



Chapter 2

Vector Spaces and Bases

Vector spaces and their ancillary structures provide the common language of linear algebra, and, as such, are an essential prerequisite for understanding contemporary applied (and pure) mathematics. The key concepts of vector space, subspace, linear independence, span, and basis will appear, not only in linear systems of algebraic equations and the geometry of n -dimensional Euclidean space, but also in the analysis of linear differential equations, linear boundary value problems, Fourier analysis, signal processing, numerical methods, and many, many other fields. Therefore, in order to master modern linear algebra and its applications, the first order of business is to acquire a firm understanding of fundamental vector space constructions.

One of the grand themes of mathematics is the recognition that many seemingly unrelated entities are, in fact, different manifestations of the same underlying abstract structure. This serves to unify and simplify the disparate special situations, at the expense of introducing an extra level of abstraction. Indeed, the history of mathematics, as well as your entire mathematical educational career, can be viewed as an evolution towards ever greater abstraction resulting in ever greater power for solving problems. Here, the abstract notion of a vector space serves to unify spaces of ordinary vectors, spaces of functions, such as polynomials, exponentials, and trigonometric functions, as well as spaces of matrices, spaces of linear operators, and so on, all in a common conceptual framework. Moreover, proofs that might appear to be complicated in a particular context often turn out to be relatively transparent when recast in the more inclusive vector space language. The price that one pays for the increased level of abstraction is that, while the underlying mathematics is not all that complicated, novices typically take a long time to assimilate the underlying concepts. In our opinion, the best way to approach the subject is to think in terms of concrete examples. First, make sure you understand what is being said in the case of ordinary Euclidean space. Once this is grasped, the next important case to consider is an elementary function space, e.g., the space of continuous scalar functions. With the two most important cases firmly in hand, the leap to the general abstract formulation should not be too painful. Patience is essential; ultimately, the only way to truly understand an abstract concept like a vector space is by working with it in real-life applications! And always keep in mind that the effort expended here will be amply rewarded later on.

Following an introduction to vector spaces and subspaces, we develop the fundamental notions of span and linear independence, first in the context of ordinary vectors, but then in more generality, with an emphasis on function spaces. These are then combined into the all-important definition of a basis of a vector space, leading to a linear algebraic characterization of its dimension. Here is where the distinction between finite-dimensional and infinite-dimensional vector spaces first becomes apparent, although the full ramifications of this dichotomy will take time to unfold. We will then study the four fundamental subspaces associated with a matrix — its image, kernel, coimage, and cokernel — and explain how they help us understand the structure and the solutions of linear algebraic systems. Of particular significance is the linear superposition principle that enables us to combine

solutions to linear systems. Superposition is the hallmark of linearity, and will apply not only to linear algebraic equations, but also to linear ordinary differential equations, linear boundary value problems, linear partial differential equations, linear integral equations, linear control systems, etc. The final section in this chapter develops some interesting applications in graph theory that serve to illustrate the fundamental matrix subspaces; these results will be developed further in our study of electrical circuits.

2.1 Real Vector Spaces

A vector space is the abstract reformulation of the quintessential properties of n -dimensional[†] *Euclidean space* \mathbb{R}^n , which is defined as the set of all real (column) vectors with n entries. The basic laws of vector addition and scalar multiplication in \mathbb{R}^n serve as the template for the following general definition.

Definition 2.1. A *vector space* is a set V equipped with two operations:

- (i) *Addition*: adding any pair of vectors $\mathbf{v}, \mathbf{w} \in V$ produces another vector $\mathbf{v} + \mathbf{w} \in V$;
- (ii) *Scalar Multiplication*: multiplying a vector $\mathbf{v} \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $c\mathbf{v} \in V$.

These are subject to the following axioms, valid for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars $c, d \in \mathbb{R}$:

- (a) *Commutativity of Addition*: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
- (b) *Associativity of Addition*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- (c) *Additive Identity*: There is a zero element $\mathbf{0} \in V$ satisfying $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$.
- (d) *Additive Inverse*: For each $\mathbf{v} \in V$ there is an element $-\mathbf{v} \in V$ such that

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = (-\mathbf{v}) + \mathbf{v}.$$
- (e) *Distributivity*: $(c + d)\mathbf{v} = (c\mathbf{v}) + (d\mathbf{v})$, and $c(\mathbf{v} + \mathbf{w}) = (c\mathbf{v}) + (c\mathbf{w})$.
- (f) *Associativity of Scalar Multiplication*: $c(d\mathbf{v}) = (cd)\mathbf{v}$.
- (g) *Unit for Scalar Multiplication*: the scalar $1 \in \mathbb{R}$ satisfies $1\mathbf{v} = \mathbf{v}$.

Remark. For most of this text, we will deal with real vector spaces, in which the scalars are ordinary real numbers, as indicated in the definition. Complex vector spaces, where complex scalars are allowed, will be introduced in Section 3.6. Vector spaces over other fields are studied in abstract algebra, [38].

In the beginning, we will refer to the individual elements of a vector space as “vectors”, even though, as we shall see, they might also be functions, or matrices, or even more general objects. Unless we are dealing with certain specific examples such as a space of functions or matrices, we will use bold face, lower case Latin letters $\mathbf{v}, \mathbf{w}, \dots$ to denote the elements of our vector space. We will usually use a bold face $\mathbf{0}$ to denote the unique[‡] zero element of our vector space, while ordinary 0 denotes the real number zero.

The following identities are elementary consequences of the vector space axioms:

- (h) $0\mathbf{v} = \mathbf{0}$;
- (i) $(-1)\mathbf{v} = -\mathbf{v}$;
- (j) $c\mathbf{0} = \mathbf{0}$;
- (k) If $c\mathbf{v} = \mathbf{0}$, then either $c = 0$ or $\mathbf{v} = \mathbf{0}$.

[†] The precise definition of dimension will appear later, in Theorem 2.29.

[‡] See Exercise 2.1.12.

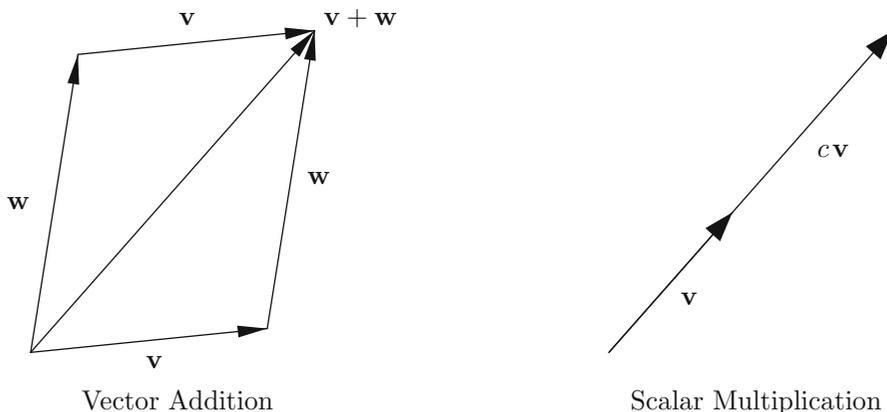


Figure 2.1. Vector Space Operations in \mathbb{R}^n .

Let us, for example, prove (h). Let $\mathbf{z} = 0\mathbf{v}$. Then, by the distributive property,

$$\mathbf{z} + \mathbf{z} = 0\mathbf{v} + 0\mathbf{v} = (0 + 0)\mathbf{v} = 0\mathbf{v} = \mathbf{z}.$$

Adding $-\mathbf{z}$ to both sides of this equation, and making use of axioms (b), (d), and then (c), we conclude that

$$\mathbf{z} = \mathbf{z} + \mathbf{0} = \mathbf{z} + (\mathbf{z} + (-\mathbf{z})) = (\mathbf{z} + \mathbf{z}) + (-\mathbf{z}) = \mathbf{z} + (-\mathbf{z}) = \mathbf{0},$$

which completes the proof. Verification of the other three properties is left as an exercise for the reader.

Let us now introduce the most important examples of (real) vector spaces.

Example 2.2. As noted above, the prototypical example of a real vector space is the Euclidean space \mathbb{R}^n , consisting of all n -tuples of real numbers $\mathbf{v} = (v_1, v_2, \dots, v_n)^T$, which we consistently write as column vectors. Vector addition and scalar multiplication are defined in the usual manner:

$$\mathbf{v} + \mathbf{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix}, \quad c\mathbf{v} = \begin{pmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{pmatrix}, \quad \text{whenever } \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}, \quad c \in \mathbb{R}.$$

The zero vector is $\mathbf{0} = (0, \dots, 0)^T$. The two vector space operations are illustrated in Figure 2.1. The fact that vectors in \mathbb{R}^n satisfy all of the vector space axioms is an immediate consequence of the laws of vector addition and scalar multiplication.

Example 2.3. Let $\mathcal{M}_{m \times n}$ denote the space of all real matrices of size $m \times n$. Then $\mathcal{M}_{m \times n}$ forms a vector space under the laws of matrix addition and scalar multiplication. The zero element is the zero matrix \mathbf{O} . (We are ignoring matrix multiplication, which is *not* a vector space operation.) Again, the vector space axioms are immediate consequences of the basic laws of matrix arithmetic. The preceding example of the vector space $\mathbb{R}^n = \mathcal{M}_{n \times 1}$ is a particular case in which the matrices have only one column.

Example 2.4. Consider the space

$$\mathcal{P}^{(n)} = \{ p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \} \quad (2.1)$$

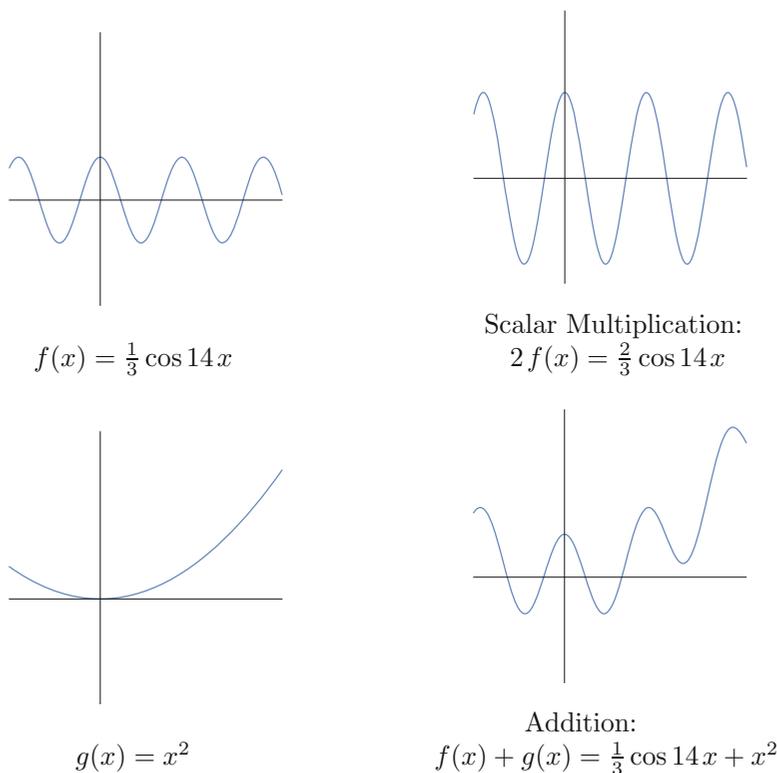


Figure 2.2. Vector Space Operations in Function Space.

consisting of all real polynomials of degree $\leq n$. Addition of polynomials is defined in the usual manner; for example,

$$(x^2 - 3x) + (2x^2 - 5x + 4) = 3x^2 - 8x + 4.$$

Note that the sum $p(x) + q(x)$ of two polynomials of degree $\leq n$ also has degree $\leq n$. The zero element of $\mathcal{P}^{(n)}$ is the zero polynomial. We can multiply polynomials by scalars — real constants — in the usual fashion; for example if $p(x) = x^2 - 2x$, then $3p(x) = 3x^2 - 6x$. The proof that $\mathcal{P}^{(n)}$ satisfies the vector space axioms is an easy consequence of the basic laws of polynomial algebra.

Warning. It is not true that the sum of two polynomials of degree n also has degree n . For example $(x^2 + 1) + (-x^2 + x) = x + 1$ has degree 1 even though the two summands have degree 2. This means that the set of polynomials of degree $= n$ is *not* a vector space.

Warning. You might be tempted to identify a scalar with a constant polynomial, but one should really regard these as two completely different objects — one is a *number*, while the other is a *constant function*. To add to the confusion, one typically uses the same notation for these two objects; for instance, 0 could mean either the real number 0 or the constant function taking the value 0 everywhere, which is the zero element, $\mathbf{0}$, of this vector space. The reader needs to exercise due care when interpreting each occurrence.

For much of analysis, including differential equations, Fourier theory, numerical methods, etc., the most important vector spaces consist of functions that have certain prescribed properties. The simplest such example is the following.

Example 2.5. Let $I \subset \mathbb{R}$ be an interval[†]. Consider the *function space* $\mathcal{F} = \mathcal{F}(I)$ whose elements are all real-valued functions $f(x)$ defined for all $x \in I$. The claim is that the function space \mathcal{F} has the structure of a vector space. Addition of functions in \mathcal{F} is defined in the usual manner: $(f+g)(x) = f(x) + g(x)$ for all $x \in I$. Multiplication by scalars $c \in \mathbb{R}$ is the same as multiplication by constants, $(cf)(x) = cf(x)$. The zero element is the zero function — the constant function that is identically 0 for all $x \in I$. The proof of the vector space axioms is straightforward. Observe that we are ignoring all additional operations that affect functions such as multiplication, division, inversion, composition, etc.; these are irrelevant as far as the vector space structure of \mathcal{F} goes.

Example 2.6. The preceding examples are all, in fact, special cases of an even more general construction. A clue is to note that the last example of a function space does not make any use of the fact that the domain of the functions is a real interval. Indeed, the same construction produces a function space $\mathcal{F}(I)$ corresponding to *any* subset $I \subset \mathbb{R}$.

Even more generally, let S be *any* set. Let $\mathcal{F} = \mathcal{F}(S)$ denote the space of all real-valued functions $f: S \rightarrow \mathbb{R}$. Then we claim that V is a vector space under the operations of function addition and scalar multiplication. More precisely, given functions f and g , we define their sum to be the function $h = f + g$ such that $h(x) = f(x) + g(x)$ for all $x \in S$. Similarly, given a function f and a real scalar $c \in \mathbb{R}$, we define the scalar multiple $g = cf$ to be the function such that $g(x) = cf(x)$ for all $x \in S$. The proof of the vector space axioms is straightforward, and the reader should be able to fill in the necessary details.

In particular, if $S \subset \mathbb{R}$ is an interval, then $\mathcal{F}(S)$ coincides with the space of scalar functions described in the preceding example. If $S \subset \mathbb{R}^n$ is a subset of Euclidean space, then the elements of $\mathcal{F}(S)$ are real-valued functions $f(x_1, \dots, x_n)$ depending upon the n variables corresponding to the coordinates of points $\mathbf{x} = (x_1, \dots, x_n) \in S$ in the domain. In this fashion, the set of real-valued functions defined on any domain in \mathbb{R}^n forms a vector space.

Another useful example is to let $S = \{x_1, \dots, x_n\} \subset \mathbb{R}$ be a finite set of real numbers. A real-valued function $f: S \rightarrow \mathbb{R}$ is defined by its values $f(x_1), f(x_2), \dots, f(x_n)$ at the specified points. In applications, these objects serve to indicate the *sample values* of a scalar function $f(x) \in \mathcal{F}(\mathbb{R})$ taken at the *sample points* x_1, \dots, x_n . For example, if $f(x) = x^2$ and the sample points are $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 3$, then the corresponding sample values are $f(x_1) = 0$, $f(x_2) = 1$, $f(x_3) = 4$, $f(x_4) = 9$. When measuring a physical quantity — velocity, temperature, pressure, etc. — one typically records only a finite set of sample values. The intermediate, non-recorded values between the sample points are then reconstructed through some form of interpolation, a topic that we shall visit in depth in Chapters 4 and 5.

Interestingly, the sample values $f(x_i)$ can be identified with the entries f_i of a vector

$$\mathbf{f} = (f_1, f_2, \dots, f_n)^T = (f(x_1), f(x_2), \dots, f(x_n))^T \in \mathbb{R}^n,$$

[†] An *interval* is a subset $I \subset \mathbb{R}$ that contains all the real numbers between $a, b \in \mathbb{R}$, where $a < b$, and can be

- *closed*, meaning that it includes its endpoints: $I = [a, b] = \{x \mid a \leq x \leq b\}$;
- *open*, which does not include either endpoint: $I = (a, b) = \{x \mid a < x < b\}$; or
- *half-open*, which includes one but not the other endpoint, so $I = [a, b) = \{x \mid a \leq x < b\}$ or $I = (a, b] = \{x \mid a < x \leq b\}$.

An open endpoint is allowed to be infinite; in particular, $(-\infty, \infty) = \mathbb{R}$ is another way of writing the entire real line.

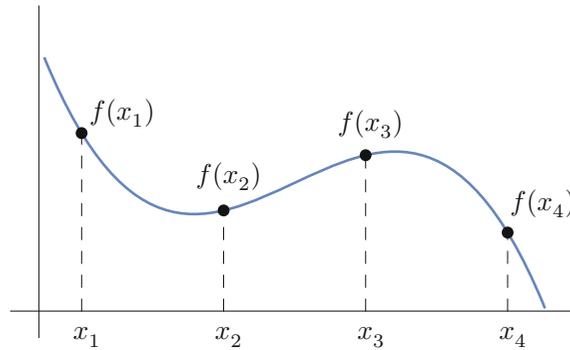


Figure 2.3. Sampled Function.

known as the *sample vector*. Every sampled function $f: S \rightarrow \mathbb{R}$ corresponds to a unique vector $\mathbf{f} \in \mathbb{R}^n$ and vice versa. (But keep in mind that different scalar functions $f(x) \in \mathcal{F}(\mathbb{R})$ may have the same sample values.) Addition of sample functions corresponds to addition of their sample vectors, as does scalar multiplication. Thus, *the vector space of sample functions $\mathcal{F}(S) = \mathcal{F}\{x_1, \dots, x_n\}$ is the same as the vector space \mathbb{R}^n* ! The identification of sampled functions as vectors is of fundamental importance in modern signal processing and data analysis, as we will see below.

Example 2.7. The above construction admits yet a further generalization. We continue to let S be an arbitrary set. Let V be a vector space. The claim is that the space $\mathcal{F}(S, V)$ consisting of all V -valued functions $\mathbf{f}: S \rightarrow V$ is a vector space. In other words, we replace the particular vector space \mathbb{R} in the preceding example by a general vector space V , and the same conclusion holds. The operations of function addition and scalar multiplication are defined in the evident manner: $(\mathbf{f} + \mathbf{g})(x) = \mathbf{f}(x) + \mathbf{g}(x)$ and $(c\mathbf{f})(x) = c\mathbf{f}(x)$ for $x \in S$, where we are using the vector addition and scalar multiplication operations on V to induce corresponding operations on V -valued functions. The proof that $\mathcal{F}(S, V)$ satisfies all of the vector space axioms proceeds as before.

The most important example of such a function space arises when $S \subset \mathbb{R}^n$ is a domain in Euclidean space and $V = \mathbb{R}^m$ is itself a Euclidean space. In this case, the elements of $\mathcal{F}(S, \mathbb{R}^m)$ consist of vector-valued functions $\mathbf{f}: S \rightarrow \mathbb{R}^m$, so that $\mathbf{f}(\mathbf{x}) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))^T$ is a column vector consisting of m functions of n variables, all defined on a common domain S . The general construction implies that addition and scalar multiplication of vector-valued functions is done componentwise; for example

$$2 \begin{pmatrix} x^2 \\ e^x - 4 \end{pmatrix} - \begin{pmatrix} \cos x \\ x \end{pmatrix} = \begin{pmatrix} 2x^2 - \cos x \\ 2e^x - x - 8 \end{pmatrix}.$$

Of particular importance are the vector fields arising in physics, including gravitational force fields, electromagnetic fields, fluid velocity fields, and many others.

Exercises

- 2.1.1. Show that the set of complex numbers $x + iy$ forms a real vector space under the operations of addition $(x + iy) + (u + iv) = (x + u) + i(y + v)$ and scalar multiplication $c(x + iy) = cx + icy$. (But complex multiplication is *not* a real vector space operation.)

2.1.2. Show that the positive quadrant $Q = \{(x, y) \mid x, y > 0\} \subset \mathbb{R}^2$ forms a vector space if we define addition by $(x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2)$ and scalar multiplication by $c(x, y) = (x^c, y^c)$.

◇ 2.1.3. Let S be any set. Carefully justify the validity of all the vector space axioms for the space $\mathcal{F}(S)$ consisting of all real-valued functions $f: S \rightarrow \mathbb{R}$.

2.1.4. Let $S = \{0, 1, 2, 3\}$. (a) Find the sample vectors corresponding to the functions $1, \cos \pi x, \cos 2\pi x, \cos 3\pi x$. (b) Is a function uniquely determined by its sample values?

2.1.5. Find two different functions $f(x)$ and $g(x)$ that have the *same* sample vectors \mathbf{f}, \mathbf{g} at the sample points $x_1 = 0, x_2 = 1, x_3 = -1$.

2.1.6. (a) Let $x_1 = 0, x_2 = 1$. Find the unique linear function $f(x) = ax + b$ that has the sample vector $\mathbf{f} = (3, -1)^T$. (b) Let $x_1 = 0, x_2 = 1, x_3 = -1$. Find the unique quadratic function $f(x) = ax^2 + bx + c$ with sample vector $\mathbf{f} = (1, -2, 0)^T$.

2.1.7. Let $\mathcal{F}(\mathbb{R}^2, \mathbb{R}^2)$ denote the vector space consisting of all functions $\mathbf{f}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

(a) Which of the following functions $\mathbf{f}(x, y)$ are elements? (i) $x^2 + y^2$, (ii) $\begin{pmatrix} x - y \\ xy \end{pmatrix}$, (iii) $\begin{pmatrix} e^x \\ \cos y \end{pmatrix}$, (iv) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, (v) $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$, (vi) $\begin{pmatrix} x \\ y \\ x + y \end{pmatrix}$. (b) Sum all of the elements of $\mathcal{F}(\mathbb{R}^2, \mathbb{R}^2)$ you identified in part (a). Then multiply your sum by the scalar -5 . (c) Carefully describe the zero element of the vector space $\mathcal{F}(\mathbb{R}^2, \mathbb{R}^2)$.

◇ 2.1.8. A *planar vector field* is a function that assigns a vector $\mathbf{v}(x, y) = \begin{pmatrix} v_1(x, y) \\ v_2(x, y) \end{pmatrix}$ to each point $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Explain why the set of all planar vector fields forms a vector space.

♡ 2.1.9. Let $h, k > 0$ be fixed. Let $S = \{(ih, jk) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ be points in a rectangular planar grid. Show that the function space $\mathcal{F}(S)$ can be identified with the vector space of $m \times n$ matrices $\mathcal{M}_{m \times n}$.

2.1.10. The space \mathbb{R}^∞ is defined as the set of all infinite real sequences $\mathbf{a} = (a_1, a_2, a_3, \dots)$, where $a_i \in \mathbb{R}$. Define addition and scalar multiplication in such a way as to make \mathbb{R}^∞ into a vector space. Explain why all the vector space axioms are valid.

2.1.11. Prove the basic vector space properties (i), (j), (k) following Definition 2.1.

◇ 2.1.12. Prove that a vector space has only one zero element $\mathbf{0}$.

◇ 2.1.13. Suppose that V and W are vector spaces. The *Cartesian product space*, denoted by $V \times W$, is defined as the set of all ordered pairs (\mathbf{v}, \mathbf{w}) , where $\mathbf{v} \in V, \mathbf{w} \in W$, with vector addition $(\mathbf{v}, \mathbf{w}) + (\hat{\mathbf{v}}, \hat{\mathbf{w}}) = (\mathbf{v} + \hat{\mathbf{v}}, \mathbf{w} + \hat{\mathbf{w}})$ and scalar multiplication $c(\mathbf{v}, \mathbf{w}) = (c\mathbf{v}, c\mathbf{w})$. (a) Prove that $V \times W$ is a vector space. (b) Explain why $\mathbb{R} \times \mathbb{R}$ is the same as \mathbb{R}^2 . (c) More generally, explain why $\mathbb{R}^m \times \mathbb{R}^n$ is the same as \mathbb{R}^{m+n} .

2.1.14. Use Exercise 2.1.13 to show that the space of pairs $(f(x), a)$, where f is a continuous scalar function and a is a real number, is a vector space. What is the zero element? Be precise! Write out the laws of vector addition and scalar multiplication.

2.2 Subspaces

In the preceding section, we were introduced to the most basic vector spaces that arise in this text. Almost all of the vector spaces used in applications appear as subsets of these prototypical examples.

Definition 2.8. A *subspace* of a vector space V is a subset $W \subset V$ that is a vector space in its own right — under the same operations of vector addition and scalar multiplication and the same zero element.

In particular, a subspace W *must* contain the zero element of V . Proving that a given subset of a vector space is a subspace is particularly easy: we need only check its *closure* under addition and scalar multiplication.

Proposition 2.9. A nonempty subset $W \subset V$ of a vector space is a subspace if and only if

- (a) for every $\mathbf{v}, \mathbf{w} \in W$, the sum $\mathbf{v} + \mathbf{w} \in W$, and
- (b) for every $\mathbf{v} \in W$ and every $c \in \mathbb{R}$, the scalar product $c\mathbf{v} \in W$.

Proof: The proof is immediate. For example, let us check commutativity. The subspace elements $\mathbf{v}, \mathbf{w} \in W$ can be regarded as elements of V , in which case $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ because V is a vector space. But the closure condition implies that the common sum also belongs to W , and so the commutativity axiom also holds for elements of W . Establishing the validity of the other axioms is equally easy. *Q.E.D.*

It is sometimes convenient to combine the two closure conditions. Thus, to prove that $W \subset V$ is a subspace, it suffices to check that $c\mathbf{v} + d\mathbf{w} \in W$ for all $\mathbf{v}, \mathbf{w} \in W$ and $c, d \in \mathbb{R}$.

Example 2.10. Let us list some examples of subspaces of the three-dimensional Euclidean space \mathbb{R}^3 .

(a) The trivial subspace $W = \{\mathbf{0}\}$. Demonstrating closure is easy: since there is only one element $\mathbf{0}$ in W , we just need to check that $\mathbf{0} + \mathbf{0} = \mathbf{0} \in W$ and $c\mathbf{0} = \mathbf{0} \in W$ for every scalar c .

(b) The entire space $W = \mathbb{R}^3$. Here closure is immediate because \mathbb{R}^3 is a vector space in its own right.

(c) The set of all vectors of the form $(x, y, 0)^T$, i.e., the xy coordinate plane. To prove closure, we check that all sums $(x, y, 0)^T + (\hat{x}, \hat{y}, 0)^T = (x + \hat{x}, y + \hat{y}, 0)^T$ and scalar multiples $c(x, y, 0)^T = (cx, cy, 0)^T$ of vectors in the xy -plane remain in the plane.

(d) The set of solutions $(x, y, z)^T$ to the homogeneous linear equation

$$3x + 2y - z = 0. \tag{2.2}$$

Indeed, if $\mathbf{x} = (x, y, z)^T$ is a solution, then so is every scalar multiple $c\mathbf{x} = (cx, cy, cz)^T$ since

$$3(cx) + 2(cy) - (cz) = c(3x + 2y - z) = 0.$$

Moreover, if $\hat{\mathbf{x}} = (\hat{x}, \hat{y}, \hat{z})^T$ is a second solution, the sum $\mathbf{x} + \hat{\mathbf{x}} = (x + \hat{x}, y + \hat{y}, z + \hat{z})^T$ is also a solution, since

$$3(x + \hat{x}) + 2(y + \hat{y}) - (z + \hat{z}) = (3x + 2y - z) + (3\hat{x} + 2\hat{y} - \hat{z}) = 0.$$

The solution space is, in fact, the two-dimensional plane passing through the origin with normal vector $(3, 2, -1)^T$.

(e) The set of all vectors lying in the plane spanned by the vectors $\mathbf{v}_1 = (2, -3, 0)^T$ and $\mathbf{v}_2 = (1, 0, 3)^T$. In other words, we consider all vectors of the form

$$\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2 = a \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2a + b \\ -3a \\ 3b \end{pmatrix},$$

where $a, b \in \mathbb{R}$ are arbitrary scalars. If $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$ and $\mathbf{w} = \widehat{a}\mathbf{v}_1 + \widehat{b}\mathbf{v}_2$ are any two vectors in the span, then so is

$$c\mathbf{v} + d\mathbf{w} = c(a\mathbf{v}_1 + b\mathbf{v}_2) + d(\widehat{a}\mathbf{v}_1 + \widehat{b}\mathbf{v}_2) = (ac + \widehat{a}d)\mathbf{v}_1 + (bc + \widehat{b}d)\mathbf{v}_2 = \widetilde{a}\mathbf{v}_1 + \widetilde{b}\mathbf{v}_2,$$

where $\widetilde{a} = ac + \widehat{a}d$, $\widetilde{b} = bc + \widehat{b}d$. This demonstrates that the span is a subspace of \mathbb{R}^3 . The reader might already have noticed that this subspace is the same plane defined by (2.2).

Example 2.11. The following subsets of \mathbb{R}^3 are *not* subspaces.

(a) The set P of all vectors of the form $(x, y, 1)^T$, i.e., the plane parallel to the xy coordinate plane passing through $(0, 0, 1)^T$. Indeed, $(0, 0, 0)^T \notin P$, which is the most basic requirement for a subspace. In fact, neither of the closure axioms hold for this subset.

(b) The nonnegative orthant $\mathcal{O}^+ = \{x \geq 0, y \geq 0, z \geq 0\}$. Although $\mathbf{0} \in \mathcal{O}^+$, and the sum of two vectors in \mathcal{O}^+ also belongs to \mathcal{O}^+ , multiplying by negative scalars takes us outside the orthant, violating closure under scalar multiplication.

(c) The unit sphere $S_1 = \{x^2 + y^2 + z^2 = 1\}$. Again, $\mathbf{0} \notin S_1$. More generally, curved surfaces, such as the paraboloid $P = \{z = x^2 + y^2\}$, are not subspaces. Although $\mathbf{0} \in P$, most scalar multiples of elements of P do not belong to P . For example, $(1, 1, 2)^T \in P$, but $2(1, 1, 2)^T = (2, 2, 4)^T \notin P$.

In fact, there are only four fundamentally different types of subspaces $W \subset \mathbb{R}^3$ of three-dimensional Euclidean space:

- (i) the entire three-dimensional space $W = \mathbb{R}^3$,
- (ii) a plane passing through the origin,
- (iii) a line passing through the origin,
- (iv) a point — the trivial subspace $W = \{\mathbf{0}\}$.

We can establish this observation by the following argument. If $W = \{\mathbf{0}\}$ contains only the zero vector, then we are in case (iv). Otherwise, $W \subset \mathbb{R}^3$ contains a nonzero vector $\mathbf{0} \neq \mathbf{v}_1 \in W$. But since W must contain all scalar multiples $c\mathbf{v}_1$ of this element, it includes the entire line in the direction of \mathbf{v}_1 . If W contains another vector \mathbf{v}_2 that does not lie in the line through \mathbf{v}_1 , then it must contain the entire plane $\{c\mathbf{v}_1 + d\mathbf{v}_2\}$ spanned by $\mathbf{v}_1, \mathbf{v}_2$. Finally, if there is a third vector \mathbf{v}_3 not contained in this plane, then we claim that $W = \mathbb{R}^3$. This final fact will be an immediate consequence of general results in this chapter, although the interested reader might try to prove it directly before proceeding.

Example 2.12. Let $I \subset \mathbb{R}$ be an interval, and let $\mathcal{F}(I)$ be the space of real-valued functions $f: I \rightarrow \mathbb{R}$. Let us look at some of the most important examples of subspaces of $\mathcal{F}(I)$. In each case, we need only verify the closure conditions to verify that the given subset is indeed a subspace. In particular, the zero function belongs to each of the subspaces.

(a) The space $\mathcal{P}^{(n)}$ of polynomials of degree $\leq n$, which we already encountered.

(b) The space $\mathcal{P}^{(\infty)} = \bigcup_{n \geq 0} \mathcal{P}^{(n)}$ consisting of all polynomials. Closure means that the sum of any two polynomials is a polynomial, as is any scalar (constant) multiple of a polynomial.

(c) The space $\mathcal{C}^0(I)$ of all continuous functions. Closure of this subspace relies on knowing that if $f(x)$ and $g(x)$ are continuous, then both $f(x) + g(x)$ and $cf(x)$, for any $c \in \mathbb{R}$, are also continuous — two basic results from calculus, [2, 78].

(d) More restrictively, we can consider the subspace $C^n(I)$ consisting of all functions $f(x)$ that have n continuous derivatives $f'(x), f''(x), \dots, f^{(n)}(x)$ on[†] I . Again, we need to know that if $f(x)$ and $g(x)$ have n continuous derivatives, then so does $cf(x) + dg(x)$ for all $c, d \in \mathbb{R}$.

(e) The space $C^\infty(I) = \bigcap_{n \geq 0} C^n(I)$ of infinitely differentiable or *smooth* functions is also a subspace. This can be proved directly, or it follows from the general fact that the intersection of subspaces is a subspace, cf. Exercise 2.2.23.

(f) The space $\mathcal{A}(I)$ of analytic functions on the interval I . Recall that a function $f(x)$ is called *analytic* at a point a if it is smooth, and, moreover, its Taylor series

$$f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad (2.3)$$

converges to $f(x)$ for all x sufficiently close to a . (The series is not required to converge on the entire interval I .) Not every smooth function is analytic, and so $\mathcal{A}(I) \subsetneq C^\infty(I)$. An explicit example of a smooth but non-analytic function can be found in Exercise 2.2.30.

(g) The set of all mean zero functions. The *mean* or *average* of an integrable function defined on a closed interval $I = [a, b]$ is the real number

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx. \quad (2.4)$$

In particular, f has *mean zero* if and only if $\int_a^b f(x) dx = 0$. Since $\overline{f+g} = \bar{f} + \bar{g}$, and $\overline{cf} = c\bar{f}$, sums and scalar multiples of mean zero functions also have mean zero, proving closure.

(h) Fix $x_0 \in I$. The set of all functions $f(x)$ that vanish at the point, $f(x_0) = 0$, is a subspace. Indeed, if $f(x_0) = 0$ and $g(x_0) = 0$, then, clearly $(cf + dg)(x_0) = cf(x_0) + dg(x_0) = 0$ for all $c, d \in \mathbb{R}$, proving closure. This example can evidently be generalized to functions that vanish at several points, or even on an entire subset $S \subset I$.

(i) The set of all solutions $u = f(x)$ to the homogeneous linear differential equation

$$u'' + 2u' - 3u = 0.$$

Indeed, if $f(x)$ and $g(x)$ are solutions, then so is $f(x) + g(x)$ and $cf(x)$ for all $c \in \mathbb{R}$. Note that we do *not* need to actually solve the equation to verify these claims! They follow directly from linearity:

$$\begin{aligned} (f+g)'' + 2(f+g)' - 3(f+g) &= (f'' + 2f' - 3f) + (g'' + 2g' - 3g) = 0, \\ (cf)'' + 2(cf)' - 3(cf) &= c(f'' + 2f' - 3f) = 0. \end{aligned}$$

Remark. In the last three examples, 0 is essential for the indicated set of functions to be a subspace. The set of functions such that $f(x_0) = 1$, say, is not a subspace. The set of functions with a given nonzero mean, say $\bar{f} = 3$, is also not a subspace. Nor is the set of solutions to an inhomogeneous ordinary differential equation, say $u'' + 2u' - 3u = x - 3$. None of these subsets contain the zero function, nor do they satisfy the closure conditions.

[†] We use one-sided derivatives at any endpoint belonging to the interval.

Exercises

- 2.2.1. (a) Prove that the set of all vectors $(x, y, z)^T$ such that $x - y + 4z = 0$ forms a subspace of \mathbb{R}^3 . (b) Explain why the set of all vectors that satisfy $x - y + 4z = 1$ does not form a subspace.
- 2.2.2. Which of the following are subspaces of \mathbb{R}^3 ? Justify your answers! (a) The set of all vectors $(x, y, z)^T$ satisfying $x + y + z + 1 = 0$. (b) The set of vectors of the form $(t, -t, 0)^T$ for $t \in \mathbb{R}$. (c) The set of vectors of the form $(r - s, r + 2s, -s)^T$ for $r, s \in \mathbb{R}$. (d) The set of vectors whose first component equals 0. (e) The set of vectors whose last component equals 1. (f) The set of all vectors $(x, y, z)^T$ with $x \geq y \geq z$. (g) The set of all solutions to the equation $z = x - y$. (h) The set of all solutions to $z = xy$. (i) The set of all solutions to $x^2 + y^2 + z^2 = 0$. (j) The set of all solutions to the system $xy = yz = xz$.
- 2.2.3. Graph the following subsets of \mathbb{R}^3 and use this to explain which are subspaces:
 (a) The line $(t, -t, 3t)^T$ for $t \in \mathbb{R}$. (b) The helix $(\cos t, \sin t, t)^T$. (c) The surface $x - 2y + 3z = 0$. (d) The unit ball $x^2 + y^2 + z^2 < 1$. (e) The cylinder $(y + 2)^2 + (z - 1)^2 = 5$. (f) The intersection of the cylinders $(x - 1)^2 + y^2 = 1$ and $(x + 1)^2 + y^2 = 1$.
- 2.2.4. Show that if $W \subset \mathbb{R}^3$ is a subspace containing the vectors $(1, 2, -1)^T$, $(2, 0, 1)^T$, $(0, -1, 3)^T$, then $W = \mathbb{R}^3$.
- 2.2.5. *True or false:* An interval is a vector space.
- 2.2.6. (a) Can you construct an example of a subset $S \subset \mathbb{R}^2$ with the property that $c\mathbf{v} \in S$ for all $c \in \mathbb{R}$, $\mathbf{v} \in S$, and yet S is not a subspace? (b) What about an example in which $\mathbf{v} + \mathbf{w} \in S$ for every $\mathbf{v}, \mathbf{w} \in S$, and yet S is not a subspace?
- 2.2.7. Determine which of the following sets of vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ are subspaces of \mathbb{R}^n : (a) all equal entries $x_1 = \dots = x_n$; (b) all positive entries: $x_i \geq 0$; (c) first and last entries equal to zero: $x_1 = x_n = 0$; (d) entries add up to zero: $x_1 + \dots + x_n = 0$; (e) first and last entries differ by one: $x_1 - x_n = 1$.
- 2.2.8. Prove that the set of all solutions \mathbf{x} of the linear system $A\mathbf{x} = \mathbf{b}$ forms a subspace if and only if the system is homogeneous.
- 2.2.9. A square matrix is called *strictly lower triangular* if all entries on or above the main diagonal are 0. Prove that the space of strictly lower triangular matrices is a subspace of the vector space of all $n \times n$ matrices.
- 2.2.10. Which of the following are subspaces of the vector space of $n \times n$ matrices $\mathcal{M}_{n \times n}$?
 The set of all (a) regular matrices; (b) nonsingular matrices; (c) singular matrices; (d) lower triangular matrices; (e) lower unitriangular matrices; (f) diagonal matrices; (g) symmetric matrices; (h) skew-symmetric matrices.
- ◇ 2.2.11. The *trace* of an $n \times n$ matrix $A \in \mathcal{M}_{n \times n}$ is defined to be the sum of its diagonal entries: $\text{tr } A = a_{11} + a_{22} + \dots + a_{nn}$. Prove that the set of trace zero matrices, $\text{tr } A = 0$, is a subspace of $\mathcal{M}_{n \times n}$.
- 2.2.12. (a) Is the set of $n \times n$ matrices with $\det A = 1$ a subspace of $\mathcal{M}_{n \times n}$?
 (b) What about the matrices with $\det A = 0$?
- 2.2.13. Let $V = C^0(\mathbb{R})$ be the vector space consisting of all continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$. Explain why the set of all functions such that $f(1) = 0$ is a subspace, but the set of functions such that $f(0) = 1$ is not. For which values of a, b does the set of functions such that $f(a) = b$ form a subspace?

- 2.2.14. Which of the following are vector spaces? Justify your answer! (a) The set of all row vectors of the form $(a, 3a)$. (b) The set of all vectors of the form $(a, a + 1)$. (c) The set of all continuous functions for which $f(-1) = 0$. (d) The set of all periodic functions of period 1, i.e., $f(x + 1) = f(x)$. (e) The set of all non-negative functions: $f(x) \geq 0$. (f) The set of all even polynomials: $p(x) = p(-x)$. (g) The set of all polynomials $p(x)$ that have $x - 1$ as a factor. (h) The set of all quadratic forms $q(x, y) = ax^2 + bxy + cy^2$.
- 2.2.15. Determine which of the following conditions describe subspaces of the vector space C^1 consisting of all continuously differentiable scalar functions $f(x)$.
 (a) $f(2) = f(3)$, (b) $f'(2) = f(3)$, (c) $f'(x) + f(x) = 0$, (d) $f(2 - x) = f(x)$,
 (e) $f(x + 2) = f(x) + 2$, (f) $f(-x) = e^x f(x)$. (g) $f(x) = a + b|x|$ for some $a, b \in \mathbb{R}$,
- 2.2.16. Let $V = C^0[a, b]$ be the vector space consisting of all functions $f(t)$ that are defined and continuous on the interval $0 \leq t \leq 1$. Which of the following conditions define subspaces of V ? Explain your answer. (a) $f(0) = 0$, (b) $f(0) = 2f(1)$, (c) $f(0)f(1) = 1$,
 (d) $f(0) = 0$ or $f(1) = 0$, (e) $f(1 - t) = -tf(t)$, (f) $f(1 - t) = 1 - f(t)$,
 (g) $f\left(\frac{1}{2}\right) = \int_0^1 f(t) dt$, (h) $\int_0^1 (t - 1)f(t) dt = 0$, (i) $\int_0^t f(s) \sin s ds = \sin t$.
- 2.2.17. Prove that the set of solutions to the second order ordinary differential equation $u'' = xu$ is a vector space.
- 2.2.18. Show that the set of solutions to $u'' = x + u$ does not form a vector space.
- 2.2.19. (a) Prove that $C^1([a, b], \mathbb{R}^2)$, which is the space of continuously differentiable parameterized plane curves $\mathbf{f}: [a, b] \rightarrow \mathbb{R}^2$, is a vector space.
 (b) Is the subset consisting of all curves that go through the origin a subspace?
- 2.2.20. A planar vector field $\mathbf{v}(x, y) = (u(x, y), v(x, y))^T$ is called *irrotational* if it has zero divergence: $\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \equiv 0$. Prove that the set of all irrotational vector fields is a subspace of the space of all planar vector fields.
- 2.2.21. Let $C \subset \mathbb{R}^\infty$ denote the set of all convergent sequences of real numbers, where \mathbb{R}^∞ was defined in Exercise 2.2.21. Is C a subspace?
- ◇ 2.2.22. Show that if W and Z are subspaces of V , then (a) their intersection $W \cap Z$ is a subspace of V , (b) their sum $W + Z = \{\mathbf{w} + \mathbf{z} \mid \mathbf{w} \in W, \mathbf{z} \in Z\}$ is also a subspace, but (c) their union $W \cup Z$ is not a subspace of V , unless $W \subset Z$ or $Z \subset W$.
- ◇ 2.2.23. Let V be a vector space. Prove that the intersection $\bigcap W_i$ of any collection (finite or infinite) of subspaces $W_i \subset V$ is a subspace.
- ♡ 2.2.24. Let $W \subset V$ be a subspace. A subspace $Z \subset V$ is called a *complementary subspace* to W if (i) $W \cap Z = \{0\}$, and (ii) $W + Z = V$, i.e., every $\mathbf{v} \in V$ can be written as $\mathbf{v} = \mathbf{w} + \mathbf{z}$ for $\mathbf{w} \in W$ and $\mathbf{z} \in Z$. (a) Show that the x - and y -axes are complementary subspaces of \mathbb{R}^2 . (b) Show that the lines $x = y$ and $x = 3y$ are complementary subspaces of \mathbb{R}^2 . (c) Show that the line $(a, 2a, 3a)^T$ and the plane $x + 2y + 3z = 0$ are complementary subspaces of \mathbb{R}^3 . (d) Prove that if $\mathbf{v} = \mathbf{w} + \mathbf{z}$, then $\mathbf{w} \in W$ and $\mathbf{z} \in Z$ are uniquely determined.
- 2.2.25. (a) Show that $V_0 = \{(\mathbf{v}, \mathbf{0}) \mid \mathbf{v} \in V\}$ and $W_0 = \{(\mathbf{0}, \mathbf{w}) \mid \mathbf{w} \in W\}$ are complementary subspaces, as in Exercise 2.2.24, of the Cartesian product space $V \times W$, as defined in Exercise 2.1.13. (b) Prove that the *diagonal* $D = \{(\mathbf{v}, \mathbf{v})\}$ and the *anti-diagonal* $A = \{(\mathbf{v}, -\mathbf{v})\}$ are complementary subspaces of $V \times V$.
- 2.2.26. Show that the set of skew-symmetric $n \times n$ matrices forms a complementary subspace to the set of symmetric $n \times n$ matrices. Explain why this implies that every square matrix can be uniquely written as the sum of a symmetric and a skew-symmetric matrix.

2.2.27. (a) Show that the set of even functions, $f(-x) = f(x)$, is a subspace of the vector space of all functions $\mathcal{F}(\mathbb{R})$. (b) Show that the set of odd functions, $g(-x) = -g(x)$, forms a complementary subspace, as defined in Exercise 2.2.24. (c) Explain why every function can be uniquely written as the sum of an even function and an odd function.

♡ 2.2.28. Let V be a vector space. A subset of the form $A = \{\mathbf{w} + \mathbf{a} \mid \mathbf{w} \in W\}$, where $W \subset V$ is a subspace and $\mathbf{a} \in V$ is a fixed vector, is known as an *affine subspace* of V . (a) Show that an affine subspace $A \subset V$ is a genuine subspace if and only if $\mathbf{a} \in W$. (b) Draw the affine subspaces $A \subset \mathbb{R}^2$ when (i) W is the x -axis and $\mathbf{a} = (2, 1)^T$, (ii) W is the line $y = \frac{3}{2}x$ and $\mathbf{a} = (1, 1)^T$, (iii) W is the line $\{(t, -t)^T \mid t \in \mathbb{R}\}$, and $\mathbf{a} = (2, -2)^T$. (c) Prove that every affine subspace $A \subset \mathbb{R}^2$ is either a point, a line, or all of \mathbb{R}^2 . (d) Show that the plane $x - 2y + 3z = 1$ is an affine subspace of \mathbb{R}^3 . (e) Show that the set of all polynomials such that $p(0) = 1$ is an affine subspace of $\mathcal{P}^{(n)}$.

♡ 2.2.29. *Quotient spaces:* Let V be a vector space and $W \subset V$ a subspace. We say that two vectors $\mathbf{u}, \mathbf{v} \in V$ are *equivalent modulo W* if $\mathbf{u} - \mathbf{v} \in W$. (a) Show that this defines an *equivalence relation*, written $\mathbf{u} \sim_W \mathbf{v}$ on V , i.e., (i) $\mathbf{v} \sim_W \mathbf{v}$ for every \mathbf{v} ; (ii) if $\mathbf{u} \sim_W \mathbf{v}$, then $\mathbf{v} \sim_W \mathbf{u}$; and (iii) if, in addition, $\mathbf{v} \sim_W \mathbf{z}$, then $\mathbf{u} \sim_W \mathbf{z}$. (b) The *equivalence class* of a vector $\mathbf{u} \in V$ is defined as the set of all equivalent vectors, written $[\mathbf{u}]_W = \{\mathbf{v} \in V \mid \mathbf{v} \sim_W \mathbf{u}\}$. Show that $[\mathbf{0}]_W = W$. (c) Let $V = \mathbb{R}^2$ and $W = \{(x, y)^T \mid x = 2y\}$. Sketch a picture of several equivalence classes as subsets of \mathbb{R}^2 . (d) Show that each equivalence class $[\mathbf{u}]_W$ for $\mathbf{u} \in V$ is an affine subspace of V , as in Exercise 2.2.28. (e) Prove that the set of equivalence classes, called the *quotient space* and denoted by $V/W = \{[\mathbf{u}] \mid \mathbf{u} \in V\}$, forms a vector space under the operations of addition, $[\mathbf{u}]_W + [\mathbf{v}]_W = [\mathbf{u} + \mathbf{v}]_W$, and scalar multiplication, $c[\mathbf{u}]_W = [c\mathbf{u}]_W$. What is the zero element? Thus, you first need to prove that these operations are well defined, and then demonstrate the vector space axioms.

◇ 2.2.30. Define $f(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$

(a) Prove that all derivatives of f vanish at the origin: $f^{(n)}(0) = 0$ for $n = 0, 1, 2, \dots$.

(b) Prove that $f(x)$ is not analytic by showing that its Taylor series at $a = 0$ does not converge to $f(x)$ when $x > 0$.

2.2.31. Let $f(x) = \frac{1}{1+x^2}$. (a) Find the Taylor series of f at $a = 0$. (b) Prove that the Taylor series converges for $|x| < 1$, but diverges for $|x| \geq 1$. (c) Prove that $f(x)$ is analytic at $x = 0$.

2.3 Span and Linear Independence

The definition of the span of a collection of elements of a vector space generalizes, in a natural fashion, the geometric notion of two vectors spanning a plane in \mathbb{R}^3 . As such, it describes the first of two universal methods for constructing subspaces of vector spaces.

Definition 2.13. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be elements of a vector space V . A sum of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \sum_{i=1}^k c_i\mathbf{v}_i, \quad (2.5)$$

where the coefficients c_1, c_2, \dots, c_k are any scalars, is known as a *linear combination* of the elements $\mathbf{v}_1, \dots, \mathbf{v}_k$. Their *span* is the subset $W = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subset V$ consisting of all possible linear combinations with scalars $c_1, \dots, c_k \in \mathbb{R}$.

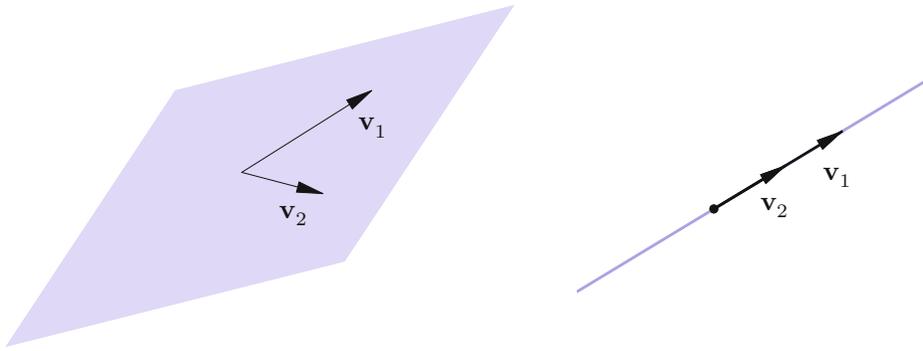


Figure 2.4. Plane and Line Spanned by Two Vectors.

For instance, $3\mathbf{v}_1 + \mathbf{v}_2 - 2\mathbf{v}_3$, $8\mathbf{v}_1 - \frac{1}{3}\mathbf{v}_3 = 8\mathbf{v}_1 + 0\mathbf{v}_2 - \frac{1}{3}\mathbf{v}_3$, $\mathbf{v}_2 = 0\mathbf{v}_1 + 1\mathbf{v}_2 + 0\mathbf{v}_3$, and $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3$ are four different linear combinations of the three vector space elements $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$.

The key observation is that the span always forms a subspace.

Proposition 2.14. The span $W = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of any finite collection of vector space elements $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ is a subspace of the underlying vector space V .

Proof: We need to show that if

$$\mathbf{v} = c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k \quad \text{and} \quad \widehat{\mathbf{v}} = \widehat{c}_1\mathbf{v}_1 + \dots + \widehat{c}_k\mathbf{v}_k$$

are any two linear combinations, then their sum is also a linear combination, since

$$\mathbf{v} + \widehat{\mathbf{v}} = (c_1 + \widehat{c}_1)\mathbf{v}_1 + \dots + (c_k + \widehat{c}_k)\mathbf{v}_k = \widetilde{c}_1\mathbf{v}_1 + \dots + \widetilde{c}_k\mathbf{v}_k,$$

where $\widetilde{c}_i = c_i + \widehat{c}_i$. Similarly, for any scalar multiple,

$$a\mathbf{v} = (ac_1)\mathbf{v}_1 + \dots + (ac_k)\mathbf{v}_k = c_1^*\mathbf{v}_1 + \dots + c_k^*\mathbf{v}_k,$$

where $c_i^* = ac_i$, which completes the proof.

Q.E.D.

Example 2.15. *Examples of subspaces spanned by vectors in \mathbb{R}^3 :*

- (i) If $\mathbf{v}_1 \neq \mathbf{0}$ is any non-zero vector in \mathbb{R}^3 , then its span is the line $\{c\mathbf{v}_1 \mid c \in \mathbb{R}\}$ consisting of all vectors *parallel* to \mathbf{v}_1 . If $\mathbf{v}_1 = \mathbf{0}$, then its span just contains the origin.
- (ii) If \mathbf{v}_1 and \mathbf{v}_2 are any two vectors in \mathbb{R}^3 , then their span is the set of all vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Typically, such a span prescribes a plane passing through the origin. However, if \mathbf{v}_1 and \mathbf{v}_2 are parallel, then their span is just a line. The most degenerate case occurs when $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{0}$, where the span is just a point — the origin.
- (iii) If we are given three non-coplanar vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, then their span is all of \mathbb{R}^3 , as we shall prove below. However, if they all lie in a plane, then their span is the plane — unless they are all parallel, in which case their span is a line — or, in the completely degenerate situation $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3 = \mathbf{0}$, a single point.

Thus, every subspace of \mathbb{R}^3 can be realized as the span of some set of vectors. One can consider subspaces spanned by four or more vectors in \mathbb{R}^3 , but these continue to be limited to being either a point (the origin), a line, a plane, or the entire three-dimensional space.

A crucial question is to determine when a given vector belongs to the span of a prescribed collection.

Example 2.16. Let $W \subset \mathbb{R}^3$ be the plane spanned by the vectors $\mathbf{v}_1 = (1, -2, 1)^T$ and $\mathbf{v}_2 = (2, -3, 1)^T$. Question: Is the vector $\mathbf{v} = (0, 1, -1)^T$ an element of W ? To answer, we need to see whether we can find scalars c_1, c_2 such that

$$\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2; \quad \text{that is,} \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2 \\ -2c_1 - 3c_2 \\ c_1 + c_2 \end{pmatrix}.$$

Thus, c_1, c_2 must satisfy the linear algebraic system

$$c_1 + 2c_2 = 0, \quad -2c_1 - 3c_2 = 1, \quad c_1 + c_2 = -1.$$

Applying Gaussian Elimination, we find the solution $c_1 = -2, c_2 = 1$, and so $\mathbf{v} = -2\mathbf{v}_1 + \mathbf{v}_2$ does belong to the span. On the other hand, $\tilde{\mathbf{v}} = (1, 0, 0)^T$ does not belong to W . Indeed, there are no scalars c_1, c_2 such that $\tilde{\mathbf{v}} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2$, because the corresponding linear system is incompatible.

Warning. It is entirely possible for different sets of vectors to span the *same* subspace. For instance, $\mathbf{e}_1 = (1, 0, 0)^T$ and $\mathbf{e}_2 = (0, 1, 0)^T$ span the xy -plane in \mathbb{R}^3 , as do the three coplanar vectors $\mathbf{v}_1 = (1, -1, 0)^T, \mathbf{v}_2 = (-1, 2, 0)^T, \mathbf{v}_3 = (2, 1, 0)^T$.

Example 2.17. Let $V = \mathcal{F}(\mathbb{R})$ denote the space of all scalar functions $f(x)$.

(a) The span of the three monomials $f_1(x) = 1, f_2(x) = x$, and $f_3(x) = x^2$ is the set of all functions of the form

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = c_1 + c_2 x + c_3 x^2,$$

where c_1, c_2, c_3 are arbitrary scalars (constants). In other words, $\text{span}\{1, x, x^2\} = \mathcal{P}^{(2)}$ is the subspace of all quadratic (degree ≤ 2) polynomials. In a similar fashion, the space $\mathcal{P}^{(n)}$ of polynomials of degree $\leq n$ is spanned by the monomials $1, x, x^2, \dots, x^n$.

(b) The next example plays a key role in many applications. Let $0 \neq \omega \in \mathbb{R}$. Consider the two basic trigonometric functions $f_1(x) = \cos \omega x, f_2(x) = \sin \omega x$ of frequency ω , and hence period $2\pi/\omega$. Their span consists of all functions of the form

$$f(x) = c_1 f_1(x) + c_2 f_2(x) = c_1 \cos \omega x + c_2 \sin \omega x. \quad (2.6)$$

For example, the function $\cos(\omega x + 2)$ lies in the span because, by the addition formula for the cosine,

$$\cos(\omega x + 2) = (\cos 2) \cos \omega x - (\sin 2) \sin \omega x$$

is a linear combination of $\cos \omega x$ and $\sin \omega x$, with respective coefficients $\cos 2, \sin 2$. Indeed, we can express a general function in the span in the alternative *phase-amplitude form*

$$f(x) = c_1 \cos \omega x + c_2 \sin \omega x = r \cos(\omega x - \delta), \quad (2.7)$$

in which $r \geq 0$ is known as the *amplitude* and $0 \leq \delta < 2\pi$ the *phase shift*. Indeed, expanding the right-hand side, we obtain

$$r \cos(\omega x - \delta) = (r \cos \delta) \cos \omega x + (r \sin \delta) \sin \omega x, \quad \text{and hence} \quad c_1 = r \cos \delta, \quad c_2 = r \sin \delta.$$

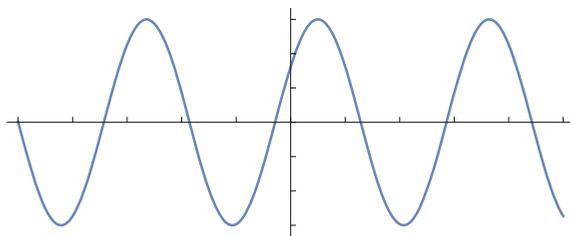


Figure 2.5. Graph of $3 \cos(2x - 1)$.

Thus, (r, δ) are the polar coordinates of the point $\mathbf{c} = (c_1, c_2) \in \mathbb{R}^2$ prescribed by the coefficients. We conclude that every linear combination of $\sin \omega x$ and $\cos \omega x$ can be rewritten as a single cosine containing an extra phase shift. [Figure 2.5](#) shows the particular function $3 \cos(2x - 1)$, which has amplitude $r = 3$, frequency $\omega = 2$, and phase shift $\delta = 1$. The first peak appears at $x = \delta/\omega = \frac{1}{2}$.

(c) The space $\mathcal{T}^{(2)}$ of *quadratic trigonometric polynomials* is spanned by the functions

$$1, \quad \cos x, \quad \sin x, \quad \cos^2 x, \quad \cos x \sin x, \quad \sin^2 x.$$

Its general element is a linear combination

$$q(x) = c_0 + c_1 \cos x + c_2 \sin x + c_3 \cos^2 x + c_4 \cos x \sin x + c_5 \sin^2 x, \quad (2.8)$$

where c_0, \dots, c_5 are arbitrary constants. A more useful spanning set for the same subspace consists of the trigonometric functions

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x. \quad (2.9)$$

Indeed, by the double-angle formulas, both

$$\cos 2x = \cos^2 x - \sin^2 x, \quad \sin 2x = 2 \sin x \cos x,$$

have the form of a quadratic trigonometric polynomial (2.8), and hence both belong to $\mathcal{T}^{(2)}$. On the other hand, we can write

$$\cos^2 x = \frac{1}{2} \cos 2x + \frac{1}{2}, \quad \cos x \sin x = \frac{1}{2} \sin 2x, \quad \sin^2 x = -\frac{1}{2} \cos 2x + \frac{1}{2},$$

in terms of the functions (2.9). Therefore, the original linear combination (2.8) can be written in the alternative form

$$\begin{aligned} q(x) &= \left(c_0 + \frac{1}{2}c_3 + \frac{1}{2}c_5\right) + c_1 \cos x + c_2 \sin x + \left(\frac{1}{2}c_3 - \frac{1}{2}c_5\right) \cos 2x + \frac{1}{2}c_4 \sin 2x \\ &= \widehat{c}_0 + \widehat{c}_1 \cos x + \widehat{c}_2 \sin x + \widehat{c}_3 \cos 2x + \widehat{c}_4 \sin 2x, \end{aligned} \quad (2.10)$$

and so the functions (2.9) do indeed span $\mathcal{T}^{(2)}$. It is worth noting that we first characterized $\mathcal{T}^{(2)}$ as the span of 6 functions, whereas the second characterization required only 5 functions. It turns out that 5 is the minimal number of functions needed to span $\mathcal{T}^{(2)}$, but the proof of this fact will be deferred until Chapter 4.

(d) The homogeneous linear ordinary differential equation

$$u'' + 2u' - 3u = 0 \quad (2.11)$$

considered in part (i) of Example 2.12 has two solutions: $f_1(x) = e^x$ and $f_2(x) = e^{-3x}$. (Now may be a good time for you to review the basic techniques for solving linear, constant

coefficient ordinary differential equations, cf. [7, 22]; see also Chapter 7.) Its general solution is, in fact, a linear combination

$$u = c_1 f_1(x) + c_2 f_2(x) = c_1 e^x + c_2 e^{-3x},$$

where c_1, c_2 are arbitrary scalars. Thus, the vector space of solutions to (2.11) is described as the span of these two basic solutions. The fact that there are no other solutions is not obvious, but relies on the basic uniqueness theorem for ordinary differential equations; further details can be found in Theorem 7.34.

Remark. One can also define the span of an infinite collection of elements of a vector space. To avoid convergence issues, one should consider only finite linear combinations (2.5). For example, the span of the monomials $1, x, x^2, x^3, \dots$ is the subspace $\mathcal{P}^{(\infty)}$ of all polynomials — *not* the space of analytic functions or convergent Taylor series. Similarly, the span of the functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x, \dots$ is the space $\mathcal{T}^{(\infty)}$ containing all *trigonometric polynomials*, of fundamental importance in the theory of Fourier series, [61].

Exercises

2.3.1. Show that $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ belongs to the subspace of \mathbb{R}^3 spanned by $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$ by writing it as a linear combination of the spanning vectors.

2.3.2. Show that $\begin{pmatrix} -3 \\ 7 \\ 6 \\ 1 \end{pmatrix}$ is in the subspace of \mathbb{R}^4 spanned by $\begin{pmatrix} 1 \\ -3 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 6 \\ 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 4 \\ 6 \\ -7 \end{pmatrix}$.

2.3.3. (a) Determine whether $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$ is in the span of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. (b) Is $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ in the span of $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$? (c) Is $\begin{pmatrix} 3 \\ 0 \\ -1 \\ -2 \end{pmatrix}$ in the span of $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix}$?

2.3.4. Which of the following sets of vectors span all of \mathbb{R}^2 ? (a) $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$; (b) $\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$; (c) $\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}$; (d) $\begin{pmatrix} 6 \\ -9 \end{pmatrix}, \begin{pmatrix} -4 \\ 6 \end{pmatrix}$; (e) $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix}$; (f) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix}$.

2.3.5. (a) Graph the subspace of \mathbb{R}^3 spanned by the vector $\mathbf{v}_1 = (3, 0, 1)^T$.
 (b) Graph the subspace spanned by the vectors $\mathbf{v}_1 = (3, -2, -1)^T, \mathbf{v}_2 = (-2, 0, -1)^T$.
 (c) Graph the span of $\mathbf{v}_1 = (1, 0, -1)^T, \mathbf{v}_2 = (0, -1, 1)^T, \mathbf{v}_3 = (1, -1, 0)^T$.

2.3.6. Let U be the subspace of \mathbb{R}^3 spanned by $\mathbf{u}_1 = (1, 2, 3)^T, \mathbf{u}_2 = (2, -1, 0)^T$. Let V be the subspace spanned by $\mathbf{v}_1 = (5, 0, 3)^T, \mathbf{v}_2 = (3, 1, 3)^T$. Is V a subspace of U ? Are U and V the same?

2.3.7. (a) Let S be the subspace of $\mathcal{M}_{2 \times 2}$ consisting of all symmetric 2×2 matrices. Show that S is spanned by the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. (b) Find a spanning set of the space of symmetric 3×3 matrices.

- 2.3.8. (a) Determine whether the polynomials $x^2 + 1, x^2 - 1, x^2 + x + 1$ span $\mathcal{P}^{(2)}$.
 (b) Do $x^3 - 1, x^2 + 1, x - 1, 1$ span $\mathcal{P}^{(3)}$? (c) What about $x^3, x^2 + 1, x^2 - x, x + 1$?
- 2.3.9. Determine whether any of the following functions lies in the subspace spanned by $1, x, \sin x, \sin^2 x$: (a) $3 - 5x$, (b) $x^2 + \sin^2 x$, (c) $\sin x - 2 \cos x$, (d) $\cos^2 x$, (e) $x \sin x$, (f) e^x .
- 2.3.10. Write the following trigonometric functions in phase–amplitude form:
 (a) $\sin 3x$, (b) $\cos x - \sin x$, (c) $3 \cos 2x + 4 \sin 2x$, (d) $\cos x \sin x$.
- 2.3.11. (a) Prove that the set of solutions to the homogeneous ordinary differential equation $u'' - 4u' + 3u = 0$ is a vector space. (b) Write the solution space as the span of a finite number of functions. (c) What is the minimal number of functions needed to span the solution space?
- 2.3.12. Explain why the functions $1, \cos x, \sin x$ span the solution space to the third order ordinary differential equation $u''' + u' = 0$.
- 2.3.13. Find a finite set of real functions that spans the solution space to the following homogeneous ordinary differential equations: (a) $u' - 2u = 0$, (b) $u'' + 4u = 0$, (c) $u'' - 3u' = 0$, (d) $u'' + u' + u = 0$, (e) $u''' - 5u'' = 0$, (f) $u^{(4)} + u = 0$.
- 2.3.14. Consider the boundary value problem $u'' + 4u = 0, 0 \leq x \leq \pi, u(0) = 0, u(\pi) = 0$.
 (a) Prove, without solving, that the set of solutions forms a vector space.
 (b) Write this space as the span of one or more functions. *Hint*: First solve the differential equation; then find out which solutions satisfy the boundary conditions.
- 2.3.15. Which of the following functions lie in the span of the vector-valued functions
 $\mathbf{f}_1(x) = \begin{pmatrix} 1 \\ x \end{pmatrix}, \quad \mathbf{f}_2(x) = \begin{pmatrix} x \\ 1 \end{pmatrix}, \quad \mathbf{f}_3(x) = \begin{pmatrix} x \\ 2x \end{pmatrix}?$
 (a) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, (b) $\begin{pmatrix} 1 - 2x \\ 1 - x \end{pmatrix}$, (c) $\begin{pmatrix} 1 - 2x \\ -1 - x \end{pmatrix}$, (d) $\begin{pmatrix} 1 + x^2 \\ 1 - x^2 \end{pmatrix}$, (e) $\begin{pmatrix} 2 - x \\ 0 \end{pmatrix}$.
- 2.3.16. *True or false*: The zero vector belongs to the span of any collection of vectors.
- 2.3.17. Prove or give a counter-example: if \mathbf{z} is a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$, then \mathbf{w} is a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{z}$.
- ◇ 2.3.18. Suppose $\mathbf{v}_1, \dots, \mathbf{v}_m$ span V . Let $\mathbf{v}_{m+1}, \dots, \mathbf{v}_n \in V$ be any other elements. Prove that the combined collection $\mathbf{v}_1, \dots, \mathbf{v}_n$ also spans V .
- ◇ 2.3.19. (a) Show that if \mathbf{v} is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_m$, and each \mathbf{v}_j is a linear combination of $\mathbf{w}_1, \dots, \mathbf{w}_n$, then \mathbf{v} is a linear combination of $\mathbf{w}_1, \dots, \mathbf{w}_n$.
 (b) Suppose $\mathbf{v}_1, \dots, \mathbf{v}_m$ span V . Let $\mathbf{w}_1, \dots, \mathbf{w}_m \in V$ be any other elements. Suppose that each \mathbf{v}_i can be written as a linear combination of $\mathbf{w}_1, \dots, \mathbf{w}_m$. Prove that $\mathbf{w}_1, \dots, \mathbf{w}_m$ also span V .
- ◇ 2.3.20. The span of an infinite collection $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots \in V$ of vector space elements is defined as the set of all *finite* linear combinations $\sum_{i=1}^n c_i \mathbf{v}_i$, where $n < \infty$ is finite but arbitrary.
 (a) Prove that the span defines a subspace of the vector space V .
 (b) What is the span of the monomials $1, x, x^2, x^3, \dots$?

Linear Independence and Dependence

Most of the time, all of the vectors used to form a span are essential. For example, we cannot use fewer than two vectors to span a plane in \mathbb{R}^3 , since the span of a single vector is at most a line. However, in degenerate situations, some of the spanning elements may be

redundant. For instance, if the two vectors are parallel, then their span is a line, but only one of the vectors is really needed to prescribe the line. Similarly, the subspace spanned by the polynomials $p_1(x) = x - 2$, $p_2(x) = 3x + 4$, $p_3(x) = -x + 1$, is the vector space $\mathcal{P}^{(1)}$ consisting of all linear polynomials. But only two of the polynomials are really required to span $\mathcal{P}^{(1)}$. (The reason will become clear soon, but you may wish to see whether you can demonstrate this on your own.) The elimination of such superfluous spanning elements is encapsulated in the following important definition.

Definition 2.18. The vector space elements $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ are called *linearly dependent* if there exist scalars c_1, \dots, c_k , *not all zero*, such that

$$c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k = \mathbf{0}. \quad (2.12)$$

Elements that are not linearly dependent are called *linearly independent*.

The restriction that not all the c_i 's are zero is essential: if $c_1 = \cdots = c_k = 0$, then the linear combination (2.12) is automatically zero. Thus, to check linear independence, one needs to show that the *only* linear combination that produces the zero vector (2.12) is this trivial one. In other words, $c_1 = \cdots = c_k = 0$ is the *one and only* solution to the vector equation (2.12).

Example 2.19. Some examples of linear independence and dependence:

(a) The vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix},$$

are linearly dependent, because

$$\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}.$$

On the other hand, the first two vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent. To see this, suppose that

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = \begin{pmatrix} c_1 \\ 2c_1 + 3c_2 \\ -c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For this to happen, c_1, c_2 must satisfy the homogeneous linear system

$$c_1 = 0, \quad 2c_1 + 3c_2 = 0, \quad -c_1 + c_2 = 0,$$

which, as you can check, has only the trivial solution $c_1 = c_2 = 0$.

(b) In general, any collection $\mathbf{v}_1, \dots, \mathbf{v}_k$ that includes the zero vector, say $\mathbf{v}_1 = \mathbf{0}$, is automatically linearly dependent, since $1\mathbf{0} + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_k = \mathbf{0}$ is a nontrivial linear combination that adds up to $\mathbf{0}$.

(c) Two vectors $\mathbf{v}, \mathbf{w} \in V$ are linearly dependent if and only if they are *parallel*, meaning that one is a scalar multiple of the other. Indeed, if $\mathbf{v} = a\mathbf{w}$, then $\mathbf{v} - a\mathbf{w} = \mathbf{0}$ is a nontrivial linear combination summing to zero. Conversely, if $c\mathbf{v} + d\mathbf{w} = \mathbf{0}$ and $c \neq 0$, then $\mathbf{v} = -(d/c)\mathbf{w}$, while if $c = 0$ but $d \neq 0$, then $\mathbf{w} = \mathbf{0}$.

(d) The polynomials

$$p_1(x) = x - 2, \quad p_2(x) = x^2 - 5x + 4, \quad p_3(x) = 3x^2 - 4x, \quad p_4(x) = x^2 - 1,$$

are linearly dependent, since

$$p_1(x) + p_2(x) - p_3(x) + 2p_4(x) \equiv 0$$

is a nontrivial linear combination that vanishes identically. On the other hand, the first three polynomials,

$$p_1(x) = x - 2, \quad p_2(x) = x^2 - 5x + 4, \quad p_3(x) = 3x^2 - 4x,$$

are linearly independent. Indeed, if the linear combination

$$c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) = (c_2 + 3c_3)x^2 + (c_1 - 5c_2 - 4c_3)x - 2c_1 + 4c_2 \equiv 0$$

is the zero polynomial, then its coefficients must vanish, and hence c_1, c_2, c_3 are required to solve the homogeneous linear system

$$c_2 + 3c_3 = 0, \quad c_1 - 5c_2 - 4c_3 = 0, \quad -2c_1 + 4c_2 = 0.$$

But this has only the trivial solution $c_1 = c_2 = c_3 = 0$, and so linear independence follows.

Remark. In the last example, we are using the basic fact that a polynomial is identically zero,

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \equiv 0 \quad \text{for all } x,$$

if and only if its coefficients all vanish: $a_0 = a_1 = \cdots = a_n = 0$. This is equivalent to the “obvious” fact that the basic monomial functions $1, x, x^2, \dots, x^n$ are linearly independent. Exercise 2.3.36 asks for a bona fide proof.

Example 2.20. The trigonometric functions

$$1, \quad \cos x, \quad \sin x, \quad \cos^2 x, \quad \cos x \sin x, \quad \sin^2 x,$$

which were used to define the vector space $\mathcal{T}^{(2)}$ of quadratic trigonometric polynomials, are, in fact, linearly dependent. This is a consequence of the basic trigonometric identity

$$\cos^2 x + \sin^2 x \equiv 1,$$

which can be rewritten as a nontrivial linear combination

$$1 + 0 \cos x + 0 \sin x + (-1) \cos^2 x + 0 \cos x \sin x + (-1) \sin^2 x \equiv 0$$

that equals the zero function. On the other hand, the alternative spanning set

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x$$

is linearly independent, since the only identically zero linear combination,

$$c_0 + c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x \equiv 0,$$

turns out to be the trivial one $c_0 = \cdots = c_4 = 0$. However, the latter fact is not as obvious, and requires a bit of work to prove directly; see Exercise 2.3.37. An easier proof, based on orthogonality, will appear in Chapter 4.

Let us now focus our attention on the linear independence or dependence of a set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ in Euclidean space. We begin by forming the $n \times k$ matrix $A = (\mathbf{v}_1 \ \dots \ \mathbf{v}_k)$ whose *columns* are the given vectors. (The fact that we use column vectors is essential here.) Our analysis is based on the very useful formula

$$A\mathbf{c} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k, \quad \text{where} \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}, \quad (2.13)$$

that expresses any linear combination in terms of matrix multiplication. For example,

$$\begin{pmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 4 & -1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 + 3c_2 \\ -c_1 + 2c_2 + c_3 \\ 4c_1 - c_2 - 2c_3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}.$$

Formula (2.13) follows directly from the rules of matrix multiplication; see also Exercise 1.2.34(c). It enables us to reformulate the notions of linear independence and span of vectors in \mathbb{R}^n in terms of linear algebraic systems. The key result is the following:

Theorem 2.21. Let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and let $A = (\mathbf{v}_1 \dots \mathbf{v}_k)$ be the corresponding $n \times k$ matrix whose columns are the given vectors.

- (a) The vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly dependent if and only if there is a non-zero solution $\mathbf{c} \neq \mathbf{0}$ to the homogeneous linear system $A\mathbf{c} = \mathbf{0}$.
- (b) The vectors are linearly independent if and only if the only solution to the homogeneous system $A\mathbf{c} = \mathbf{0}$ is the trivial one, $\mathbf{c} = \mathbf{0}$.
- (c) A vector \mathbf{b} lies in the span of $\mathbf{v}_1, \dots, \mathbf{v}_k$ if and only if the linear system $A\mathbf{c} = \mathbf{b}$ is compatible, i.e., has at least one solution.

Proof: We prove the first statement, leaving the other two as exercises for the reader. The condition that $\mathbf{v}_1, \dots, \mathbf{v}_k$ be linearly dependent is that there exists a nonzero vector

$$\mathbf{c} = (c_1, c_2, \dots, c_k)^T \neq \mathbf{0} \quad \text{such that} \quad A\mathbf{c} = c_1 \mathbf{v}_1 + \dots + c_k \mathbf{v}_k = \mathbf{0}.$$

Therefore, linear dependence requires the existence of a nontrivial solution to the homogeneous linear system $A\mathbf{c} = \mathbf{0}$. *Q.E.D.*

Example 2.22. Let us determine whether the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} \quad (2.14)$$

are linearly independent or linearly dependent. We combine them as column vectors into a single matrix

$$A = \begin{pmatrix} 1 & 3 & 1 & 4 \\ 2 & 0 & -4 & 2 \\ -1 & 4 & 6 & 3 \end{pmatrix}.$$

According to Theorem 2.21, we need to figure out whether there are any nontrivial solutions to the homogeneous equation $A\mathbf{c} = \mathbf{0}$; this can be done by reducing A to row echelon form

$$U = \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & -6 & -6 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (2.15)$$

The general solution to the homogeneous system $A\mathbf{c} = \mathbf{0}$ is $\mathbf{c} = (2c_3 - c_4, -c_3 - c_4, c_3, c_4)^T$, where c_3, c_4 — the free variables — are arbitrary. Any nonzero choice of c_3, c_4 will produce a nontrivial linear combination

$$(2c_3 - c_4)\mathbf{v}_1 + (-c_3 - c_4)\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$$

that adds up to the zero vector. We conclude that the vectors (2.14) are linearly dependent.

In fact, in this particular case, we didn't even need to complete the row reduction if we only need to check linear (in)dependence. According to Theorem 1.47, any coefficient matrix with more columns than rows automatically has a nontrivial solution to the associated homogeneous system. This implies the following result:

Lemma 2.23. Any collection of $k > n$ vectors in \mathbb{R}^n is linearly dependent.

Warning. The converse to this lemma is *not* true. For example, $\mathbf{v}_1 = (1, 2, 3)^T$ and $\mathbf{v}_2 = (-2, -4, -6)^T$ are two linearly dependent vectors in \mathbb{R}^3 , since $2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$. For a collection of n or fewer vectors in \mathbb{R}^n , one needs to analyze the homogeneous linear system.

Lemma 2.23 is a particular case of the following general characterization of linearly independent vectors.

Proposition 2.24. A set of k vectors in \mathbb{R}^n is linearly independent if and only if the corresponding $n \times k$ matrix A has rank k . In particular, this requires $k \leq n$.

Or, to state the result another way, the vectors are linearly independent if and only if the homogeneous linear system $A\mathbf{c} = \mathbf{0}$ has no free variables. Proposition 2.24 is an immediate corollary of Theorems 2.21 and 1.47.

Example 2.22 (continued). Let us now see which vectors $\mathbf{b} \in \mathbb{R}^3$ lie in the span of the vectors (2.14). According to Theorem 2.21, this will be the case if and only if the linear system $A\mathbf{c} = \mathbf{b}$ has a solution. Since the resulting row echelon form (2.15) has a row of all zeros, there will be a compatibility condition on the entries of \mathbf{b} , and hence not every vector lies in the span. To find the precise condition, we augment the coefficient matrix, and apply the same row operations, leading to the reduced augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 4 & b_1 \\ 0 & -6 & -6 & -6 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + \frac{7}{6}b_2 - \frac{4}{3}b_1 \end{array} \right).$$

Therefore, $\mathbf{b} = (b_1, b_2, b_3)^T$ lies in the span if and only if $-\frac{4}{3}b_1 + \frac{7}{6}b_2 + b_3 = 0$. Thus, these four vectors span only a plane in \mathbb{R}^3 .

The same method demonstrates that a collection of vectors will span all of \mathbb{R}^n if and only if the row echelon form of the associated matrix contains no all-zero rows, or, equivalently, the rank is equal to n , the number of rows in the matrix.

Proposition 2.25. A collection of k vectors spans \mathbb{R}^n if and only if their $n \times k$ matrix has rank n . In particular, this requires $k \geq n$.

Warning. Not every collection of n or more vectors in \mathbb{R}^n will span all of \mathbb{R}^n . A counterexample was already provided by the vectors (2.14).

Exercises

2.3.21. Determine whether the given vectors are linearly independent or linearly dependent:

(a) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix},$ (b) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -6 \end{pmatrix},$ (c) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix},$ (d) $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix},$

(e) $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix},$ (f) $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix},$

(g) $\begin{pmatrix} 4 \\ 2 \\ 0 \\ -6 \end{pmatrix}, \begin{pmatrix} -6 \\ -3 \\ 0 \\ 9 \end{pmatrix},$ (h) $\begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 2 \\ -3 \end{pmatrix},$ (i) $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix},$

2.3.22. (a) Show that the vectors $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \\ -1 \end{pmatrix}$ are linearly independent. (b) Which

of the following vectors are in their span? (i) $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix},$ (ii) $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$ (iii) $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$ (iv) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$

(c) Suppose $\mathbf{b} = (a, b, c, d)^T$ lies in their span. What conditions must a, b, c, d satisfy?

2.3.23. (a) Show that the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$ are linearly independent.

(b) Show that they also span \mathbb{R}^4 . (c) Write $(1, 0, 0, 1)^T$ as a linear combination of them.

2.3.24. Determine whether the given row vectors are linearly independent or linearly dependent:

(a) $(2, 1), (-1, 3), (5, 2),$ (b) $(1, 2, -1), (2, 4, -2),$ (c) $(1, 2, 3), (1, 4, 8), (1, 5, 7),$
 (d) $(1, 1, 0), (1, 0, 3), (2, 2, 1), (1, 3, 4),$ (e) $(1, 2, 0, 3), (-3, -1, 2, -2), (3, -4, -4, 5),$
 (f) $(2, 1, -1, 3), (-1, 3, 1, 0), (5, 1, 2, -3).$

2.3.25. *True or false:* The six 3×3 permutation matrices (1.30) are linearly independent.

2.3.26. *True or false:* A set of vectors is linearly dependent if the zero vector belongs to their span.

2.3.27. Does a single vector ever define a linearly dependent set?

2.3.28. Let \mathbf{x} and \mathbf{y} be linearly independent elements of a vector space V . Show that $\mathbf{u} = a\mathbf{x} + b\mathbf{y}$, and $\mathbf{v} = c\mathbf{x} + d\mathbf{y}$ are linearly independent if and only if $ad - bc \neq 0$. Is the entire collection $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}$ linearly independent?

2.3.29. Prove or give a counterexample to the following statement: If $\mathbf{v}_1, \dots, \mathbf{v}_k$ are elements of a vector space V that do not span V , then $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent.

◇ 2.3.30. Prove parts (b) and (c) of Theorem 2.21.

◇ 2.3.31. (a) Prove that if $\mathbf{v}_1, \dots, \mathbf{v}_m$ are linearly independent, then every subset, e.g., $\mathbf{v}_1, \dots, \mathbf{v}_k$ with $k < m$, is also linearly independent. (b) Does the same hold true for linearly dependent vectors? Prove or give a counterexample.

2.3.32. (a) Determine whether the polynomials $f_1(x) = x^2 - 3$, $f_2(x) = 2 - x$, $f_3(x) = (x - 1)^2$, are linearly independent or linearly dependent.

(b) Do they span the vector space of all quadratic polynomials?

2.3.33. Determine whether the given functions are linearly independent or linearly dependent:

- (a) $2 - x^2$, $3x$, $x^2 + x - 2$, (b) $3x - 1$, $x(2x + 1)$, $x(x - 1)$; (c) e^x , e^{x+1} ; (d) $\sin x$, $\sin(x + 1)$; (e) e^x , e^{x+1} , e^{x+2} ; (f) $\sin x$, $\sin(x + 1)$, $\sin(x + 2)$; (g) e^x , xe^x , x^2e^x ; (h) e^x , e^{2x} , e^{3x} ; (i) $x + y$, $x - y + 1$, $x + 3y + 2$ — these are functions of two variables.

2.3.34. Show that the functions $f(x) = x$ and $g(x) = |x|$ are linearly independent when considered as functions on all of \mathbb{R} , but are linearly dependent when considered as functions defined only on $\mathbb{R}^+ = \{x > 0\}$.

♡ 2.3.35. (a) Prove that the polynomials $p_i(x) = \sum_{j=0}^n a_{ij}x^j$ for $i = 1, \dots, k$ are linearly

independent if and only if the $k \times (n + 1)$ matrix A whose entries are their coefficients a_{ij} , $1 \leq i \leq k$, $0 \leq j \leq n$, has rank k . (b) Formulate a similar matrix condition for testing whether another polynomial $q(x)$ lies in their span. (c) Use (a) to determine whether $p_1(x) = x^3 - 1$, $p_2(x) = x^3 - 2x + 4$, $p_3(x) = x^4 - 4x$, $p_4(x) = x^2 + 1$, $p_5(x) = -x^4 + 4x^3 + 2x + 1$ are linearly independent or linearly dependent. (d) Does the polynomial $q(x) = x^3$ lie in their span? If so find a linear combination that adds up to $q(x)$.

◇ 2.3.36. The Fundamental Theorem of Algebra, [26], states that a non-zero polynomial of degree n has at most n distinct real roots, that is, real numbers x such that $p(x) = 0$. Use this fact to prove linear independence of the monomial functions $1, x, x^2, \dots, x^n$.

Remark. An elementary proof of the latter fact can be found in Exercise 5.5.38.

♡ 2.3.37. (a) Let x_1, x_2, \dots, x_n be a set of distinct sample points. Prove that the functions $f_1(x), \dots, f_k(x)$ are linearly independent if their sample vectors $\mathbf{f}_1, \dots, \mathbf{f}_k$ are linearly independent vectors in \mathbb{R}^n . (b) Give an example of linearly independent functions that have linearly dependent sample vectors. (c) Use this method to prove that the functions $1, \cos x, \sin x, \cos 2x, \sin 2x$, are linearly independent. *Hint:* You need at least 5 sample points.

2.3.38. Suppose $\mathbf{f}_1(t), \dots, \mathbf{f}_k(t)$ are vector-valued functions from \mathbb{R} to \mathbb{R}^n . (a) Prove that if $\mathbf{f}_1(t_0), \dots, \mathbf{f}_k(t_0)$ are linearly independent vectors in \mathbb{R}^n at one point t_0 , then $\mathbf{f}_1(t), \dots, \mathbf{f}_k(t)$ are linearly independent functions. (b) Show that $\mathbf{f}_1(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$ and $\mathbf{f}_2(t) = \begin{pmatrix} 2t - 1 \\ 2t^2 - t \end{pmatrix}$ are linearly independent functions, even though at each t_0 , the vectors $\mathbf{f}_1(t_0), \mathbf{f}_2(t_0)$ are linearly dependent. Therefore, the converse to the result in part (a) is not valid.

♡ 2.3.39. The *Wronskian* of a pair of differentiable functions $f(x), g(x)$ is the scalar function

$$W[f(x), g(x)] = \det \begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} = f(x)g'(x) - f'(x)g(x). \quad (2.16)$$

(a) Prove that if f, g are linearly dependent, then $W[f(x), g(x)] \equiv 0$. Hence, if $W[f(x), g(x)] \not\equiv 0$, then f, g are linearly independent. (b) Let $f(x) = x^3$, $g(x) = |x|^3$. Prove that $f, g \in C^2$ are twice continuously differentiable and linearly independent, but $W[f(x), g(x)] \equiv 0$. Thus, the Wronskian is *not* a fool-proof test for linear independence.

Remark. It can be proved, [7], that if f, g both satisfy a second order linear ordinary differential equation, then f, g are linearly dependent if and only if $W[f(x), g(x)] \equiv 0$.

2.4 Basis and Dimension

In order to span a vector space or subspace, we must employ a sufficient number of distinct elements. On the other hand, including too many elements in the spanning set will violate linear independence, and cause redundancies. The optimal spanning sets are those that are

also linearly independent. By combining the properties of span and linear independence, we arrive at the all-important concept of a “basis”.

Definition 2.26. A *basis* of a vector space V is a finite collection of elements $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$ that (a) spans V , and (b) is linearly independent.

Bases are absolutely fundamental in all areas of linear algebra and linear analysis, including matrix algebra, Euclidean geometry, statistical analysis, solutions to linear differential equations — both ordinary and partial — linear boundary value problems, Fourier analysis, signal and image processing, data compression, control systems, and many others.

Example 2.27. The *standard basis* of \mathbb{R}^n consists of the n vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \dots \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad (2.17)$$

so that \mathbf{e}_i is the vector with 1 in the i^{th} slot and 0's elsewhere. We already encountered these vectors — they are the columns of the $n \times n$ identity matrix. They clearly span \mathbb{R}^n , since we can write any vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \dots + x_n \mathbf{e}_n \quad (2.18)$$

as a linear combination, whose coefficients are its entries. Moreover, the only linear combination that yields the zero vector $\mathbf{x} = \mathbf{0}$ is the trivial one $x_1 = \dots = x_n = 0$, which shows that $\mathbf{e}_1, \dots, \mathbf{e}_n$ are linearly independent.

In the three-dimensional case \mathbb{R}^3 , a common physical notation for the standard basis is

$$\mathbf{i} = \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j} = \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{k} = \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (2.19)$$

This is but one of many possible bases for \mathbb{R}^3 . Indeed, any three non-coplanar vectors can be used to form a basis. This is a consequence of the following general characterization of bases in Euclidean space as the columns of a nonsingular matrix.

Theorem 2.28. Every basis of \mathbb{R}^n consists of exactly n vectors. Furthermore, a set of n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$ is a basis if and only if the $n \times n$ matrix $A = (\mathbf{v}_1 \ \dots \ \mathbf{v}_n)$ is nonsingular: $\text{rank } A = n$.

Proof: This is a direct consequence of Theorem 2.21. Linear independence requires that the only solution to the homogeneous system $A\mathbf{c} = \mathbf{0}$ be the trivial one $\mathbf{c} = \mathbf{0}$. On the other hand, a vector $\mathbf{b} \in \mathbb{R}^n$ will lie in the span of $\mathbf{v}_1, \dots, \mathbf{v}_n$ if and only if the linear system $A\mathbf{c} = \mathbf{b}$ has a solution. For $\mathbf{v}_1, \dots, \mathbf{v}_n$ to span all of \mathbb{R}^n , this must hold for all possible right-hand sides \mathbf{b} . Theorem 1.7 tells us that both results require that A be nonsingular, i.e., have maximal rank n . Q.E.D.

Thus, every basis of n -dimensional Euclidean space \mathbb{R}^n contains the same number of vectors, namely n . This is a general fact, that motivates a linear algebraic characterization of dimension.

Theorem 2.29. Suppose the vector space V has a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ for some $n \in \mathbb{N}$. Then every other basis of V has the same number, n , of elements in it. This number is called the *dimension* of V , and written $\dim V = n$.

The proof of Theorem 2.29 rests on the following lemma.

Lemma 2.30. Suppose $\mathbf{v}_1, \dots, \mathbf{v}_n$ span a vector space V . Then every set of $k > n$ elements $\mathbf{w}_1, \dots, \mathbf{w}_k \in V$ is linearly dependent.

Proof: Let us write each element

$$\mathbf{w}_j = \sum_{i=1}^n a_{ij} \mathbf{v}_i, \quad j = 1, \dots, k,$$

as a linear combination of the spanning set. Then

$$c_1 \mathbf{w}_1 + \dots + c_k \mathbf{w}_k = \sum_{i=1}^n \sum_{j=1}^k a_{ij} c_j \mathbf{v}_i.$$

This linear combination will be zero whenever $\mathbf{c} = (c_1, c_2, \dots, c_k)^T$ solves the homogeneous linear system

$$\sum_{j=1}^k a_{ij} c_j = 0, \quad i = 1, \dots, n,$$

consisting of n equations in $k > n$ unknowns. Theorem 1.47 guarantees that every homogeneous system with more unknowns than equations always has a non-trivial solution $\mathbf{c} \neq \mathbf{0}$, and this immediately implies that $\mathbf{w}_1, \dots, \mathbf{w}_k$ are linearly dependent. *Q.E.D.*

Proof of Theorem 2.29: Suppose we have two bases containing a different number of elements. By definition, the smaller basis spans the vector space. But then Lemma 2.30 tell us that the elements in the larger purported basis must be linearly dependent, which contradicts our initial assumption that the latter is a basis. *Q.E.D.*

As a direct consequence, we can now give a precise meaning to the optimality of bases.

Theorem 2.31. Suppose V is an n -dimensional vector space. Then

- Every set of more than n elements of V is linearly dependent.
- No set of fewer than n elements spans V .
- A set of n elements forms a basis if and only if it spans V .
- A set of n elements forms a basis if and only if it is linearly independent.

In other words, once we know the dimension of a vector space, to check that a collection having the correct number of elements forms a basis, we only need establish one of the two defining properties: span or linear independence. Thus, n elements that span an n -dimensional vector space are automatically linearly independent and hence form a basis; conversely, n linearly independent elements of an n -dimensional vector space automatically span the space and so form a basis.

Example 2.32. The standard basis of the space $\mathcal{P}^{(n)}$ of polynomials of degree $\leq n$ is given by the $n + 1$ monomials $1, x, x^2, \dots, x^n$. We conclude that the vector space $\mathcal{P}^{(n)}$

has dimension $n + 1$. Any other basis of $\mathcal{P}^{(n)}$ must contain precisely $n + 1$ polynomials. But, not every collection of $n + 1$ polynomials in $\mathcal{P}^{(n)}$ is a basis — they must be linearly independent. We conclude that no set of n or fewer polynomials can span $\mathcal{P}^{(n)}$, while any collection of $n + 2$ or more polynomials of degree $\leq n$ is automatically linearly dependent.

By definition, every vector space of dimension $1 \leq n < \infty$ has a basis. If a vector space V has no basis, it is either the trivial vector space $V = \{\mathbf{0}\}$, which by convention has dimension 0, or its dimension is infinite. An infinite-dimensional vector space contains an infinite collection of linearly independent elements, and hence no (finite) basis. Examples of infinite-dimensional vector spaces include most spaces of functions, such as the spaces of continuous, differentiable, or mean zero functions, as well as the space of *all* polynomials, and the space of solutions to a linear homogeneous partial differential equation. (On the other hand, the solution space for a homogeneous linear ordinary differential equation turns out to be a finite-dimensional vector space.) There is a well-developed concept of a “complete basis” of certain infinite-dimensional function spaces, [67, 68], but this requires more delicate analytical considerations that lie beyond our present abilities. Thus, in this book, the term “basis” *always* means a finite collection of vectors in a finite-dimensional vector space.

Proposition 2.33. If $\mathbf{v}_1, \dots, \mathbf{v}_m$ span the vector space V , then $\dim V \leq m$.

Thus, every vector space spanned by a finite number of elements is necessarily finite-dimensional, and so, if non-zero, admits a basis. Indeed, one can find the basis by successively looking at the members of a collection of spanning vectors, and retaining those that cannot be expressed as linear combinations of their predecessors in the list. Therefore, $n = \dim V$ is the maximal number of linearly independent vectors in the set $\mathbf{v}_1, \dots, \mathbf{v}_m$. The details of the proof are left to the reader; see Exercise 2.4.22.

Lemma 2.34. The elements $\mathbf{v}_1, \dots, \mathbf{v}_n$ form a basis of V if and only if every $\mathbf{x} \in V$ can be written *uniquely* as a linear combination of the basis elements:

$$\mathbf{x} = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n = \sum_{i=1}^n c_i \mathbf{v}_i. \quad (2.20)$$

Proof: The fact that a basis spans V implies that every $\mathbf{x} \in V$ can be written as some linear combination of the basis elements. Suppose we can write an element

$$\mathbf{x} = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n = \widehat{c}_1 \mathbf{v}_1 + \cdots + \widehat{c}_n \mathbf{v}_n \quad (2.21)$$

as two different combinations. Subtracting one from the other, we obtain

$$(c_1 - \widehat{c}_1) \mathbf{v}_1 + \cdots + (c_n - \widehat{c}_n) \mathbf{v}_n = \mathbf{0}.$$

The left-hand side is a linear combination of the basis elements, and hence vanishes if and only if all its coefficients $c_i - \widehat{c}_i = 0$, meaning that the two linear combinations (2.21) are one and the same. *Q.E.D.*

One sometimes refers to the coefficients (c_1, \dots, c_n) in (2.20) as the *coordinates* of the vector \mathbf{x} with respect to the given basis. For the standard basis (2.17) of \mathbb{R}^n , the coordinates of a vector $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ are its entries, i.e., its usual Cartesian coordinates, cf. (2.18).

Example 2.35. *A Wavelet Basis.* The vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad (2.22)$$

form a basis of \mathbb{R}^4 . This is verified by performing Gaussian Elimination on the corresponding 4×4 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}$$

to check that it is nonsingular. This is a very simple example of a *wavelet basis*. Wavelets play an increasingly central role in modern signal and digital image processing; see Section 9.7 and [18, 88].

How do we find the coordinates of a vector, say $\mathbf{x} = (4, -2, 1, 5)^T$, relative to the wavelet basis? We need to find the coefficients c_1, c_2, c_3, c_4 such that

$$\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 + c_4 \mathbf{v}_4.$$

We use (2.13) to rewrite this equation in matrix form $\mathbf{x} = A\mathbf{c}$, where $\mathbf{c} = (c_1, c_2, c_3, c_4)^T$. Solving the resulting linear system by Gaussian Elimination produces

$$c_1 = 2, \quad c_2 = -1, \quad c_3 = 3, \quad c_4 = -2,$$

which are the coordinates of

$$\mathbf{x} = \begin{pmatrix} 4 \\ -2 \\ 1 \\ 5 \end{pmatrix} = 2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3 - 2\mathbf{v}_4 = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

in the wavelet basis. See Section 9.7 for the general theory of wavelet bases.

In general, to find the coordinates of a vector \mathbf{x} with respect to a new basis of \mathbb{R}^n requires the solution of a linear system of equations, namely

$$A\mathbf{c} = \mathbf{x} \quad \text{for} \quad \mathbf{c} = A^{-1}\mathbf{x}. \quad (2.23)$$

The columns of $A = (\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n)$ are the basis vectors, $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ are the Cartesian coordinates of \mathbf{x} , with respect to the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_n$, while $\mathbf{c} = (c_1, c_2, \dots, c_n)^T$ contains its coordinates with respect to the new basis $\mathbf{v}_1, \dots, \mathbf{v}_n$. In practice, one finds the coordinates \mathbf{c} by Gaussian Elimination, *not* matrix inversion.

Why would one want to change bases? The answer is *simplification* and *speed* — many computations and formulas become much easier, and hence faster, to perform in a basis that is adapted to the problem at hand. In signal processing, wavelet bases are particularly appropriate for denoising, compression, and efficient storage of signals, including audio, still images, videos, medical and geophysical images, and so on. These processes would be quite time-consuming — if not impossible in complicated situations like video and three-dimensional image processing — to accomplish in the standard basis. Additional examples will appear throughout the text.

Exercises

2.4.1. Determine which of the following sets of vectors are bases of \mathbb{R}^2 : (a) $\begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \end{pmatrix}$;
 (b) $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$; (c) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$; (d) $\begin{pmatrix} 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; (e) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

2.4.2. Determine which of the following are bases of \mathbb{R}^3 : (a) $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$; (b) $\begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$; (c) $\begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -8 \\ 1 \end{pmatrix}$; (d) $\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$.

2.4.3. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$. (a) Do $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span \mathbb{R}^3 ? Why or why not? (b) Are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ linearly independent? Why or why not? (c) Do $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ form a basis for \mathbb{R}^3 ? Why or why not? If not, is it possible to choose some subset that is a basis? (d) What is the dimension of the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$? Justify your answer.

2.4.4. Answer Exercise 2.4.3 when $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$.

2.4.5. Find a basis for (a) the plane given by the equation $z - 2y = 0$ in \mathbb{R}^3 ; (b) the plane given by the equation $4x + 3y - z = 0$ in \mathbb{R}^3 ; (c) the hyperplane $x + 2y + z - w = 0$ in \mathbb{R}^4 .

2.4.6. (a) Show that $\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ are two different bases for the plane $x - 2y - 4z = 0$. (b) Show how to write both elements of the second basis as linear combinations of the first. (c) Can you find a third basis?

♥ 2.4.7. A basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of \mathbb{R}^n is called *right-handed* if the $n \times n$ matrix $A = (\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n)$ whose columns are the basis vectors has positive determinant: $\det A > 0$. If $\det A < 0$, the basis is called *left-handed*. (a) Which of the following form right-handed bases of \mathbb{R}^3 ?

(i) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, (ii) $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, (iii) $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$,

(iv) $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$. (b) Show that if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a left-handed basis of \mathbb{R}^3 , then $\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3$ and $-\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are both right-handed bases. (c) What sort of basis has $\det A = 0$?

2.4.8. Find a basis for and the dimension of the following subspaces: (a) The space of solutions to the linear system $A\mathbf{x} = \mathbf{0}$, where $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 3 & 0 & 2 & -1 \end{pmatrix}$. (b) The set of all quadratic polynomials $p(x) = ax^2 + bx + c$ that satisfy $p(1) = 0$. (c) The space of all solutions to the homogeneous ordinary differential equation $u''' - u'' + 4u' - 4u = 0$.

2.4.9. (a) Prove that $1 + t^2, t + t^2, 1 + 2t + t^2$ is a basis for the space of quadratic polynomials $\mathcal{P}^{(2)}$. (b) Find the coordinates of $p(t) = 1 + 4t + 7t^2$ in this basis.

2.4.10. Find a basis for and the dimension of the span of

$$(a) \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -6 \\ -2 \\ 2 \end{pmatrix}, \quad (b) \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \quad (c) \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}.$$

2.4.11. (a) Show that $1, 1-t, (1-t)^2, (1-t)^3$ is a basis for $\mathcal{P}^{(3)}$.

(b) Write $p(t) = 1 + t^3$ in terms of the basis elements.

2.4.12. Let $\mathcal{P}^{(4)}$ denote the vector space consisting of all polynomials $p(x)$ of degree ≤ 4 .

(a) Are $x^3 - 3x + 1$, $x^4 - 6x + 3$, $x^4 - 2x^3 + 1$ linearly independent elements of $\mathcal{P}^{(4)}$?

(b) What is the dimension of the subspace of $\mathcal{P}^{(4)}$ they span?

2.4.13. Let $S = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$. (a) Show that the sample vectors corresponding to the functions $1, \cos \pi x, \cos 2\pi x$, and $\cos 3\pi x$ form a basis for the vector space of all sample functions on S .

(b) Write the sampled version of the function $f(x) = x$ in terms of this basis.

2.4.14. (a) Prove that the vector space of all 2×2 matrices is a four-dimensional vector space by exhibiting a basis. (b) Generalize your result and prove that the vector space $\mathcal{M}_{m \times n}$ consisting of all $m \times n$ matrices has dimension mn .

2.4.15. Determine all values of the scalar k for which the following four matrices form a basis

$$\text{for } \mathcal{M}_{2 \times 2}: A_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} k & -3 \\ 1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 \\ -k & 2 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & k \\ -1 & -2 \end{pmatrix}.$$

2.4.16. Prove that the space of diagonal $n \times n$ matrices is an n -dimensional vector space.

2.4.17. (a) Find a basis for and the dimension of the space of upper triangular 2×2 matrices.

(b) Can you generalize your result to upper triangular $n \times n$ matrices?

2.4.18. (a) What is the dimension of the vector space of 2×2 symmetric matrices? Of skew-symmetric matrices? (b) Generalize to the 3×3 case. (c) What about $n \times n$ matrices?

♡ 2.4.19. A matrix is said to be a *semi-magic square* if its row sums and column sums (i.e., the sum of entries in an individual row or column) all add up to the same number. An example

is $\begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix}$, whose row and column sums are all equal to 15. (a) Explain why the set

of all semi-magic squares is a subspace of the vector space of 3×3 matrices. (b) Prove that the 3×3 permutation matrices (1.30) span the space of semi-magic squares. What is its dimension? (c) A *magic square* also has the diagonal and *anti-diagonal* (running from top right to bottom left) add up to the common row and column sum; the preceding 3×3 example is magic. Does the set of 3×3 magic squares form a vector space? If so, what is its dimension? (d) Write down a formula for all 3×3 magic squares.

◇ 2.4.20. (a) Prove that if $\mathbf{v}_1, \dots, \mathbf{v}_m$ forms a basis for $V \subsetneq \mathbb{R}^n$, then $m < n$. (b) Under the hypothesis of part (a), prove that there exist vectors $\mathbf{v}_{m+1}, \dots, \mathbf{v}_n \in \mathbb{R}^n \setminus V$ such that the complete collection $\mathbf{v}_1, \dots, \mathbf{v}_n$ forms a basis for \mathbb{R}^n . (c) Illustrate by constructing bases of \mathbb{R}^3 that include (i) the basis $(1, 1, \frac{1}{2})^T$ of the line $x = y = 2z$; (ii) the basis $(1, 0, -1)^T$, $(0, 1, -2)^T$ of the plane $x + 2y + z = 0$.

◇ 2.4.21. Suppose that $\mathbf{v}_1, \dots, \mathbf{v}_n$ form a basis for \mathbb{R}^n . Let A be a nonsingular matrix. Prove that $A\mathbf{v}_1, \dots, A\mathbf{v}_n$ also form a basis for \mathbb{R}^n . What is this basis if you start with the standard basis: $\mathbf{v}_i = \mathbf{e}_i$?

◇ 2.4.22. Show that if $\mathbf{v}_1, \dots, \mathbf{v}_n$ span $V \neq \{\mathbf{0}\}$, then one can choose a subset $\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_m}$ that forms a basis of V . Thus, $\dim V = m \leq n$. Under what conditions is $\dim V = n$?

◇ 2.4.23. Prove that if $\mathbf{v}_1, \dots, \mathbf{v}_n$ are a basis of V , then every subset thereof, e.g., $\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_k}$, is linearly independent.

- ◇ 2.4.24. Show, by example, how the uniqueness result in Lemma 2.34 fails if one has a linearly dependent set of vectors.
- ◇ 2.4.25. Let $W \subset V$ be a subspace. (a) Prove that $\dim W \leq \dim V$.
 (b) Prove that if $\dim W = \dim V = n < \infty$, then $W = V$. Equivalently, if $W \subsetneq V$ is a proper subspace of a finite-dimensional vector space, then $\dim W < \dim V$.
 (c) Give an example in which the result is false if $\dim V = \infty$.
- ◇ 2.4.26. Let $W, Z \subset V$ be complementary subspaces in a finite-dimensional vector space V , as in Exercise 2.2.24. (a) Prove that if $\mathbf{w}_1, \dots, \mathbf{w}_j$ form a basis for W and $\mathbf{z}_1, \dots, \mathbf{z}_k$ a basis for Z , then $\mathbf{w}_1, \dots, \mathbf{w}_j, \mathbf{z}_1, \dots, \mathbf{z}_k$ form a basis for V . (b) Prove that $\dim W + \dim Z = \dim V$.
- ◇ 2.4.27. Let V be a finite-dimensional vector space and $W \subset V$ a subspace. Prove that the quotient space, as defined in Exercise 2.2.29, has dimension $\dim(V/W) = \dim V - \dim W$.
- ◇ 2.4.28. Let $f_1(x), \dots, f_n(x)$ be scalar functions. Suppose that *every* set of sample points $x_1, \dots, x_m \in \mathbb{R}$, for all finite $m \geq 1$, leads to linearly dependent sample vectors $\mathbf{f}_1, \dots, \mathbf{f}_n \in \mathbb{R}^m$. Prove that $f_1(x), \dots, f_n(x)$ are linearly dependent functions.
Hint: Given sample points x_1, \dots, x_m , let $V_{x_1, \dots, x_m} \subset \mathbb{R}^n$ be the subspace consisting of all vectors $\mathbf{c} = (c_1, c_2, \dots, c_n)^T$ such that $c_1 \mathbf{f}_1 + \dots + c_n \mathbf{f}_n = \mathbf{0}$. First, show that one can select sample points x_1, x_2, x_3, \dots such that $\mathbb{R}^n \supsetneq V_{x_1} \supsetneq V_{x_1, x_2} \supsetneq \dots$. Then, apply Exercise 2.4.25 to conclude that $V_{x_1, \dots, x_n} = \{\mathbf{0}\}$.

2.5 The Fundamental Matrix Subspaces

Let us now return to the general study of linear systems of equations, which we write in our usual matrix form

$$A\mathbf{x} = \mathbf{b}. \quad (2.24)$$

As before, A is an $m \times n$ matrix, where m is the number of equations, so $\mathbf{b} \in \mathbb{R}^m$, and n is the number of unknowns, i.e., the entries of $\mathbf{x} \in \mathbb{R}^n$. We already know how to solve the system, at least when the coefficient matrix is not too large: just apply a variant of Gaussian Elimination. Our goal now is to better understand the solution(s) and thereby prepare ourselves for more sophisticated problems and solution techniques.

Kernel and Image

There are four important vector subspaces associated with any matrix. The first two are defined as follows.

Definition 2.36. The *image* of an $m \times n$ matrix A is the subspace $\text{img } A \subset \mathbb{R}^m$ spanned by its columns. The *kernel* of A is the subspace $\ker A \subset \mathbb{R}^n$ consisting of all vectors that are annihilated by A , so

$$\ker A = \{ \mathbf{z} \in \mathbb{R}^n \mid A\mathbf{z} = \mathbf{0} \} \subset \mathbb{R}^n. \quad (2.25)$$

The image is also known as the *column space* or the *range*[†] of the matrix. By definition,

[†] The latter term can be confusing, since some authors call all of \mathbb{R}^m the range of the (function defined by the) matrix, hence our preference to use image here, and, later, *codomain* to refer to the space \mathbb{R}^n . On the other hand, the space \mathbb{R}^m will be called the *domain* of the (function defined by the) matrix.

a vector $\mathbf{b} \in \mathbb{R}^m$ belongs to $\text{img } A$ if it can be written as a linear combination,

$$\mathbf{b} = x_1 \mathbf{v}_1 + \cdots + x_n \mathbf{v}_n,$$

of the columns of $A = (\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n)$. By our basic matrix multiplication formula (2.13), the right-hand side of this equation equals the product $A \mathbf{x}$ of the matrix A with the column vector $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$, and hence $\mathbf{b} = A \mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. Thus,

$$\text{img } A = \{ A \mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n \} \subset \mathbb{R}^m, \quad (2.26)$$

and so a vector \mathbf{b} lies in the image of A if and only if the linear system $A \mathbf{x} = \mathbf{b}$ has a solution. The compatibility conditions for linear systems can thereby be re-interpreted as the requirements for a vector to lie in the image of the coefficient matrix.

A common alternative name for the kernel is the *null space*. The kernel or null space of A is the set of solutions \mathbf{z} to the homogeneous system $A \mathbf{z} = \mathbf{0}$. The proof that $\ker A$ is a subspace requires us to verify the usual closure conditions: suppose that $\mathbf{z}, \mathbf{w} \in \ker A$, so that $A \mathbf{z} = \mathbf{0} = A \mathbf{w}$. Then, by the compatibility of scalar and matrix multiplication, for any scalars c, d ,

$$A(c\mathbf{z} + d\mathbf{w}) = cA\mathbf{z} + dA\mathbf{w} = \mathbf{0},$$

which implies that $c\mathbf{z} + d\mathbf{w} \in \ker A$. Closure of $\ker A$ can be re-expressed as the following important *superposition principle* for solutions to a homogeneous system of linear equations.

Theorem 2.37. If $\mathbf{z}_1, \dots, \mathbf{z}_k$ are individual solutions to the same homogeneous linear system $A \mathbf{z} = \mathbf{0}$, then so is every linear combination $c_1 \mathbf{z}_1 + \cdots + c_k \mathbf{z}_k$.

Warning. The set of solutions to an inhomogeneous linear system $A \mathbf{x} = \mathbf{b}$ with $\mathbf{b} \neq \mathbf{0}$ is *not* a subspace. Linear combinations of solutions are not, in general, solutions to the same inhomogeneous system.

Superposition is the reason why linear systems are so much easier to solve, since one needs to find only relatively few solutions in order to construct the general solution as a linear combination. In Chapter 7 we shall see that superposition applies to completely general linear systems, including linear differential equations, both ordinary and partial; linear boundary value problems; linear integral equations; linear control systems; etc.

Example 2.38. Let us compute the kernel of the matrix

$$A = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & -3 & -1 & -4 \\ 3 & -5 & -1 & -1 \end{pmatrix}.$$

Our task is to solve the homogeneous system $A \mathbf{x} = \mathbf{0}$, so we need only perform the elementary row operations on A itself. The resulting row echelon form

$$U = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & -1 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

corresponds to the equations $x - 2y + 3w = 0$, $y - z - 10w = 0$. The free variables are z, w , and the general solution is

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2z + 17w \\ z + 10w \\ z \\ w \end{pmatrix} = z \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 17 \\ 10 \\ 0 \\ 1 \end{pmatrix}.$$

The result describes the most general vector in $\ker A$, which is thus the two-dimensional subspace of \mathbb{R}^4 spanned by the linearly independent vectors $(2, 1, 1, 0)^T$, $(17, 10, 0, 1)^T$. This example is indicative of a general method for finding a basis for $\ker A$, to be developed in more detail below.

Once we know the kernel of the coefficient matrix A , i.e., the space of solutions to the homogeneous system $A\mathbf{z} = \mathbf{0}$, we are able to completely characterize the solutions to the inhomogeneous linear system (2.24).

Theorem 2.39. The linear system $A\mathbf{x} = \mathbf{b}$ has a solution \mathbf{x}^* if and only if \mathbf{b} lies in the image of A . If this occurs, then \mathbf{x} is a solution to the linear system if and only if

$$\mathbf{x} = \mathbf{x}^* + \mathbf{z}, \quad (2.27)$$

where $\mathbf{z} \in \ker A$ is an element of the kernel of the coefficient matrix.

Proof: We already demonstrated the first part of the theorem. If $A\mathbf{x} = \mathbf{b} = A\mathbf{x}^*$ are any two solutions, then their difference $\mathbf{z} = \mathbf{x} - \mathbf{x}^*$ satisfies

$$A\mathbf{z} = A(\mathbf{x} - \mathbf{x}^*) = A\mathbf{x} - A\mathbf{x}^* = \mathbf{b} - \mathbf{b} = \mathbf{0},$$

and hence \mathbf{z} is in the kernel of A . Therefore, \mathbf{x} and \mathbf{x}^* are related by formula (2.27), which proves the second part of the theorem. *Q.E.D.*

Therefore, to construct the most general solution to an inhomogeneous system, we need only know one *particular solution* \mathbf{x}^* , along with the general solution $\mathbf{z} \in \ker A$ to the corresponding homogeneous system. This construction should remind the reader of the method for solving inhomogeneous linear ordinary differential equations. Indeed, both linear algebraic systems and linear ordinary differential equations are but two particular instances in the general theory of linear systems, to be developed in Chapter 7.

Example 2.40. Consider the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ 1 & -2 & 3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

where the right-hand side of the system will remain unspecified for the moment. Applying our usual Gaussian Elimination procedure to the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ -1 & 1 & -1 & b_2 \\ 1 & -2 & 3 & b_3 \end{array} \right) \text{ leads to the row echelon form } \left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 0 & 1 & -2 & b_2 \\ 0 & 0 & 0 & b_3 + 2b_2 + b_1 \end{array} \right).$$

Therefore, the system has a solution if and only if the compatibility condition

$$b_1 + 2b_2 + b_3 = 0 \quad (2.28)$$

holds. This equation serves to characterize the vectors \mathbf{b} that belong to the image of the matrix A , which is therefore a plane in \mathbb{R}^3 .

To characterize the kernel of A , we take $\mathbf{b} = \mathbf{0}$, and solve the homogeneous system

$A\mathbf{z} = \mathbf{0}$. The row echelon form corresponds to the reduced system

$$z_1 - z_3 = 0, \quad z_2 - 2z_3 = 0.$$

The free variable is z_3 , and the equations are solved to give

$$z_1 = c, \quad z_2 = 2c, \quad z_3 = c,$$

where c is an arbitrary scalar. Thus, the general solution to the homogeneous system is $\mathbf{z} = (c, 2c, c)^T = c(1, 2, 1)^T$, and so the kernel is the line in the direction of the vector $(1, 2, 1)^T$.

If we take $\mathbf{b} = (3, -2, 1)^T$ — which satisfies (2.28) and hence lies in the image of A — then the general solution to the inhomogeneous system $A\mathbf{x} = \mathbf{b}$ is

$$x_1 = 3 + c, \quad x_2 = 1 + 2c, \quad x_3 = c,$$

where c is arbitrary. We can write the solution in the form (2.27), namely

$$\mathbf{x} = \begin{pmatrix} 3 + c \\ 1 + 2c \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \mathbf{x}^* + \mathbf{z}, \quad (2.29)$$

where, as in (2.27), $\mathbf{x}^* = (3, 1, 0)^T$ plays the role of the particular solution, while $\mathbf{z} = c(1, 2, 1)^T$ is the general element of the kernel.

Finally, we remark that the particular solution is not uniquely defined — any individual solution to the system will serve the purpose. Thus, in this example, we could choose, for instance, $\mathbf{x}^{**} = (-2, -9, -5)^T$ instead, corresponding to $c = -5$ in the preceding formula (2.29). The general solution can be expressed in the alternative form

$$\mathbf{x} = \mathbf{x}^{**} + \mathbf{z} = \begin{pmatrix} -2 \\ -9 \\ -5 \end{pmatrix} + \tilde{c} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \text{where} \quad \mathbf{z} = \tilde{c} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \in \ker A,$$

which agrees with (2.29) when we identify $\tilde{c} = c + 5$.

We can characterize the situations in which the linear system has a unique solution in any of the following equivalent ways.

Proposition 2.41. If A is an $m \times n$ matrix, then the following conditions are equivalent:

- (i) $\ker A = \{\mathbf{0}\}$, i.e., the homogeneous system $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$.
- (ii) $\text{rank } A = n$.
- (iii) The linear system $A\mathbf{x} = \mathbf{b}$ has no free variables.
- (iv) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \text{img } A$.

Thus, while existence of a solution may depend upon the particularities of the right-hand side \mathbf{b} , uniqueness is universal: if for any one \mathbf{b} , e.g., $\mathbf{b} = \mathbf{0}$, the system admits a unique solution, then all $\mathbf{b} \in \text{img } A$ also admit unique solutions. Specializing even further to square matrices, we can now characterize invertible matrices by looking at either their kernels or their images.

Proposition 2.42. If A is a square $n \times n$ matrix, then the following four conditions are equivalent: (i) A is nonsingular; (ii) $\text{rank } A = n$; (iii) $\ker A = \{\mathbf{0}\}$; (iv) $\text{img } A = \mathbb{R}^n$.

Exercises

2.5.1. Characterize the image and kernel of the following matrices:

$$(a) \begin{pmatrix} 8 & -4 \\ -6 & 3 \end{pmatrix}, (b) \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}, (c) \begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & 1 \\ 4 & 0 & 5 \end{pmatrix}, (d) \begin{pmatrix} 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & -2 & 1 & 1 \\ 1 & 2 & -3 & 1 \end{pmatrix}.$$

2.5.2. For the following matrices, write the kernel as the span of a finite number of vectors.

Is the kernel a point, line, plane, or all of \mathbb{R}^3 ? (a) $(2 \ -1 \ 5)$, (b) $\begin{pmatrix} 1 & 2 & -1 \\ 3 & -2 & 0 \end{pmatrix}$,

(c) $\begin{pmatrix} 2 & 6 & -4 \\ -1 & -3 & 2 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 2 & 5 \\ 0 & 4 & 8 \\ 1 & -6 & -11 \end{pmatrix}$, (e) $\begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & -2 \\ 3 & -1 & 1 \end{pmatrix}$, (f) $\begin{pmatrix} 1 & -2 & 3 \\ -3 & 6 & -9 \\ -2 & 4 & -6 \\ 3 & 0 & -1 \end{pmatrix}$.

2.5.3. (a) Find the kernel and image of the coefficient matrix for the system $x - 3y + 2z = a$, $2x - 6y + 2w = b$, $z - 3w = c$. (b) Write down compatibility conditions on a, b, c for a solution to exist.

2.5.4. Suppose $\mathbf{x}^* = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a particular solution to the equation $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{x} = \mathbf{b}$.

(a) What is \mathbf{b} ? (b) Find the general solution.

2.5.5. Prove that the average of all the entries in each row of A is 0 if and only if $(1, 1, \dots, 1)^T \in \ker A$.

2.5.6. *True or false:* If A is a square matrix, then $\ker A \cap \text{img } A = \{\mathbf{0}\}$.

2.5.7. Write the general solution to the following linear systems in the form (2.27). Clearly identify the particular solution \mathbf{x}^* and the element \mathbf{z} of the kernel. (a) $x - y + 3z = 1$,

$$(b) \begin{pmatrix} 1 & -2 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, (c) \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & -4 \\ 2 & -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \\ -4 \end{pmatrix},$$

$$(d) \begin{pmatrix} 2 & -1 & 1 \\ 4 & -1 & 2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, (e) \begin{pmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 3 \\ 1 \end{pmatrix},$$

$$(f) \begin{pmatrix} 1 & -3 & 2 & 0 \\ -1 & 5 & 1 & 1 \\ 2 & -8 & 1 & -1 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix}, (g) \begin{pmatrix} 0 & -1 & 2 & -1 \\ 1 & -3 & 0 & 1 \\ -2 & 5 & 2 & -3 \\ 1 & 1 & -8 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 4 \\ 5 \end{pmatrix}.$$

2.5.8. Given $a, r \neq 0$, characterize the kernel and the image of the matrix

$$\begin{pmatrix} a & ar & \dots & ar^{n-1} \\ ar^n & ar^{n+1} & \dots & ar^{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ ar^{(n-1)n} & ar^{(n-1)n+1} & \dots & ar^{n^2-1} \end{pmatrix}. \quad \text{Hint: See Exercise 1.8.17.}$$

◇ 2.5.9. Let the square matrix P be idempotent, meaning that $P^2 = P$. (a) Prove that $\mathbf{w} \in \text{img } P$ if and only if $P\mathbf{w} = \mathbf{w}$. (b) Show that $\text{img } P$ and $\ker P$ are complementary subspaces, as defined in Exercise 2.2.24, so every $\mathbf{v} \in \mathbb{R}^n$ can be uniquely written as $\mathbf{v} = \mathbf{w} + \mathbf{z}$ where $\mathbf{w} \in \text{img } P$, $\mathbf{z} \in \ker P$.

◇ 2.5.10. Let A be an $m \times n$ matrix. Suppose that $C = \begin{pmatrix} A \\ B \end{pmatrix}$ is an $(m+k) \times n$ matrix whose first m rows are the same as those of A . Prove that $\ker C \subseteq \ker A$. Thus, appending more rows cannot increase the size of a matrix's kernel. Give an example in which $\ker C \neq \ker A$.

- ◇ 2.5.11. Let A be an $m \times n$ matrix. Suppose that $C = (A \ B)$ is an $m \times (n + k)$ matrix whose first n columns are the same as those of A . Prove that $\text{img } C \supseteq \text{img } A$. Thus, appending more columns cannot decrease the size of a matrix's image. Give an example in which $\text{img } C \neq \text{img } A$.

The Superposition Principle

The principle of superposition lies at the heart of linearity. For homogeneous systems, superposition allows one to generate new solutions by combining known solutions. For inhomogeneous systems, superposition combines the solutions corresponding to different inhomogeneities.

Suppose we know particular solutions \mathbf{x}_1^* and \mathbf{x}_2^* to two inhomogeneous linear systems

$$A \mathbf{x} = \mathbf{b}_1, \quad A \mathbf{x} = \mathbf{b}_2,$$

that have the *same* coefficient matrix A . Consider the system

$$A \mathbf{x} = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2,$$

whose right-hand side is a linear combination, or *superposition*, of the previous two. Then a particular solution to the combined system is given by the *same* superposition of the previous solutions:

$$\mathbf{x}^* = c_1 \mathbf{x}_1^* + c_2 \mathbf{x}_2^*.$$

The proof is easy:

$$A \mathbf{x}^* = A(c_1 \mathbf{x}_1^* + c_2 \mathbf{x}_2^*) = c_1 A \mathbf{x}_1^* + c_2 A \mathbf{x}_2^* = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2.$$

In physical applications, the inhomogeneities $\mathbf{b}_1, \mathbf{b}_2$ typically represent external forces, and the solutions $\mathbf{x}_1^*, \mathbf{x}_2^*$ represent the respective responses of the physical apparatus. The linear superposition principle says that if we know how the system responds to the individual forces, we immediately know its response to any combination thereof. The precise details of the system are irrelevant — all that is required is its linearity.

Example 2.43. For example, the system

$$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

models the mechanical response of a pair of masses connected by springs, subject to external forcing. The solution $\mathbf{x} = (x_1, x_2)^T$ represents the displacements of the masses, while the entries of the right-hand side $\mathbf{f} = (f_1, f_2)^T$ are the applied forces. (Details can be found in Chapter 6.) We can directly determine the response of the system $\mathbf{x}_1^* = (\frac{4}{15}, -\frac{1}{15})^T$ to a unit force $\mathbf{e}_1 = (1, 0)^T$ on the first mass, and the response $\mathbf{x}_2^* = (-\frac{1}{15}, \frac{4}{15})^T$ to a unit force $\mathbf{e}_2 = (0, 1)^T$ on the second mass. Superposition gives the response of the system to a general force, since we can write

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = f_1 \mathbf{e}_1 + f_2 \mathbf{e}_2 = f_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + f_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and hence

$$\mathbf{x} = f_1 \mathbf{x}_1^* + f_2 \mathbf{x}_2^* = f_1 \begin{pmatrix} \frac{4}{15} \\ -\frac{1}{15} \end{pmatrix} + f_2 \begin{pmatrix} -\frac{1}{15} \\ \frac{4}{15} \end{pmatrix} = \begin{pmatrix} \frac{4}{15} f_1 - \frac{1}{15} f_2 \\ -\frac{1}{15} f_1 + \frac{4}{15} f_2 \end{pmatrix}.$$

The preceding construction is easily extended to several inhomogeneities, and the result is the general *Superposition Principle* for inhomogeneous linear systems.

Theorem 2.44. Suppose that $\mathbf{x}_1^*, \dots, \mathbf{x}_k^*$ are particular solutions to each of the inhomogeneous linear systems

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \quad \dots \quad A\mathbf{x} = \mathbf{b}_k, \quad (2.30)$$

all having the same coefficient matrix, and where $\mathbf{b}_1, \dots, \mathbf{b}_k \in \text{img } A$. Then, for any choice of scalars c_1, \dots, c_k , a particular solution to the combined system

$$A\mathbf{x} = c_1\mathbf{b}_1 + \dots + c_k\mathbf{b}_k \quad (2.31)$$

is the corresponding superposition

$$\mathbf{x}^* = c_1\mathbf{x}_1^* + \dots + c_k\mathbf{x}_k^* \quad (2.32)$$

of individual solutions. The general solution to (2.31) is

$$\mathbf{x} = \mathbf{x}^* + \mathbf{z} = c_1\mathbf{x}_1^* + \dots + c_k\mathbf{x}_k^* + \mathbf{z}, \quad (2.33)$$

where $\mathbf{z} \in \ker A$ is the general solution to the homogeneous system $A\mathbf{z} = \mathbf{0}$.

For instance, if we know particular solutions $\mathbf{x}_1^*, \dots, \mathbf{x}_m^*$ to

$$A\mathbf{x} = \mathbf{e}_i, \quad \text{for each } i = 1, \dots, m, \quad (2.34)$$

where $\mathbf{e}_1, \dots, \mathbf{e}_m$ are the standard basis vectors of \mathbb{R}^m , then we can reconstruct a particular solution \mathbf{x}^* to the general linear system $A\mathbf{x} = \mathbf{b}$ by first writing

$$\mathbf{b} = b_1\mathbf{e}_1 + \dots + b_m\mathbf{e}_m$$

as a linear combination of the basis vectors, and then using superposition to form

$$\mathbf{x}^* = b_1\mathbf{x}_1^* + \dots + b_m\mathbf{x}_m^*. \quad (2.35)$$

However, for linear algebraic systems, the practical value of this insight is rather limited. Indeed, in the case that A is square and nonsingular, the superposition formula (2.35) is merely a reformulation of the method of computing the inverse of the matrix. Indeed, the vectors $\mathbf{x}_1^*, \dots, \mathbf{x}_m^*$ that satisfy (2.34) are just the columns of A^{-1} (why?), while (2.35) is precisely the solution formula $\mathbf{x}^* = A^{-1}\mathbf{b}$ that we abandoned in practical computations, in favor of the more efficient Gaussian Elimination process. Nevertheless, this idea turns out to have important implications in more general situations, such as linear differential equations and boundary value problems.

Exercises

2.5.12. Find the solution \mathbf{x}_1^* to the system $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and the solution \mathbf{x}_2^* to

$\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Express the solution to $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ as a linear combination of \mathbf{x}_1^* and \mathbf{x}_2^* .

2.5.13. Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 3 & 2 \end{pmatrix}$. Given that $\mathbf{x}_1^* = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$ solves $A\mathbf{x} = \mathbf{b}_1 = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$ and $\mathbf{x}_2^* = \begin{pmatrix} -11 \\ 5 \\ -1 \end{pmatrix}$ solves $A\mathbf{x} = \mathbf{b}_2 = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$, find a solution to $A\mathbf{x} = 2\mathbf{b}_1 + \mathbf{b}_2 = \begin{pmatrix} 2 \\ 10 \\ 14 \end{pmatrix}$.

2.5.14. (a) Show that $\mathbf{x}_1^* = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{x}_2^* = \begin{pmatrix} -3 \\ 3 \\ -2 \end{pmatrix}$ are particular solutions to the system $\begin{pmatrix} 2 & -1 & -5 \\ 1 & -4 & -6 \\ 3 & 2 & -4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$. (b) Find the general solution.

2.5.15. A physical apparatus moves 2 meters under a force of 4 newtons. Assuming linearity, how far will it move under a force of 10 newtons?

2.5.16. Applying a unit external force in the horizontal direction moves a mass 3 units to the right, while applying a unit force in the vertical direction moves it up 2 units. Assuming linearity, where will the mass move under the applied force $\mathbf{f} = (2, -3)^T$?

2.5.17. Suppose \mathbf{x}_1^* and \mathbf{x}_2^* are both solutions to $A\mathbf{x} = \mathbf{b}$. List all linear combinations of \mathbf{x}_1^* and \mathbf{x}_2^* that solve the system.

◇ 2.5.18. Let A be a nonsingular $m \times m$ matrix. (a) Explain in detail why the solutions $\mathbf{x}_1^*, \dots, \mathbf{x}_m^*$ to the systems (2.34) are the columns of the matrix inverse A^{-1} .

(b) Illustrate your argument in the case $A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$.

2.5.19. *True or false:* If \mathbf{x}_1^* solves $A\mathbf{x} = \mathbf{c}$, and \mathbf{x}_2^* solves $B\mathbf{x} = \mathbf{d}$, then $\mathbf{x}^* = \mathbf{x}_1^* + \mathbf{x}_2^*$ solves $(A+B)\mathbf{x} = \mathbf{c} + \mathbf{d}$.

◇ 2.5.20. Under what conditions on the coefficient matrix A will the systems in (2.34) all have a solution?

Adjoint Systems, Cokernel, and Coimage

A linear system of m equations in n unknowns is based on an $m \times n$ coefficient matrix A . The transposed matrix A^T will be of size $n \times m$, and forms the coefficient matrix of an associated linear system, consisting of n equations in m unknowns.

Definition 2.45. The *adjoint*[†] to a linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns is the linear system

$$A^T \mathbf{y} = \mathbf{f} \tag{2.36}$$

consisting of n equations in m unknowns $\mathbf{y} \in \mathbb{R}^m$ with right-hand side $\mathbf{f} \in \mathbb{R}^n$.

Example 2.46. Consider the linear system

$$\begin{aligned} x_1 - 3x_2 - 7x_3 + 9x_4 &= b_1, \\ x_2 + 5x_3 - 3x_4 &= b_2, \\ x_1 - 2x_2 - 2x_3 + 6x_4 &= b_3, \end{aligned} \tag{2.37}$$

[†] **Warning.** Some texts misuse the term “adjoint” to describe the *adjugate* or *cofactor matrix*, [80]. The constructions are completely unrelated, and the adjugate will play no role in this book.

of three equations in four unknowns. Its coefficient matrix

$$A = \begin{pmatrix} 1 & -3 & -7 & 9 \\ 0 & 1 & 5 & -3 \\ 1 & -2 & -2 & 6 \end{pmatrix} \quad \text{has transpose} \quad A^T = \begin{pmatrix} 1 & 0 & 1 \\ -3 & 1 & -2 \\ -7 & 5 & -2 \\ 9 & -3 & 6 \end{pmatrix}.$$

Thus, the adjoint system to (2.37) is the following system of four equations in three unknowns:

$$\begin{aligned} y_1 + y_3 &= f_1, \\ -3y_1 + y_2 - 2y_3 &= f_2, \\ -7y_1 + 5y_2 - 2y_3 &= f_3, \\ 9y_1 - 3y_2 + 6y_3 &= f_4. \end{aligned} \tag{2.38}$$

On the surface, there appears to be no direct connection between the solutions to a linear system and its adjoint. Nevertheless, as we shall soon see (and then in even greater depth in Sections 4.4 and 8.7), the two are linked in a number of remarkable, but subtle ways. As a first step in this direction, we use the adjoint system to define the remaining two fundamental subspaces associated with a coefficient matrix A .

Definition 2.47. The *coimage* of an $m \times n$ matrix A is the image of its transpose,

$$\text{coimg } A = \text{img } A^T = \{ A^T \mathbf{y} \mid \mathbf{y} \in \mathbb{R}^m \} \subset \mathbb{R}^n. \tag{2.39}$$

The *cokernel* of A is the kernel of its transpose,

$$\text{coker } A = \ker A^T = \{ \mathbf{w} \in \mathbb{R}^m \mid A^T \mathbf{w} = \mathbf{0} \} \subset \mathbb{R}^m, \tag{2.40}$$

that is, the set of solutions to the homogeneous adjoint system.

The coimage coincides with the subspace of \mathbb{R}^n spanned by the rows[†] of A , and is thus often referred to as the *row space*. As a direct consequence of Theorem 2.39, the adjoint system $A^T \mathbf{y} = \mathbf{f}$ has a solution if and only if $\mathbf{f} \in \text{img } A^T = \text{coimg } A$. The cokernel is also sometimes called the *left null space* of A , since it can be identified with the set of all row vectors \mathbf{r} satisfying $\mathbf{r}A = \mathbf{0}^T$, where $\mathbf{0}^T$ is the row vector with m zero entries. Indeed, we can identify $\mathbf{r} = \mathbf{w}^T$ and so, taking the transpose of the preceding equation, deduce $A^T \mathbf{w} = (\mathbf{w}^T A)^T = (\mathbf{r}A)^T = \mathbf{0}$, and so $\mathbf{w} = \mathbf{r}^T \in \text{coker } A$.

Example 2.48. To solve the linear system (2.37) just presented, we perform Gaussian

Elimination on the augmented matrix $\left(\begin{array}{cccc|c} 1 & -3 & -7 & 9 & b_1 \\ 0 & 1 & 5 & -3 & b_2 \\ 1 & -2 & -2 & 6 & b_3 \end{array} \right)$, reducing it to the row

echelon form $\left(\begin{array}{cccc|c} 1 & -3 & -7 & 9 & b_1 \\ 0 & 1 & 5 & -3 & b_2 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array} \right)$. Thus, the system has a solution if and only if

$$-b_1 - b_2 + b_3 = 0,$$

[†] Or, more precisely, the column vectors obtained by transposing the rows.

which is required in order that $\mathbf{b} \in \text{img } A$. For such vectors, the general solution is

$$\mathbf{x} = \begin{pmatrix} b_1 + 3b_2 - 8x_3 \\ b_2 - 5x_3 + 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 + 3b_2 \\ b_2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -8 \\ -5 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}.$$

In the second expression, the first vector represents a particular solution, while the two remaining terms constitute the general element of $\ker A$.

The solution to the adjoint system (2.38) is also obtained by Gaussian Elimination,

starting with its augmented matrix $\left(\begin{array}{ccc|c} 1 & 0 & 1 & f_1 \\ -3 & 1 & -2 & f_2 \\ -7 & 5 & -2 & f_3 \\ 9 & -3 & 6 & f_4 \end{array} \right)$. The resulting row echelon

form is $\left(\begin{array}{ccc|c} 1 & 0 & 1 & f_1 \\ 0 & 1 & 1 & f_2 + 3f_1 \\ 0 & 0 & 0 & f_3 - 5f_2 - 8f_1 \\ 0 & 0 & 0 & f_4 + 3f_2 \end{array} \right)$. Thus, there are two consistency constraints re-

quired for a solution to the adjoint system:

$$-8f_1 - 5f_2 + f_3 = 0, \quad 3f_2 + f_4 = 0.$$

These are the conditions required for the right-hand side to belong to the coimage: $\mathbf{f} \in \text{img } A^T = \text{coimg } A$. If these conditions are satisfied, the adjoint system has the following general solution depending on the single free variable y_3 :

$$\mathbf{y} = \begin{pmatrix} f_1 - y_3 \\ 3f_1 + f_2 - y_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ 3f_1 + f_2 \\ 0 \end{pmatrix} + y_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

In the latter formula, the first term represents a particular solution, while the second is the general element of the cokernel $\ker A^T = \text{coker } A$.

The Fundamental Theorem of Linear Algebra

The four fundamental subspaces associated with an $m \times n$ matrix A , then, are its image, coimage, kernel, and cokernel. The image and cokernel are subspaces of \mathbb{R}^m , while the kernel and coimage are subspaces of \mathbb{R}^n . The *Fundamental Theorem of Linear Algebra*[†] states that their dimensions are determined by the rank (and size) of the matrix.

Theorem 2.49. Let A be an $m \times n$ matrix, and let r be its rank. Then

$$\begin{aligned} \dim \text{coimg } A &= \dim \text{img } A = \text{rank } A = \text{rank } A^T = r, \\ \dim \ker A &= n - r, \quad \dim \text{coker } A = m - r. \end{aligned} \tag{2.41}$$

Thus, the rank of a matrix, i.e., the number of pivots, indicates the number of linearly independent columns, which, remarkably, is always the same as the number of linearly independent rows. A matrix and its transpose are guaranteed to have the same rank, i.e.,

[†] Not to be confused with the Fundamental Theorem of Algebra, which states that every (nonconstant) polynomial has a complex root; see [26].

the same number of pivots, despite the fact that their row echelon forms are quite different, and are almost never transposes of each other. Theorem 2.49 also establishes our earlier contention that the rank of a matrix is an *intrinsic* quantity, since it equals the common dimension of its image and coimage, and so does not depend on which specific elementary row operations are employed during the reduction process, nor on the final row echelon form.

Let us turn to the proof of the Fundamental Theorem 2.49. Since the dimension of a subspace is prescribed by the number of vectors in any basis, we need to relate bases of the fundamental subspaces to the rank of the matrix. Before trying to digest the general argument, it is better first to understand how to construct the required bases in a particular example. Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 & 2 \\ -8 & 4 & -6 & -4 \\ 4 & -2 & 3 & 2 \end{pmatrix}. \quad \text{Its row echelon form } U = \begin{pmatrix} 2 & -1 & 1 & 2 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.42)$$

is obtained in the usual manner. There are two pivots, and thus the rank of A is $r = 2$.

Kernel: The general solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$ can be expressed as a linear combination of $n - r$ linearly independent vectors, whose coefficients are the free variables for the system corresponding to the $n - r$ columns without pivots. In fact, these vectors form a basis for the kernel, which thus has dimension $n - r$.

In our example, the pivots are in columns 1 and 3, and so the free variables are x_2, x_4 . Applying Back Substitution to the reduced homogeneous system $U\mathbf{x} = \mathbf{0}$, we obtain the general solution

$$\mathbf{x} = \begin{pmatrix} \frac{1}{2}x_2 - 2x_4 \\ x_2 \\ 2x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 2 \\ 1 \end{pmatrix} \quad (2.43)$$

written as a linear combination of the vectors

$$\mathbf{z}_1 = \left(\frac{1}{2}, 1, 0, 0\right)^T, \quad \mathbf{z}_2 = (-2, 0, 2, 1)^T.$$

We claim that $\mathbf{z}_1, \mathbf{z}_2$ form a basis of $\ker A$. By construction, they span the kernel, and linear independence follows easily, since the only way in which the linear combination (2.43) could vanish is if both free variables vanish: $x_2 = x_4 = 0$.

Coimage: The coimage is the subspace of \mathbb{R}^n spanned by the rows[†] of A . As we prove below, applying an elementary row operation to a matrix does not alter its coimage. Since the row echelon form U is obtained from A by a sequence of elementary row operations, we conclude that $\text{coimg } A = \text{coimg } U$. Moreover, the row echelon structure implies that the r nonzero rows of U are necessarily linearly independent, and hence form a basis of both $\text{coimg } U$ and $\text{coimg } A$, which therefore have dimension $r = \text{rank } A$. In our example, then, a basis for $\text{coimg } A$ consists of the vectors

$$\mathbf{s}_1 = (2, -1, 1, 2)^T, \quad \mathbf{s}_2 = (0, 0, -2, 4)^T,$$

[†] Or, more correctly, the transposes of the rows, since the elements of \mathbb{R}^n are supposed to be column vectors.

coming from the nonzero rows of U . The reader can easily check their linear independence, as well as the fact that every row of A lies in their span.

Image: There are two methods for computing a basis of the image, or column space. The first proves that it has dimension equal to the rank. This has the important, and remarkable consequence that the space spanned by the rows of a matrix and the space spanned by its columns always have the same dimension, even though they are usually different subspaces of different vector spaces.

Now, the row echelon structure implies that the columns of U that contain the pivots form a basis for its image, i.e., $\text{img } U$. In our example, these are its first and third columns, and you can check that they are linearly independent and span the full column space. But the image of A is *not* the same as the image of U , and so, unlike the coimage, we cannot directly use a basis for $\text{img } U$ as a basis for $\text{img } A$. However, the linear dependencies among the columns of A and U are the same, and this implies that the r columns of A that end up containing the pivots will form a basis for $\text{img } A$. In our example (2.42), the pivots lie in the first and third columns of U , and hence the first and third columns of A ; namely,

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ -8 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix},$$

form a basis for $\text{img } A$. This means that every column of A can be written uniquely as a linear combination of its first and third columns. Again, skeptics may wish to check this.

An alternative method to find a basis for the image is to recall that $\text{img } A = \text{coimg } A^T$, and hence we can employ the previous algorithm to compute $\text{coimg } A^T$. In our example, applying Gaussian Elimination to

$$A^T = \begin{pmatrix} 2 & -8 & 4 \\ -1 & 4 & -2 \\ 1 & -6 & 3 \\ 2 & -4 & 2 \end{pmatrix} \text{ leads to the row echelon form } \hat{U} = \begin{pmatrix} 2 & -8 & 4 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.44)$$

Note that the row echelon form of A^T is *not* the transpose of the row echelon form of A . However, they do have the same number of pivots, since, as we now know, both A and A^T have the same rank, namely 2. The two nonzero rows of \hat{U} (again transposed to be column vectors) form a basis for $\text{coimg } A^T$, and therefore

$$\mathbf{y}_1 = \begin{pmatrix} 2 \\ -8 \\ 4 \end{pmatrix}, \quad \mathbf{y}_2 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix},$$

forms an alternative basis for $\text{img } A$.

Cokernel: Finally, to determine a basis for the cokernel, we apply the algorithm for finding a basis for $\ker A^T = \text{coker } A$. Since the ranks of A and A^T coincide, there are now $m - r$ free variables, which is the same as the dimension of $\ker A^T$. In our particular example, using the reduced form (2.44), the only free variable is y_3 , and the general solution to the homogeneous adjoint system $A^T \mathbf{y} = \mathbf{0}$ is

$$\mathbf{y} = \begin{pmatrix} 0 \\ \frac{1}{2}y_3 \\ y_3 \end{pmatrix} = y_3 \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix}.$$

We conclude that $\text{coker } A$ is one-dimensional, with basis $(0, \frac{1}{2}, 1)^T$.

Summarizing, given an $m \times n$ matrix A with row echelon form U , to find a basis for

- $\text{img } A$: choose the r columns of A in which the pivots appear in U ;
- $\text{ker } A$: write the general solution to $A\mathbf{x} = \mathbf{0}$ as a linear combination of the $n - r$ basis vectors whose coefficients are the free variables;
- $\text{coimg } A$: choose the r nonzero rows of U ;
- $\text{coker } A$: write the general solution to the adjoint system $A^T\mathbf{y} = \mathbf{0}$ as a linear combination of the $m - r$ basis vectors whose coefficients are the free variables. (An alternative method — one that does not require solving the adjoint system — can be found on page 223.)

Let us conclude this section by justifying these constructions for general matrices, and thereby complete the proof of the Fundamental Theorem 2.49.

Kernel: If A has rank r , then the general element of the kernel, i.e., solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$, can be written as a linear combination of $n - r$ vectors whose coefficients are the free variables, and hence these vectors span $\text{ker } A$. Moreover, the only combination that yields the zero solution $\mathbf{x} = \mathbf{0}$ is when all the free variables are zero, since any nonzero value for a free variable, say $x_i \neq 0$, gives a solution $\mathbf{x} \neq \mathbf{0}$ whose i^{th} entry (at least) is nonzero. Thus, the only linear combination of the $n - r$ kernel basis vectors that sums to $\mathbf{0}$ is the trivial one, which implies their linear independence.

Coimage: We need to prove that elementary row operations do not change the coimage. To see this for row operations of the first type, suppose, for instance, that \hat{A} is obtained by adding b times the first row of A to the second row. If $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \dots, \mathbf{r}_m$ are the rows of A , then the rows of \hat{A} are $\mathbf{r}_1, \hat{\mathbf{r}}_2 = \mathbf{r}_2 + b\mathbf{r}_1, \mathbf{r}_3, \dots, \mathbf{r}_m$. If

$$\mathbf{v} = c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 + \dots + c_m\mathbf{r}_m$$

is any vector belonging to $\text{coimg } A$, then

$$\mathbf{v} = \hat{c}_1\mathbf{r}_1 + c_2\hat{\mathbf{r}}_2 + c_3\mathbf{r}_3 + \dots + c_m\mathbf{r}_m, \quad \text{where} \quad \hat{c}_1 = c_1 - bc_2,$$

is also a linear combination of the rows of the new matrix, and hence lies in $\text{coimg } \hat{A}$. The converse is also valid — $\mathbf{v} \in \text{coimg } \hat{A}$ implies $\mathbf{v} \in \text{coimg } A$ — and we conclude that elementary row operations of type #1 do not change $\text{coimg } A$. The proofs for the other two types of elementary row operations are even easier, and are left to the reader.

The basis for $\text{coimg } A$ will be the first r nonzero pivot rows $\mathbf{s}_1, \dots, \mathbf{s}_r$ of U . Since the other rows, if any, are all $\mathbf{0}$, the pivot rows clearly span $\text{coimg } U = \text{coimg } A$. To prove their linear independence, suppose

$$c_1\mathbf{s}_1 + \dots + c_r\mathbf{s}_r = \mathbf{0}. \tag{2.45}$$

Let $u_{1k} \neq 0$ be the first pivot. Since all entries of U lying below the pivot are zero, the k^{th} entry of (2.45) is $c_1u_{1k} = 0$, which implies that $c_1 = 0$. Next, suppose $u_{2l} \neq 0$ is the second pivot. Again, using the row echelon structure of U , the l^{th} entry of (2.45) is found to be $c_1u_{1l} + c_2u_{2l} = 0$, and so $c_2 = 0$, since we already know $c_1 = 0$. Continuing in this manner, we deduce that only the trivial linear combination $c_1 = \dots = c_r = 0$ will satisfy (2.45), proving linear independence. Thus, $\mathbf{s}_1, \dots, \mathbf{s}_r$ form a basis for $\text{coimg } U = \text{coimg } A$, which therefore has dimension $r = \text{rank } A$.

Image: In general, a vector $\mathbf{b} \in \text{img } A$ if and only if it can be written as a linear combination of the columns: $\mathbf{b} = A\mathbf{x}$. But, as we know, the general solution to the linear

system $A\mathbf{x} = \mathbf{b}$ is expressed in terms of the free and basic variables; in particular, we are allowed to set all the free variables to zero, and so end up writing \mathbf{b} in terms of the basic variables alone. This effectively expresses \mathbf{b} as a linear combination of the pivot columns of A only, which proves that they span $\text{im} A$. To prove their linear independence, suppose some linear combination of the pivot columns adds up to $\mathbf{0}$. Interpreting the coefficients as basic variables, this would correspond to a vector \mathbf{x} , all of whose free variables are zero, satisfying $A\mathbf{x} = \mathbf{0}$. But our solution to this homogeneous system expresses the basic variables as combinations of the free variables, which, if the latter are all zero, are also zero when the right-hand sides all vanish. This shows that, under these assumptions, $\mathbf{x} = \mathbf{0}$, and hence the pivot columns are linearly independent.

Cokernel: By the preceding arguments, $\text{rank } A = \text{rank } A^T = r$, and hence the general element of $\text{coker } A = \ker A^T$ can be written as a linear combination of $m - r$ basis vectors whose coefficients are the free variables in the homogeneous adjoint system $A^T\mathbf{y} = \mathbf{0}$. Linear independence of the basis elements follows as in the case of the kernel.

Exercises

2.5.21. For each of the following matrices find bases for the (i) image, (ii) coimage, (iii) kernel, and (iv) cokernel.

$$(a) \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}, \quad (b) \begin{pmatrix} 0 & 0 & -8 \\ 1 & 2 & -1 \\ 2 & 4 & 6 \end{pmatrix}, \quad (c) \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 3 \\ 2 & 3 & 7 & 0 \end{pmatrix}, \quad (d) \begin{pmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & -6 & 0 & -2 \\ 2 & -3 & -2 & 4 & 0 \\ 3 & -3 & -6 & 6 & 3 \\ 1 & 0 & -4 & 2 & 3 \end{pmatrix}.$$

2.5.22. Find a set of columns of the matrix $\begin{pmatrix} -1 & 2 & 0 & -3 & 5 \\ 2 & -4 & 1 & 1 & -4 \\ -3 & 6 & 2 & 0 & 8 \end{pmatrix}$ that form a basis for its image. Then express each column as a linear combination of the basis columns.

2.5.23. For each of the following matrices A : (a) Determine the rank and the dimensions of the four fundamental subspaces. (b) Find bases for both the kernel and cokernel. (c) Find explicit conditions on vectors \mathbf{b} that guarantee that the system $A\mathbf{x} = \mathbf{b}$ has a solution. (d) Write down a specific *nonzero* vector \mathbf{b} that satisfies your conditions, and then find all possible solutions \mathbf{x} .

$$(i) \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}, \quad (ii) \begin{pmatrix} 3 & -1 & -2 \\ -6 & 2 & 4 \end{pmatrix}, \quad (iii) \begin{pmatrix} 1 & 5 \\ -2 & 3 \\ 2 & 7 \end{pmatrix}, \quad (iv) \begin{pmatrix} 2 & -5 & -1 \\ 1 & -6 & -4 \\ 3 & -4 & 2 \end{pmatrix},$$

$$(v) \begin{pmatrix} 2 & 5 & 7 \\ 6 & 13 & 19 \\ 3 & 8 & 11 \\ 1 & 2 & 3 \end{pmatrix}, \quad (vi) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \\ 1 & -2 & 2 & 7 \\ 3 & 6 & 5 & -2 \end{pmatrix}, \quad (vii) \begin{pmatrix} 2 & 4 & 0 & -6 & 0 \\ 1 & 2 & 3 & 15 & 0 \\ 3 & 6 & -1 & 15 & 5 \\ -3 & -6 & 2 & 21 & -6 \end{pmatrix}.$$

2.5.24. Find the dimension of and a basis for the subspace spanned by the following sets of vectors. *Hint:* First identify the subspace with the image of a certain matrix.

$$(a) \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \quad (b) \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix}, \quad (c) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \\ -3 \end{pmatrix},$$

$$(d) \begin{pmatrix} 1 \\ 0 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -3 \\ -4 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -8 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -6 \\ 9 \end{pmatrix}, \quad (e) \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \\ 0 \end{pmatrix}.$$

2.5.25. Show that the set of all vectors $\mathbf{v} = (a - 3b, a + 2c + 4d, b + 3c - d, c - d)^T$, where a, b, c, d are real numbers, forms a subspace of \mathbb{R}^4 , and find its dimension.

2.5.26. Find a basis of the solution space of the following homogeneous linear systems.

$$\begin{array}{lll} \text{(a)} & x_1 - 2x_3 = 0, & \\ & x_2 + x_4 = 0. & \\ \text{(b)} & 2x_1 + x_2 - 3x_3 + x_4 = 0, & \\ & 2x_1 - x_2 - x_3 - x_4 = 0. & \\ \text{(c)} & & x_1 - x_2 - 2x_3 + 4x_4 = 0, \\ & & 2x_1 + x_2 - x_4 = 0, \\ & & -2x_1 + 2x_3 - 2x_4 = 0. \end{array}$$

2.5.27. Find bases for the image and coimage of $\begin{pmatrix} 1 & -3 & 0 \\ 2 & -6 & 4 \\ -3 & 9 & 1 \end{pmatrix}$. Make sure they have the same number of elements. Then write each row and column as a linear combination of the appropriate basis vectors.

2.5.28. Find bases for the image of $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 2 & -4 & 6 \\ 1 & 5 & -4 \end{pmatrix}$ using both of the indicated methods.

Demonstrate that they are indeed both bases for the same subspace by showing how to write each basis in terms of the other.

2.5.29. Show that $\mathbf{v}_1 = (1, 2, 0, -1)^T$, $\mathbf{v}_2 = (-3, 1, 1, -1)^T$, $\mathbf{v}_3 = (2, 0, -4, 3)^T$ and $\mathbf{w}_1 = (3, 2, -4, 2)^T$, $\mathbf{w}_2 = (2, 3, -7, 4)^T$, $\mathbf{w}_3 = (0, 3, -3, 1)^T$ are two bases for the same three-dimensional subspace $V \subset \mathbb{R}^4$.

2.5.30. (a) Prove that if A is a symmetric matrix, then $\ker A = \text{coker } A$ and $\text{img } A = \text{coimg } A$.

(b) Use this observation to produce bases for the four fundamental subspaces associated with $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 2 \end{pmatrix}$. (c) Is the converse to part (a) true?

2.5.31. (a) Write down a matrix of rank r whose first r rows do *not* form a basis for its row space. (b) Can you find an example that can be reduced to row echelon form without any row interchanges?

2.5.32. Let A be a 4×4 matrix and let U be its row echelon form. (a) Suppose columns 1, 2, 4 of U form a basis for its image. Do columns 1, 2, 4 of A form a basis for its image? If so, explain why; if not, construct a counterexample. (b) Suppose rows 1, 2, 3 of U form a basis for its coimage. Do rows 1, 2, 3 of A form a basis for its coimage? If so, explain why; if not, construct a counterexample. (c) Suppose you find a basis for $\ker U$. Is it also a basis for $\ker A$? (d) Suppose you find a basis for $\text{coker } U$. Is it also a basis for $\text{coker } A$?

2.5.33. Can you devise a nonzero matrix whose row echelon form is the same as the row echelon form of its transpose?

◇ 2.5.34. Explain why the elementary row operations of types #2 and #3 do not change the coimage of a matrix.

2.5.35. Let A be an $m \times n$ matrix. Prove that $\text{img } A = \mathbb{R}^m$ if and only if $\text{rank } A = m$.

2.5.36. Prove or give a counterexample: If U is the row echelon form of A , then $\text{img } U = \text{img } A$.

◇ 2.5.37. (a) Devise an alternative method for finding a basis of the coimage of a matrix.

Hint: Look at the two methods for finding a basis for the image. (b) Use your method

to find a basis for the coimage of $\begin{pmatrix} 1 & 3 & -5 & 2 \\ 2 & -1 & 1 & -4 \\ 4 & 5 & -9 & 2 \end{pmatrix}$. Is it the same basis as found

by the method in the text?

◇ 2.5.38. Prove that $\ker A \subseteq \ker A^2$. More generally, prove $\ker A \subseteq \ker B A$ for every compatible matrix B .

- ◇ 2.5.39. Prove that $\text{img } A \supseteq \text{img } A^2$. More generally, prove $\text{img } A \supseteq \text{img } (AB)$ for every compatible matrix B .
- 2.5.40. Suppose A is an $m \times n$ matrix, and B and C are nonsingular matrices of sizes $m \times m$ and $n \times n$, respectively. Prove that $\text{rank } A = \text{rank } BA = \text{rank } AC = \text{rank } BAC$.
- 2.5.41. *True or false:* If $\ker A = \ker B$, then $\text{rank } A = \text{rank } B$.
- ◇ 2.5.42. Let A and B be matrices of respective sizes $m \times n$ and $n \times p$.
- (a) Prove that $\dim \ker(AB) \leq \dim \ker A + \dim \ker B$.
- (b) Prove the *Sylvester Inequalities* $\text{rank } A + \text{rank } B - n \leq \text{rank}(AB) \leq \min\{\text{rank } A, \text{rank } B\}$.
- ◇ 2.5.43. Suppose A is a nonsingular $n \times n$ matrix. (a) Prove that every $n \times (n+k)$ matrix of the form $(A \ B)$, where B has size $n \times k$, has rank n . (b) Prove that every $(n+k) \times n$ matrix of the form $\begin{pmatrix} A \\ C \end{pmatrix}$, where C has size $k \times n$, has rank n .
- ◇ 2.5.44. Let A be an $m \times n$ matrix of rank r . Suppose $\mathbf{v}_1, \dots, \mathbf{v}_n$ are a basis for \mathbb{R}^n such that $\mathbf{v}_{r+1}, \dots, \mathbf{v}_n$ form a basis for $\ker A$. Prove that $\mathbf{w}_1 = A\mathbf{v}_1, \dots, \mathbf{w}_r = A\mathbf{v}_r$ form a basis for $\text{img } A$.
- ◇ 2.5.45. (a) Suppose A, B are $m \times n$ matrices such that $\ker A = \ker B$. Prove that there is a nonsingular $m \times m$ matrix M such that $MA = B$. *Hint:* Use Exercise 2.5.44. (b) Use this to conclude that if $A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} = \mathbf{c}$ have the same solutions then they are equivalent linear systems, i.e., one can be obtained from the other by a sequence of elementary row operations.
- ◇ 2.5.46. (a) Let A be an $m \times n$ matrix and let V be a subspace of \mathbb{R}^n . Show that $W = AV = \{A\mathbf{v} \mid \mathbf{v} \in V\}$ forms a subspace of $\text{img } A$. (b) If $\dim V = k$, show that $\dim W \leq \min\{k, r\}$, where $r = \text{rank } A$. Give an example in which $\dim(AV) < \dim V$. *Hint:* Use Exercise 2.4.25.
- ◇ 2.5.47. (a) Show that an $m \times n$ matrix has a left inverse if and only if it has rank n . *Hint:* Use Exercise 2.5.46. (b) Show that it has a right inverse if and only if it has rank m . (c) Conclude that only nonsingular square matrices have both left and right inverses.

2.6 Graphs and Digraphs

We now present an intriguing application of linear algebra to graph theory. A *graph* consists of a finite number of points, called *vertices*, and finitely many lines or curves connecting them, called *edges*. Each edge connects exactly two vertices, which are its endpoints. To avoid technicalities, we will always assume that the graph is *simple*, which means that every edge connects two *distinct* vertices, so no edge forms a *loop* that connects a vertex to itself, and, moreover, two distinct vertices are connected by at most one edge. Some examples of graphs appear in [Figure 2.6](#); the vertices are the black dots and the edges are the lines connecting them.

Graphs arise in a multitude of applications. A particular case that will be considered in depth is electrical networks, where the edges represent wires, and the vertices represent the nodes where the wires are connected. Another example is the framework for a building — the edges represent the beams, and the vertices the joints where the beams are connected. In each case, the graph encodes the topology — meaning interconnectedness — of the system, but not its geometry — lengths of edges, angles, etc.

In a planar representation of a graph, the edges are allowed to cross over each other at non-nodal points without meeting — think of a network where the (insulated) wires lie

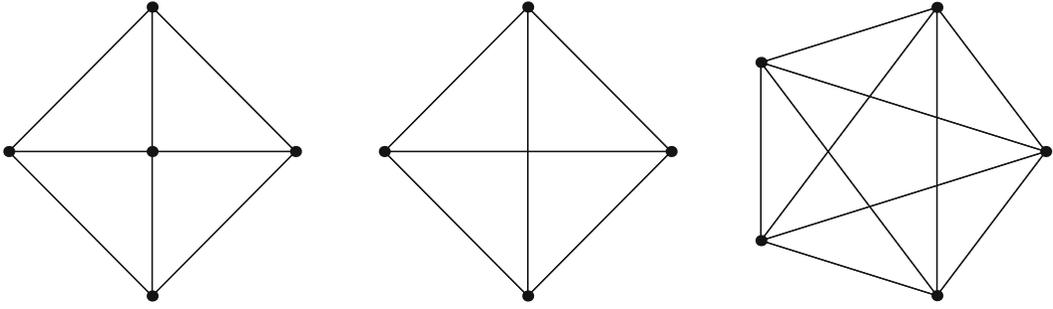


Figure 2.6. Three Different Graphs.

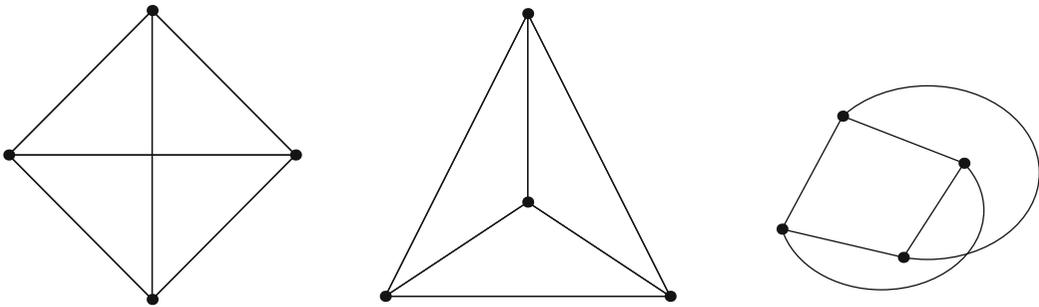


Figure 2.7. Three Versions of the Same Graph.

on top of each other, but do not interconnect. Thus, the first graph in [Figure 2.6](#) has 5 vertices and 8 edges; the second has 4 vertices and 6 edges — the two central edges do not meet; the final graph has 5 vertices and 10 edges.

Two graphs are considered to be the same if there is a one-to-one correspondence between their edges and their vertices, so that matched edges connect matched vertices. In an electrical network, moving the nodes and wires around without cutting or rejoining will have no effect on the underlying graph. Consequently, there are many ways to draw a given graph; three representations of one and the same graph appear in [Figure 2.7](#).

A *path* in a graph is an ordered list of distinct edges e_1, \dots, e_k connecting (not necessarily distinct) vertices v_1, \dots, v_{k+1} so that edge e_i connects vertex v_i to v_{i+1} . For instance, in the graph in [Figure 2.8](#), one path starts at vertex 1, then goes in order along the edges labeled as 1, 4, 3, 2, successively passing through the vertices 1, 2, 4, 1, 3. Observe that while an edge cannot be repeated in a path, a vertex may be. A graph is *connected* if you can get from any vertex to any other vertex by a path, which is the most important case for applications. We note that every graph can be decomposed into a disconnected collection of connected subgraphs.

A *circuit* is a path that ends up where it began, i.e., $v_{k+1} = v_1$. For example, the circuit in [Figure 2.8](#) consisting of edges 1, 4, 5, 2 starts at vertex 1, then goes to vertices 2, 4, 3 in order, and finally returns to vertex 1. In a closed circuit, the choice of starting vertex is not important, and we identify circuits that go around the edges in the same order. Thus, for example, the edges 4, 5, 2, 1 represent the same circuit as above.

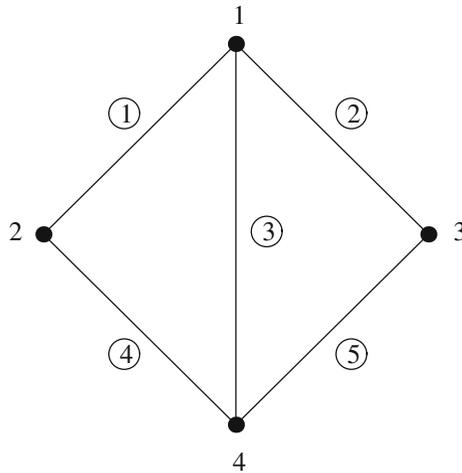


Figure 2.8. A Simple Graph.

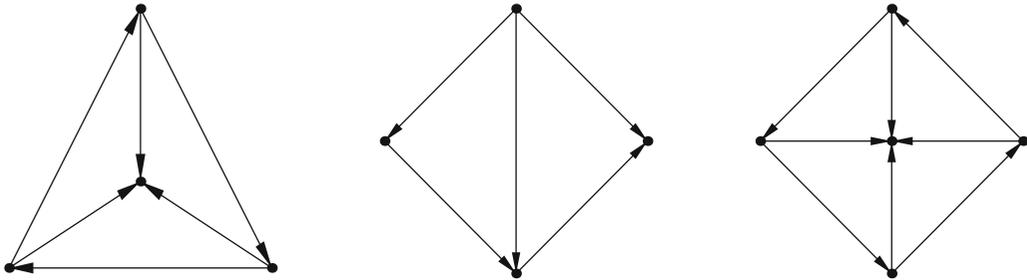


Figure 2.9. Digraphs.

In electrical circuits, one is interested in measuring currents and voltage drops along the wires in the network represented by the graph. Both of these quantities have a direction, and therefore we need to specify an orientation on each edge in order to quantify how the current moves along the wire. The orientation will be fixed by specifying the vertex the edge “starts” at, and the vertex it “ends” at. Once we assign a direction to an edge, a current along that wire will be positive if it moves in the same direction, i.e., goes from the starting vertex to the ending one, and negative if it moves in the opposite direction. The direction of the edge does *not* dictate the direction of the current — it just fixes what directions positive and negative values of current represent. A graph with directed edges is known as a *directed graph*, or *digraph* for short. The edge directions are represented by arrows; examples of digraphs can be seen in [Figure 2.9](#). Again, the underlying graph is always assumed to be simple. For example, at any instant in time, the internet can be viewed as a gigantic digraph, in which each vertex represents a web page, and each edge represents an existing link from one page to another.

Consider a digraph D consisting of n vertices connected by m edges. The *incidence matrix* associated with D is an $m \times n$ matrix A whose rows are indexed by the edges and whose columns are indexed by the vertices. If edge k starts at vertex i and ends at vertex j , then row k of the incidence matrix will have $+1$ in its (k, i) entry and -1 in its (k, j) entry; all other entries in the row are zero. Thus, our convention is that $+1$ represents the

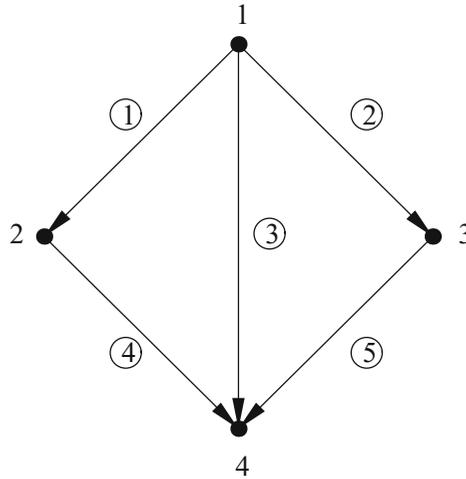


Figure 2.10. A Simple Digraph.

outgoing vertex at which the edge starts and -1 the incoming vertex at which it ends.

A simple example is the digraph in [Figure 2.10](#), which consists of five edges joined at four different vertices. Its 5×4 incidence matrix is

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \quad (2.46)$$

Thus the first row of A tells us that the first edge starts at vertex 1 and ends at vertex 2. Similarly, row 2 says that the second edge goes from vertex 1 to vertex 3, and so on. Clearly, one can completely reconstruct any digraph from its incidence matrix.

Example 2.50. The matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \quad (2.47)$$

qualifies as an incidence matrix of a simple graph because each row contains a single $+1$, a single -1 , and the other entries are 0; moreover, to ensure simplicity, no two rows are identical or -1 times each other. Let us construct the digraph corresponding to A . Since A has five columns, there are five vertices in the digraph, which we label by the numbers 1, 2, 3, 4, 5. Since it has seven rows, there are 7 edges. The first row has its $+1$ in column 1 and its -1 in column 2, and so the first edge goes from vertex 1 to vertex 2. Similarly, the second edge corresponds to the second row of A and so goes from vertex 3 to vertex 1. The third row of A indicates an edge from vertex 3 to vertex 2; and so on. In this manner, we construct the digraph drawn in [Figure 2.11](#).

The incidence matrix serves to encode important geometric information about the digraph it represents. In particular, its kernel and cokernel have topological significance.

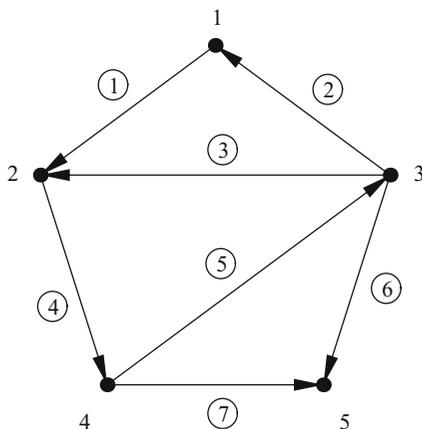


Figure 2.11. Another Digraph.

For example, the kernel of the incidence matrix (2.47) is spanned by the single vector $\mathbf{z} = (1, 1, 1, 1, 1)^T$, and represents the fact that the sum of the entries in any given row of A is zero. This observation holds in general for connected digraphs.

Proposition 2.51. If A is the incidence matrix for a connected digraph, then $\ker A$ is one-dimensional, with basis $\mathbf{z} = (1, 1, \dots, 1)^T$.

Proof: If edge k connects vertex i to vertex j , then the k^{th} equation in $A\mathbf{z} = \mathbf{0}$ is $z_i - z_j = 0$, or, equivalently, $z_i = z_j$. The same equality holds, by a simple induction, if the vertices i and j are connected by a path. Therefore, if D is connected, then all the entries of \mathbf{z} are equal, and the result follows. *Q.E.D.*

Remark. In general, $\dim \ker A$ equals the number of connected components in the digraph D . See Exercise 2.6.12.

Applying the Fundamental Theorem 2.49, we immediately deduce the following:

Corollary 2.52. If A is the incidence matrix for a connected digraph with n vertices, then $\text{rank } A = n - 1$.

Next, let us look at the cokernel of an incidence matrix. Consider the particular example (2.46) corresponding to the digraph in Figure 2.10. We need to compute the kernel of the transposed incidence matrix

$$A^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix}. \quad (2.48)$$

Solving the homogeneous system $A^T \mathbf{y} = \mathbf{0}$ by Gaussian Elimination, we discover that $\text{coker } A = \ker A^T$ is spanned by the two vectors

$$\mathbf{y}_1 = (1, 0, -1, 1, 0)^T, \quad \mathbf{y}_2 = (0, 1, -1, 0, 1)^T.$$

Each of these vectors represents a *circuit* in the digraph. Keep in mind that their entries are indexed by the edges, so a nonzero entry indicates the direction to traverse the corresponding edge. For example, \mathbf{y}_1 corresponds to the circuit that starts out along edge 1,

then goes along edge 4 and finishes by going along edge 3 in the reverse direction, which is indicated by the minus sign in its third entry. Similarly, \mathbf{y}_2 represents the circuit consisting of edge 2, followed by edge 5, and then edge 3, backwards. The fact that \mathbf{y}_1 and \mathbf{y}_2 are linearly independent vectors says that the two circuits are “independent”.

The general element of coker A is a linear combination $c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2$. Certain values of the constants lead to other types of circuits; for example, $-\mathbf{y}_1$ represents the same circuit as \mathbf{y}_1 , but traversed in the opposite direction. Another example is

$$\mathbf{y}_1 - \mathbf{y}_2 = (1, -1, 0, 1, -1)^T,$$

which represents the square circuit going around the outside of the digraph along edges 1, 4, 5, 2, the fifth and second edges taken in the reverse direction. We can view this circuit as a combination of the two triangular circuits; when we add them together, the middle edge 3 is traversed once in each direction, which effectively “cancels” its contribution. (A similar cancellation occurs in the calculus of line integrals, [2, 78].) Other combinations represent “virtual” circuits; for instance, one can “interpret” $2\mathbf{y}_1 - \frac{1}{2}\mathbf{y}_2$ as two times around the first triangular circuit plus one-half of the other triangular circuit, taken in the reverse direction — whatever that might mean.

Let us summarize the preceding discussion.

Theorem 2.53. Each circuit in a digraph D is represented by a vector in the cokernel of its incidence matrix A , whose entries are +1 if the edge is traversed in the correct direction, -1 if in the opposite direction, and 0 if the edge is not in the circuit. The dimension of the cokernel of A equals the number of independent circuits in D .

Remark. A full proof that the cokernel of the incidence matrix of a general digraph has a basis consisting entirely of independent circuits requires a more in depth analysis of the properties of graphs than we can provide in this abbreviated treatment. Full details can be found in [6; §II.3].

The preceding two theorems have an important and remarkable consequence. Suppose D is a connected digraph with m edges and n vertices and A its $m \times n$ incidence matrix. Corollary 2.52 implies that A has rank $r = n - 1 = n - \dim \ker A$. On the other hand, Theorem 2.53 tells us that $l = \dim \text{coker } A$ equals the number of independent circuits in D . The Fundamental Theorem 2.49 says that $r = m - l$. Equating these two formulas for the rank, we obtain $r = n - 1 = m - l$, or $n + l = m + 1$. This celebrated result is known as *Euler’s formula* for graphs, first discovered by the extraordinarily prolific and influential eighteenth-century Swiss mathematician Leonhard Euler[†].

Theorem 2.54. If G is a connected graph, then

$$\# \text{ vertices} + \# \text{ independent circuits} = \# \text{ edges} + 1. \quad (2.49)$$

Remark. If the graph is *planar*, meaning that it can be graphed in the plane without any edges crossing over each other, then the number of independent circuits is equal to the number of “holes”, i.e., the number of distinct polygonal regions bounded by the edges of the graph. For example, the pentagonal digraph in Figure 2.11 bounds three triangles, and so has three independent circuits.

[†] Pronounced “Oiler”. Euler spent most of his career in Russia and Germany.



Figure 2.12. A Cubical Digraph.

Example 2.55. Consider the graph corresponding to the edges of a cube, as illustrated in Figure 2.12, where the second figure represents the same graph squashed down onto a plane. The graph has 8 vertices and 12 edges. Euler’s formula (3.92) tells us that there are 5 independent circuits. These correspond to the interior square and four trapezoids in the planar version of the digraph, and hence to circuits around 5 of the 6 faces of the cube. The “missing” face does indeed define a circuit, but it can be represented as the sum of the other five circuits, and so is not independent. In Exercise 2.6.6, the reader is asked to write out the incidence matrix for the cubical digraph and explicitly identify the basis of its kernel with the circuits.

Further development of the many remarkable connections between graph theory and linear algebra will be developed in the later chapters. The applications to very large graphs, e.g., with millions or billions of vertices, is playing an increasingly important role in modern computer science and data analysis. One example is the dominant internet search engine run by Google, which is based on viewing the entire internet as a gigantic (time-dependent) digraph. The vertices are the web pages, and a directed edge represents a link from one web page to another. (The resulting digraph is not simple according to our definition, since web pages can link in both directions.) Ranking web pages by importance during a search relies on analyzing the internet digraph; see Section 9.3 for further details.

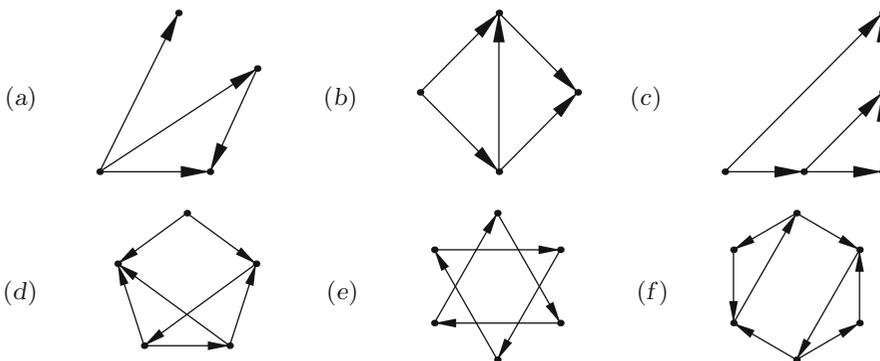
Exercises

2.6.1. (a) Draw the graph corresponding to the 6×7 incidence matrix whose nonzero (i, j) entries equal 1 if $j = i$ and -1 if $j = i + 1$, for $i = 1$ to 6. (b) Find a basis for its kernel and cokernel. (c) How many circuits are in the digraph?

2.6.2. Draw the digraph represented by the following incidence matrices:

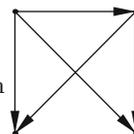
$$\begin{aligned}
 (a) \quad & \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, & (b) \quad & \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, & (c) \quad & \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 \end{pmatrix}, \\
 (d) \quad & \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}, & (e) \quad & \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.
 \end{aligned}$$

2.6.3. Write out the incidence matrix of the following digraphs.



2.6.4. For each of the digraphs in Exercise 2.6.3, see whether you can predict a collection of independent circuits. Verify your prediction by constructing a suitable basis of the cokernel of the incidence matrix and identifying each basis vector with a circuit.

♡ 2.6.5. (a) Write down the incidence matrix A for the indicated digraph. (b) What is the rank of A ? (c) Determine the dimensions of its four fundamental subspaces. (d) Find a basis for its kernel and cokernel. (e) Determine explicit conditions on vectors \mathbf{b} that guarantee that the system $A\mathbf{x} = \mathbf{b}$ has a solution. (f) Write down a specific nonzero vector \mathbf{b} that satisfies your conditions, and then find all possible solutions.

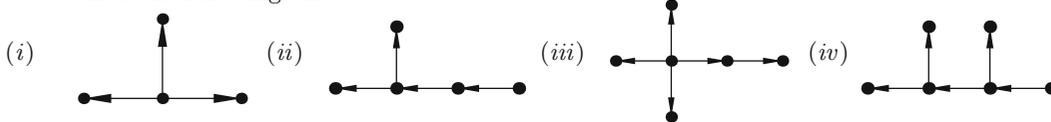


◇ 2.6.6. (a) Write out the incidence matrix for the cubical digraph and identify the basis of its cokernel with the circuits. (b) Find three circuits that do not correspond to any of your basis elements, and express them as a linear combination of the basis circuit vectors.

♡ 2.6.7. Write out the incidence matrix for the other Platonic solids: (a) tetrahedron, (b) octahedron, (c) dodecahedron, and (d) icosahedron. (You will need to choose an orientation for the edges.) Show that, in each case, the number of independent circuits equals the number of faces minus 1.

◇ 2.6.8. Prove that a graph with n vertices and n edges must have at least one circuit.

♡ 2.6.9. A connected graph is called a *tree* if it has no circuits. (a) Find the incidence matrix for each of the following directed trees:

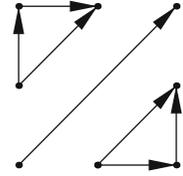


(b) Draw all distinct trees with 4 vertices. Assign a direction to the edges, and write down the corresponding incidence matrices. (c) Prove that a connected graph on n vertices is a tree if and only if it has precisely $n - 1$ edges.

♡ 2.6.10. A *complete graph* G_n on n vertices has one edge joining every distinct pair of vertices. (a) Draw G_3 , G_4 and G_5 . (b) Choose an orientation for each edge and write out the resulting incidence matrix of each digraph. (c) How many edges does G_n have? (d) How many independent circuits?

♡ 2.6.11. The *complete bipartite digraph* $G_{m,n}$ is based on two disjoint sets of, respectively, m and n vertices. Each vertex in the first set is connected to each vertex in the second set by a single edge. (a) Draw $G_{2,3}$, $G_{2,4}$, and $G_{3,3}$. (b) Write the incidence matrix of each digraph. (c) How many edges does $G_{m,n}$ have? (d) How many independent circuits?

- ♡ 2.6.12. (a) Construct the incidence matrix A for the disconnected digraph D in the figure. (b) Verify that $\dim \ker A = 3$, which is the same as the number of connected components, meaning the maximal connected subgraphs in D . (c) Can you assign an interpretation to your basis for $\ker A$? (d) Try proving the general statement that $\dim \ker A$ equals the number of connected components in the digraph D .



2.6.13. How does altering the direction of the edges of a digraph affect its incidence matrix? The cokernel of its incidence matrix? Can you realize this operation by matrix multiplication?

- ♡ 2.6.14. (a) Explain why two digraphs are equivalent under relabeling of vertices and edges if and only if their incidence matrices satisfy $PAQ = B$, where P, Q are permutation matrices. (b) Decide which of the following incidence matrices produce the equivalent digraphs:

$$\begin{aligned}
 (i) \quad & \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, & (ii) \quad & \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix}, & (iii) \quad & \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \\
 (iv) \quad & \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, & (v) \quad & \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, & (vi) \quad & \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}.
 \end{aligned}$$

(c) How are the cokernels of equivalent incidence matrices related?

2.6.15. *True or false:* If A and B are incidence matrices of the same size and $\text{coker } A = \text{coker } B$, then the corresponding digraphs are equivalent.

- ◇ 2.6.16. (a) Explain why the incidence matrix for a disconnected graph can be written in block diagonal matrix form $A = \begin{pmatrix} B & O \\ O & C \end{pmatrix}$ under an appropriate labeling of the vertices. (b) Show how to label the vertices of the digraph in Exercise 2.6.3e so that its incidence matrix is in block form.
-