mathbf{u}. \tag{10.82}$$

The mass matrix M is always positive definite (and, almost always, diagonal, although this is not required by the general theory), while the stiffness matrix $K = A^T C\,A$ is either positive definite or, in the unstable situation when $\ker A \neq \{\mathbf{0}\}$, positive semi-definite. The coefficient matrix

$$P = M^{-1}K = M^{-1}A^T C\,A \tag{10.83}$$

is *not* in general symmetric, and so we cannot directly apply the preceding constructions. However, P does have the more general self-adjoint form (7.85) based on the weighted inner products

$$\langle\, \mathbf{u}\,,\widetilde{\mathbf{u}}\,\rangle = \mathbf{u}^T M\,\widetilde{\mathbf{u}}, \qquad\qquad \langle\!\langle\, \mathbf{v}\,,\widetilde{\mathbf{v}}\,\rangle\!\rangle = \mathbf{v}^T C\,\widetilde{\mathbf{v}}, \tag{10.84}$$

on, respectively, the domain and codomain of the (reduced) incidence matrix A. Moreover, in the stable case when $\ker A = \{\mathbf{0}\}$, the matrix P is positive definite in the generalized sense of Definition 7.59.

To solve the system of differential equations, we substitute the same trigonometric solution ansatz $\mathbf{u}(t) = \cos(\omega\,t)\,\mathbf{v}$. This results in a *generalized eigenvalue equation*

$$K\,\mathbf{v} = \lambda\,M\,\mathbf{v}, \qquad \text{or, equivalently,} \qquad P\,\mathbf{v} = \lambda\,\mathbf{v}, \qquad \text{with} \qquad \lambda = \omega^2. \tag{10.85}$$

The matrix M assumes the role of the identity matrix in the standard eigenvalue equation (8.13), and λ is a generalized eigenvalue if and only if it satisfies the generalized characteristic equation

$$\det(K - \lambda\,M) = 0. \tag{10.86}$$

According to Exercise 8.5.8, if $M > 0$ and $K > 0$, then all the generalized eigenvalues are real and non-negative. Moreover the generalized eigenvectors form an orthogonal basis of

$\mathbb{R}^n$, but now with respect to the weighted inner product (10.84) prescribed by the mass matrix M. The general solution is a quasi-periodic linear combination of the eigensolutions, of the same form as in (10.75). In the unstable case, when $K \geq 0$ (but M necessarily remains positive definite), one must include enough generalized null eigenvectors to span $\ker K$, each of which leads to an unstable mode of the form (10.80). Further details are relegated to the exercises.

Exercises

10.5.16. Find the general solution to the following systems. Distinguish between the vibrational and unstable modes. What constraints on the initial conditions ensure that the unstable modes are not excited? (*a*) $\dfrac{d^2u}{dt^2} = -4u - 2v$, $\dfrac{d^2v}{dt^2} = -2u - v$. (*b*) $\dfrac{d^2u}{dt^2} = -u - 3v$, $\dfrac{d^2v}{dt^2} = -3u - 9v$. (*c*) $\dfrac{d^2u}{dt^2} = -2u + v - 2w$, $\dfrac{d^2v}{dt^2} = u - v$, $\dfrac{d^2w}{dt^2} = -2u - 4w$. (*d*) $\dfrac{d^2u}{dt^2} = -u + v - 2w$, $\dfrac{d^2v}{dt^2} = u - v + 2w$, $\dfrac{d^2w}{dt^2} = -2u + 2v - 4w$.

10.5.17. Let $K = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$. (*a*) Find an orthogonal matrix Q and a diagonal matrix Λ such that $K = Q\,\Lambda\,Q^T$. (*b*) Is K positive definite? (*c*) Solve the second order system $\dfrac{d^2\mathbf{u}}{dt^2} = A\,\mathbf{u}$ subject to the initial conditions $\mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\dfrac{d\mathbf{u}}{dt}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.
(*d*) Is your solution periodic? If your answer is yes, indicate the period.
(*e*) Is the general solution to the system periodic?

10.5.18. Answer Exercise 10.5.17 when $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

10.5.19. Compare the solutions to the mass–spring system (10.65) with tiny spring constant $k = \varepsilon \ll 1$ to those of the completely unrestrained system (10.78). Are they close? Discuss.

♡ 10.5.20. Discuss the three-dimensional motions of the triatomic molecule of Example 10.41. Are the vibrational frequencies the same as those of the one-dimensional model?

♡ 10.5.21. So far, our mass–spring chains have been allowed to move only in the vertical direction. (*a*) Set up the system governing the planar motions of a mass–spring chain consisting of two unit masses attached to top and bottom supports by unit springs, where the masses are allowed to move in the longitudinal and transverse directions. Compare the resulting vibrational frequencies with the one-dimensional case. (*b*) Repeat the analysis when the bottom support is removed. (*c*) Can you make any conjectures concerning the planar motions of general mass–spring chains?

♠ 10.5.22. Find the vibrational frequencies and instabilities of the following structures, assuming they have unit masses at all the nodes. Explain in detail how each normal mode moves the structure: (*a*) the three bar planar structure in Figure 6.13; (*b*) its reinforced version in Figure 6.16; (*c*) the swing set in Figure 6.18.

♠ 10.5.23. Assuming unit masses at the nodes, find the vibrational frequencies and describe the normal modes for the following planar structures. What initial conditions will not excite its instabilities? (*a*) An equilateral triangle; (*b*) a square; (*c*) a regular hexagon.

♠ 10.5.24. Answer Exercise 10.5.23 for the three-dimensional motions of a regular tetrahedron.

♡ 10.5.25. (*a*) Show that if a structure contains all unit masses and bars with unit stiffness, $c_i = 1$, then its frequencies of vibration are the nonzero singular values of the reduced incidence matrix. (*b*) How would you recognize when a structure is close to being unstable?

10.5.26. Prove that if the initial velocity satisfies $\dot{\mathbf{u}}(t_0) = \mathbf{b} \in \operatorname{coimg} A$, then the solution to the initial value problem (10.70, 76) remains bounded.

10.5.27. Find the general solution to the system (10.82) for the following matrix pairs:

(*a*) $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $K = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$, (*b*) $M = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, $K = \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$,

(*c*) $M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, (*d*) $M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$, $K = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 6 & 3 \\ -1 & 3 & 9 \end{pmatrix}$,

(*e*) $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $K = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$, (*f*) $M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $K = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 1 \end{pmatrix}$.

10.5.28. A mass–spring chain consists of two masses, $m_1 = 1$ and $m_2 = 2$, connected to top and bottom supports by identical springs with unit stiffness. The upper mass is displaced by a unit distance. Find the subsequent motion of the system.

10.5.29. Answer Exercise 10.5.28 when the bottom support is removed.

♣ 10.5.30. (*a*) A water molecule consists of two hydrogen atoms connected at an angle of 105° to an oxygen atom whose relative mass is 16 times that of each of the hydrogen atoms. If the molecular bonds are modeled as linear unit springs, determine the fundamental frequencies and describe the corresponding vibrational modes. (*b*) Do the same for a carbon tetrachloride molecule, in which the chlorine atoms, with atomic weight 35, are positioned on the vertices of a regular tetrahedron and the carbon atom, with atomic weight 12, is at the center. (*c*) Finally try a benzene molecule, consisting of 6 carbon atoms arranged in a regular hexagon. In this case, every other bond is double strength because two electrons are shared. (Ignore the six extra hydrogen atoms for simplicity.)

♡ 10.5.31. Repeat Exercise 10.5.21 for fully 3-dimensional motions of the chain.

♠ 10.5.32. Suppose you have masses $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, connected to top and bottom supports by identical unit springs. Does rearranging the order of the masses change the fundamental frequencies? If so, which order produces the fastest vibrations?

♢ 10.5.33. Suppose M is a nonsingular matrix. Prove that λ is a generalized eigenvalue of the matrix pair K, M if and only if it is an ordinary eigenvalue of the matrix $P = M^{-1}K$. How are the eigenvectors related? How are the characteristic equations related?

10.5.34. Suppose that $\mathbf{u}(t)$ is a solution to (10.82). Let $N = \sqrt{M}$ denote the positive definite square root of the mass matrix M, as defined in Exercise 8.5.27. (*a*) Prove that the "weighted" displacement vector $\widetilde{\mathbf{u}}(t) = N\,\mathbf{u}(t)$ solves $d^2\widetilde{\mathbf{u}}/dt^2 = -\widetilde{K}\,\widetilde{\mathbf{u}}$, where $\widetilde{K} = N^{-1}K\,N^{-1}$ is a symmetric, positive semi-definite matrix. (*b*) Explain in what sense this can serve as an alternative to the generalized eigenvector solution method.

♢ 10.5.35. Provide the details of the proof of Theorem 10.42.

♢ 10.5.36. State and prove the counterpart of Theorem 10.42 for the variable mass system (10.82).

Friction and Damping

We have not yet allowed friction to affect the motion of our dynamical equations. In the standard physical model, the frictional force on a mass in motion is directly proportional

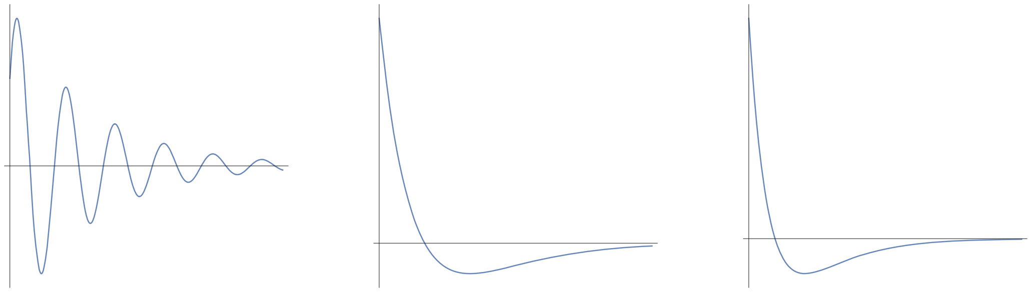


Figure 10.10. Damped Vibrations.

to its velocity, [**31**]. For the simplest case of a single mass attached to a spring, one amends the balance of forces in the undamped Newtonian equation (10.65) to obtain

$$m \frac{d^2 u}{dt^2} + \beta \frac{du}{dt} + k\, u = 0. \tag{10.87}$$

As before, $m > 0$ is the mass, and $k > 0$ the spring stiffness, while $\beta > 0$ measures the effect of a velocity-dependent frictional force — the larger the value of β, the greater the frictional force.

The solution of this more general second order homogeneous linear ordinary differential equation is found by substituting the usual exponential ansatz $u(t) = e^{\lambda t}$, reducing it to the quadratic characteristic equation

$$m\,\lambda^2 + \beta\,\lambda + k = 0. \tag{10.88}$$

Assuming that $m, \beta, k > 0$, there are three possible cases:

Underdamped: If $0 < \beta < 2\sqrt{m\,k}$, then (10.88) has two complex-conjugate roots:

$$\lambda \;=\; -\,\frac{\beta}{2m} \;\pm\; \mathrm{i}\;\frac{\sqrt{4\,m\,k - \beta^2}}{2m} \;=\; -\,\mu \pm \mathrm{i}\,\nu\,. \tag{10.89}$$

The general solution to the differential equation,

$$u(t) = e^{-\mu t}\big(\,c_1 \cos \nu\, t + c_2 \sin \nu\, t\,\big) = r\,e^{-\mu t} \cos(\nu\, t - \delta), \tag{10.90}$$

represents a damped periodic motion. The mass continues to oscillate at a fixed frequency

$$\nu \;=\; \frac{\sqrt{4\,m\,k - \beta^2}}{2m} \;=\; \sqrt{\frac{k}{m} - \frac{\beta^2}{4m^2}}\;, \tag{10.91}$$

but the vibrational amplitude $r\,e^{-\mu t}$ decays to zero at an exponential rate as $t \to \infty$. Observe that, in a rigorous mathematical sense, the mass never quite returns to equilibrium, although in the real world, after a sufficiently long time the residual vibrations are not noticeable, and equilibrium is physically (but not mathematically) achieved. The rate of decay, $\mu = \beta/(2\,m)$, is directly proportional to the friction, and inversely proportional to the mass. Thus, greater friction and/or less mass will accelerate the return to equilibrium. The friction also has an effect on the vibrational frequency (10.91); the larger β is, the slower the oscillations become and the more rapid the damping effect. As the friction approaches the critical threshold $\beta_\star = 2\sqrt{m\,k}$, the vibrational frequency goes to zero, $\nu \to 0$, and so the oscillatory period $2\,\pi/\nu$ becomes longer and longer.

Overdamped: If $\beta > 2\sqrt{m\,k}$, then the characteristic equation (10.88) has two negative real roots

$$\lambda_1 \;=\; -\,\frac{\beta + \sqrt{\beta^2 - 4\,m\,k}}{2\,m} \;<\; \lambda_2 \;=\; -\,\frac{\beta - \sqrt{\beta^2 - 4\,m\,k}}{2\,m} \;<\; 0.$$

The solution

$$u(t) = c_1\,e^{\lambda_1 t} + c_2\,e^{\lambda_2 t} \tag{10.92}$$

is a linear combination of two decaying exponentials. An overdamped system models the motion of, say, a mass in a vat of molasses. Its "vibration" is so slow that it can pass at most once through the equilibrium position, and then only when its initial velocity is relatively large. In the long term, the first exponential in the solution (10.92) will go to zero faster, and hence, as long as $c_2 \neq 0$, the overall decay rate of the solution is governed by the dominant (least negative) eigenvalue λ_2.

Critically Damped: The borderline case occurs when $\beta = \beta_\star = 2\sqrt{m\,k}$, which means that the characteristic equation (10.88) has only a single negative real root:

$$\lambda_1 = -\,\frac{\beta}{2\,m}\,.$$

In this case, our ansatz supplies only one exponential solution $e^{\lambda_1 t} = e^{-\beta\,t/(2\,m)}$. A second independent solution is obtained by multiplication by t, leading to the general solution

$$u(t) = (c_1\,t + c_2)e^{-\beta\,t/(2\,m)}. \tag{10.93}$$

Even though the formula looks quite different, its qualitative behavior is very similar to the overdamped case. The factor t plays an unimportant role, since the asymptotics of this solution are almost entirely governed by the decaying exponential function. This represents a non-vibrating solution that has the slowest possible decay rate, since any further reduction of the frictional coefficient will allow a damped, slowly oscillatory vibration to appear.

In all three cases, the zero equilibrium solution is globally asymptotically stable. Physically, no matter how small the frictional contribution, all solutions to the unforced system eventually return to equilibrium as friction eventually overwhelms the motion.

This concludes our discussion of the scalar case. Similar considerations apply to mass–spring chains, and to two- and three-dimensional structures. A frictionally damped structure is modeled by a second order system of the form

$$M\,\frac{d^2\mathbf{u}}{dt^2} + B\,\frac{d\mathbf{u}}{dt} + K\,\mathbf{u} = \mathbf{0}, \tag{10.94}$$

where the mass matrix M and the matrix of frictional coefficients B are both diagonal and positive definite, while the stiffness matrix $K = A^T C\,A \geq 0$ is a positive semi-definite Gram matrix constructed from the (reduced) incidence matrix A. Under these assumptions, it can be proved that the zero equilibrium solution is globally asymptotically stable. However, the mathematical details in this case are sufficiently intricate that we shall leave their analysis as an advanced project for the highly motivated student.

Exercises

10.5.37. Consider the overdamped mass–spring equation $\ddot{u} + 6\,\dot{u} + 5\,u = 0$. If the mass starts out a distance 1 away from equilibrium, how large must the initial velocity be in order that it pass through equilibrium once?

10.5.38. Solve the following mass–spring initial value problems, and classify as to (i) overdamped, (ii) critically damped, (iii) underdamped, or (iv) undamped:
(a) $\ddot{u} + 6\dot{u} + 9u = 0,\ u(0) = 0,\ \dot{u}(0) = 1.$ (b) $\ddot{u} + 2\dot{u} + 10u = 0,\ u(0) = 1,\ \dot{u}(0) = 1.$
(c) $\ddot{u} + 16u = 0,\ u(1) = 0,\ \dot{u}(1) = 1.$ (d) $\ddot{u} + 3\dot{u} + 9u = 0,\ u(0) = 0,\ \dot{u}(0) = 1.$
(e) $2\ddot{u} + 3\dot{u} + u = 0,\ u(0) = 2,\ \dot{u}(0) = 0.$ (f) $\ddot{u} + 6\dot{u} + 10u = 0,\ u(0) = 3,\ \dot{u}(0) = -2.$

10.5.39. (a) A mass weighing 16 pounds stretches a spring 6.4 feet. Assuming no friction, determine the equation of motion and the natural frequency of vibration of the mass–spring system. Use the value $g = 32\,\mathrm{ft/sec}^2$ for the gravitational acceleration. (b) The mass–spring system is placed in a jar of oil, whose frictional resistance equals the speed of the mass. Assume the spring is stretched an additional 2 feet from its equilibrium position and let go. Determine the motion of the mass. (c) Is the system overdamped or underdamped? Are the vibrations more rapid or less rapid than in the undamped system?

10.5.40. Suppose you convert the second order equation (10.87) into its phase plane equivalent. What are the phase portraits corresponding to (a) undamped, (b) underdamped, (c) critically damped, and (d) overdamped motion?

♢ 10.5.41. (a) Prove that, given a non-constant solution to an overdamped mass–spring system, there is at most one time where $u(t_\star) = 0$. (b) Is this statement also valid in the critically damped case?

10.5.42. Discuss the possible behaviors of a mass moving in a frictional medium that is not attached to a spring, i.e., set $k = 0$ in (10.87).

10.6 Forcing and Resonance

Up until now, our physical system has been left free to vibrate on its own. Let us investigate what happens when we shake it. In this section, we will consider the effects of periodic external forcing on both undamped and damped systems.

The simplest case is that of a single mass connected to a spring that has no frictional damping. We append an external forcing function $f(t)$ to the homogeneous (unforced) equation (10.65), leading to the inhomogeneous second order equation

$$m\,\frac{d^2u}{dt^2} + k\,u = f(t), \tag{10.95}$$

in which $m > 0$ is the mass and $k > 0$ the spring stiffness. We are particularly interested in the case of periodic forcing

$$f(t) = \alpha\cos\gamma t \tag{10.96}$$

of frequency $\gamma > 0$ and amplitude α. To find a particular solution to (10.95–96), we use the method of undetermined coefficients[†] which tells us to guess a trigonometric solution ansatz of the form

$$u^\star(t) = a\cos\gamma t + b\sin\gamma t, \tag{10.97}$$

where a, b are constants to be determined. Substituting (10.97) into the differential equation produces

$$m\,\frac{d^2u^\star}{dt^2} + k\,u^\star = a\,(k - m\gamma^2)\cos\gamma t + b\,(k - m\gamma^2)\sin\gamma t = \alpha\cos\gamma t.$$

[†] One can also use variation of parameters, although the intervening calculations are slightly more complicated.

We can solve for
$$a = \frac{\alpha}{k - m\gamma^2} = \frac{\alpha}{m(\omega^2 - \gamma^2)}\,, \qquad\qquad b = 0, \tag{10.98}$$
where
$$\omega = \sqrt{\frac{k}{m}} \tag{10.99}$$
refers to the natural, unforced vibrational frequency of the system. The solution (10.98) is valid provided its denominator is nonzero:
$$k - m\gamma^2 = m(\omega^2 - \gamma^2) \neq 0.$$
Therefore, as long as the forcing frequency is *not* equal to the system's natural frequency, i.e., $\gamma \neq \omega$, there exists a particular solution
$$u^\star(t) = a\,\cos\gamma\,t = \frac{\alpha}{m(\omega^2 - \gamma^2)}\,\cos\gamma\,t \tag{10.100}$$
that vibrates at the same frequency as the forcing function.

The general solution to the inhomogeneous system (10.95) is found, as usual, by adding in an arbitrary solution (10.66) to the homogeneous equation, yielding
$$u(t) = \frac{\alpha}{m(\omega^2 - \gamma^2)}\,\cos\gamma\,t \;+\; r\,\cos(\omega\,t - \delta), \tag{10.101}$$
where r and δ are determined by the initial conditions. The solution is therefore a quasi-periodic combination of two simple periodic motions — the second, vibrating with frequency ω, represents the internal or natural vibrations of the system, while the first, with frequency γ, represents the response to the periodic forcing. Due to the factor $\omega^2 - \gamma^2$ in the denominator of the latter, the closer the forcing frequency is to the natural frequency, the larger the overall amplitude of the response.

Suppose we start the mass at equilibrium at the initial time $t_0 = 0$, so the initial conditions are
$$u(0) = 0, \qquad\qquad \dot u(0) = 0. \tag{10.102}$$
Substituting (10.101) and solving for r, δ, we find that
$$r = -\,\frac{\alpha}{m(\omega^2 - \gamma^2)}\,, \qquad\qquad \delta = 0.$$
Thus, the solution to the initial value problem can be written in the form
$$u(t) = \frac{\alpha}{m(\omega^2 - \gamma^2)}\,\bigl(\cos\gamma\,t - \cos\omega\,t\bigr) = \frac{2\,\alpha}{m(\omega^2 - \gamma^2)}\,\sin\left(\frac{\omega + \gamma}{2}\,t\right)\,\sin\left(\frac{\omega - \gamma}{2}\,t\right), \tag{10.103}$$
where we have employed a standard trigonometric identity, cf. Exercise 3.6.17. The first trigonometric factor, $\sin\frac{1}{2}(\omega + \gamma)\,t$, represents a periodic motion at a frequency equal to the average of the natural and the forcing frequencies. If the forcing frequency γ is close to the natural frequency ω, then the second factor, $\sin\frac{1}{2}(\omega - \gamma)\,t$, has a much smaller frequency, and so oscillates on a much longer time scale. As a result, it *modulates* the amplitude of the more rapid vibrations, and is responsible for the phenomenon of *beats*, in which a rapid vibration is subject to a slowly varying amplitude. An everyday illustration of beats is two tuning forks that have nearby pitches. When they vibrate close to each other, the sound

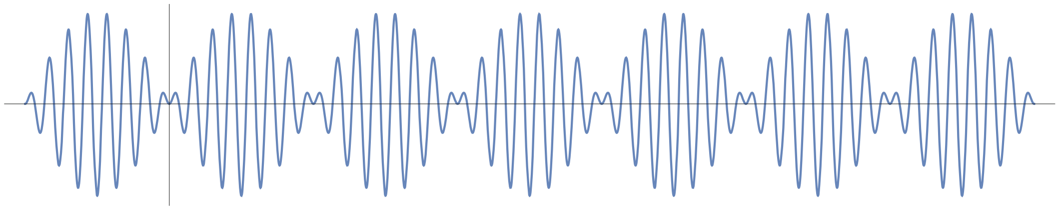


Figure 10.11. Beats in a Periodically Forced Vibration.

you hear waxes and wanes in intensity. As a mathematical example, Figure 10.11 displays the graph of the particular function

$$\cos 14\,t - \cos 16\,t = 2\,\sin t\ \sin 15\,t$$

on the interval $-\pi \le t \le 6\,\pi$. The slowly varying amplitude $2\,\sin t$ is clearly visible as the envelope of the relatively rapid vibrations at frequency 15.

When we force the system at exactly the natural frequency $\gamma = \omega$, the trigonometric ansatz (10.97) no longer works. This is because both terms are now solutions to the homogeneous equation, and so cannot be combined to form a solution to the inhomogeneous version. In this situation, there is a simple modification to the ansatz, namely multiplication by t, that does the trick. Substituting

$$u^\star(t) = a\,t\,\cos\omega\,t + b\,t\,\sin\omega\,t \tag{10.104}$$

into the differential equation (10.95), we obtain

$$m\,\frac{d^2u^\star}{dt^2} + k\,u^\star = -\,2\,a\,m\,\omega\,\sin\omega\,t + 2\,b\,m\,\omega\,\cos\omega\,t = \alpha\cos\omega\,t,$$

provided

$$a = 0, \qquad b = \frac{\alpha}{2\,m\,\omega}, \qquad \text{and so} \qquad u^\star(t) = \frac{\alpha}{2\,m\,\omega}\,t\,\sin\omega\,t.$$

Combining the resulting particular solution with the solution to the homogeneous equation leads to the general solution

$$u(t) = \frac{\alpha}{2\,m\,\omega}\,t\,\sin\omega\,t \ + \ r\cos(\omega\,t - \delta). \tag{10.105}$$

Both terms vibrate with frequency ω, but the amplitude of the first grows larger and larger as $t \to \infty$. As illustrated in Figure 10.12, the mass will oscillate more and more wildly. In this situation, the system is said to be in *resonance*, and the increasingly large oscillations are provoked by forcing it at its natural frequency ω. In a physical apparatus, once the amplitude of resonant vibrations stretches the spring beyond its elastic limits, the linear Hooke's Law model is no longer applicable, and either the spring breaks or the system enters a nonlinear regime.

Furthermore, if we are very close to resonance, the oscillations induced by the particular solution (10.103) will have extremely large, although bounded, amplitude. The lesson is, never force a system at or close to its natural frequency (or frequencies) of vibration. A classic example was the 1831 collapse of a bridge when a British infantry regiment marched in unison across it, apparently inducing a resonant vibration of the structure. The bridge in question was an early example of the suspension style, similar to that pictured in Figure 10.13. Learning their lesson, soldiers nowadays no longer march in step across

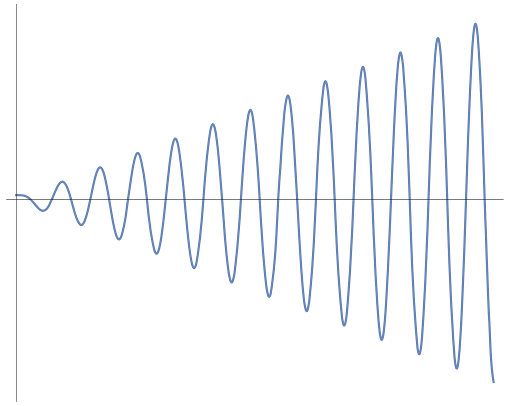


Figure 10.12. Resonance.

Figure 10.13. The Albert Bridge in London.

bridges — as reminded by the sign in the photo in Figure 10.13. An even more dramatic case is the 1940 Tacoma Narrows Bridge disaster, when the vibrations due to a strong wind caused the bridge to oscillate wildly and break apart! The collapse was caught on film, which can be found on YouTube, and is extremely impressive. The traditional explanation was the excitement of the bridge's resonant frequencies, although later studies revealed a more sophisticated mathematical explanation of the collapse, [**22**; p. 118]. But resonance is not exclusively harmful. In a microwave oven, the electromagnetic waves are tuned to the resonant frequencies of water molecules so as to excite them into large vibrations and thereby heat up your dinner. Blowing into a clarinet or other wind instrument excites the resonant frequencies in the column of air contained within it, and this produces the musical sound vibrations that we hear.

Frictional effects can partially mollify the extreme behavior near the resonant frequency. The frictionally damped vibrations of a mass on a spring, when subject to periodic forcing, are described by the inhomogeneous differential equation

$$m\,\frac{d^2u}{dt^2} + \beta\,\frac{du}{dt} + k\,u = \alpha \cos \gamma\, t. \tag{10.106}$$

Let us assume that the friction is sufficiently small so as to be in the underdamped regime $\beta < 2\sqrt{m\,k}$. Since neither summand solves the homogeneous system, we can use the trigonometric solution ansatz (10.97) to construct the particular solution

$$u^\star(t) = \frac{\alpha}{\sqrt{m^2(\omega^2-\gamma^2)^2+\beta^2\gamma^2}}\,\cos(\gamma\, t - \varepsilon), \qquad \text{where} \qquad \omega = \sqrt{\frac{k}{m}} \tag{10.107}$$

continues to denote the undamped resonant frequency (10.99), while ε, defined by

$$\tan\varepsilon = \frac{\beta\,\gamma}{m(\omega^2-\gamma^2)}\,, \tag{10.108}$$

represents a frictionally induced *phase lag*. Thus, the larger the friction β, the more pronounced the phase lag ε in the response of the system to the external forcing. As the forcing frequency γ increases, so does the phase lag, which attains the value $\frac{1}{2}\pi$ at the resonant frequency $\gamma = \omega$, meaning that the system lags a quarter period behind the forcing, and converges to its maximum $\varepsilon = \pi$ as $\gamma \to \infty$. Thus, the response to a very high frequency forcing is almost exactly out of phase — the mass is moving downwards when the force is pulling it upwards, and vice versa! The amplitude of the persistent response (10.107) is at a maximum at the resonant frequency $\gamma = \omega$, where it takes the value $\alpha/(\beta\,\omega)$. Thus, the smaller the frictional coefficient β (or the slower the resonant frequency ω), the more likely the breakdown of the system due to an overly large response.

The general solution is

$$u(t) = \frac{\alpha}{\sqrt{m^2(\omega^2-\gamma^2)^2+\beta^2\gamma^2}}\,\cos(\gamma\, t - \varepsilon) \;+\; r\,e^{-\mu\, t}\cos(\nu\, t - \delta), \tag{10.109}$$

where $\lambda = \mu \pm \mathrm{i}\,\nu$ are the roots of the characteristic equation, while r, δ are determined by the initial conditions, cf. (10.89). The second term — the solution to the homogeneous equation — is known as the *transient*, since it decays exponentially fast to zero. Thus, at large times, any internal motions of the system that might have been excited by the initial conditions die out, and only the particular solution (10.107) incited by the continued forcing persists.

Exercises

10.6.1. Graph the following functions. Describe the fast oscillatory and beat frequencies: (*a*) $\cos 8t - \cos 9t$, (*b*) $\cos 26t - \cos 24t$, (*c*) $\cos 10t + \cos 9.5t$, (*d*) $\cos 5t - \sin 5.2t$.

10.6.2. Solve the following initial value problems: (*a*) $\ddot u + 36u = \cos 3t$, $u(0) = 0$, $\dot u(0) = 0$. (*b*) $\ddot u + 6\dot u + 9u = \cos t$, $u(0) = 0$, $\dot u(0) = 1$. (*c*) $\ddot u + \dot u + 4u = \cos 2t$, $u(0) = 1$, $\dot u(0) = -1$. (*d*) $\ddot u + 9u = 3\sin 3t$, $u(0) = 1$, $\dot u(0) = -1$. (*e*) $2\ddot u + 3\dot u + u = \cos\frac{1}{2}t$, $u(0) = 3$, $\dot u(0) = -2$. (*f*) $3\ddot u + 4\dot u + u = \cos t$, $u(0) = 0$, $\dot u(0) = 0$.

10.6.3. Solve the following initial value problems. In each case, graph the solution and explain what type of motion is represented. (*a*) $\ddot u + 4\dot u + 40u = 125\cos 5t$, $u(0) = 0$, $\dot u(0) = 0$, (*b*) $\ddot u + 25u = 3\cos 4t$, $u(0) = 1$, $\dot u(0) = 1$, (*c*) $\ddot u + 16u = \sin 4t$, $u(0) = 0$, $\dot u(0) = 0$, (*d*) $\ddot u + 6\dot u + 5u = 25\sin 5t$, $u(0) = 4$, $\dot u(0) = 2$.

10.6.4. A mass $m = 25$ is attached to a unit spring with $k = 1$, and frictional coefficient $\beta = .01$. The spring will break when it moves more than 1 unit. Ignoring the effect of the transient, what is the maximum allowable amplitude α of periodic forcing at frequency $\gamma =$ (*a*) .19? (*b*) .2? (*c*) .21?

10.6.5. (*a*) For what range of frequencies γ can you force the mass in Exercise 10.6.4 with amplitude $\alpha = .5$ without breaking the spring? (*b*) How large should the friction be so that you can safely force the mass at any frequency?

10.6.6. Suppose the mass–spring–oil system of Exercise 10.5.39(*b*) is subject to a periodic external force $2\cos 2t$. Discuss, in as much detail as you can, the long-term motion of the mass.

♢ 10.6.7. Write down the solution $u(t,\gamma)$ to the initial value problem $m\,\dfrac{d^2u}{dt^2} + k\,u = \alpha\cos\gamma t$, $u(0) = \dot u(0) = 0$, for (a) a non-resonant forcing function at frequency $\gamma \neq \omega$; (*b*) a resonant forcing function at frequency $\gamma = \omega$. (*c*) Show that, as $\gamma \to \omega$, the limit of the non-resonant solution equals the resonant solution. Conclude that the solution $u(t,\gamma)$ depends continuously on the frequency γ even though its mathematical formula changes significantly at resonance.

♢ 10.6.8. Justify the solution formulas (10.107) and (10.108).

♠ 10.6.9. (*a*) Does a function of the form $u(t) = a\cos\gamma t - b\cos\omega t$ still exhibit beats when $\gamma \approx \omega$, but $a \neq b$? Use a computer to graph some particular cases and discuss what you observe. (*b*) Explain to what extent the conclusions based on (10.103) do not depend upon the choice of initial conditions (10.102).

Electrical Circuits

The Electrical–Mechanical Correspondence outlined in Section 6.2 will continue to operate in the dynamical universe. The equations governing the equilibria of simple electrical circuits and the mechanical systems such as mass–spring chains and structures all have the same underlying mathematical structure. In a similar manner, although they are based on a completely different set of physical principles, circuits with dynamical currents and voltages are modeled by second order linear dynamical systems of the Newtonian form presented earlier.

In this section, we briefly analyze the very simplest situation: a single loop containing a *resistor* R, an *inductor* L, and a *capacitor* C, as illustrated in Figure 10.14. This basic *RLC circuit* serves as the prototype for more general electrical networks linking various resistors, inductors, capacitors, batteries, voltage sources, etc. (Extending the mathematical analysis to more complicated circuits would make an excellent in-depth student research project.) Let $u(t)$ denote the current in the circuit at time t. We use v_R, v_L, v_C to denote the induced voltages in the three circuit elements; these are prescribed by the fundamental laws of electrical circuitry.

(*a*) First, as we learned in Section 6.2, the resistance $R \geq 0$ is the proportionality factor between voltage and current, so $v_R = R\,u$.

(*b*) The voltage passing through an inductor is proportional to the rate of change in the current. Thus, $v_L = L\,\dot u$, where $L > 0$ is the *inductance*, and the dot indicates time derivative.

(*c*) On the other hand, the current passing through a capacitor is proportional to the rate of change in the voltage, and so $u = C\,\dot v_C$, where $C > 0$ denotes the *capacitance*. We integrate[†] this relation to produce the capacitor voltage $v_C = \displaystyle\int \frac{u(t)}{C}\,dt$.

[†] The integration constant is not important, since we will differentiate the resulting equation.

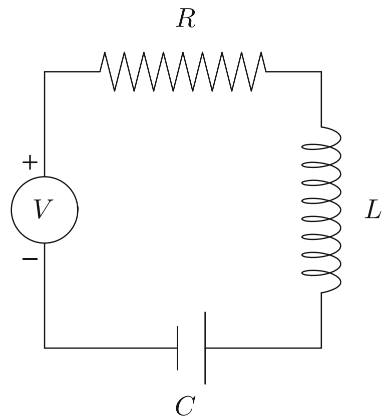


Figure 10.14. The Basic RLC Circuit.

The *Voltage Balance Law* tells us that the total of these individual voltages must equal any externally applied voltage $v_E = F(t)$ coming from, say, a battery or generator. Therefore,

$$v_R + v_L + v_C = v_E.$$

Substituting the preceding formulas, we deduce that the current $u(t)$ in our circuit satisfies the following linear integro-differential equation:

$$L \frac{du}{dt} + R\, u + \int \frac{u}{C}\, dt = F(t). \tag{10.110}$$

We can convert this into a differential equation by differentiating both sides with respect to t. Assuming, for simplicity, that L, R, and C are constant, the result is the linear second order ordinary differential equation

$$L \frac{d^2u}{dt^2} + R \frac{du}{dt} + \frac{1}{C}\, u = f(t) = F'(t). \tag{10.111}$$

The current will be uniquely specified by the initial conditions $u(t_0) = a, \dot{u}(t_0) = b$.

Comparing (10.111) with the equation (10.87) for a mechanically vibrating mass, we see that the correspondence between electrical circuits and mechanical structures developed in Chapter 6 continues to hold in the dynamical regime. The current u corresponds to the displacement. The inductance L plays the role of mass, the resistance R corresponds, as before, to friction, while the reciprocal $1/C$ of capacitance is analogous to the spring stiffness. Thus, all of our analytical conclusions regarding stability of equilibria, qualitative behavior, solution formulas, etc., that we established in the mechanical context can, suitably re-interpreted, be immediately applied to electrical circuit theory.

In particular, an RLC circuit is *underdamped* if $R^2 < 4L/C$, and the current $u(t)$ oscillates with frequency

$$\nu = \sqrt{\frac{1}{C\,L} - \frac{R^2}{4L^2}}\,, \tag{10.112}$$

while dying off to zero at an exponential rate $e^{-Rt/(2L)}$. In the overdamped and critically damped cases $R^2 \geq 4L/C$, the resistance in the circuit is so large that the current merely decays to zero at an exponential rate and no longer exhibits any oscillatory behavior. Attaching an alternating current source $F(t) = \alpha \cos \gamma\, t$ to the circuit can induce a catastrophic resonance if there is no resistance and the forcing frequency is equal to the circuit's natural frequency.

Exercises

10.6.10. Classify the following RLC circuits as (i) underdamped, (ii) critically damped, or (iii) overdamped: (a) $R=1,\ L=2,\ C=4$, (b) $R=4,\ L=3,\ C=1$, (c) $R=2,\ L=3,\ C=3$, (d) $R=4,\ L=10,\ C=2$, (e) $R=1,\ L=1,\ C=3$.

10.6.11. Find the current in each of the unforced RLC circuits in Exercise 10.6.10 induced by the initial data $u(0)=1,\ \dot u(0)=0$.

10.6.12. A circuit with $R=1$, $L=2$, $C=4$, includes an alternating current source $F(t)=25\cos 2t$. Find the solution to the initial value problem $u(0)=1,\ \dot u(0)=0$.

10.6.13. A superconducting LC circuit has no resistance: $R=0$. Discuss what happens when the circuit is wired to an alternating current source $F(t)=\alpha\cos\gamma t$.

10.6.14. A circuit with $R=.002$, $L=12.5$, and $C=50$ can carry a maximum current of 250. Ignoring the effect of the transient, what is the maximum allowable amplitude α of an applied periodic current $F(t)=\alpha\cos\gamma t$ at frequency $\gamma=$ (a) $.04$? (b) $.05$? (c) $.1$?

10.6.15. Given the circuit in Exercise 10.6.14, over what range of frequencies γ can you supply a unit amplitude periodic current source?

10.6.16. How large should the resistance in the circuit in Exercise 10.6.14 be so that you can safely apply any unit amplitude periodic current?

Forcing and Resonance in Systems

Let us conclude by briefly discussing the effect of periodic forcing on a system of second order ordinary differential equations. Periodically forcing an undamped mass–spring chain or structure, or a resistanceless electrical network, leads to a second order system of the form

$$M\,\frac{d^2\mathbf{u}}{dt^2}+K\,\mathbf{u}=\cos(\gamma\,t)\,\mathbf{a}. \tag{10.113}$$

Here $M>0$ and $K\geq 0$ are $n\times n$ matrices as above, cf. (10.82), while $\mathbf{a}\in\mathbb{R}^n$ is a constant vector representing both a magnitude and a "direction" of the forcing and γ is the forcing frequency. Superposition is used to determine the effect of several such forcing functions. As always, the solution to the inhomogeneous system is composed of one particular response to the external force combined with the general solution to the homogeneous system, which, in the stable case $K>0$, is a quasi-periodic combination of the normal vibrational modes.

To find a particular solution to the inhomogeneous system, let us try the trigonometric ansatz

$$\mathbf{u}^\star(t)=\cos(\gamma\,t)\,\mathbf{w} \tag{10.114}$$

in which $\mathbf{w}$ is a constant vector. Substituting into (10.113) leads to a linear algebraic system

$$(K-\mu\,M)\,\mathbf{w}=\mathbf{a},\qquad\text{where}\qquad \mu=\gamma^2. \tag{10.115}$$

If the linear system (10.115) has a solution, then our ansatz (10.114) is valid, and we have produced a particular vibration of the system (10.113) possessing the same frequency as the forcing vibration. In particular, if $\mu=\gamma^2$ is *not* a generalized eigenvalue of the matrix pair K,M, as described in (10.85), then the coefficient matrix $K-\mu\,M$ is nonsingular, and so (10.115) can be uniquely solved for any right-hand side $\mathbf{a}$. The general solution, then, will be a quasi-periodic combination of this particular solution coupled with the normal mode vibrations at the natural, unforced frequencies of the system.

The more interesting case occurs when $\gamma^2 = \mu$ is a generalized eigenvalue, and so $K - \mu M$ is singular, its kernel being equal to the generalized eigenspace $V_\mu = \ker(K - \mu M)$. In this case, (10.115) will have a solution $\mathbf{w}$ if and only if $\mathbf{a}$ lies in the image of $K - \mu M$. According to the Fredholm Alternative Theorem 4.46, the image is the orthogonal complement of the cokernel, which, since the coefficient matrix is symmetric, is the same as the kernel. Therefore, (10.115) will have a solution if and only if $\mathbf{a}$ is orthogonal to V_μ, i.e., $\mathbf{a} \cdot \mathbf{v} = \mathbf{0}$ for every generalized eigenvector $\mathbf{v} \in V_\mu$. Thus, one can force a system at a natural frequency without inciting resonance, provided that the "direction" of forcing, as determined by the vector $\mathbf{a}$, is orthogonal — in the linear algebraic sense — to the natural directions of motion of the system, as governed by the eigenvectors for that particular frequency.

If the orthogonality condition is not satisfied, then the periodic solution ansatz (10.114) does not apply, and we are in a truly resonant situation. Inspired by the scalar solution, let us try a *resonant solution ansatz*

$$\mathbf{u}^\star(t) = t \sin(\gamma t)\, \mathbf{y} + \cos(\gamma t)\, \mathbf{w}. \tag{10.116}$$

Since

$$\frac{d^2 \mathbf{u}^\star}{dt^2} = -\gamma^2\, t \sin(\gamma t)\, \mathbf{y} + \cos(\gamma t)\, (2\gamma \mathbf{y} - \gamma^2\, \mathbf{w}),$$

the function (10.116) will solve the differential equation (10.113) provided

$$(K - \mu M)\mathbf{y} = \mathbf{0}, \qquad (K - \mu M)\mathbf{w} = \mathbf{a} - 2\gamma \mathbf{y}, \qquad \mu = \gamma^2. \tag{10.117}$$

The first equation requires that $\mathbf{y} \in V_\mu$ be a generalized eigenvector of the matrix pair K, M. Again, the Fredholm Alternative implies that, since the coefficient matrix $K - \mu M$ is symmetric, the second equation will be solvable for $\mathbf{w}$ if and only if $\mathbf{a} - 2\gamma\mathbf{y}$ is orthogonal to the generalized eigenspace $V_\mu = \operatorname{coker}(K - \mu M) = \ker(K - \mu M)$. Thus, the vector $2\gamma\mathbf{y}$ is required to be the orthogonal projection of $\mathbf{a}$ onto the eigenspace V_μ. With this choice of $\mathbf{y}$ and $\mathbf{w}$, formula (10.116) defines the resonant solution to the system.

Theorem 10.43. An undamped vibrational system will be periodically forced into resonance if and only if the forcing $\mathbf{f} = \cos(\gamma t)\, \mathbf{a}$ is at a natural frequency of the system and the direction of forcing $\mathbf{a}$ is not orthogonal to the natural direction(s) of motion of the system at that frequency.

Example 10.44. Consider the periodically forced system

$$\frac{d^2 \mathbf{u}}{dt^2} + \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \mathbf{u} = \begin{pmatrix} \cos t \\ 0 \end{pmatrix}.$$

The eigenvalues of the coefficient matrix are $\lambda_1 = 5,\ \lambda_2 = 1$, with corresponding orthogonal eigenvectors $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The internal frequencies are $\omega_1 = \sqrt{\lambda_1} = \sqrt{5}$,

† We can safely ignore the arbitrary multiple of the generalized eigenvector that can be added to $\mathbf{w}$ as we only need find one particular solution; these will reappear anyway once we assemble the general solution to the system.

$\omega_2 = \sqrt{\lambda_2} = 1$, and hence we are forcing at a resonant frequency. To obtain the resonant solution (10.116), we first note that $\mathbf{a} = (1, 0)^T$ has orthogonal projection $\mathbf{p} = \left(\frac{1}{2}, \frac{1}{2}\right)^T$ onto the eigenline spanned by $\mathbf{v}_2$, and hence $\mathbf{y} = \frac{1}{2}\mathbf{p} = \left(\frac{1}{4}, \frac{1}{4}\right)^T$. We can then solve

$$(K - \mathrm{I})\mathbf{w} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \mathbf{w} = \mathbf{a} - \mathbf{p} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \qquad \text{for}^\dagger \qquad \mathbf{w} = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}.$$

Therefore, the particular resonant solution is

$$\mathbf{u}^\star(t) = (t \sin t)\,\mathbf{y} + (\cos t)\,\mathbf{w} = \begin{pmatrix} \frac{1}{4}\,t \sin t + \frac{1}{4}\cos t \\ \frac{1}{4}\,t \sin t \end{pmatrix}.$$

The general solution to the system is

$$\mathbf{u}(t) = \begin{pmatrix} \frac{1}{4}\,t \sin t + \frac{1}{4}\cos t \\ \frac{1}{4}\,t \sin t \end{pmatrix} + r_1 \cos\left(\sqrt{5}\,t - \delta_1\right) \begin{pmatrix} -1 \\ 1 \end{pmatrix} + r_2 \cos(t - \delta_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where the amplitudes r_1, r_2 and phase shifts δ_1, δ_2, are fixed by the initial conditions. Eventually the resonant terms involving $t \sin t$ dominate the solution, inducing progressively larger and larger oscillations.

Exercises

10.6.17. Find the general solution to the following forced second order systems:

(a) $\dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 7 & -2 \\ -2 & 4 \end{pmatrix} \mathbf{u} = \begin{pmatrix} \cos t \\ 0 \end{pmatrix}$, (b) $\dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 5 & -2 \\ -2 & 3 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 0 \\ 5 \sin 3t \end{pmatrix}$,

(c) $\dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 13 & -6 \\ -6 & 8 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 5 \cos 2t \\ \cos 2t \end{pmatrix}$, (d) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{u} = \begin{pmatrix} \cos \frac{1}{2} t \\ -\cos \frac{1}{2} t \end{pmatrix}$,

(e) $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix} \mathbf{u} = \begin{pmatrix} \cos t \\ 11 \sin 2t \end{pmatrix}$, (f) $\dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 6 & -4 & 1 \\ -4 & 6 & -1 \\ 1 & -1 & 11 \end{pmatrix} \mathbf{u} = \begin{pmatrix} \cos t \\ 0 \\ \cos t \end{pmatrix}$.

10.6.18. (a) Find the resonant frequencies of a mass–spring chain consisting of two masses, $m_1 = 1$ and $m_2 = 2$, connected to top and bottom supports by identical springs with unit stiffness. (b) Write down an explicit forcing function that will excite the resonance.

10.6.19. Suppose one of the fixed supports is removed from the mass–spring chain of Exercise 10.6.18. Does your forcing function still excite the resonance? Do the internal vibrations of the masses (i) speed up, (ii) slow down, or (iii) remain the same? Does your answer depend upon which of the two supports is removed?

♣ 10.6.20. Find the resonant frequencies of the following structures, assuming the nodes all have unit mass. Then find a means of forcing the structure at one of the resonant frequencies, and yet not exciting the resonance. Can you also force the structure without exciting any mechanism or rigid motion? (a) The square truss of Exercise 6.3.5; (b) the joined square truss of Exercise 6.3.6; (c) the house of Exercise 6.3.8; (d) the triangular space station of Example 6.6; (e) the triatomic molecule of Example 10.41; (f) the water molecule of Exercise 10.5.30.