Chapter 6

Equilibrium

In this chapter, we will apply what we have learned so far to the analysis of equilibrium configurations and stability of mechanical structures and electrical networks. Both physical problems fit into a common, and surprisingly general, mathematical framework. The physical laws of equilibrium mechanics and circuits lead to linear algebraic systems whose coefficient matrix is of positive (semi-)definite Gram form. The positive definite cases correspond to stable structures and networks, which can support any applied forcing or external current, producing a unique, stable equilibrium solution that can be characterized by an energy minimization principle. On the other hand, systems with semi-definite coefficient matrices model unstable structures and networks that are unable to remain in equilibrium except under very special configurations of external forces. In the case of mechanical structures, the instabilities are of two types: rigid motions, in which the structure moves while maintaining its overall geometrical shape, and mechanisms, in which it spontaneously deforms in the absence of any applied force. The same linear algebra framework, but now reformulated for infinite-dimensional function space, also characterizes the boundary value problems for both ordinary and partial differential equation that model the equilibria of continuous media, including bars, beams, solid bodies, and many other systems arising throughout physics and engineering, [**61**, **79**].

The starting point is a linear chain of masses interconnected by springs and constrained to move only in the longitudinal direction. Our general mathematical framework is already manifest in this rather simple mechanical system. In the second section, we discuss simple electrical networks consisting of resistors, current sources and/or batteries, interconnected by a network of wires. Here, the resulting Gram matrix is known as the graph Laplacian, which plays an increasingly important role in modern data analysis and network theory. Finally, we treat small (so as to remain in a linear modeling regime) displacements of two- and three-dimensional structures constructed out of elastic bars. In all cases, we consider only the equilibrium solutions. Dynamical (time-varying) processes for each of these physical systems are governed by linear systems of ordinary differential equations, to be formulated and analyzed in Chapter 10.

6.1 Springs and Masses

A *mass–spring chain* consists of n masses $m_1,\ m_2,\ \ldots\ m_n$ arranged in a straight line. Each mass is connected to its immediate neighbor(s) by springs. Moreover, the chain may be connected at one or both ends to a fixed support by a spring — or may even be completely free, e.g., floating in outer space. For specificity, let us first look at the case when both ends of the chain are attached to unmoving supports, as illustrated in Figure 6.1

We assume that the masses are arranged in a vertical line, and order them from top to bottom. For simplicity, we will only allow the masses to move in the vertical direction, that is, we restrict to a one-dimensional motion. (Section 6.3 deals with the more complicated two- and three-dimensional situations.)

P. J. Olver, C. Shakiban, *Applied Linear Algebra*, Undergraduate Texts in Mathematics,
https://doi.org/10.1007/978-3-319-91041-3_6

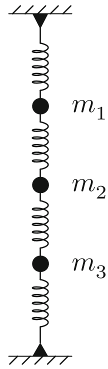

Figure 6.1. A Mass–Spring Chain with Fixed Ends.

If we subject some or all of the masses to an external force, e.g., gravity, then the system will move† to a new equilibrium position. The resulting position of the i^{th} mass is measured by its *displacement* u_i from its original position, which, since we are only allowing vertical motion, is a scalar quantity. Referring to Figure 6.1, we use the convention that $u_i > 0$ if the mass has moved downwards, and $u_i < 0$ if it has moved upwards. Our goal is to determine the new equilibrium configuration of the chain under the prescribed forcing, that is, to set up and solve a system of equations for the displacements $u_1, \ldots, u_n$.

As sketched in Figure 6.2, let e_j denote the *elongation* of the j^{th} spring, which connects mass m_{j-1} to mass m_j. By "elongation", we mean how far the spring has been stretched, so that $e_j > 0$ if the spring is longer than its reference length, while $e_j < 0$ if the spring has been compressed. The elongations of the internal springs can be determined directly from the displacements of the masses at each end according to the geometric formula

$$e_j = u_j - u_{j-1}, \qquad j = 2, \ldots, n, \tag{6.1}$$

while, for the top and bottom springs,

$$e_1 = u_1, \qquad e_{n+1} = -u_n, \tag{6.2}$$

since the supports are not allowed to move. We write the elongation equations (6.1–2) in matrix form

$$\mathbf{e} = A\mathbf{u}, \tag{6.3}$$

where $\mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{n+1} \end{pmatrix}$ is the *elongation vector*, $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ is the *displacement vector*, and

† The differential equations governing its dynamical behavior during the motion will be the subject of Chapter 10. Damping or frictional effects will cause the system to eventually settle down into a stable equilibrium configuration, if such exists.

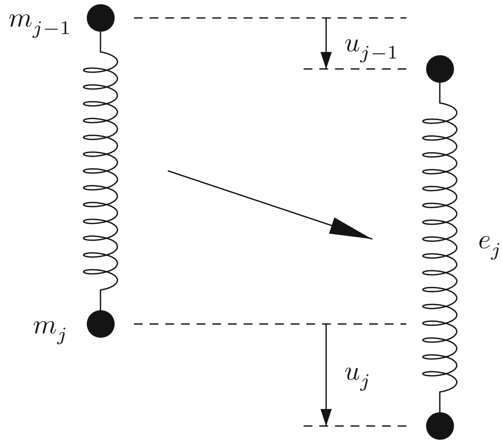

Figure 6.2. Elongation of a Spring.

the coefficient matrix

$$A = \begin{pmatrix} 1 & & & & & & \\ -1 & 1 & & & & & \\ & -1 & 1 & & & & \\ & & -1 & 1 & & & \\ & & & \ddots & \ddots & & \\ & & & & -1 & 1 \\ & & & & & -1 \end{pmatrix} \tag{6.4}$$

has size $(n+1) \times n$, with only its non-zero entries being indicated. We refer to A as the *reduced incidence matrix*† for the mass–spring chain. The incidence matrix effectively encodes the underlying geometry of the system, including the fixed "boundary conditions" at the top and the bottom.

The next step is to relate the elongation e_j experienced by the j^{th} spring to its internal force y_j. This is the basic *constitutive assumption*, which relates geometry to kinematics. In the present case, we suppose that the springs are not stretched (or compressed) particularly far. Under this assumption, *Hooke's Law*, named in honor of the seventeenth-century English scientist and inventor Robert Hooke, states that the internal force is directly proportional to the elongation — the more you stretch a spring, the more it tries to pull you back. Thus,

$$y_j = c_j \, e_j, \tag{6.5}$$

where the constant of proportionality $c_j > 0$ measures the spring's *stiffness*. Hard springs have large stiffness and so takes a large force to stretch, whereas soft springs have a small, but still positive, stiffness. We will also write the constitutive equations (6.5) in matrix form

$$\mathbf{y} = C\,\mathbf{e}, \tag{6.6}$$

† The connection with the incidence matrix of a graph, as introduced in Section 2.6, will become evident in the following Section 6.2.

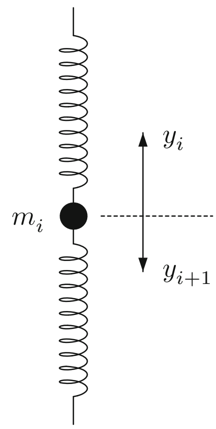

Figure 6.3. Force Balance.

where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{pmatrix}, \qquad C = \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_{n+1} \end{pmatrix}$$

are the internal force vector and the matrix of spring stiffnesses. Note particularly that C is a diagonal matrix, and, more importantly, positive definite, $C > 0$, since all its diagonal entries are strictly positive.

Finally, the forces must balance if the system is to remain in equilibrium. In this simplified model, the external forces act only on the masses, and not on the springs. Let f_i denote the external force on the i^{th} mass m_i. We also measure force in the downward direction, so $f_i > 0$ means that the force is pulling the i^{th} mass downward. (In particular, gravity would induce a positive force on each mass.) If the i^{th} spring is stretched, it will exert an upward force on m_i, while if the $(i+1)^{\text{st}}$ spring is stretched, it will pull m_i downward. Therefore, the balance of forces on m_i requires that

$$f_i = y_i - y_{i+1}. \tag{6.7}$$

The vectorial form of the force balance law is

$$\mathbf{f} = A^T \mathbf{y}, \tag{6.8}$$

where $\mathbf{f} = (f_1, \ldots, f_n)^T$. The remarkable fact is that the force balance coefficient matrix

$$A^T = \begin{pmatrix} 1 & -1 & & & & & \\ & 1 & -1 & & & & \\ & & 1 & -1 & & & \\ & & & 1 & -1 & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & -1 \end{pmatrix} \tag{6.9}$$

is the *transpose* of the reduced incidence matrix (6.4) for the chain. This connection between geometry and force balance turns out to be of almost universal applicability, and is the reason underlying the positivity of the final coefficient matrix in the resulting system of equilibrium equations.

Summarizing, the basic geometrical and physical properties of our mechanical system lead us to the full system of equilibrium equations (6.3, 6, 8) relating its displacements $\mathbf{u}$, elongations $\mathbf{e}$, internal forces $\mathbf{y}$, and external forces $\mathbf{f}$:

$$\mathbf{e} = A\,\mathbf{u}, \qquad \mathbf{y} = C\,\mathbf{e}, \qquad \mathbf{f} = A^T\,\mathbf{y}. \tag{6.10}$$

These equations imply $\mathbf{f} = A^T\,\mathbf{y} = A^T C\,\mathbf{e} = A^T C\,A\,\mathbf{u}$, and hence can be combined into a single linear system

$$K\,\mathbf{u} = \mathbf{f}, \qquad \text{where} \qquad K = A^T C\,A \tag{6.11}$$

is called the *stiffness matrix* associated with the entire mass–spring chain. In the particular case under consideration,

$$K = \begin{pmatrix} c_1 + c_2 & -c_2 & & & & & \\ -c_2 & c_2 + c_3 & -c_3 & & & & \\ & -c_3 & c_3 + c_4 & -c_4 & & & \\ & & -c_4 & c_4 + c_5 & -c_5 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -c_{n-1} & c_{n-1} + c_n & -c_n \\ & & & & & -c_n & c_n + c_{n+1} \end{pmatrix} \tag{6.12}$$

has a very simple symmetric, tridiagonal form. As such, we can use the tridiagonal solution algorithm of Section 1.7 to rapidly solve the linear system (6.11) for the displacements of the masses. Once we have solved (6.11) for the displacements $\mathbf{u}$ we can then compute the resulting elongations $\mathbf{e}$ and internal forces $\mathbf{y}$ by substituting into the original system (6.10).

Example 6.1. Let us consider the particular case of $n = 3$ masses connected by identical springs with unit spring constant. Thus, $c_1 = c_2 = c_3 = c_4 = 1$, and $C = \operatorname{diag}(1,1,1,1) = \mathrm{I}$ is the 4×4 identity matrix. The 3×3 stiffness matrix is then

$$K = A^T A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

A straightforward Gaussian Elimination produces the $K = L\,D\,L^T$ factorization

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.13}$$

With this in hand, we can solve the basic equilibrium equations $K\,\mathbf{u} = \mathbf{f}$ by the usual Forward and Back Substitution algorithm.

Remark. Even though we construct $K = A^T C\,A$ and then factor it as $K = L\,D\,L^T$, there is no direct algorithm to get from A and C to L and D, which, typically, are matrices of different sizes.

Suppose, for example, we pull the middle mass downwards with a unit force, so $f_2 = 1$ while $f_1 = f_3 = 0$. Then $\mathbf{f} = (\,0,1,0\,)^T$, and the solution to the equilibrium equations

(6.11) is $\mathbf{u} = \left(\frac{1}{2}, 1, \frac{1}{2}\right)^T$, whose entries prescribe the mass displacements. Observe that all three masses have moved down, with the middle mass moving twice as far as the other two. The corresponding spring elongations and internal forces are obtained by matrix multiplication

$$\mathbf{y} = \mathbf{e} = A\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix},$$

since $C = \mathrm{I}$. Thus, the top two springs are elongated, while the bottom two are compressed, all by an equal amount.

Similarly, if all the masses are equal, $m_1 = m_2 = m_3 = m$, then the solution under a constant downwards gravitational force $\mathbf{f} = (m\,g, m\,g, m\,g)^T$ is

$$\mathbf{u} = K^{-1}\begin{pmatrix} m\,g \\ m\,g \\ m\,g \end{pmatrix} = \begin{pmatrix} \frac{3}{2}\,m\,g \\ 2\,m\,g \\ \frac{3}{2}\,m\,g \end{pmatrix}, \qquad \text{and} \qquad \mathbf{y} = \mathbf{e} = A\mathbf{u} = \begin{pmatrix} \frac{3}{2}\,m\,g \\ \frac{1}{2}\,m\,g \\ -\frac{1}{2}\,m\,g \\ -\frac{3}{2}\,m\,g \end{pmatrix}.$$

Now, the middle mass has only moved 33% farther than the others, whereas the top and bottom springs are experiencing three times as much elongation/compression as the middle two springs.

An important observation is that we *cannot* determine the internal forces $\mathbf{y}$ or elongations $\mathbf{e}$ directly from the force balance law (6.8), because the transposed matrix A^T is not square, and so the system $\mathbf{f} = A^T\mathbf{y}$ does not have a unique solution. We must first compute the displacements $\mathbf{u}$ by solving the full equilibrium equations (6.11), and then use the resulting displacements to reconstruct the elongations and internal forces. Such systems are referred to as *statically indeterminate*.

The behavior of the system will depend on both the forcing and the boundary conditions. Suppose, by way of contrast, that we fix only the top of the chain to a support, and leave the bottom mass hanging freely, as in Figure 6.4. The geometric relation between the displacements and the elongations has the same form (6.3) as before, but the reduced incidence matrix is slightly altered:

$$A = \begin{pmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & -1 & 1 \end{pmatrix}. \tag{6.14}$$

This matrix has size $n \times n$ and is obtained from the preceding example (6.4) by eliminating the last row corresponding to the missing bottom spring. The constitutive equations are still governed by Hooke's law $\mathbf{y} = C\mathbf{e}$, as in (6.6), with $C = \operatorname{diag}(c_1, \ldots, c_n)$ the $n \times n$ diagonal matrix of spring stiffnesses. Finally, the force balance equations are also found to have the same general form $\mathbf{f} = A^T\mathbf{y}$ as in (6.8), but with the transpose of the revised incidence matrix (6.14). In conclusion, the equilibrium equations $K\mathbf{u} = \mathbf{f}$ have an identical

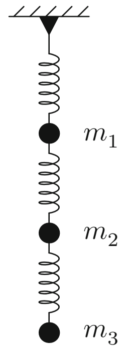

Figure 6.4. A Mass–Spring Chain with One Free End.

form (6.11), based on the revised stiffness matrix

$$K = A^T C A = \begin{pmatrix} c_1 + c_2 & -c_2 & & & & & \\ -c_2 & c_2 + c_3 & -c_3 & & & & \\ & -c_3 & c_3 + c_4 & -c_4 & & & \\ & & -c_4 & c_4 + c_5 & -c_5 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -c_{n-1} & c_{n-1} + c_n & -c_n \\ & & & & & -c_n & c_n \end{pmatrix} \qquad (6.15)$$

Only the bottom right entry differs from the fixed end matrix (6.12).

This system is called *statically determinate*, because the incidence matrix A is square and nonsingular, and so it is possible to solve the force balance law (6.8) directly for the internal forces $\mathbf{y} = A^{-T}\mathbf{f}$ without having to solve the full equilibrium equations for the displacements $\mathbf{u}$ before computing the internal forces $\mathbf{y} = C\,A\mathbf{u}$.

Example 6.2. For a three mass chain with one free end and equal unit spring constants $c_1 = c_2 = c_3 = 1$, the stiffness matrix is

$$K = A^T A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Pulling the middle mass downwards with a unit force, whereby $\mathbf{f} = (\,0, 1, 0\,)^T$, results in the displacements

$$\mathbf{u} = K^{-1}\mathbf{f} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \qquad \text{so that} \qquad \mathbf{y} = \mathbf{e} = A\,\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

In this configuration, the bottom two masses have moved by the same amount, which is twice as far as the top mass. Because we are pulling only on the middle mass, the bottom spring hangs free and experiences no elongation, whereas the top two springs are stretched by the same amount.

Similarly, for a chain of equal masses subject to a constant downwards gravitational force $\mathbf{f} = (\, mg, mg, mg\,)^T$, the equilibrium position is

$$\mathbf{u} = K^{-1} \begin{pmatrix} mg \\ mg \\ mg \end{pmatrix} = \begin{pmatrix} 3mg \\ 5mg \\ 6mg \end{pmatrix}, \qquad \text{and} \qquad \mathbf{y} = \mathbf{e} = A\mathbf{u} = \begin{pmatrix} 3mg \\ 2mg \\ mg \end{pmatrix}.$$

Note how much farther the masses have moved now that the restraining influence of the bottom support has been removed. The top spring is experiencing the most elongation, and is thus the most likely to break, because it must support all three masses.

Exercises

6.1.1. A mass–spring chain consists of two masses connected to two fixed supports. The spring constants are $c_1 = c_3 = 1$ and $c_2 = 2$. (*a*) Find the stiffness matrix K. (*b*) Solve the equilibrium equations $K\mathbf{u} = \mathbf{f}$ when $\mathbf{f} = (\,4, 3\,)^T$. (*c*) Which mass moved the farthest? (*d*) Which spring has been stretched the most? Compressed the most?

6.1.2. Solve Exercise 6.1.1 when the first and second springs are interchanged, $c_1 = 2$, $c_2 = c_3 = 1$. Which of your conclusions changed?

6.1.3. Redo Exercises 6.1.1–2 when the bottom support and spring are removed.

6.1.4. A mass–spring chain consists of four masses suspended between two fixed supports. The spring stiffnesses are $c_1 = 1$, $c_2 = \frac{1}{2}$, $c_3 = \frac{2}{3}$, $c_4 = \frac{1}{2}$, $c_5 = 1$. (a) Determine the equilibrium positions of the masses and the elongations of the springs when the external force is $\mathbf{f} = (\,0, 1, 1, 0\,)^T$. Is your solution unique? (*b*) Suppose we fix only the top support. Solve the problem with the same data and compare your results.

6.1.5. (*a*) Show that, in a mass–spring chain with two fixed ends, under any external force, the average elongation of the springs is zero: $\dfrac{1}{n+1}(e_1 + \cdots + e_{n+1}) = 0$. (*b*) What can you say about the average elongation of the springs in a chain with one fixed end?

♢ 6.1.6. Suppose we subject the i^{th} mass (and no others) in a chain to a unit force, and then measure the resulting displacement of the j^{th} mass. Prove that this is the *same* as the displacement of the i^{th} mass when the chain is subject to a unit force on the j^{th} mass. *Hint*: See Exercise 1.6.20.

♣ 6.1.7. Find the displacements $u_1, u_2, \ldots, u_{100}$ of 100 masses connected in a row by identical springs, with spring constant $c = 1$. Consider the following three types of force functions: (a) Constant force: $f_1 = \cdots = f_{100} = .01$; (*b*) Linear force: $f_i = .0002\, i$; (*c*) Quadratic force: $f_i = 6 \cdot 10^{-6}\, i\,(100 - i)$. Also consider two different boundary conditions at the bottom: (*i*) spring 101 connects the last mass to a support; (*ii*) mass 100 hangs free at the end of the line of springs. Graph the displacements and elongations in all six cases. Discuss your results; in particular, comment on whether they agree with your physical intuition.

6.1.8. (*a*) Suppose you are given three springs with respective stiffnesses $c = 1$, $c' = 2$, $c'' = 3$. In what order should you connect them to three masses and a top support so that the bottom mass goes down the farthest under a uniform gravitational force?

(*b*) Answer Exercise 6.1.8 when the springs connect two masses to top and bottom supports.

♣ 6.1.9. Generalizing Exercise 6.1.8, suppose you are given n different springs. (a) In which order should you connect them to n masses and a top support so that the bottom mass goes down the farthest under a uniform gravitational force? Does your answer depend upon the relative sizes of the spring constants? (*b*) Answer the same question when the springs connect $n - 1$ masses to both top and bottom supports.

6.1.10. Find the $L D L^T$ factorization of an $n \times n$ tridiagonal matrix whose diagonal entries are all equal to 2 and whose sub- and super-diagonal entries are all equal to -1. *Hint*: Start with the 3×3 case (6.13), and then analyze a slightly larger one to spot the pattern.

♡ 6.1.11. In a statically indeterminate situation, the equations $A^T \mathbf{y} = \mathbf{f}$ do not have a unique solution for the internal forces $\mathbf{y}$ in terms of the external forces $\mathbf{f}$. (*a*) Prove that, nevertheless, if $C = \mathrm{I}$, the internal forces are the *unique* solution of minimal Euclidean norm, as given by Theorem 4.50. (*b*) Use this method to directly find the internal force for the system in Example 6.1. Make sure that your values agree with those in the example.

Positive Definiteness and the Minimization Principle

You may have already observed that the stiffness matrix $K = A^T C A$ of a mass–spring chain has the form of a Gram matrix, cf. (3.64), for the weighted inner product $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T C \mathbf{w}$ induced by the diagonal matrix of spring stiffnesses. Moreover, since A has linearly independent columns (which should be checked), and C is positive definite, Theorem 3.37 tells us that the stiffness matrix is positive definite: $K > 0$. In particular, Theorem 3.43 guarantees that K is nonsingular, and hence the linear system (6.11) has a unique solution $\mathbf{u} = K^{-1}\mathbf{f}$. We can therefore conclude that the mass–spring chain assumes a unique equilibrium position under an arbitrary external force. However, one must keep in mind that this is a mathematical result and may not hold in all physical situations. Indeed, we should anticipate that a very large force will take us outside the regime covered by the linear Hooke's law relation (6.5), and render our simple mathematical model physically irrelevant.

According to Theorem 5.2, when the coefficient matrix of a linear system is positive definite, the equilibrium solution can be characterized by a minimization principle. For mass–spring chains, the quadratic function to be minimized has a physical interpretation: it is the potential energy of the system. Nature is parsimonious with energy, so a physical system seeks out an energy-minimizing equilibrium configuration. Energy minimization principles are of almost universal validity, and can be advantageously used for the construction of mathematical models, as well as their solutions, both analytical and numerical.

The energy function to be minimized can be determined directly from physical principles. For a mass–spring chain, the potential energy of the i^{th} mass equals the product of the applied force and the displacement: $-f_i u_i$. The minus sign is the result of our convention that a positive displacement $u_i > 0$ means that the mass has moved down, and hence decreased its potential energy. Thus, the total potential energy due to external forcing on all the masses in the chain is

$$-\sum_{i=1}^{n} f_i u_i = -\mathbf{u}^T \mathbf{f}.$$

Next, we calculate the internal energy of the system. In a single spring elongated by an amount e, the work done by the internal forces $y = c e$ is stored as potential energy, and so is calculated by integrating the force over the elongated distance:

$$\int_0^e y \, de = \int_0^e c\, e \, de = \tfrac{1}{2} c\, e^2.$$

Totaling the contributions from each spring, we find the internal spring energy to be

$$\frac{1}{2}\sum_{i=1}^{n} c_i\, e_i^2 = \tfrac{1}{2}\,\mathbf{e}^T C\,\mathbf{e} = \tfrac{1}{2}\,\mathbf{u}^T A^T C A\,\mathbf{u} = \tfrac{1}{2}\,\mathbf{u}^T K\,\mathbf{u},$$

where we used the incidence equation $\mathbf{e} = A\,\mathbf{u}$ relating elongation and displacement. Therefore, the total potential energy is

$$p(\mathbf{u}) = \tfrac{1}{2}\,\mathbf{u}^T K\,\mathbf{u} - \mathbf{u}^T\mathbf{f}. \tag{6.16}$$

Since $K > 0$, Theorem 5.2 implies that this quadratic function has a unique minimizer that satisfies the equilibrium equation $K\mathbf{u} = \mathbf{f}$.

Example 6.3. For the three mass chain with two fixed ends described in Example 6.1, the potential energy function (6.16) has the explicit form

$$\begin{aligned} p(\mathbf{u}) &= \frac{1}{2}\,(\,u_1 \quad u_2 \quad u_3\,)\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} - (\,u_1 \quad u_2 \quad u_3\,)\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \\ &= u_1^2 - u_1\,u_2 + u_2^2 - u_2\,u_3 + u_3^2 - f_1\,u_1 - f_2\,u_2 - f_3\,u_3, \end{aligned}$$

where $\mathbf{f} = (\,f_1, f_2, f_3\,)^T$ is the external forcing. The minimizer of this particular quadratic function gives the equilibrium displacements $\mathbf{u} = (\,u_1, u_2, u_3\,)^T$ of the three masses.

Exercises

6.1.12. Prove directly that the stiffness matrices in Examples 6.1 and 6.2 are positive definite.

6.1.13. Write down the potential energy for the following mass–spring chains with identical unit springs when subject to a uniform gravitational force: (*a*) three identical masses connected to only a top support. (*b*) four identical masses connected to top and bottom supports. (*c*) four identical masses connected only to a top support.

6.1.14. (*a*) Find the total potential energy of the equilibrium configuration of the mass–spring chain in Exercise 6.1.1. (*b*) Test the minimum principle by substituting three other possible displacements of the masses and checking that they all have larger potential energy.

6.1.15. Answer Exercise 6.1.14 for the mass–spring chain in Exercise 6.1.4.

6.1.16. Describe the mass–spring chains that gives rise to the following potential energy functions, and find their equilibrium configuration: (*a*) $3u_1^2 - 4u_1\,u_2 + 3u_2^2 + u_1 - 3u_2$, (*b*) $5u_1^2 - 6u_1\,u_2 + 3u_2^2 + 2u_2$, (*c*) $2u_1^2 - 3u_1\,u_2 + 4u_2^2 - 5u_2\,u_3 + \frac{5}{2}u_3^2 - u_1 - u_2 + u_3$, (*d*) $2u_1^2 - u_1\,u_2 + u_2^2 - u_2\,u_3 + u_3^2 - u_3\,u_4 + 2u_4^2 + u_1 - 2u_3$.

6.1.17. Explain why the columns of the reduced incidence matrices (6.4) and (6.14) are linearly independent.

6.1.18. Suppose that when subject to a nonzero external force $\mathbf{f} \neq \mathbf{0}$, a mass–spring chain has equilibrium position $\mathbf{u}^\star$. Prove that the potential energy is strictly negative at equilibrium: $p(\mathbf{u}^\star) < 0$.

♡ 6.1.19. Return to the situation investigated in Exercise 6.1.8. How should you arrange the springs in order to minimize the potential energy in the resulting mass–spring chain?

6.1.20. *True or false*: The potential energy function uniquely determines the mass–spring chain.

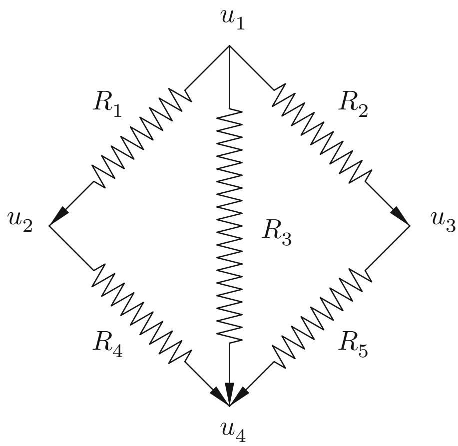

Figure 6.5. A Simple Electrical Network.

6.2 Electrical Networks

By an electrical *network*, we mean a collection of (insulated) wires that are joined together at their ends. The junctions connecting the ends of one or more wires are called *nodes*. Mathematically, we can view any such electrical network as a graph, the wires being the edges and the nodes the vertices. As before, to avoid technicalities, we will assume that the underlying graph is *simple*, meaning that there are no loops and at most one edge connecting any two vertices. To begin with, we further assume that there are no electrical devices (batteries, inductors, capacitors, etc.) in the network, and so the only impediments to the current flowing through the network are the resistances in the wires. As we shall see, resistance (or rather its reciprocal) plays a very similar role to that of spring stiffness. Thus, the network corresponds to a *weighted graph* in which the weight of an edge is the number representing the resistance of the corresponding wire. We shall feed a current into the network at one or more of the nodes, and would like to determine how the induced current flows through the wires. The basic equations governing the equilibrium voltages and currents in such a network follow from the three fundamental laws of electricity, named after the pioneering nineteenth-century German physicists Gustav Kirchhoff and Georg Ohm, two of the founders of electric circuit theory, [**58**].

Voltage is defined as the electromotive force that moves electrons through a wire. An individual wire's voltage is determined by the difference in the voltage potentials at its two ends — just as the gravitational force on a mass is induced by a difference in gravitational potential. To quantify voltage, we need to fix an orientation for the wire. A positive voltage will mean that the electrons move in the chosen direction, while a negative voltage causes them to move in reverse. The original choice of orientation is arbitrary, but once assigned will pin down the sign conventions to be used by voltages, currents, etc. To this end, we draw a digraph to represent the network, whose edges represent wires and whose vertices represent nodes. Each edge is assigned an orientation that indicates the wire's starting and ending nodes. A simple example consisting of five wires joined at four different nodes can be seen in Figure 6.5. The arrows indicate the selected directions for the wires, the wavy lines are the standard electrical symbols for resistance, while the resistances provide the edge weights in the resulting *weighted digraph*.

In an electrical network, each node will have a voltage potential, denoted by u_i. If wire k starts at node i and ends at node j under its assigned orientation, then its voltage v_k

equals the difference between the voltage potentials at its ends:

$$v_k = u_i - u_j. \tag{6.17}$$

Note that $v_k > 0$ if $u_i > u_j$, indicating that the electrons flow from the starting node i to the ending node j. In our particular illustrative example, the five wires have respective voltages

$$v_1 = u_1 - u_2, \qquad v_2 = u_1 - u_3, \qquad v_3 = u_1 - u_4, \qquad v_4 = u_2 - u_4, \qquad v_5 = u_3 - u_4.$$

Let us rewrite this linear system of equations in vector form

$$\mathbf{v} = A\mathbf{u}, \tag{6.18}$$

where

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \tag{6.19}$$

The alert reader will recognize the *incidence matrix* (2.46) for the digraph defined by the network. This is true in general — *the voltages along the wires of an electrical network are related to the potentials at the nodes by a linear system of the form* (6.18)*, in which A is the incidence matrix of the network digraph.* The rows of the incidence matrix are indexed by the wires, and the columns by the nodes. Each row of the matrix A has a single $+1$ in the column indexed by the starting node of the associated wire, and a single -1 in the column of the ending node.

Kirchhoff's Voltage Law states that the sum of the voltages around each closed circuit in the network is zero. For example, in the network under consideration, summing the voltages around the left-hand triangular circuit gives

$$v_1 + v_4 - v_3 = (u_1 - u_2) + (u_2 - u_4) - (u_1 - u_4) = 0.$$

Note that v_3 appears with a minus sign, since we must traverse wire 3 in the opposite direction to its assigned orientation when going around the circuit in the counterclockwise direction. The voltage law is a direct consequence of (6.18). Indeed, as discussed in Section 2.6, the circuits can be identified with vectors $\ell \in \operatorname{coker} A = \ker A^T$ in the cokernel of the incidence matrix, and so

$$\ell \cdot \mathbf{v} = \ell^T \mathbf{v} = \ell^T A\mathbf{u} = 0. \tag{6.20}$$

Therefore, orthogonality of the voltage vector $\mathbf{v}$ to the circuit vector ℓ is the mathematical formalization of Kirchhoff's Voltage Law.

Given a prescribed set of voltages $\mathbf{v}$ along the wires, can one find corresponding voltage potentials $\mathbf{u}$ at the nodes? To answer this question, we need to solve $\mathbf{v} = A\mathbf{u}$, which requires $\mathbf{v} \in \operatorname{img} A$. According to the Fredholm Alternative Theorem 4.46, the necessary and sufficient condition for this to hold is that $\mathbf{v}$ be orthogonal to $\operatorname{coker} A$. Theorem 2.53 says that the cokernel of an incidence matrix is spanned by the circuit vectors, and so $\mathbf{v}$ is a possible set of voltages if and only if $\mathbf{v}$ is orthogonal to all the circuit vectors $\ell \in \operatorname{coker} A$, i.e., the Voltage Law is necessary and sufficient for the given voltages to be physically realizable in the network.

Kirchhoff's Law is related to the topology of the network — how the different wires are connected together. *Ohm's Law* is a constitutive relation, indicating what the wires are made of. The resistance along a wire (including any added resistors) prescribes the relation between voltage and current or the rate of flow of electric charge. The law reads

$$v_k = R_k\, y_k, \tag{6.21}$$

where v_k is the voltage, R_k is the resistance, and y_k (often denoted by I_k in the engineering literature) denotes the current along wire k. Thus, for a fixed voltage, the larger the resistance of the wire, the smaller the current that flows through it. The direction of the current is also prescribed by our choice of orientation of the wire, so that $y_k > 0$ if the current is flowing from the starting to the ending node. We combine the individual equations (6.21) into a single vector equation

$$\mathbf{v} = R\,\mathbf{y}, \tag{6.22}$$

where the *resistance matrix* $R = \operatorname{diag}(R_1, \ldots, R_n) > 0$ is diagonal and positive definite. We shall, in analogy with (6.6), replace (6.22) by the inverse relationship

$$\mathbf{y} = C\,\mathbf{v}, \tag{6.23}$$

where $C = R^{-1}$ is the *conductance matrix*, again diagonal, positive definite, whose entries are the *conductances* $c_k = 1/R_k$ of the wires. For the particular network in Figure 6.5,

$$C = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & c_4 & 0 \\ 0 & 0 & 0 & 0 & c_5 \end{pmatrix} = \begin{pmatrix} 1/R_1 & 0 & 0 & 0 & 0 \\ 0 & 1/R_2 & 0 & 0 & 0 \\ 0 & 0 & 1/R_3 & 0 & 0 \\ 0 & 0 & 0 & 1/R_4 & 0 \\ 0 & 0 & 0 & 0 & 1/R_5 \end{pmatrix} = R^{-1}. \tag{6.24}$$

Finally, we stipulate that electric current is not allowed to accumulate at any node, i.e., every electron that arrives at a node must leave along one of the wires. Let $y_k, y_l, \ldots, y_m$ denote the currents along all the wires $k, l, \ldots, m$ that meet at node i in the network, and f_i an external current source, if any, applied at node i. *Kirchhoff's Current Law* requires that the net current leaving the node along the wires equals the external current coming into the node, and so

$$\pm\, y_k \,\pm\, y_l \,\pm\, \cdots \,\pm\, y_m = f_i. \tag{6.25}$$

Each $\pm$ sign is determined by the orientation of the wire, with $+$ if node i is its starting node and $-$ if it is its ending node.

In our particular example, suppose that we send a 1 amp current source into the first node. Then Kirchhoff's Current Law requires

$$y_1 + y_2 + y_3 = 1, \qquad -y_1 + y_4 = 0, \qquad -y_2 + y_5 = 0, \qquad -y_3 - y_4 - y_5 = 0,$$

the four equations corresponding to the four nodes in our network. The vector form of this linear system is

$$A^T\,\mathbf{y} = \mathbf{f}, \tag{6.26}$$

where $\mathbf{y} = (\,y_1, y_2, y_3, y_4, y_5\,)^T$ are the currents along the five wires, and $\mathbf{f} = (\,1, 0, 0, 0\,)^T$ represents the current sources at the four nodes. The coefficient matrix

$$A^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix} \tag{6.27}$$

is the *transpose* of the incidence matrix (6.19). As in the mass–spring chain, this is a remarkable general fact, which follows directly from Kirchhoff's two laws. *The coefficient matrix for the Current Law is the transpose of the incidence matrix for the Voltage Law.*

Let us assemble the full system of equilibrium equations (6.18, 23, 26):

$$\mathbf{v} = A\,\mathbf{u}, \qquad \mathbf{y} = C\,\mathbf{v}, \qquad \mathbf{f} = A^T\mathbf{y}. \tag{6.28}$$

Remarkably, we arrive at a system of linear relations that has an identical form to the mass–spring chain system (6.10), albeit with different physical quantities and different coefficient matrices. As before, they combine into a single linear system

$$K\,\mathbf{u} = \mathbf{f}, \qquad \text{where} \qquad K = A^T C\,A \tag{6.29}$$

is known as the *resistivity matrix* associated with the network. In our particular example, combining (6.19, 24, 27) produces the resistivity matrix

$$K = A^T C\,A = \begin{pmatrix} c_1+c_2+c_3 & -\,c_1 & -\,c_2 & -\,c_3 \\ -\,c_1 & c_1+c_4 & 0 & -\,c_4 \\ -\,c_2 & 0 & c_2+c_5 & -\,c_5 \\ -\,c_3 & -\,c_4 & -\,c_5 & c_3+c_4+c_5 \end{pmatrix}, \tag{6.30}$$

whose entries depend on the conductances of the five wires in the network.

Remark. There is a simple pattern to the resistivity matrix, evident in (6.30). The diagonal entries k_{ii} equal the sum of the conductances of all the wires having node i at one end. The non-zero off-diagonal entries k_{ij}, $i \neq j$, equal $-\,c_k$, the conductance of the wire† joining node i to node j, while $k_{ij} = 0$ if there is no wire joining the two nodes.

Consider the case in which all the wires in our network have equal unit resistance, and so $c_k = 1/R_k = 1$ for $k = 1, \ldots, 5$. Then the resistivity matrix is

$$K = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}. \tag{6.31}$$

However, when trying to solve the linear system (6.29), we run into an immediate difficulty: *there is no solution*! The matrix (6.31) is *not* positive definite — it is a singular matrix. Moreover, the particular current source vector $\mathbf{f} = (\,1, 0, 0, 0\,)^T$ does not lie in the image of K. Something is clearly amiss.

Before getting discouraged, let us sit back and use a little physical intuition. We are trying to put a 1 amp current into the network at node 1. Where can the electrons go? The answer is nowhere — they are all trapped in the network and, as they accumulate, something drastic will happen — sparks will fly! This is clearly an unstable situation, and so the fact that the equilibrium equations do not have a solution is trying to tell us that the physical system cannot remain in a steady state. The physics rescues the mathematics, or, vice versa, the mathematics elucidates the underlying physical processes.

In order to achieve equilibrium in an electrical network, we must remove as much current as we put in. Thus, if we feed a 1 amp current into node 1, then we must extract a total of

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† This assumes that there is only one wire joining the two nodes.

1 amp's worth of current from the other nodes. In other words, the sum of all the external current sources must vanish:

$$f_1 + f_2 + \cdots + f_n = 0,$$

and so there is no net current being fed into the network. Suppose we also extract a 1 amp current from node 4; then the modified current source vector $\mathbf{f} = (1, 0, 0, -1)^T$ indeed lies in the image of K, as you can check, and the equilibrium system (6.29) has a solution.

This is all well and good, but we are not out of the woods yet. As we know, if a linear system has a singular coefficient matrix, then either it has no solutions — the case we already rejected — or it has infinitely many solutions — the case we are considering now. In the particular network under consideration, the general solution to the linear system

$$\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

is found by Gaussian Elimination:

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2} + t \\ \frac{1}{4} + t \\ \frac{1}{4} + t \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \tag{6.32}$$

where $t = u_4$ is the free variable. The resulting nodal voltage potentials

$$u_1 = \tfrac{1}{2} + t, \qquad u_2 = \tfrac{1}{4} + t, \qquad u_3 = \tfrac{1}{4} + t, \qquad u_4 = t,$$

depend on a free parameter t.

The ambiguity arises because voltage potential is a mathematical abstraction that cannot be measured directly; only relative potential differences have physical import. To resolve the inherent ambiguity, we need to assign a baseline value for the voltage potentials. In terrestrial electricity, the Earth is assumed to have zero potential. Specifying a particular node to have zero potential is physically equivalent to grounding that node. For our example, suppose we ground node 4 by setting $u_4 = 0$. This fixes the free variable $t = 0$ in our solution (6.32), and so uniquely specifies all the other voltage potentials: $u_1 = \frac{1}{2}$, $u_2 = \frac{1}{4}$, $u_3 = \frac{1}{4}$, $u_4 = 0$.

On the other hand, even without specification of a baseline potential level, the corresponding physical voltages and currents along the wires are uniquely specified. In our example, computing $\mathbf{y} = \mathbf{v} = A\mathbf{u}$ gives

$$y_1 = v_1 = \tfrac{1}{4}, \qquad y_2 = v_2 = \tfrac{1}{4}, \qquad y_3 = v_3 = \tfrac{1}{2}, \qquad y_4 = v_4 = \tfrac{1}{4}, \qquad y_5 = v_5 = \tfrac{1}{4},$$

independent of the value of t in (6.32). Thus, the nonuniqueness of the voltage potential solution $\mathbf{u}$ is an inessential feature. All physical quantities that we can measure — currents and voltages — *are* uniquely specified by the solution to the equilibrium system.

Remark. Although they have no real physical meaning, we *cannot* dispense with the nonmeasurable (and nonunique) voltage potentials $\mathbf{u}$. Most networks are *statically indeterminate*, since their incidence matrices are rectangular and hence not invertible, so the linear system $A^T\mathbf{y} = \mathbf{f}$ cannot be solved directly for the currents in terms of the voltage sources since the system does not have a unique solution. Only by first solving the full equilibrium system (6.29) for the potentials, and then using the relation $\mathbf{y} = CA\mathbf{u}$ between the potentials and the currents, can we determine their actual values.

Let us analyze what is going on in the context of our general mathematical framework. Proposition 3.36 says that the resistivity matrix $K = A^T CA$ is a positive semi-definite Gram matrix, which is positive definite (and hence nonsingular) if and only if A has linearly independent columns, or, equivalently, $\ker A = \{\mathbf{0}\}$. But Proposition 2.51 says that the incidence matrix A of a directed graph *never* has a trivial kernel. Therefore, the resistivity matrix K is only positive semi-definite, and hence singular. If the network is connected, then $\ker A = \ker K = \operatorname{coker} K$ is one-dimensional, spanned by the vector $\mathbf{z} = (1, 1, 1, \ldots, 1)^T$. According to the Fredholm Alternative Theorem 4.46, the fundamental network equation $K\mathbf{u} = \mathbf{f}$ has a solution if and only if $\mathbf{f}$ is orthogonal to $\operatorname{coker} K$, and so the current source vector must satisfy

$$\mathbf{z} \cdot \mathbf{f} = f_1 + f_2 + \cdots + f_n = 0, \tag{6.33}$$

as we already observed. Therefore, the linear algebra reconfirms our physical intuition: a connected network admits an equilibrium configuration, obtained by solving (6.29), if and only if the nodal current sources add up to zero, i.e., there is no net influx of current into the network.

Grounding one of the nodes is equivalent to nullifying the value of its voltage potential: $u_i = 0$. This variable is now fixed, and can be safely eliminated from our system. To accomplish this, we let $A^\star$ denote the $m \times (n-1)$ matrix obtained by deleting the i^{th} column from A. For example, grounding node 4 in our sample network, so $u_4 = 0$, allows us to erase the fourth column of the incidence matrix (6.19), leading to the *reduced incidence matrix*

$$A^\star = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.34}$$

The key observation is that $A^\star$ has trivial kernel, $\ker A^\star = \{\mathbf{0}\}$, and therefore the reduced network resistivity matrix

$$K^\star = (A^\star)^T C\, A^\star = \begin{pmatrix} c_1 + c_2 + c_3 & -c_1 & -c_2 \\ -c_1 & c_1 + c_4 & 0 \\ -c_2 & 0 & c_2 + c_5 \end{pmatrix} \tag{6.35}$$

is positive definite. Note that we can obtain $K^\star$ directly from K in (6.30) by deleting both its fourth row and fourth column. Let $\mathbf{f}^\star = (1, 0, 0)^T$ denote the reduced current source vector obtained by deleting the fourth entry from $\mathbf{f}$. Then the reduced linear system is

$$K^\star \mathbf{u}^\star = \mathbf{f}^\star, \tag{6.36}$$

where $\mathbf{u}^\star = (u_1, u_2, u_3)^T$ is the reduced voltage potential vector. Positive definiteness of $K^\star$ implies that (6.36) has a unique solution $\mathbf{u}^\star$, from which we can reconstruct the voltages $\mathbf{v} = A^\star \mathbf{u}^\star$ and currents $\mathbf{y} = C\,\mathbf{v} = CA^\star \mathbf{u}^\star$ along the wires. In our example, if all the wires have unit resistance, then the reduced system (6.36) is

$$\begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

and has unique solution $\mathbf{u}^\star = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)^T$. The voltage potentials are

$$u_1 = \tfrac{1}{2}, \qquad u_2 = \tfrac{1}{4}, \qquad u_3 = \tfrac{1}{4}, \qquad u_4 = 0,$$

and correspond to the earlier solution (6.32) when $t = 0$. The corresponding voltages and currents along the wires are the same as before.

Remark. When $C = \mathrm{I}$, the matrix $K = A^T A$ constructed from the incidence matrix of a directed graph is known in the mathematical literature as the *graph Laplacian* associated with the graph. The graph Laplacian matrix can be easily constructed directly: its rows and columns are indexed by the vertices. The diagonal entry k_{ii} equals the degree of the i^{th} vertex, meaning the number of edges that have vertex i as one of their endpoints. The off-diagonal entries k_{ij} are equal to -1 if there is an edge connecting vertices i and j and 0 otherwise. This is often written as $K = D - J$, where D is the diagonal *degree matrix*, whose diagonal entries are the degrees of the nodes, and J is the symmetric *adjacency matrix*, which contains a 1 in every off-diagonal entry corresponding to two adjacent nodes, that is two nodes connected by a single edge; all other entries are 0. Observe that the graph Laplacian is independent of the direction assigned to the edges; it depends only on the underlying graph. The term "Laplacian" is used because this matrix represents the discrete analogue of the *Laplacian* differential operator, described in Examples 7.36 and 7.52 below. In particular, if the graph comes from an n-dimensional square grid, the corresponding graph Laplacian coincides with the standard finite difference numerical discretization of the Laplacian differential operator.

Batteries, Power, and the Electrical–Mechanical Correspondence

So far, we have considered only the effect of current sources at the nodes. Suppose now that the network contains one or more batteries. Each *battery* serves as a voltage source along a wire, and we let b_k denote the voltage of a battery connected to wire k. The sign of b_k indicates the relative orientation of the battery's terminals with respect to the wire, with $b_k > 0$ if the current produced by the battery runs in the same direction as our chosen orientation of the wire. The battery's voltage is included in the voltage balance equation (6.17):

$$v_k = u_i - u_j + b_k.$$

The corresponding vector equation (6.18) becomes

$$\mathbf{v} = A\,\mathbf{u} + \mathbf{b}, \tag{6.37}$$

where $\mathbf{b} = (\,b_1, b_2, \ldots, b_m\,)^T$ is the *battery vector*, whose entries are indexed by the wires. (If there is no battery on wire k, the corresponding entry is $b_k = 0$.) The remaining two equations are as before, so $\mathbf{y} = C\,\mathbf{v}$ are the currents in the wires, and, in the absence of external current sources, Kirchhoff's Current Law implies $A^T\mathbf{y} = \mathbf{0}$. Using the modified formula (6.37) for the voltages, these combine into the following equilibrium system:

$$K\,\mathbf{u} = A^T C\,A\,\mathbf{u} = -\,A^T C\,\mathbf{b}. \tag{6.38}$$

Remark. Interestingly, the voltage potentials satisfy the weighted normal equations (5.36) that characterize the least squares solution to the system $A\,\mathbf{u} = -\,\mathbf{b}$ for the weighted norm

$$\|\,\mathbf{v}\,\|^2 = \mathbf{v}^T C\,\mathbf{v} \tag{6.39}$$

determined by the network's conductance matrix C. It is a striking fact that Nature solves a least squares problem in order to make the weighted norm of the voltages $\mathbf{v}$ as small as possible. A similar remark holds for the mass–spring chains considered above.

Batteries have exactly the same effect on the voltage potentials as if we imposed the current source vector

$$\mathbf{f} = -A^T C\,\mathbf{b}. \tag{6.40}$$

Namely, placing a battery of voltage b_k on wire k is exactly the same as introducing additional current sources of $-c_k\,b_k$ at the starting node and $c_k\,b_k$ at the ending node. Note that the induced current vector $\mathbf{f} \in \operatorname{coimg} A = \operatorname{img} K$ (see Exercise 3.4.32) continues to satisfy the network constraint (6.33). Conversely, a system of allowed current sources $\mathbf{f} \in \operatorname{img} K$ has the same effect as any collection of batteries $\mathbf{b}$ that satisfies (6.40).

In the absence of external current sources, a network with batteries always admits a solution for the voltage potentials and currents. Although the currents are uniquely determined, the voltage potentials are not. As before, to eliminate the ambiguity, we can ground one of the nodes and use the reduced incidence matrix $A^\star$ and reduced current source vector $\mathbf{f}^\star$ obtained by eliminating the column, respectively entry, corresponding to the grounded node. The details are left to the interested reader.

Example 6.4. Consider an electrical network running along the sides of a cube, where each wire contains a 2 ohm resistor and there is a 9 volt battery source on one wire. The problem is to determine how much current flows through the wire directly opposite the battery. Orienting the wires and numbering them as indicated in Figure 6.6, the incidence matrix is

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

We connect the battery along wire 1 and measure the resulting current along wire 12. To avoid the ambiguity in the voltage potentials, we ground the last node and erase the final column from A to obtain the reduced incidence matrix $A^\star$. Since the resistance matrix R has all 2's along the diagonal, the conductance matrix is $C = \frac{1}{2}\,\mathrm{I}$. Therefore, the network resistivity matrix is one-half the cubical graph Laplacian:

$$K^\star = (A^\star)^T C A^\star = \tfrac{1}{2}\,(A^\star)^T A^\star = \frac{1}{2}\begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & -1 & -1 & 0 \\ -1 & 0 & 3 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 3 & 0 & -1 & -1 \\ 0 & -1 & -1 & 0 & 3 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 3 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 3 \end{pmatrix}.$$

Alternatively, it can be found by eliminating the final row and column, representing the grounded node, from the graph Laplacian matrix constructed by the above recipe. The reduced current source vector

$$\mathbf{b} = (\,9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\,)^T$$

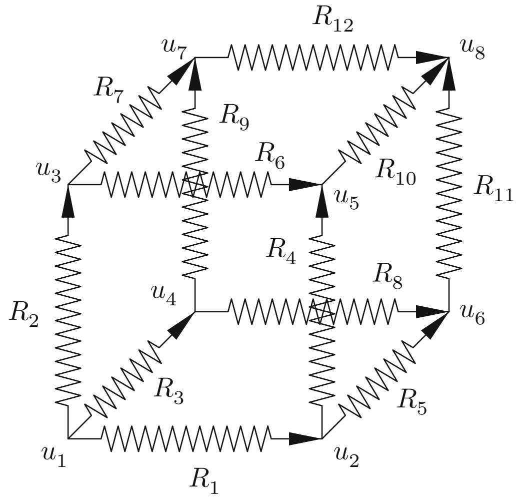


Figure 6.6. Cubical Electrical Network.

corresponding to the battery situated on the first wire is

$$\mathbf{f}^\star = -(A^\star)^T C\,\mathbf{b} = \left(-\tfrac{9}{2}, \tfrac{9}{2}, 0, 0, 0, 0, 0\right)^T.$$

Solving the resulting linear system $K^\star \mathbf{u}^\star = \mathbf{f}^\star$ by Gaussian Elimination yields the voltage potentials

$$\mathbf{u}^\star = \left(-3,\ \tfrac{9}{4},\ -\tfrac{9}{8},\ -\tfrac{9}{8},\ \tfrac{3}{8},\ \tfrac{3}{8},\ -\tfrac{3}{4}\right)^T.$$

Thus, the induced currents along the sides of the cube are

$$\mathbf{y} = C\,\mathbf{v} = \tfrac{1}{2}(A^\star \mathbf{u}^\star + \mathbf{b}) = \left(\tfrac{15}{8},\ -\tfrac{15}{16},\ -\tfrac{15}{16},\ \tfrac{15}{16},\ \tfrac{15}{16},\ -\tfrac{3}{4},\ -\tfrac{3}{16},\ -\tfrac{3}{4},\ -\tfrac{3}{16},\ \tfrac{3}{16},\ \tfrac{3}{16},\ -\tfrac{3}{8}\right)^T.$$

In particular, the current on the wire that is opposite the battery is $y_{12} = -\frac{3}{8}$, flowing in the opposite direction to its orientation. The largest current flows through the battery wire, while wires $7, 9, 10, 11$ transmit the least.

As with a mass–spring chain, the voltage potentials in such a resistive electrical network can be characterized by a minimization principle. The *power* in a single conducting wire is defined as the product of its current y_j and voltage v_j,

$$P_j = y_j\,v_j = R_j\,y_j^2 = c_j\,v_j^2, \tag{6.41}$$

where R_j is the resistance, $c_j = 1/R_j$ the conductance, and we are using Ohm's Law (6.21) to relate voltage and current. Physically, the power quantifies the rate at which electrical energy is converted into heat by the wire's resistance. Summing over all wires in the system, the internal power[†] of the network

$$P_{int} = \sum_j P_j = \sum_j c_j\,v_j^2 = \|\,\mathbf{v}\,\|^2$$

is identified as the square of the weighted norm (6.39).

[†] So far, we have not considered the effect of batteries or current sources on the network.

The Electrical–Mechanical Correspondence

Structures	Variables	Networks
Displacements	$\mathbf{u}$	Voltage potentials
Prestressed bars/springs	$\mathbf{b}$	Batteries
Elongations†	$\mathbf{v} = A\,\mathbf{u} + \mathbf{b}$	Voltages
Spring stiffnesses	C	Conductivities
Internal Forces	$\mathbf{y} = C\,\mathbf{v}$	Currents
External forcing	$\mathbf{f} = A^T\mathbf{y}$	Current sources
Stiffness matrix	$K = A^T C\,A$	Resistivity matrix
Potential energy	$p(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T K\mathbf{u} - \mathbf{u}^T\mathbf{f}$	$\frac{1}{2} \times$ Power

Consider a network that contains batteries, but no external current sources. Summing over all the wires in the network, the total power due to internal and external sources can be identified as the product of the current and voltage vectors:

$$\begin{aligned} P = y_1 v_1 + \;\cdots\; + y_m v_m &= \mathbf{y}^T\mathbf{v} = \mathbf{v}^T C\,\mathbf{v} = (A\,\mathbf{u} + \mathbf{b})^T C\,(A\,\mathbf{u} + \mathbf{b}) \\ &= \mathbf{u}^T A^T C\,A\,\mathbf{u} + 2\,\mathbf{u}^T A^T C\,\mathbf{b} + \mathbf{b}^T C\,\mathbf{b}, \end{aligned}$$

and is thus a quadratic function of the voltage potentials, which we rewrite in our usual form‡

$$\tfrac{1}{2}P = p(\mathbf{u}) = \tfrac{1}{2}\,\mathbf{u}^T K\,\mathbf{u} - \mathbf{u}^T\mathbf{f} + c, \tag{6.42}$$

where $K = A^T C\,A$ is the network resistivity matrix, while $\mathbf{f} = -\,A^T C\,\mathbf{b}$ are the equivalent current sources at the nodes (6.40) that correspond to the batteries. The last term $c = \frac{1}{2}\,\mathbf{b}^T C\,\mathbf{b}$ is one-half the internal power of the batteries, and is not affected by the currents/voltages in the wires. In deriving (6.42), we have ignored external current sources at the nodes. By the preceding discussion, external current sources can be viewed as an equivalent collection of batteries, and so contribute to the linear terms $\mathbf{u}^T\mathbf{f}$ in the power, which will then represent the combined effect of all batteries and external current sources.

In general, the resistivity matrix K is only positive semi-definite, and so the quadratic power function (6.42) does not, in general, possess a minimizer. As argued above, to ensure equilibrium, we need to ground one or more of the nodes. The resulting reduced power function

$$p^\star(\mathbf{u}^\star) = \tfrac{1}{2}\,(\mathbf{u}^\star)^T K^\star\mathbf{u}^\star - (\mathbf{u}^\star)^T\mathbf{f}^\star, \tag{6.43}$$

has a positive definite coefficient matrix: $K^\star > 0$. Its unique minimizer is the voltage potential $\mathbf{u}^\star$ that solves the reduced linear system (6.36). We conclude that the electrical network adjusts itself so as to *minimize the power or total energy loss* throughout the network. As in mechanics, Nature solves a minimization problem in an effort to conserve energy.

† Here, we use $\mathbf{v}$ instead of $\mathbf{e}$ to represent elongation.

‡ For alternating currents, the power is reduced by a factor of $\frac{1}{2}$, so $p(\mathbf{u})$ equals the power.

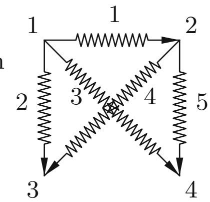
We have now discovered the remarkable correspondence between the equilibrium equations for electrical networks (6.10) and those of mass–spring chains (6.28). This *Electrical–Mechanical Correspondence* is summarized in the above table. In the following section, we will see that the analogy extends to more general mechanical structures.

Exercises

6.2.1. Draw the electrical networks corresponding to the following incidence matrices.

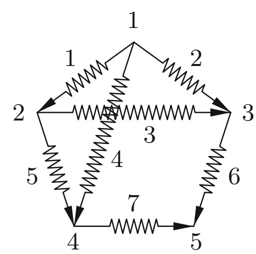
$$(a)\ \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad (b)\ \begin{pmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \quad (c)\ \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 \end{pmatrix},$$

$$(d)\ \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}, \quad (e)\ \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$



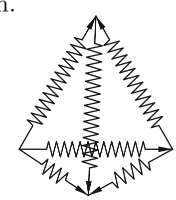
6.2.2. Suppose that all wires in the illustrated network have unit resistivity. (*a*) Write down the incidence matrix A. (*b*) Write down the equilibrium system for the network when node 4 is grounded and there is a current source of magnitude 3 at node 1. (*c*) Solve the system for the voltage potentials at the ungrounded nodes. (*d*) If you connect a light bulb to the network, which wire should you connect it to so that it shines the brightest?

6.2.3. What happens in the network in Figure 6.5 if we ground both nodes 3 and 4? Set up and solve the system and compare the currents for the two cases.



6.2.4. (*a*) Write down the incidence matrix A for the illustrated electrical network. (*b*) Suppose all the wires contain unit resistors, except for $R_4 = 2$. Let there be a unit current source at node 1, and assume node 5 is grounded. Find the voltage potentials at the nodes and the currents through the wires. (*c*) Which wire would shock you the most?

6.2.5. Answer Exercise 6.2.4 if, instead of the current source, you put a 1.5 volt battery on wire 1.



♠ 6.2.6. Consider an electrical network running along the sides of a tetrahedron. Suppose that each wire contains a 3 ohm resistor and there is a 10 volt battery source on one wire. Determine how much current flows through the wire directly opposite the battery.

♠ 6.2.7. Now suppose that each wire in the tetrahedral network in Exercise 6.2.6 contains a 1 ohm resistor and there are two 5 volt battery sources located on two non-adjacent wires. Determine how much current flows through the wires in the network.

♠ 6.2.8. (*a*) How do the currents change if the resistances in the wires in the cubical network in Example 6.4 are all equal to 1 ohm? (*b*) What if wire k has resistance $R_k = k$ ohms?

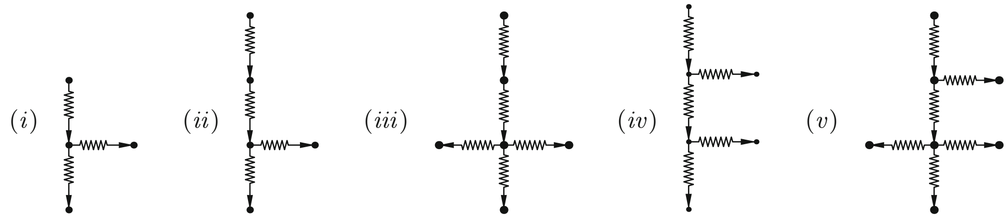
♣ 6.2.9. Suppose you are given six resistors with respective resistances $1, 2, 3, 4, 5,$ and 6. How should you connect them in a tetrahedral network (one resistor per wire) so that a light bulb on the wire opposite the battery burns the brightest?

♣ 6.2.10. The nodes in an electrical network lie on the vertices $\left(\frac{i}{n}, \frac{j}{n}\right)$ for $-n \le i, j \le n$ in a square grid centered at the origin; the wires run along the grid lines. The boundary nodes, when x or $y = \pm 1$, are all grounded. A unit current source is introduced at the origin.
(a) Compute the potentials at the nodes and currents along the wires for $n = 2, 3, 4$.
(b) Investigate and compare the solutions for large n, i.e., as the grid size becomes small. Do you detect any form of limiting behavior?

6.2.11. Show that, in a network with all unit resistors, the currents $\mathbf{y}$ can be characterized as the unique solution to the Kirchhoff equations $A^T \mathbf{y} = \mathbf{f}$ of minimum Euclidean norm.

6.2.12. *True or false*: (a) The nodal voltage potentials in a network with batteries $\mathbf{b}$ are the same as in the same network with the current sources $\mathbf{f} = -A^T C \mathbf{b}$. ($b$) Are the currents the same?

6.2.13. (a) Assuming all wires have unit resistance, find the voltage potentials at all the nodes and the currents along the wires of the following trees when the bottom node is grounded and a unit current source is introduced at the top node.



(b) Can you make any general predictions about electrical currents in trees?

6.2.14. A node in a tree is called *terminating* if it has only one edge. Repeat the preceding exercise when all terminating nodes except for the top one are grounded.

6.2.15. Suppose the graph of an electrical network is a tree, as in Exercise 2.6.9. Show that if one of the nodes in the tree is grounded, the system is statically determinate.

6.2.16. Suppose two wires in a network join the *same* pair of nodes. Explain why their effect on the rest of the network is the same as a single wire whose conductance $c = c_1 + c_2$ is the sum of the individual conductances. How are the resistances related?

6.2.17. (a) Write down the equilibrium equations for a network that contains both batteries and current sources. (b) Formulate a general superposition principle for such situations. (c) Write down a formula for the power in the network.

♢ 6.2.18. Prove that the voltage potential at node i due to a unit current source at node j is the *same* as the voltage potential at node j due to a unit current source at node i. Can you give a physical explanation of this *reciprocity relation*?

6.2.19. What is the analogue of condition (6.33) for a disconnected graph?

6.3 Structures

A *structure* (sometimes called a *truss*) is a mathematical idealization of a framework for a building. Think of a radio tower or a skyscraper when just the I-beams are connected — before the walls, floors, ceilings, roof, and ornamentation are added. An ideal structure is constructed of elastic bars connected at *joints*. By a *bar*, we mean a straight, rigid rod that can be (slightly) elongated, but not bent. (Beams, which are allowed to bend, are more complicated and are modeled by boundary value problems for ordinary and partial differential equations, [**61**, **79**]. See also our discussion of splines in Section 5.5.) When a bar is stretched, it obeys Hooke's law — at least in the linear regime we are modeling

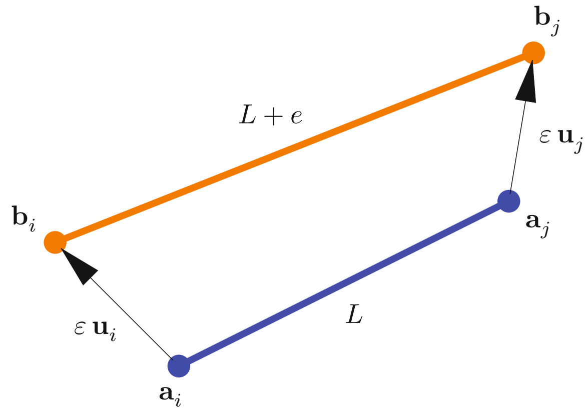


Figure 6.7. Displacement of a Bar.

— and so, for all practical purposes, behaves like a spring with a very large stiffness. As a result, a structure can be regarded as a two- or three-dimensional generalization of a mass–spring chain.

The joints will allow the bar to rotate in any direction. Of course, this is an idealization; in a building, the rivets and bolts will (presumably) prevent rotation to a significant degree. However, under moderate stress — for example, if the wind is blowing on our skyscraper, the bolts can be expected only to keep the structure connected, and the resulting motions will induce stresses on the joints that must be taken into account when designing the structure. Of course, under extreme stress, the structure will fall apart — a disaster that its designers must avoid. The purpose of this section is to derive conditions that will guarantee that a structure is rigidly stable under moderate forcing, or, alternatively, help us to understand the processes that might lead to its collapse.

The first order of business is to understand how an individual bar reacts to motion. We have already encountered the basic idea in our treatment of springs. The key complication here is that the ends of the bar are not restricted to a single direction of motion, but can move in either two- or three-dimensional space. We use d to denote the dimension of the underlying space. In the $d=1$-dimensional case, the structure reduces to a mass–spring chain that we analyzed in Section 6.1. Here we concentrate on structures in $d=2$ and 3 dimensions.

Consider an unstressed bar with one end at position $\mathbf{a}_1\in\mathbb{R}^d$ and the other end at position $\mathbf{a}_2\in\mathbb{R}^d$. In $d=2$ dimensions, we write $\mathbf{a}_i=(\,a_i,b_i\,)^T$, while in $d=3$-dimensional space, $\mathbf{a}_i=(\,a_i,b_i,c_i\,)^T$. The length of the bar is $L=\|\,\mathbf{a}_1-\mathbf{a}_2\,\|$, where we use the standard Euclidean norm to measure distance on $\mathbb{R}^d$ throughout this section.

Suppose we move the ends of the bar a little, sending $\mathbf{a}_i$ to $\mathbf{b}_i=\mathbf{a}_i+\varepsilon\,\mathbf{u}_i$ and, simultaneously, $\mathbf{a}_j$ to $\mathbf{b}_j=\mathbf{a}_j+\varepsilon\,\mathbf{u}_j$, moving the blue bar in Figure 6.7 to the displaced orange bar. The vectors $\mathbf{u}_i,\mathbf{u}_j\in\mathbb{R}^d$ indicate the respective directions of displacement of the two ends, and we use ε to represent the relative magnitude of the displacement. How much has this motion stretched the bar? Since we are assuming that the bar can't bend, the length of the displaced bar is

$$\begin{aligned} L+e=\|\,\mathbf{b}_i-\mathbf{b}_j\,\| &= \|\,(\mathbf{a}_i+\varepsilon\,\mathbf{u}_i)-(\mathbf{a}_j+\varepsilon\,\mathbf{u}_j)\,\| = \|\,(\mathbf{a}_i-\mathbf{a}_j)+\varepsilon\,(\mathbf{u}_i-\mathbf{u}_j)\,\| \\ &= \sqrt{\|\,\mathbf{a}_i-\mathbf{a}_j\,\|^2+2\,\varepsilon\,(\mathbf{a}_i-\mathbf{a}_j)\cdot(\mathbf{u}_i-\mathbf{u}_j)+\varepsilon^2\,\|\,\mathbf{u}_i-\mathbf{u}_j\,\|^2}\,. \end{aligned} \tag{6.44}$$

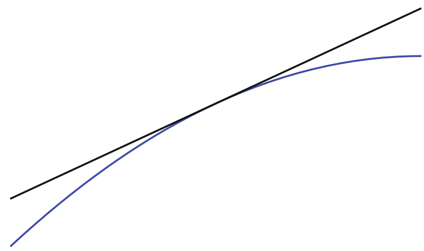


Figure 6.8. Tangent Line Approximation.

The difference between the new length and the original length, namely

$$e \;=\; \sqrt{\|\,\mathbf{a}_i - \mathbf{a}_j\,\|^2 + 2\,\varepsilon\,(\mathbf{a}_i - \mathbf{a}_j)\cdot(\mathbf{u}_i - \mathbf{u}_j) + \varepsilon^2\,\|\,\mathbf{u}_i - \mathbf{u}_j\,\|^2} \;-\; \|\,\mathbf{a}_i - \mathbf{a}_j\,\|, \tag{6.45}$$

is, by definition, the bar's *elongation*.

If the underlying dimension d is 2 or more, the elongation (6.45) is a *nonlinear* function of the displacement vectors $\mathbf{u}_i, \mathbf{u}_j$. Thus, an exact, geometrical treatment of structures in equilibrium requires dealing with complicated nonlinear systems of equations. In some situations, e.g., the design of robotic mechanisms, [**57**, **75**], analysis of the nonlinear system is crucial, but this lies beyond the scope of this text. However, in many practical situations, the displacements are fairly small, so $|\,\varepsilon\,| \ll 1$. For example, when a building moves, the lengths of bars are in meters, but the displacements are, barring catastrophes, typically in centimeters if not millimeters. In such situations, we can replace the geometrically exact elongation by a much simpler linear approximation.

As you learned in calculus, the most basic linear approximation to a nonlinear function $g(\varepsilon)$ near $\varepsilon = 0$ is given by its tangent line or linear Taylor polynomial

$$g(\varepsilon) \;\approx\; g(0) + g'(0)\,\varepsilon, \qquad |\,\varepsilon\,| \ll 1, \tag{6.46}$$

as sketched in Figure 6.8. In the case of small displacements of a bar, the elongation (6.45) is a square root function of the particular form

$$g(\varepsilon) = \sqrt{a^2 + 2\,\varepsilon\,b + \varepsilon^2\,c^2} \;-a,$$

where

$$a = \|\,\mathbf{a}_i - \mathbf{a}_j\,\|, \qquad b = (\mathbf{a}_i - \mathbf{a}_j)\cdot(\mathbf{u}_i - \mathbf{u}_j), \qquad c = \|\,\mathbf{u}_i - \mathbf{u}_j\,\|,$$

are independent of ε. Since $g(0) = 0$ and $g'(0) = b/a$, the linear approximation (6.46) is

$$\sqrt{a^2 + 2\,\varepsilon\,b + \varepsilon^2\,c^2} \;-a \;\approx\; \varepsilon\,\frac{b}{a} \qquad \text{for} \qquad |\,\varepsilon\,| \ll 1.$$

In this manner, we arrive at the linear approximation to the bar's elongation

$$e \;\approx\; \varepsilon\,\frac{(\mathbf{a}_i - \mathbf{a}_j)\cdot(\mathbf{u}_i - \mathbf{u}_j)}{\|\,\mathbf{a}_i - \mathbf{a}_j\,\|} \;=\; \mathbf{n}\cdot(\varepsilon\,\mathbf{u}_i - \varepsilon\,\mathbf{u}_j), \qquad \text{where} \qquad \mathbf{n} = \frac{\mathbf{a}_i - \mathbf{a}_j}{\|\,\mathbf{a}_i - \mathbf{a}_j\,\|}$$

is the unit vector, $\|\,\mathbf{n}\,\| = 1$, that points in the direction of the bar from node j to node i.

The factor ε was merely a mathematical device used to derive the linear approximation. It can now be safely discarded, so that the displacement of the i^{th} node is now $\mathbf{u}_i$ instead of $\varepsilon\,\mathbf{u}_i$, and we assume $\|\,\mathbf{u}_i\,\|$ is small. If bar k connects node i to node j, then its (approximate)

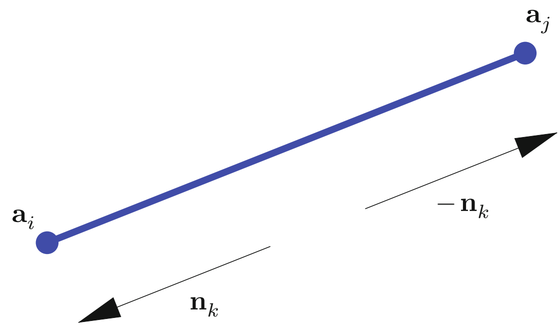


Figure 6.9. Unit Vectors for a Bar.

elongation is equal to

$$e_k = \mathbf{n}_k \cdot (\mathbf{u}_i - \mathbf{u}_j) = \mathbf{n}_k \cdot \mathbf{u}_i - \mathbf{n}_k \cdot \mathbf{u}_j, \qquad \text{where} \qquad \mathbf{n}_k = \frac{\mathbf{a}_i - \mathbf{a}_j}{\| \mathbf{a}_i - \mathbf{a}_j \|}. \tag{6.47}$$

The elongation e_k is the sum of two terms: the first, $\mathbf{n}_k \cdot \mathbf{u}_i$, is the component of the displacement vector for node i in the direction of the unit vector $\mathbf{n}_k$ that points along the bar *towards* node i, whereas the second, $-\mathbf{n}_k \cdot \mathbf{u}_j$, is the component of the displacement vector for node j in the direction of the unit vector $-\mathbf{n}_k$ that points in the opposite direction along the bar *toward* node j; see Figure 6.9. Their sum equals the total elongation of the bar.

We assemble all the linear equations (6.47) relating nodal displacements to bar elongations in matrix form

$$\mathbf{e} = A\,\mathbf{u}. \tag{6.48}$$

Here $\mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix} \in \mathbb{R}^m$ is the vector of elongations, while $\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \end{pmatrix} \in \mathbb{R}^{dn}$ is the vector of displacements. Each $\mathbf{u}_i \in \mathbb{R}^d$ is itself a column vector with d entries, and so $\mathbf{u}$ has a total of dn entries. For example, in the planar case $d = 2$, we have $\mathbf{u}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$, since each node's displacement has both an x and y component, and so

$$\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_n \\ y_n \end{pmatrix} \in \mathbb{R}^{2n}.$$

In three dimensions, $d = 3$, we have $\mathbf{u}_i = (x_i, y_i, z_i)^T$, and so each node will contribute three components to the displacement vector

$$\mathbf{u} = (x_1, y_1, z_1, x_2, y_2, z_2, \ \ldots \ , x_n, y_n, z_n)^T \in \mathbb{R}^{3n}.$$

The *incidence matrix* A connecting the displacements and elongations will be of size $m \times (dn)$. The k^{th} row of A will have (at most) $2d$ nonzero entries. The entries in the d

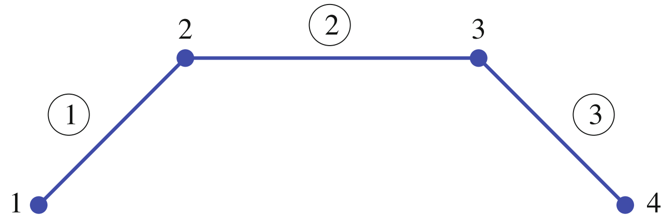


Figure 6.10. Three Bar Planar Structure.

slots corresponding to node i will be the components of the (transposed) unit bar vector $\mathbf{n}_k^T$ pointing towards node i, as given in (6.47), while the entries in the d slots corresponding to node j will be the components of its negative $-\mathbf{n}_k^T$, which is the unit bar vector pointing towards node j. All other entries are 0. The general mathematical formulation is best appreciated by working through an explicit example.

Example 6.5. Consider the planar structure pictured in Figure 6.10. The four nodes are at positions

$$\mathbf{a}_1 = (0,0)^T, \qquad \mathbf{a}_2 = (1,1)^T, \qquad \mathbf{a}_3 = (3,1)^T, \qquad \mathbf{a}_4 = (4,0)^T,$$

so the two side bars are at 45° angles and the center bar is horizontal. Implementing our construction, the associated incidence matrix is

$$A = \left(\begin{array}{cc|cc|cc|cc} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{array} \right). \tag{6.49}$$

The three rows of A refer to the three bars in our structure. The columns come in pairs, as indicated by the vertical lines in the matrix: the first two columns refer to the x and y displacements of the first node; the third and fourth columns refer to the second node; and so on. The first two entries of the first row of A indicate the unit vector

$$\mathbf{n}_1 = \frac{\mathbf{a}_1 - \mathbf{a}_2}{\|\mathbf{a}_1 - \mathbf{a}_2\|} = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)^T$$

that points along the first bar towards the first node, while the third and fourth entries have the opposite signs, and form the unit vector

$$-\mathbf{n}_1 = \frac{\mathbf{a}_2 - \mathbf{a}_1}{\|\mathbf{a}_2 - \mathbf{a}_1\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T$$

along the same bar that points in the opposite direction — towards the second node. The remaining entries are zero because the first bar connects only the first two nodes. Similarly, the unit vector along the second bar pointing towards node 2 is

$$\mathbf{n}_2 = \frac{\mathbf{a}_2 - \mathbf{a}_3}{\|\mathbf{a}_2 - \mathbf{a}_3\|} = (-1, 0)^T,$$

and this gives the third and fourth entries of the second row of A; the fifth and sixth entries are their negatives, corresponding to the unit vector $-\mathbf{n}_2$ pointing towards node 3. The last row is constructed from the unit vectors along bar #3 in the same fashion.

Remark. Interestingly, the incidence matrix for a structure depends only on the directions of the bars and not their lengths. This is analogous to the fact that the incidence matrix for an electrical network depends only on the connectivity properties of the wires and not on their overall lengths. One can regard the incidence matrix for a structure as a kind of d-dimensional generalization of the incidence matrix for a directed graph.

The next phase of our procedure is to introduce the constitutive relations for the bars that determine their internal forces or stresses. As we remarked at the beginning of the section, each bar is viewed as a hard spring, subject to a linear Hooke's law equation

$$y_k = c_k\, e_k \tag{6.50}$$

that relates its elongation e_k to its internal force y_k. The bar stiffness $c_k > 0$ is a positive scalar, and so $y_k > 0$ if the bar is in tension, while $y_k < 0$ if the bar is compressed. We write (6.50) in matrix form

$$\mathbf{y} = C\,\mathbf{e}, \tag{6.51}$$

where $C = \operatorname{diag}(c_1, \ldots, c_m) > 0$ is a diagonal, positive definite matrix.

Finally, we need to balance the forces at each node in order to achieve equilibrium. If bar k terminates at node i, then it exerts a force $-\,y_k\,\mathbf{n}_k$ on the node, where $\mathbf{n}_k$ is the unit vector pointing towards the node in the direction of the bar, as in (6.47). The minus sign comes from physics: if the bar is under tension, so $y_k > 0$, then it is trying to contract back to its unstressed state, and so will pull the node towards it — in the opposite direction to $\mathbf{n}_k$ — while a bar in compression will push the node away. In addition, we may have an externally applied force vector, denoted by $\mathbf{f}_i$, on node i, which might be some combination of gravity, weights, mechanical forces, and so on. (In this admittedly simplified model, external forces act only on the nodes and not directly on the bars.) Force balance at equilibrium requires that all the nodal forces, external and internal, cancel; thus,

$$\mathbf{f}_i \;+\; \sum_k (-\,y_k\,\mathbf{n}_k) = \mathbf{0}, \qquad \text{or} \qquad \sum_k y_k\,\mathbf{n}_k = \mathbf{f}_i,$$

where the sum is over all the bars that are attached to node i. The matrix form of the force balance equations is (and this should no longer come as a surprise)

$$\mathbf{f} = A^T\mathbf{y}, \tag{6.52}$$

where A^T is the transpose of the incidence matrix, and $\mathbf{f} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_n \end{pmatrix} \in \mathbb{R}^{dn}$ is the vector containing all external forces on the nodes. Putting everything together, (6.48, 51, 52),

$$\mathbf{e} = A\,\mathbf{u}, \qquad \mathbf{y} = C\,\mathbf{e}, \qquad \mathbf{f} = A^T\mathbf{y},$$

we are once again led to a by now familiar linear system of equations:

$$K\,\mathbf{u} = \mathbf{f}, \qquad \text{where} \qquad K = A^T C\,A \tag{6.53}$$

is the stiffness matrix for our structure.

The stiffness matrix K is a positive (semi-)definite Gram matrix (3.64) associated with the weighted inner product on the space of elongations prescribed by the diagonal matrix C. As we know, K will be positive definite if and only if the kernel of the incidence matrix is trivial: $\ker A = \{\mathbf{0}\}$. However, the preceding example does not enjoy this property, because we have not tied down (or "grounded") our structure. In essence, we are considering a structure floating in outer space, which is free to move around in any direction. Each rigid

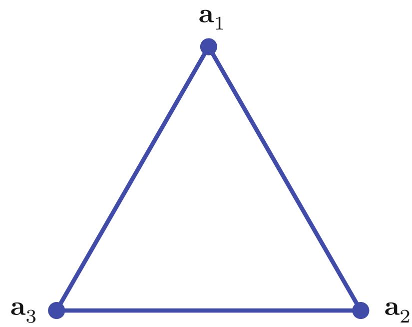


Figure 6.11. A Triangular Structure.

motion[†] of the structure will correspond to an element of the kernel of its incidence matrix, and thereby preclude positive definiteness of its stiffness matrix.

Example 6.6. Consider a planar space station in the shape of a unit equilateral triangle, as in Figure 6.11. Placing the nodes at positions

$$\mathbf{a}_1 = \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \right)^T, \qquad \mathbf{a}_2 = (1, 0)^T, \qquad \mathbf{a}_3 = (0, 0)^T,$$

we use the preceding algorithm to construct the incidence matrix

$$A = \left(\begin{array}{cc|cc|cc} -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array} \right),$$

whose rows are indexed by the bars, and whose columns are indexed in pairs by the three nodes. The kernel of A is three-dimensional, with basis

$$\mathbf{z}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{z}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{z}_3 = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{6.54}$$

We claim that these three displacement vectors represent three different planar rigid motions: the first two correspond to translations, and the third to a rotation.

The translations are easy to discern. Translating the space station in a horizontal direction means that we move all three nodes the same amount, and so the displacements are $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}_3 = \mathbf{a}$ for some vector $\mathbf{a}$. In particular, a rigid unit horizontal translation has $\mathbf{a} = \mathbf{e}_1 = (1, 0)^T$, and corresponds to the first kernel basis vector. Similarly, a unit vertical translation of all three nodes corresponds to $\mathbf{a} = \mathbf{e}_2 = (0, 1)^T$, and corresponds to the second kernel basis vector. Every other translation is a linear combination of these two. Translations do not alter the lengths of any of the bars, and so do not induce any stress in the structure.

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† See Section 7.2 for an extended discussion of rigid motions.

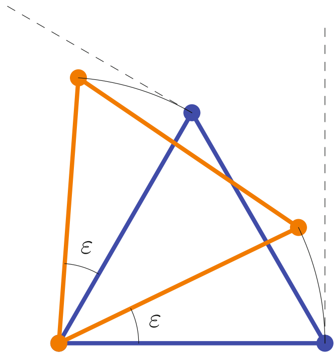


Figure 6.12. Rotating a Space Station.

The rotations are a little more subtle, owing to the linear approximation that we used to compute the elongations. Referring to Figure 6.12, we see that rotating the space station through a small angle ε around the node $\mathbf{a}_3 = (0,0)^T$ will move the other two nodes to positions

$$\mathbf{b}_1 = \begin{pmatrix} \frac{1}{2}\cos\varepsilon - \frac{\sqrt{3}}{2}\sin\varepsilon \\ \frac{1}{2}\sin\varepsilon + \frac{\sqrt{3}}{2}\cos\varepsilon \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} \cos\varepsilon \\ \sin\varepsilon \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.55}$$

However, the corresponding displacements

$$\begin{gathered} \mathbf{u}_1 = \mathbf{b}_1 - \mathbf{a}_1 = \begin{pmatrix} \frac{1}{2}(\cos\varepsilon - 1) - \frac{\sqrt{3}}{2}\sin\varepsilon \\ \frac{1}{2}\sin\varepsilon + \frac{\sqrt{3}}{2}(\cos\varepsilon - 1) \end{pmatrix}, \\ \mathbf{u}_2 = \mathbf{b}_2 - \mathbf{a}_2 = \begin{pmatrix} \cos\varepsilon - 1 \\ \sin\varepsilon \end{pmatrix}, \qquad \mathbf{u}_3 = \mathbf{b}_3 - \mathbf{a}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{gathered} \tag{6.56}$$

do *not* combine into a vector that belongs to $\ker A$. The problem is that, under a rotation, the nodes move along circles, while the kernel displacements $\mathbf{u} = \varepsilon\,\mathbf{z} \in \ker A$ correspond to straight line motion! In order to maintain consistency, we must adopt a similar linear approximation of the nonlinear circular motion of the nodes. Thus, we replace the nonlinear displacements $\mathbf{u}_j(\varepsilon)$ in (6.56) by their linear tangent approximations† $\varepsilon\,\mathbf{u}_j'(0)$, so

$$\mathbf{u}_1 \approx \varepsilon \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \qquad \mathbf{u}_2 \approx \varepsilon \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The resulting displacements *do* combine to produce the displacement vector

$$\mathbf{u} = \varepsilon \left(-\tfrac{\sqrt{3}}{2},\ \tfrac{1}{2},\ 0,\ 1,\ 0,\ 0 \right)^T = \varepsilon\,\mathbf{z}_3$$

that moves the space station in the direction of the third kernel basis vector. Thus, as claimed, $\mathbf{z}_3$ represents the linear approximation to a rigid rotation around the first node.

Remarkably, the rotations around the other two nodes, although distinct nonlinear motions, can be linearly approximated by particular combinations of the three kernel basis

† Note that $\mathbf{u}_j(0) = \mathbf{0}$.

elements $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$, and so already appear in our description of $\ker A$. For example, the displacement vector

$$\mathbf{u} = \varepsilon \left(\tfrac{\sqrt{3}}{2}\, \mathbf{z}_1 + \tfrac{1}{2}\, \mathbf{z}_2 - \mathbf{z}_3 \right) = \varepsilon \left(0,\ 0,\ \tfrac{\sqrt{3}}{2},\ -\tfrac{1}{2},\ \tfrac{\sqrt{3}}{2},\ \tfrac{1}{2} \right)^T \tag{6.57}$$

represents the linear approximation to a rigid rotation around the first node. We conclude that the three-dimensional kernel of the incidence matrix represents the sum total of all possible rigid motions of the space station, or, more correctly, their linear approximations.

Which types of forces will maintain the space station in equilibrium? This will happen if and only if we can solve the force balance equations $A^T \mathbf{y} = \mathbf{f}$ for the internal forces $\mathbf{y}$. The Fredholm Alternative Theorem 4.46 implies that this system has a solution if and only if $\mathbf{f}$ is orthogonal to $\operatorname{coker} A^T = \ker A$. Therefore, $\mathbf{f} = (\, f_1, g_1, f_2, g_2, f_3, g_3 \,)^T$ must be orthogonal to the kernel basis vectors (6.54), and so must satisfy the three linear constraints

$$\begin{aligned} \mathbf{z}_1 \cdot \mathbf{f} &= f_1 + f_2 + f_3 = 0, \\ \mathbf{z}_2 \cdot \mathbf{f} &= g_1 + g_2 + g_3 = 0, \\ \mathbf{z}_3 \cdot \mathbf{f} &= -\tfrac{\sqrt{3}}{2}\, f_1 + \tfrac{1}{2}\, g_1 + g_2 = 0. \end{aligned} \tag{6.58}$$

The first constraint requires that there be no net horizontal force on the space station. The second requires no net vertical force. The last constraint requires that the *moment* of the forces around the third node vanishes. The vanishing of the force moments around each of the other two nodes is a consequence of these three conditions, since the associated kernel vectors can be expressed as linear combinations of the three basis elements. The corresponding physical requirements are clear. If there is a net horizontal or vertical force, the space station will rigidly translate in that direction; if there is a non-zero force moment, the station will rigidly rotate. In any event, unless the force balance constraints (6.58) are satisfied, the space station cannot remain in equilibrium. A freely floating space station is an unstable structure that can easily be set into motion with a tiny external force.

Since there are three independent rigid motions, we must impose three constraints on the structure in order to fully stabilize it under general external forcing. "Grounding" one of the nodes, i.e., preventing it from moving by attaching it to a fixed support, will serve to eliminate the two translational instabilities. For example, setting $\mathbf{u}_3 = \mathbf{0}$ has the effect of fixing the third node of the space station to a support. With this specification, we can eliminate the variables associated with that node, and thereby delete the corresponding columns of the incidence matrix — leaving the *reduced incidence matrix*

$$A^\star = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The kernel of $A^\star$ is one-dimensional, spanned by the single vector $\mathbf{z}_3^\star = \left(\frac{\sqrt{3}}{2},\ \frac{1}{2},\ 0,\ 1 \right)^T$, which corresponds to (the linear approximation of) the rotations around the fixed node. To prevent the structure from rotating, we can also fix the second node, by further requiring $\mathbf{u}_2 = \mathbf{0}$. This serves to also eliminate the third and fourth columns of the original incidence matrix. The resulting "doubly reduced" incidence matrix

$$A^{\star\star} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 \end{pmatrix}$$

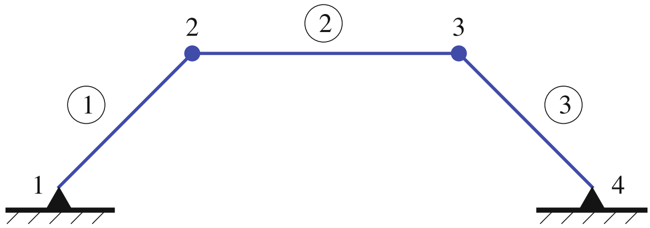


Figure 6.13. Three Bar Structure with Fixed Supports.

has trivial kernel: $\ker A^{\star\star} = \{\mathbf{0}\}$. Therefore, the corresponding reduced stiffness matrix

$$K^{\star\star} = (A^{\star\star})^T A^{\star\star} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

is positive definite. A planar triangle with two fixed nodes is a stable structure, which can now support an arbitrary external forcing on the remaining free node. (Forces on the fixed nodes have no effect, since they are no longer allowed to move.)

In general, a planar structure without any fixed nodes will have at least a three-dimensional kernel, corresponding to the rigid motions of translations and (linear approximations to) rotations. To stabilize the structure, one must fix two (non-coincident) nodes. A three-dimensional structure that is not tied to any fixed supports will admit 6 independent rigid motions in its kernel. Three of these correspond to rigid translations in the three coordinate directions, while the other three correspond to linear approximations to the rigid rotations around the three coordinate axes. To eliminate the rigid motion instabilities of the structure, we need to fix three non-collinear nodes. Indeed, fixing one node will eliminate translations; fixing two nodes will still leave the rotations around the axis through the fixed nodes. Details can be found in the exercises.

Even after a sufficient number of nodes have been attached to fixed supports so as to eliminate all possible rigid motions, there may still remain nonzero vectors in the kernel of the reduced incidence matrix of the structure. These indicate additional instabilities that allow the shape of the structure to deform without any applied force. Such non-rigid motions are known as *mechanisms* of the structure. Since a mechanism moves the nodes without elongating any of the bars, it does not induce any internal forces. A structure that admits a mechanism is unstable — even tiny external forces may provoke a large motion.

Example 6.7. Consider the three-bar structure of Example 6.5, but now with its two ends attached to supports, as pictured in Figure 6.13. Since we are fixing nodes 1 and 4, we set $\mathbf{u}_1 = \mathbf{u}_4 = \mathbf{0}$. Hence, we should remove the first and last column pairs from the incidence matrix (6.49), leading to the reduced incidence matrix

$$A^{\star} = \left(\begin{array}{cc|cc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right).$$

The structure no longer admits any rigid motions. However, the kernel of $A^{\star}$ is one-dimensional, spanned by reduced displacement vector $\mathbf{z}^{\star} = (\,1, -1, 1, 1\,)^T$, which corresponds to the unstable mechanism that displaces the second node in the direction $\mathbf{u}_2 =$

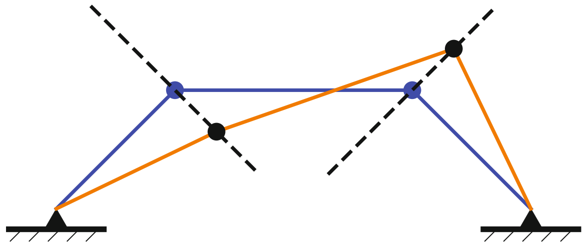


Figure 6.14. Unstable Mechanism of the Three Bar Structure.

$(1,-1)^T$ and the third node in the direction $\mathbf{u}_3 = (1,1)^T$. Geometrically, then, $\mathbf{z}^\star$ represents the displacement whereby node 2 moves down and to the right at a 45° angle, while node 3 moves simultaneously up and to the right at a 45° angle; the result of the mechanism is sketched in Figure 6.14. This mechanism does not alter the lengths of the three bars (at least in our linear approximation regime) and so requires no net force to be set into motion.

As with the rigid motions of the space station, an external forcing vector $\mathbf{f}^\star$ will maintain equilibrium only when it lies in the coimage of $A^\star$, and hence, by the Fredholm Alternative, must be orthogonal to all the mechanisms in $\ker A^\star$. Thus, the nodal forces $\mathbf{f}_2 = (f_2, g_2)^T$ and $\mathbf{f}_3 = (f_3, g_3)^T$ must satisfy the balance law

$$\mathbf{z}^\star \cdot \mathbf{f}^\star = f_2 - g_2 + f_3 + g_3 = 0.$$

If this fails, the equilibrium equation has no solution, and the structure will be set into motion. For example, a uniform horizontal force $f_2 = f_3 = 1$, $g_2 = g_3 = 0$, will induce the mechanism, whereas a uniform vertical force, $f_2 = f_3 = 0$, $g_2 = g_3 = 1$, will maintain equilibrium. In the latter case, the equilibrium equations

$$K^\star \mathbf{u}^\star = \mathbf{f}^\star, \qquad \text{where} \qquad K^\star = (A^\star)^T A^\star = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -1 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

have an indeterminate solution

$$\mathbf{u}^\star = (-3, 5, -2, 0)^T + t\,(1, -1, 1, 1)^T,$$

since we can add in any element of $\ker K^\star = \ker A^\star$. In other words, the equilibrium position is not unique, since the structure can still be displaced in the direction of the unstable mechanism while maintaining the overall force balance. On the other hand, the elongations and internal forces

$$\mathbf{y} = \mathbf{e} = A^\star \mathbf{u}^\star = \left(\sqrt{2},\ 1,\ \sqrt{2} \right)^T,$$

are well defined, indicating that, under our stabilizing uniform vertical (upwards) force, all three bars are elongated, with the two diagonals experiencing 41.4% more elongation than the horizontal bar.

Remark. Just like the rigid rotations, the mechanisms described here are linear approximations to the actual nonlinear motions. In a physical structure, the vertices will move

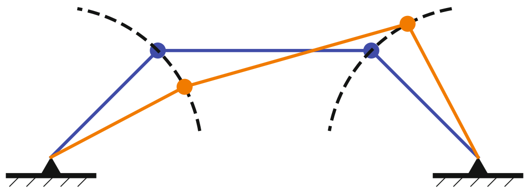


Figure 6.15. Nonlinear Mechanism of the Three Bar Structure.

along curves whose tangents at the initial configuration are the directions indicated by the mechanism vector. In the linear approximation illustrated in Figure 6.14, the lengths of the bars will change slightly. In the true nonlinear mechanism, illustrated in Figure 6.15, the nodes must move along circles so as to rigidly preserve the lengths of all three bars. In certain cases, a structure can admit a linear mechanism, but one that cannot be physically realized due to the nonlinear constraints imposed by the geometrical configurations of the bars. Nevertheless, such a structure is at best borderline stable, and should not be used in any real-world constructions.

We can always stabilize a structure by first fixing nodes to eliminate rigid motions, and then adding in a sufficient number of extra bars to prevent mechanisms. In the preceding example, suppose we attach an additional bar connecting nodes 2 and 4, leading to the reinforced structure in Figure 6.16. The revised incidence matrix is

$$A = \left(\begin{array}{cc|cc|cc|cc} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 & 0 & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{array} \right),$$

and is obtained from (6.49) by appending another row representing the added bar. When nodes 1 and 4 are fixed, the reduced incidence matrix

$$A^\star = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 & 0 \end{pmatrix}$$

has trivial kernel, $\ker A^\star = \{\mathbf{0}\}$, and hence the reinforced structure is stable. It admits no mechanisms, and can support any configuration of forces (within reason — mathematically the structure will support an arbitrarily large external force, but very large forces will take us outside the linear regime described by the model, and the structure may be crushed).

This particular case is *statically determinate* owing to the fact that the incidence matrix is square and nonsingular, which implies that one can solve the force balance equations (6.52) directly for the internal forces. For instance, a uniform downwards vertical force $f_2 = f_3 = 0$, $g_2 = g_3 = -1$, e.g., gravity, will produce the internal forces

$$y_1 = -\sqrt{2}, \qquad y_2 = -1, \qquad y_3 = -\sqrt{2}, \qquad y_4 = 0,$$

indicating that bars 1, 2 and 3 are compressed, while, interestingly, the reinforcing bar 4 remains unchanged in length and hence experiences no internal force. Assuming that the

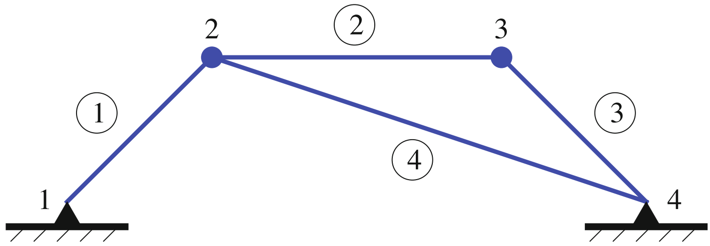


Figure 6.16. Reinforced Planar Structure.

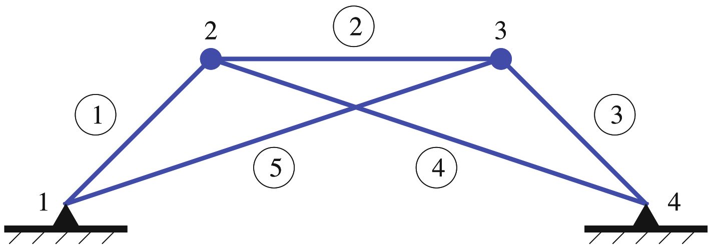


Figure 6.17. Doubly Reinforced Planar Structure.

bars are all of the same material, and taking the elastic constant to be 1, so $C = \mathrm{I}$, then the reduced stiffness matrix is

$$K^\star = (A^\star)^T A^\star = \begin{pmatrix} \frac{12}{5} & \frac{1}{5} & -1 & 0 \\ \frac{1}{5} & \frac{3}{5} & 0 & 0 \\ -1 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The solution to the reduced equilibrium equations is

$$\mathbf{u}^\star = \left(-\tfrac{1}{2}, \, -\tfrac{3}{2}, \, -\tfrac{3}{2}, \, -\tfrac{7}{2} \right)^T, \qquad \text{so} \qquad \mathbf{u}_2 = \left(-\tfrac{1}{2}, \, -\tfrac{3}{2} \right)^T, \quad \mathbf{u}_3 = \left(-\tfrac{3}{2}, \, -\tfrac{7}{2} \right)^T,$$

give the displacements of the two nodes under the applied force. Both are moving down and to the left, with node 3 moving relatively farther owing to its lack of reinforcement.

Suppose we reinforce the structure yet further by adding in a bar connecting nodes 1 and 3, as in Figure 6.17. The resulting reduced incidence matrix

$$A^\star = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 & 0 \\ 0 & 0 & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}$$

again has trivial kernel, $\ker A^\star = \{\mathbf{0}\}$, and hence the structure is stable. Indeed, adding extra bars to a stable structure cannot cause it to lose stability. (In the language of linear algebra, appending additional rows to a matrix cannot increase the size of its kernel, cf. Exercise 2.5.10.) Since the incidence matrix is rectangular, the structure is now *statically indeterminate*, and we cannot determine the internal forces without first solving the full

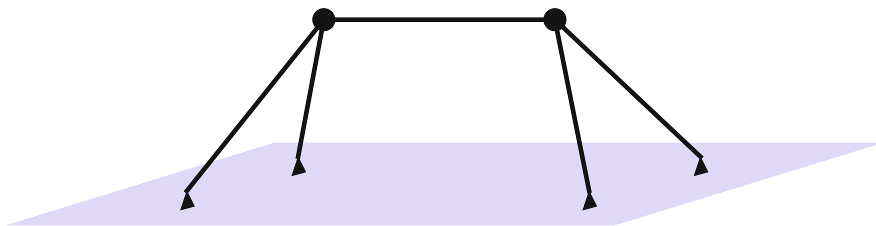


Figure 6.18. A Swing Set.

equilibrium equations (6.53) for the displacements. The stiffness matrix is

$$K^\star = (A^\star)^T A^\star = \begin{pmatrix} \frac{12}{5} & \frac{1}{5} & -1 & 0 \\ \frac{1}{5} & \frac{3}{5} & 0 & 0 \\ -1 & 0 & \frac{12}{5} & -\frac{1}{5} \\ 0 & 0 & -\frac{1}{5} & \frac{3}{5} \end{pmatrix}.$$

Under the same uniform vertical force, the displacement $\mathbf{u}^\star = \left(\frac{1}{10}, \, -\frac{17}{10}, \, -\frac{1}{10}, \, -\frac{17}{10} \right)^T$ indicates that the free nodes now move symmetrically down and towards the center of the structure. The internal forces on the bars are

$$y_1 = -\tfrac{4}{5}\sqrt{2}, \qquad y_2 = -\tfrac{1}{5}, \qquad y_3 = -\tfrac{4}{5}\sqrt{2}, \qquad y_4 = -\sqrt{\tfrac{2}{5}}, \qquad y_5 = -\sqrt{\tfrac{2}{5}}.$$

All five bars are now experiencing compression, with the two outside bars being the most stressed. This relatively simple computation should already indicate to the practicing construction engineer which of the bars in the structure are more likely to collapse under an applied external force.

Summarizing our discussion, we have established the following fundamental result characterizing the stability and equilibrium of structures.

Theorem 6.8. A structure is stable, and so will maintain its equilibrium under arbitrary external forcing, if and only if its reduced incidence matrix $A^\star$ has linearly independent columns, or, equivalently, $\ker A^\star = \{\mathbf{0}\}$. More generally, an external force $\mathbf{f}^\star$ on a structure will maintain equilibrium if and only if $\mathbf{f}^\star \in \operatorname{coimg} A^\star = (\ker A^\star)^\perp$, which requires that the external force be orthogonal to all rigid motions and all mechanisms admitted by the structure.

Example 6.9. A three-dimensional swing set is to be constructed, consisting of two diagonal supports at each end joined by a horizontal cross bar. Is this configuration stable, i.e., can a child swing on it without it collapsing? The movable joints are at positions

$$\mathbf{a}_1 = (1, 1, 3)^T, \qquad \mathbf{a}_2 = (4, 1, 3)^T,$$

while the four fixed supports are at

$$\mathbf{a}_3 = (0, 0, 0)^T, \qquad \mathbf{a}_4 = (0, 2, 0)^T, \qquad \mathbf{a}_5 = (5, 0, 0)^T, \qquad \mathbf{a}_6 = (5, 2, 0)^T.$$

The reduced incidence matrix for the structure is calculated in the usual manner:

$$A^\star = \left(\begin{array}{ccc|ccc} \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{11}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{11}} & -\frac{1}{\sqrt{11}} & \frac{3}{\sqrt{11}} & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{11}} & \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{11}} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{11}} & -\frac{1}{\sqrt{11}} & \frac{3}{\sqrt{11}} \end{array} \right).$$

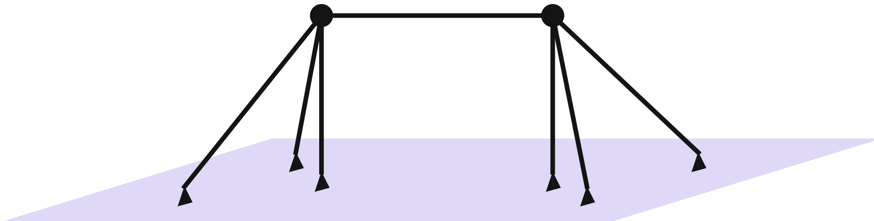


Figure 6.19. Reinforced Swing Set.

For instance, the first three entries contained in the first row refer to the unit vector $\mathbf{n}_1 = \dfrac{\mathbf{a}_1 - \mathbf{a}_3}{\|\mathbf{a}_1 - \mathbf{a}_3\|}$ in the direction of the bar going from $\mathbf{a}_3$ to $\mathbf{a}_1$. Suppose the five bars have the same stiffness $c_1 = \cdots = c_5 = 1$, so the reduced stiffness matrix for the structure is

$$K^\star = (A^\star)^T A^\star = \begin{pmatrix} \frac{13}{11} & 0 & \frac{6}{11} & -1 & 0 & 0 \\ 0 & \frac{2}{11} & 0 & 0 & 0 & 0 \\ \frac{6}{11} & 0 & \frac{18}{11} & 0 & 0 & 0 \\ -1 & 0 & 0 & \frac{13}{11} & 0 & -\frac{6}{11} \\ 0 & 0 & 0 & 0 & \frac{2}{11} & 0 \\ 0 & 0 & 0 & -\frac{6}{11} & 0 & \frac{18}{11} \end{pmatrix}.$$

Solving $A^\star \mathbf{z}^\star = \mathbf{0}$, we find $\ker A^\star = \ker K^\star$ is one-dimensional, spanned by

$$\mathbf{z}^\star = (\,3, 0, -1, 3, 0, 1\,)^T .$$

This indicates a mechanism that can cause the swing set to collapse: the first node moves down and to the right, while the second node moves up and to the right, the horizontal motion being three times as large as the vertical. The swing set can only support forces $\mathbf{f}_1 = (\,f_1, g_1, h_1\,)^T$, $\mathbf{f}_2 = (\,f_2, g_2, h_2\,)^T$ on the free nodes whose combined force vector $\mathbf{f}^\star$ is orthogonal to the mechanism vector $\mathbf{z}^\star$, and so

$$3\,(f_1 + f_2) - h_1 + h_2 = 0.$$

Otherwise, a reinforcing bar, say from node 1 to node 6 (although this will interfere with the swinging!) or another bar connecting one of the nodes to a new ground support, will be required to completely stabilize the swing.

For a uniform downwards unit vertical force, $\mathbf{f} = (\,0, 0, -1, 0, 0, -1\,)^T$, a particular solution to (6.11) is $\mathbf{u}^\star = \left(\frac{13}{6}, 0, -\frac{4}{3}, \frac{11}{6}, 0, 0\right)^T$ and the general solution $\mathbf{u} = \mathbf{u}^\star + t\,\mathbf{z}^\star$ is obtained by adding in an arbitrary element of the kernel. The resulting forces/elongations are uniquely determined,

$$\mathbf{y} = \mathbf{e} = A^\star\,\mathbf{u} = A^\star\,\mathbf{u}^\star = \left(-\frac{\sqrt{11}}{6},\ -\frac{\sqrt{11}}{6},\ -\frac{1}{3},\ -\frac{\sqrt{11}}{6},\ -\frac{\sqrt{11}}{6}\right)^T,$$

so that every bar is compressed, the middle one experiencing slightly more than half the stress of the outer supports.

If we add in two vertical supports at the nodes, as in Figure 6.19, then the corresponding reduced incidence matrix

$$A^\star = \begin{pmatrix} \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{11}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{11}} & -\frac{1}{\sqrt{11}} & \frac{3}{\sqrt{11}} & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{11}} & \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{11}} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{11}} & -\frac{1}{\sqrt{11}} & \frac{3}{\sqrt{11}} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

has trivial kernel, indicating stabilization of the structure. The reduced stiffness matrix

$$K^\star = \begin{pmatrix} \frac{13}{11} & 0 & \frac{6}{11} & -1 & 0 & 0 \\ 0 & \frac{2}{11} & 0 & 0 & 0 & 0 \\ \frac{6}{11} & 0 & \frac{29}{11} & 0 & 0 & 0 \\ -1 & 0 & 0 & \frac{13}{11} & 0 & -\frac{6}{11} \\ 0 & 0 & 0 & 0 & \frac{2}{11} & 0 \\ 0 & 0 & 0 & -\frac{6}{11} & 0 & \frac{29}{11} \end{pmatrix}$$

is now only slightly different, but this is enough to make it positive definite, $K^\star > 0$, and so allow arbitrary external forcing without collapse. Under the same uniform vertical force, the internal forces are

$$\mathbf{y} = \mathbf{e} = A^\star \mathbf{u} = \left(-\frac{\sqrt{11}}{10},\ -\frac{\sqrt{11}}{10},\ -\frac{1}{5},\ -\frac{\sqrt{11}}{10},\ -\frac{\sqrt{11}}{10},\ -\frac{2}{5},\ -\frac{2}{5} \right)^T.$$

Note the overall reductions in stress in the original bars; the two reinforcing vertical bars are now experiencing the largest compression.

Further developments in the mathematical analysis of structures can be found in the references [**33**, **79**].

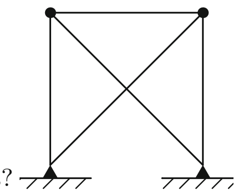
Exercises

6.3.1. If a bar in a structure compresses 2 cm under a force of 5 newtons applied to a node, how far will it compress under a force of 20 newtons applied at the same node?

6.3.2. An individual bar in a structure experiences a stress of 3 under a unit horizontal force applied to all the nodes and a stress of -2 under a unit vertical force applied to all nodes. What combinations of horizontal and vertical forces will make the bar stress-free?

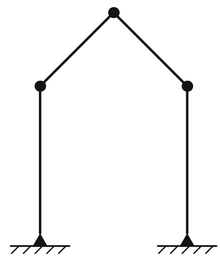
6.3.3. (*a*) For the reinforced structure illustrated in Figure 6.16, determine the displacements of the nodes and the stresses in the bars under a uniform horizontal force, and interpret physically. (*b*) Answer the same question for the doubly reinforced structure in Figure 6.17.

6.3.4. Discuss the effect of a uniform horizontal force in the direction of the horizontal bar on the swing set and its reinforced version in Example 6.9.

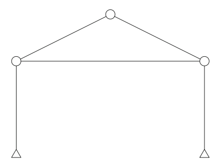


♡ 6.3.5. All the bars in the illustrated square planar structure have unit stiffness. (*a*) Write down the reduced incidence matrix A. (*b*) Write down the equilibrium equations for the structure when subjected to external forces at the free nodes. (*c*) Is the structure stable? statically determinate? Explain in detail. (*d*) Find a set of external forces with the property that the upper left node moves horizontally, while the upper right node stays in place. Which bar is under the most stress?

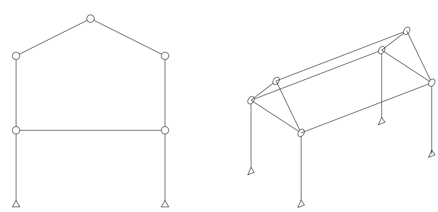
♡ 6.3.6. In the square structure of Exercise 6.3.5, the diagonal struts simply cross each other. We could also try joining them at an additional central node. Compare the stresses in the two structures under a uniform horizontal and a uniform vertical force at the two upper nodes, and discuss what you observe.



6.3.7. (*a*) Write down the reduced incidence matrix $A^\star$ for the pictured structure with 4 bars and 2 fixed supports. The width and the height of the vertical sides are each 1 unit, while the top node is 1.5 units above the base. (*b*) Predict the number of independent solutions to $A^\star\,\mathbf{u} = \mathbf{0}$, and then solve to describe them both numerically and geometrically. (*c*) What condition(s) must be imposed on the external forces to maintain equilibrium in the structure? (*d*) Add in just enough additional bars so that the resulting reinforced structure has only the trivial solution to $A^\star\,\mathbf{u} = \mathbf{0}$. Is your reinforced structure stable?



♡ 6.3.8. Consider the two-dimensional "house" constructed out of bars, as in the accompanying picture. The bottom nodes are fixed. The width of the house is 3 units, the height of the vertical sides 1 unit, and the peak is 1.5 units above the base.
(*a*) Determine the reduced incidence matrix A for this structure.
(*b*) How many distinct modes of instability are there? Describe them geometrically, and indicate whether they are mechanisms or rigid motions.
(*c*) Suppose we apply a combination of forces to each non-fixed node in the structure. Determine conditions such that the structure can support the forces. Write down an explicit nonzero set of external forces that satisfy these conditions, and compute the corresponding elongations of the individual bars. Which bar is under the most stress?
(*d*) Add in a *minimal* number of bars so that the resulting structure can support any force. Before starting, decide, from general principles, how many bars you need to add.
(*e*) With your new stable configuration, use the same force as before, and recompute the forces on the individual bars. Which bar now has the most stress? How much have you reduced the maximal stress in your reinforced building?



♣ 6.3.9. Answer Exercise 6.3.8 for the illustrated two- and three-dimensional houses. In the two-dimensional case, the width and total height of the vertical bars is 2 units, and the peak is an additional .5 unit higher. In the three-dimensional house, the width and vertical heights are equal to 1 unit, the length is 3 units, while the peaks are 1.5 units above the base.

♡ 6.3.10. Consider a structure consisting of three bars joined in a vertical line hanging from a top support. (*a*) Write down the equilibrium equations for this system when only forces and displacements in the vertical direction are allowed, i.e., a one-dimensional structure. Is the problem statically determinate, statically indeterminate, or unstable? If the latter, describe all possible mechanisms and the constraints on the forces required to maintain equilibrium. (*b*) Answer part (*a*) when the structure is two-dimensional, i.e., is allowed to move in a plane. (*c*) Answer the same question for the fully three-dimensional version.

♣ 6.3.11. A space station is built in the shape of a three-dimensional *simplex* whose nodes are at the positions $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$, and each pair of nodes is connected by a bar. (*a*) Sketch the space station and find its incidence matrix A. (*b*) Show that $\ker A$ is six-dimensional, and find a basis. (*c*) Explain which three basis vectors correspond to rigid translations. (*d*) Find three basis vectors that correspond to linear approximations to rotations around the three coordinate axes. (*e*) Suppose the bars all have unit stiffness. Compute the full stiffness matrix for the space station. (*f*) What constraints on external forces at the four nodes are required to maintain equilibrium? Can you interpret them physically? (*g*) How many nodes do you need to fix to stabilize the structure? (*h*) Suppose you fix the three nodes in the xy-plane. How much internal force does each bar experience under a unit vertical force on the upper vertex?

♣ 6.3.12. Suppose a space station is built in the shape of a regular tetrahedron with all sides of unit length. Answer all questions in Exercise 6.3.11.

♡ 6.3.13. A *mass–spring ring* consists of n masses connected in a circle by n identical springs, and the masses are allowed only to move in the angular direction. (*a*) Derive the equations of equilibrium. (*b*) Discuss stability, and characterize the external forces that will maintain equilibrium. (*c*) Find such a set of nonzero external forces in the case of a four-mass ring and solve the equilibrium equations. What does the nonuniqueness of the solution represent?

6.3.14. A structure in $\mathbb{R}^3$ has n movable nodes, admits no rigid motions, and is statically determinate. (*a*) How many bars must it have? (*b*) Find an example with $n = 3$.

♢ 6.3.15. Prove that if we apply a unit force to node i in a structure and measure the displacement of node j in the direction of the force, then we obtain the same value if we apply the force to node j and measure the displacement at node i in the same direction. *Hint*: First, solve Exercise 6.1.6.

6.3.16. *True or false*: A structure in $\mathbb{R}^3$ will admit no rigid motions if and only if at least 3 nodes are fixed.

6.3.17. Suppose all bars have unit stiffness. Explain why the internal forces in a structure form the solution of minimal Euclidean norm among all solutions to $A^T\mathbf{y} = \mathbf{f}$.

♢ 6.3.18. Let A be the reduced incidence matrix for a structure and C the diagonal bar stiffness matrix. Suppose $\mathbf{f}$ is a set of external forces that maintain equilibrium of the structure. (*a*) Prove that $\mathbf{f} = A^T C\,\mathbf{g}$ for some $\mathbf{g}$. (*b*) Prove that an allowable displacement $\mathbf{u}$ is a least squares solution to the system $A\mathbf{u} = \mathbf{g}$ with respect to the weighted norm $\|\mathbf{v}\|^2 = \mathbf{v}^T C\,\mathbf{v}$.

♡ 6.3.19. Suppose an *unstable* structure admits no rigid motions — only mechanisms. Let $\mathbf{f}$ be an external force on the structure that maintains equilibrium. Suppose that you stabilize the structure by adding in the *minimal* number of reinforcing bars. Prove that the given force $\mathbf{f}$ induces the *same* stresses in the original bars, while the reinforcing bars experience no stress. Are the displacements necessarily the same? Does the result continue to hold when more reinforcing bars are added to the structure? *Hint*: Use Exercise 6.3.18.

♡ 6.3.20. When a node is fixed to a *roller*, it is permitted to move only along a straight line — the direction of the roller. Consider the three-bar structure in Example 6.5. Suppose node 1 is fixed, but node 4 is attached to a roller that permits it to move only in the horizontal direction. (*a*) Construct the reduced incidence matrix and the equilibrium equations in this situation. You should have a system of 5 equations in 5 unknowns — the horizontal and vertical displacements of nodes 2 and 3 and the horizontal displacement of node 4. (*b*) Is your structure stable? If not, how many rigid motions and how many mechanisms does it permit?

♡ 6.3.21. Answer Exercise 6.3.20 when the roller at node 4 allows it to move in only the vertical direction.

♡ 6.3.22. Redo Exercises 6.3.20–21 for the reinforced structure in Figure 6.16.

6.3.23. (*a*) Suppose that we fix one node in a planar structure and put a second node on a roller. Does the structure admit any rigid motions? (*b*) How many rollers are needed to prevent all rigid motions in a three-dimensional structure? Are there any restrictions on the directions of the rollers?

6.3.24. *True or false*: If a structure is statically indeterminate, then every non-zero applied force will result in (*a*) one or more nodes having a non-zero displacement; (*b*) one or more bars having a non-zero elongation.

6.3.25. *True or false*: If a structure constructed out of bars with identical stiffnesses is stable, then the same structure constructed out of bars with differing stiffnesses is also stable.