Chapter 6
Generalized Functions and Green's Functions

Boundary value problems, involving both ordinary and partial differential equations, can be profitably viewed as the infinite-dimensional function space versions of finite-dimensional systems of linear algebraic equations. As a result, linear algebra not only provides us with important insights into their underlying mathematical structure, but also motivates both analytical and numerical solution techniques. In the present chapter, we develop the method of Green's functions, pioneered by the early-nineteenth-century self-taught English mathematician (and miller!) George Green, whose famous Theorem you already encountered in multivariable calculus. We begin with the simpler case of ordinary differential equations, and then move on to solving the two-dimensional Poisson equation, where the Green's function provides a powerful alternative to the method of separation of variables.

For inhomogeneous linear systems, the basic Superposition Principle says that the response to a combination of external forces is the self-same combination of responses to the individual forces. In a finite-dimensional system, any forcing function can be decomposed into a linear combination of unit impulse forces, each applied to a single component of the system, and so the full solution can be obtained by combining the solutions to the individual impulse problems. This simple idea will be adapted to boundary value problems governed by differential equations, where the response of the system to a concentrated impulse force is known as the Green's function. With the Green's function in hand, the solution to the inhomogeneous system with a general forcing function can be reconstructed by superimposing the effects of suitably scaled impulses. Understanding this construction will become increasingly important as we progress to partial differential equations, where direct analytic solution techniques are far harder to come by.

The obstruction blocking a direct implementation of this idea is that there is no ordinary function that represents an idealized concentrated impulse! Indeed, while this approach was pioneered by Green and Cauchy in the early 1800s, and then developed into an effective computational tool by Heaviside in the 1880s, it took another 60 years before mathematicians were able to develop a completely rigorous theory of *generalized functions*, also known as *distributions*. In the language of generalized functions, a unit impulse is represented by a *delta function*.† While we do not have the analytic tools to completely develop the mathematical theory of generalized functions in its full, rigorous glory, we will spend the first section learning the basic concepts and developing the practical computational skills, including Fourier methods, required for applications. The second

† *Warning*: We follow common practice and refer to the "delta distribution" as a function, even though, as we will see, it is most definitely not a function in the usual sense.

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section will discuss the method of Green's functions in the context of one-dimensional boundary value problems governed by ordinary differential equations. In the final section, we develop the Green's function method for solving basic boundary value problems for the two-dimensional Poisson equation, which epitomizes the class of planar elliptic boundary value problems.

6.1 Generalized Functions

Our goal is to solve inhomogeneous linear boundary value problems by first determining the effect of a concentrated impulse force. The response to a general forcing function is then found by linear superposition. But before diving in, let us first review the relevant constructions in the case of linear systems of algebraic equations.

Consider a system of n linear equations in n unknowns[†] $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$, written in matrix form

$$A\mathbf{u} = \mathbf{f}. \tag{6.1}$$

Here A is a fixed $n \times n$ matrix, assumed to be nonsingular, which ensures the existence of a unique solution $\mathbf{u}$ for any choice of right-hand side $\mathbf{f} = (f_1, f_2, \ldots, f_n)^T \in \mathbb{R}^n$. We regard the linear system (6.1) as representing the equilibrium equations of some physical system, e.g., a system of masses interconnected by springs. In this context, the right hand side $\mathbf{f}$ represents an external forcing, so that its i^{th} entry, f_i, represents the amount of force exerted on the i^{th} mass, while the i^{th} entry of the solution vector, u_i, represents the i^{th} mass' induced displacement.

Let

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \qquad \ldots \qquad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{6.2}$$

denote the *standard basis vectors* of $\mathbb{R}^n$, so that $\mathbf{e}_j$ has a single 1 in its j^{th} entry and all other entries 0. We interpret each $\mathbf{e}_j$ as a concentrated *unit impulse force* that is applied solely to the j^{th} mass in our physical system. Let $\mathbf{u}_j = (u_{j,1}, \ldots, u_{j,n})^T$ be the induced response of the system, that is, the solution to

$$A\mathbf{u}_j = \mathbf{e}_j. \tag{6.3}$$

Let us suppose that we have calculated the response vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ to each such impulse force. We can express any other force vector as a linear combination,

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + \cdots + f_n\mathbf{e}_n, \tag{6.4}$$

† All vectors are column vectors, but we sometimes write the transpose, which is a row vector, to save space.

of the impulse forces. The Superposition Principle of Theorem 1.7 then implies that the solution to the inhomogeneous system (6.1) is the selfsame linear combination of the individual impulse responses:

$$\mathbf{u} = f_1 \mathbf{u}_1 + f_2 \mathbf{u}_2 + \cdots + f_n \mathbf{u}_n. \tag{6.5}$$

Thus, knowing how the linear system responds to each impulse force allows us to immediately calculate its response to a general external force.

Remark: The alert reader will recognize that $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are the columns of the inverse matrix, A^{-1}, and so formula (6.5) is, in fact, reconstructing the solution to the linear system (6.1) by inverting its coefficient matrix: $\mathbf{u} = A^{-1}\mathbf{f}$. Thus, this observation is merely a restatement of a standard linear algebraic system solution technique.

The Delta Function

The aim of this chapter is to adapt the preceding algebraic solution technique to boundary value problems. Suppose we want to solve a linear boundary value problem governed by an ordinary differential equation on an interval $a < x < b$, the boundary conditions being imposed at the endpoints. The key issue is how to characterize an impulse force that is concentrated at a single point.

In general, a *unit impulse* at position $a < \xi < b$ will be described by something called the *delta function*, and denoted by $\delta_\xi(x)$. Since the impulse is supposed to be concentrated solely at $x = \xi$, our first requirement is

$$\delta_\xi(x) = 0 \qquad \text{for} \qquad x \neq \xi. \tag{6.6}$$

Moreover, since the delta function represents a *unit* impulse, we want the total amount of force to be equal to one. Since we are dealing with a continuum, the total force is represented by an integral over the entire interval, and so we also require that the delta function satisfy

$$\int_a^b \delta_\xi(x)\, dx = 1, \qquad \text{provided} \qquad a < \xi < b. \tag{6.7}$$

Alas, there is no bona fide function that enjoys both of the required properties! Indeed, according to the basic facts of Riemann (or even Lebesgue) integration, two functions that are the same everywhere except at a single point have exactly the same integral, [**96**, **98**]. Thus, since δ_ξ is zero except at one point, its integral should be 0, not 1. The mathematical conclusion is that the two requirements, (6.6–7) are inconsistent!

This unfortunate fact stopped mathematicians dead in their tracks. It took the imagination of a British engineer, Oliver Heaviside, who was not deterred by the lack of rigorous justification, to start utilizing delta functions in practical applications — with remarkable effect. Despite his success, Heaviside was ridiculed by the mathematicians of his day, and eventually succumbed to mental illness. But, some thirty years later, the great British theoretical physicist Paul Dirac resurrected the delta function for quantum-mechanical applications, and this finally made the mathematicians sit up and take notice. (Indeed, the term "Dirac delta function" is quite common, even though Heaviside should rightly have priority.) In 1944, the French mathematician Laurent Schwartz finally established a rigorous theory of *distributions* that incorporated such useful but nonstandard objects, [**103**]. Thus, to be more accurate, we should really refer to the *delta distribution*; however, we will retain the more common, intuitive designation "delta function" throughout. It is beyond the scope of this introductory text to develop a fully rigorous theory of distributions. Rather, in the spirit of Heaviside, we shall concentrate on learning, through practice with computations and applications, how to make effective use of these exotic mathematical creatures.

There are two possible ways to introduce the delta distribution. Both are important and worth understanding.

Method #1. Limits: The first approach is to regard the delta function $\delta_\xi(x)$ as a limit of a sequence of ordinary smooth functions[†] $g_n(x)$. These will represent progressively more and more concentrated unit forces, which, in the limit, converge to the desired unit impulse concentrated at a single point, $x = \xi$. Thus, we require

$$\lim_{n \to \infty} g_n(x) = 0, \qquad x \neq \xi, \tag{6.8}$$

while the total amount of force remains fixed at

$$\int_a^b g_n(x)\, dx = 1 \qquad \text{for all} \quad n. \tag{6.9}$$

On a formal level, the limit "function"

$$\delta_\xi(x) = \lim_{n \to \infty} g_n(x)$$

will satisfy the key properties (6.6–7).

An explicit example of such a sequence is provided by the rational functions

$$g_n(x) = \frac{n}{\pi(1 + n^2 x^2)}\,. \tag{6.10}$$

These functions satisfy

$$\lim_{n \to \infty} g_n(x) = \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0, \end{cases} \tag{6.11}$$

while[‡]

$$\int_{-\infty}^{\infty} g_n(x)\, dx = \frac{1}{\pi} \tan^{-1} n\,x \,\Bigg|_{x=-\infty}^{\infty} = 1. \tag{6.12}$$

Therefore, formally, we identify the limiting function

$$\lim_{n \to \infty} g_n(x) = \delta(x) = \delta_0(x) \tag{6.13}$$

with the unit-impulse delta function concentrated at $x = 0$. As sketched in Figure 6.1, as n gets larger and larger, each successive function $g_n(x)$ forms a more and more concentrated spike, while maintaining a unit total area under its graph. Thus, the limiting delta function can be thought of as an infinitely tall spike of zero width, entirely concentrated at the origin.

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† To keep the notation compact, we suppress the dependence of the functions g_n on the point ξ where the limiting delta function is concentrated.

‡ For the moment, it will be slightly simpler to consider the entire real line $-\infty < x < \infty$. Exercise 6.1.8 discusses how to adapt the construction to a finite interval.

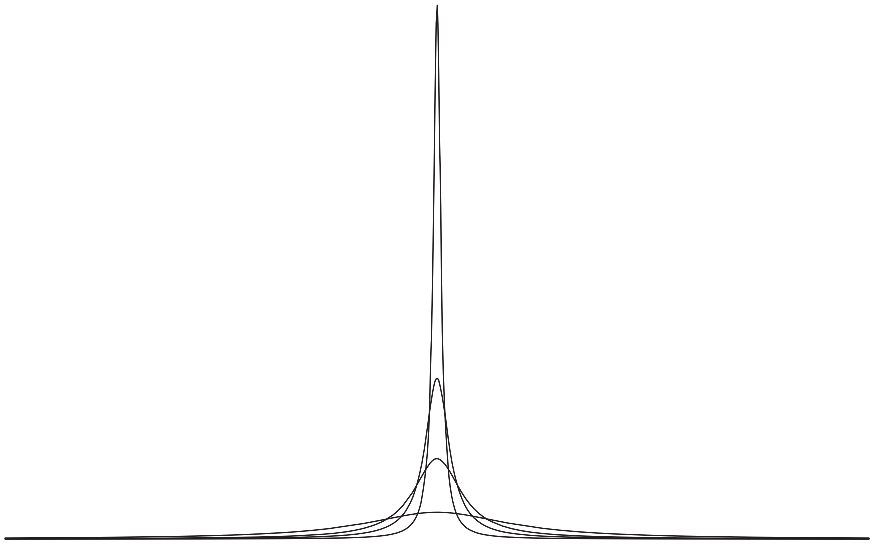


Figure 6.1. Delta function as limit.

Remark: There are many other possible choices for the limiting functions $g_n(x)$. See Exercise 6.1.7 for another important example.

Remark: This construction of the delta function highlights the perils of interchanging limits and integrals without rigorous justification. In any standard theory of integration (Riemann, Lebesgue, etc.), the limit of the functions g_n would be indistinguishable from the zero function, so the limit of their integrals (6.12) would *not* equal the integral of their limit:

$$1 = \lim_{n \to \infty} \int_{-\infty}^{\infty} g_n(x)\, dx \;\neq\; \int_{-\infty}^{\infty} \lim_{n \to \infty} g_n(x)\, dx = 0.$$

The delta function is, in a sense, a means of sidestepping this analytic inconvenience. The full ramifications and theoretical constructions underlying such limits must, however, be deferred to a rigorous course in real analysis, [**96**, **98**].

Once we have defined the basic delta function $\delta(x) = \delta_0(x)$ concentrated at the origin, we can obtain the delta function concentrated at any other position ξ by a simple translation:

$$\delta_\xi(x) = \delta(x - \xi). \tag{6.14}$$

Thus, $\delta_\xi(x)$ can be realized as the limit, as $n \to \infty$, of the translated functions

$$\widehat{g}_n(x) = g_n(x - \xi) = \frac{n}{\pi \left[\, 1 + n^2 (x - \xi)^2 \,\right]}\,. \tag{6.15}$$

Method #2. Duality: The second approach is a bit more abstract, but much closer in spirit to the proper rigorous formulation of the theory of distributions like the delta function. The critical property is that if $u(x)$ is any continuous function, then

$$\int_a^b \delta_\xi(x)\, u(x)\, dx = u(\xi), \qquad \text{for} \qquad a < \xi < b. \tag{6.16}$$

Indeed, since $\delta_\xi(x) = 0$ for $x \neq \xi$, the integrand depends only on the value of u at the point $x = \xi$, and so

$$\int_a^b \delta_\xi(x)\, u(x)\, dx = \int_a^b \delta_\xi(x)\, u(\xi)\, dx = u(\xi) \int_a^b \delta_\xi(x)\, dx = u(\xi).$$

Equation (6.16) serves to define a linear functional[†] $L_\xi \colon \mathrm{C}^0[a,b] \to \mathbb{R}$ that maps a continuous function $u \in \mathrm{C}^0[a,b]$ to its value at the point $x = \xi$:

$$L_\xi[u] = u(\xi). \tag{6.17}$$

The basic linearity requirements (1.11) are immediately established:

$$L_\xi[u+v] = u(\xi) + v(\xi) = L_\xi[u] + L_\xi[v], \qquad L_\xi[cu] = c\,u(\xi) = c\, L_\xi[u],$$

for any functions $u(x), v(x)$. In the dual approach to generalized functions, the delta function is, in fact, *defined* as this particular linear functional (6.17). The function $u(x)$ is sometimes referred to as a *test function*, since it serves to "test" the form of the linear functional L_ξ.

Remark: If the impulse point ξ lies outside the integration domain, then

$$\int_a^b \delta_\xi(x)\, u(x)\, dx = 0 \qquad \text{whenever} \qquad \xi < a \qquad \text{or} \qquad \xi > b, \tag{6.18}$$

because the integrand is identically zero on the entire interval. For technical reasons, we will not attempt to define the integral (6.18) if the impulse point $\xi = a$ or $\xi = b$ lies on the boundary of the interval of integration.

The interpretation of the linear functional L_ξ as representing a kind of function $\delta_\xi(x)$ is based on the following line of thought. According to Corollary B.34, every scalar-valued linear function $L \colon \mathbb{R}^n \to \mathbb{R}$ on the finite-dimensional vector space $\mathbb{R}^n$ is obtained by taking the dot product with a fixed element $\mathbf{a} \in \mathbb{R}^n$, so

$$L[\mathbf{u}] = \mathbf{a} \cdot \mathbf{u}.$$

In this sense, linear functions on $\mathbb{R}^n$ are the "same" as vectors. Similarly, on the infinite-dimensional function space $\mathrm{C}^0[a,b]$, the L^2 inner product

$$L_g[u] = \langle\, g\,, u \,\rangle = \int_a^b g(x)\, u(x)\, dx, \tag{6.19}$$

taken with a fixed continuous function $g \in \mathrm{C}^0[a,b]$, defines a real-valued linear functional $L_g \colon \mathrm{C}^0[a,b] \to \mathbb{R}$. However, unlike the finite-dimensional situation, *not* every real-valued linear functional is of this form! In particular, there is no bona fide function $\delta_\xi(x)$ such that the identity

$$L_\xi[u] = \langle\, \delta_\xi\,, u \,\rangle = \int_a^b \delta_\xi(x)\, u(x)\, dx = u(\xi) \tag{6.20}$$

holds for every continuous function $u(x)$. The bottom line is that every (continuous) function defines a linear functional, but not every linear functional arises in this manner.

[†] The term "functional" is used to refer to a linear function whose domain is a function space, thus avoiding confusion with the functions it acts on.

But the dual interpretation of generalized functions acts as if this were true. *Generalized functions are, in actuality, real-valued linear functionals on function space, but intuitively interpreted as a kind of function via the* L^2 *inner product*. Although this identification is not to be taken too literally, one can, with some care, manipulate generalized functions as if they were actual functions, but always keeping in mind that a rigorous justification of such computations must ultimately rely on their innate characterization as linear functionals.

The two approaches — limits and duality — are completely compatible. Indeed, one can recover the dual formula (6.20) as the limit

$$u(\xi) = \lim_{n\to\infty} \langle\, g_n\,, u\,\rangle = \lim_{n\to\infty} \int_a^b g_n(x)\, u(x)\, dx = \int_a^b \delta_\xi(x)\, u(x)\, dx = \langle\, \delta_\xi\,, u\,\rangle \tag{6.21}$$

of the inner products of the function u with the approximating concentrated impulse functions $g_n(x)$ satisfying (6.8–9). In this manner, the limiting linear functional represents the delta function:

$$u(\xi) = L_\xi[\,u\,] = \lim_{n\to\infty} L_n[\,u\,], \qquad \text{where} \qquad L_n[\,u\,] = \int_0^\ell g_n(x)\, u(x)\, dx.$$

The choice of interpretation of the generalized delta function is, at least on an operational level, a matter of taste. For the beginner, the limit version is perhaps easier to digest initially. However, the dual, linear functional interpretation has stronger connections with the rigorous theory and, even in applications, offers some significant advantages.

Although the delta function might strike you as somewhat bizarre, its utility throughout modern applied mathematics and mathematical physics more than justifies including it in your analytical toolbox. While probably not yet comfortable with either definition, you are advised to press on and familiarize yourself with its basic properties. With a little care, you usually won't go far wrong by treating it as if it were a genuine function. After you gain more practical experience, you can, if desired, return to contemplate just exactly what kind of creature the delta function really is.

Calculus of Generalized Functions

In order to make use of the delta function, we need to understand how it behaves under the basic operations of linear algebra and calculus. First, we can take linear combinations of delta functions. For example,

$$h(x) = 2\,\delta(x) - 3\,\delta(x-1) = 2\,\delta_0(x) - 3\,\delta_1(x)$$

represents a combination of an impulse of magnitude 2 concentrated at $x = 0$ and one of magnitude -3 concentrated at $x = 1$. In the dual interpretation, h defines the linear functional

$$L_h[\,u\,] = \langle\, h\,, u\,\rangle = \langle\, 2\,\delta_0 - 3\,\delta_1\,, u\,\rangle = 2\,\langle\, \delta_0\,, u\,\rangle - 3\,\langle\, \delta_1\,, u\,\rangle = 2\,u(0) - 3\,u(1),$$

or, more explicitly, provided $a < 0$ and $b > 1$,

$$\begin{aligned} L_h[\,u\,] = \int_a^b h(x)\, u(x)\, dx &= \int_a^b \big[\, 2\,\delta(x) - 3\,\delta(x-1)\,\big]\, u(x)\, dx \\ &= 2\int_a^b \delta(x)\, u(x)\, dx - 3\int_a^b \delta(x-1)\, u(x)\, dx = 2\,u(0) - 3\,u(1). \end{aligned}$$

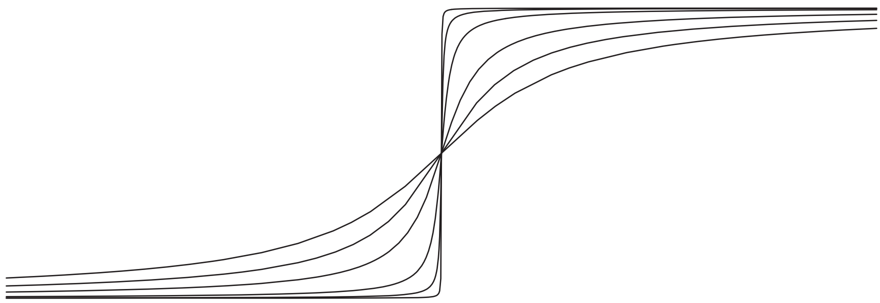


Figure 6.2. Step function as limit.

Next, since $\delta_\xi(x) = 0$ for any $x \neq \xi$, multiplying the delta function by an ordinary function is the same as multiplying by a constant:

$$g(x)\,\delta_\xi(x) = g(\xi)\,\delta_\xi(x), \tag{6.22}$$

provided $g(x)$ is continuous at $x = \xi$. For example, $x\,\delta(x) \equiv 0$ is the same as the constant zero function.

Warning: Since they are inherently *linear* functionals, it is *not* permissible to multiply delta functions together, or to apply more complicated *nonlinear* operations to them. Expressions like $\delta(x)^2$, $1/\delta(x)$, $e^{\delta(x)}$, etc., are *not* well defined in the theory of generalized functions — although this makes their application to nonlinear differential equations problematic.

The integral of the delta function is the *unit step function*:

$$\int_a^x \delta_\xi(t)\,dt = \sigma_\xi(x) = \sigma(x-\xi) = \begin{cases} 0, & x < \xi, \\ 1, & x > \xi, \end{cases} \qquad \text{provided} \qquad a < \xi. \tag{6.23}$$

Unlike the delta function, the step function $\sigma_\xi(x)$ is an ordinary function. It is continuous — indeed constant — except at $x = \xi$. The value of the step function at the discontinuity $x = \xi$ is left unspecified, although a wise choice — compatible with Fourier theory — is to set $\sigma_\xi(y) = \frac{1}{2}$, the average of its left- and right-hand limits.

We note that the integration formula (6.23) is compatible with our characterization of the delta function as the limit of highly concentrated forces. Integrating the approximating functions (6.10), we obtain

$$f_n(x) = \int_{-\infty}^x g_n(t)\,dt = \frac{1}{\pi}\,\tan^{-1} n\,x + \frac{1}{2}\,.$$

Since

$$\lim_{y\to\infty} \tan^{-1} y = \tfrac{1}{2}\pi, \qquad \text{while} \qquad \lim_{y\to-\infty} \tan^{-1} y = -\,\tfrac{1}{2}\pi,$$

these functions converge (nonuniformly) to the step function:

$$\lim_{n\to\infty} f_n(x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & x = 0, \\ 1, & x > 0. \end{cases} \tag{6.24}$$

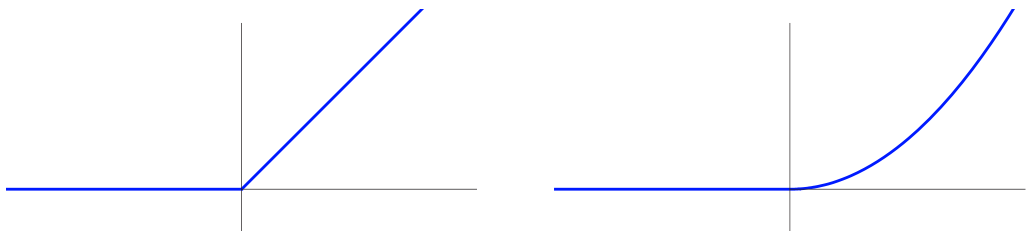


Figure 6.3. First and second-order ramp functions.

A graphical illustration of this limiting process appears in Figure 6.2.

The integral of the discontinuous step function (6.23) is the continuous *ramp function*

$$\int_a^x \sigma_\xi(t)\,dt = \rho_\xi(x) = \rho(x-\xi) = \begin{cases} 0, & x<\xi, \\ x-\xi, & x>\xi, \end{cases} \qquad \text{provided} \qquad a<\xi, \tag{6.25}$$

which is graphed in Figure 6.3. Note that $\rho_\xi(x)$ has a corner at $x=\xi$, and so is not differentiable there; indeed, its derivative $\rho'(x-\xi)=\sigma(x-\xi)$ has a jump discontinuity. We can continue to integrate; the $(n+1)^{\text{st}}$ integral of the delta function is the n^{th} *order ramp function*

$$\rho_{n,\xi}(x) = \rho_n(x-\xi) = \begin{cases} 0, & x<\xi, \\ \dfrac{(x-\xi)^n}{n!}, & x>\xi. \end{cases} \tag{6.26}$$

Note that $\rho_{n,\xi} \in \mathrm{C}^{n-1}$ has only $n-1$ continuous derivatives.

What about differentiation? Motivated by the Fundamental Theorem of Calculus, we shall use formula (6.23) to identify the derivative of the step function with the delta function

$$\frac{d\sigma}{dx} = \delta\,. \tag{6.27}$$

This fact is highly significant. In elementary calculus, one is not allowed to differentiate a discontinuous function. Here, we discover that the derivative can be defined, not as an ordinary function, but rather as a generalized delta function!

In general, the derivative of a piecewise C^1 function with jump discontinuities is a generalized function that includes a delta function concentrated at each discontinuity, whose magnitude equals the jump magnitude. More explicitly, suppose that $f(x)$ is differentiable, in the usual calculus sense, everywhere except at a point ξ, where it has a jump discontinuity of magnitude β. Using the step function (3.47), we can re-express

$$f(x) = g(x) + \beta\,\sigma(x-\xi), \tag{6.28}$$

where $g(x)$ is continuous everywhere, with a removable discontinuity at $x=\xi$, and differentiable except possibly at the jump. Differentiating (6.28), we find that

$$f'(x) = g'(x) + \beta\,\delta(x-\xi) \tag{6.29}$$

has a delta spike of magnitude β at the discontinuity. Thus, the derivatives of f and g coincide everywhere except at the discontinuity.

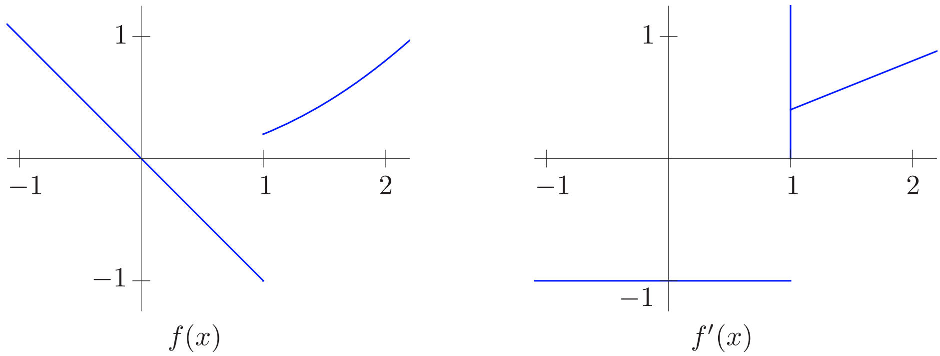


Figure 6.4. The derivative of the discontinuous function in Example 6.1.

Example 6.1. Consider the function

$$f(x) = \begin{cases} -x, & x < 1, \\ \frac{1}{5}x^2, & x > 1, \end{cases} \tag{6.30}$$

which we graph in Figure 6.4. We note that f has a single jump discontinuity at $x = 1$ of magnitude

$$f(1^+) - f(1^-) = \tfrac{1}{5} - (-1) = \tfrac{6}{5}.$$

This means that

$$f(x) = g(x) + \tfrac{6}{5}\,\sigma(x-1), \qquad \text{where} \qquad g(x) = \begin{cases} -x, & x < 1, \\ \frac{1}{5}x^2 - \frac{6}{5}, & x > 1, \end{cases}$$

is continuous everywhere, since its right- and left-hand limits at the original discontinuity are equal: $g(1^+) = g(1^-) = -1$. Therefore,

$$f'(x) = g'(x) + \tfrac{6}{5}\,\delta(x-1), \qquad \text{where} \qquad g'(x) = \begin{cases} -1, & x < 1, \\ \frac{2}{5}x, & x > 1, \end{cases}$$

while $g'(1)$ and $f'(1)$ are not defined. In Figure 6.4, the delta spike in the derivative of f is symbolized by a vertical line, although this pictorial device fails to indicate its magnitude of $\frac{6}{5}$.

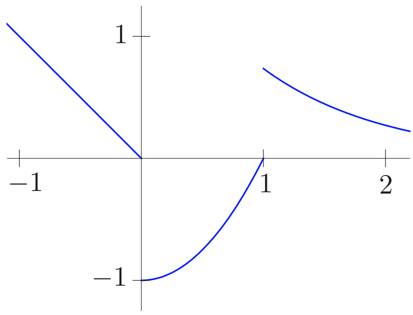
Note that in this particular example, $g'(x)$ can be found by directly differentiating the formula for $f(x)$. Indeed, in general, once we determine the magnitude and location of the jump discontinuities of $f(x)$, we can compute its derivative without introducing the auxiliary function $g(x)$.

Example 6.2. As a second, more streamlined, example, consider the function

$$f(x) = \begin{cases} -x, & x < 0, \\ x^2 - 1, & 0 < x < 1, \\ 2\,e^{-x}, & x > 1, \end{cases}$$

which is plotted in Figure 6.5. This function has jump discontinuities of magnitude -1 at $x = 0$, and of magnitude $2/e$ at $x = 1$. Therefore, in light of the preceding remark,

$$f'(x) = -\,\delta(x) + \frac{2}{e}\,\delta(x-1) + \begin{cases} -1, & x < 0, \\ 2x, & 0 < x < 1, \\ -2\,e^{-x}, & x > 1, \end{cases}$$



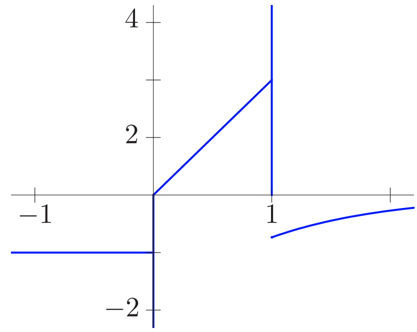


Figure 6.5. The derivative of the discontinuous function in Example 6.2.

where the final terms are obtained by directly differentiating $f(x)$.

Example 6.3. The derivative of the absolute value function

$$a(x) = |\,x\,| = \begin{cases} x, & x > 0, \\ -x, & x < 0, \end{cases}$$

is the *sign function*

$$a'(x) = \operatorname{sign} x = \begin{cases} +1, & x > 0, \\ -1, & x < 0. \end{cases} \tag{6.31}$$

Note that there is no delta function in $a'(x)$ because $a(x)$ is continuous everywhere. Since $\operatorname{sign} x$ has a jump of magnitude 2 at the origin and is otherwise constant, its derivative is twice the delta function:

$$a''(x) = \frac{d}{dx} \operatorname{sign} x = 2\,\delta(x).$$

Example 6.4. We are even allowed to differentiate the delta function. Its first derivative $\delta'(x)$ can be interpreted in two ways. First, as the limit of the derivatives of the approximating functions (6.10):

$$\frac{d\delta}{dx} = \lim_{n\to\infty} \frac{dg_n}{dx} = \lim_{n\to\infty} \frac{-\,2\,n^3\,x}{\pi(1+n^2\,x^2)^2}\,. \tag{6.32}$$

The graphs of these rational functions take the form of more and more concentrated spiked "doublets", as illustrated in Figure 6.6. To determine the effect of the derivative on a test function $u(x)$, we compute the limiting integral

$$\begin{aligned} \langle\, \delta'\,, u\,\rangle &= \int_{-\infty}^{\infty} \delta'(x)\,u(x)\,dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n'(x)\,u(x)\,dx \\ &= -\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)\,u'(x)\,dx = -\int_{-\infty}^{\infty} \delta(x)\,u'(x)\,dx = -\,u'(0). \end{aligned} \tag{6.33}$$

The middle step is the result of an integration by parts, noting that the boundary terms at $\pm\infty$ vanish, provided that $u(x)$ is continuously differentiable and bounded as $|\,x\,| \to \infty$. Pay attention to the minus sign in the final answer.

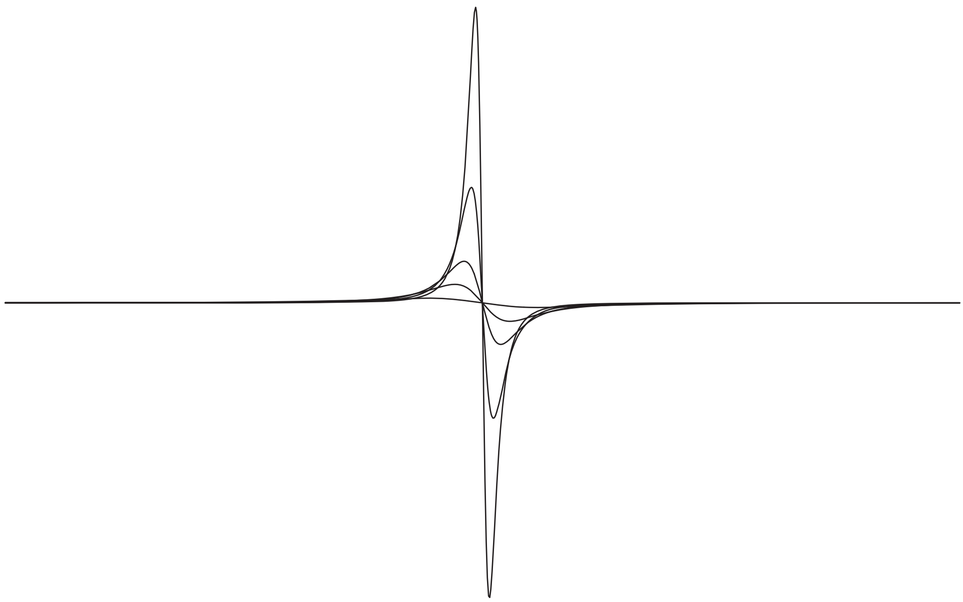


Figure 6.6. Derivative of delta function as limit of doublets.

In the dual interpretation, the generalized function $\delta'(x)$ corresponds to the linear functional

$$L'[u] = -\,u'(0) = \langle\, \delta'\,, u \,\rangle = \int_a^b \delta'(x)\, u(x)\, dx, \qquad \text{where} \qquad a < 0 < b, \tag{6.34}$$

which maps a continuously differentiable function $u(x)$ to *minus* its derivative at the origin. We note that (6.34) is compatible with a formal integration by parts:

$$\int_a^b \delta'(x)\, u(x)\, dx = \delta(x)\, u(x) \,\Big|_{x=a}^{b} - \int_a^b \delta(x)\, u'(x)\, dx = -\,u'(0).$$

The boundary terms at $x = a$ and $x = b$ automatically vanish, since $\delta(x) = 0$ for $x \neq 0$.

Remark: While we can test the delta function with any continuous function, we are permitted to test its derivative only on continuously differentiable functions. To avoid keeping track of such technicalities, one often restricts to only infinitely differentiable test functions.

Warning: The functions $\widetilde{g}_n(x) = g_n(x) + g_n'(x)$, cf. (6.10, 32), satisfy $\lim_{n\to\infty} \widetilde{g}_n(x) = 0$ for all $x \neq 0$, while $\int_{-\infty}^{\infty} \widetilde{g}_n(x)\, dx = 1$. However, $\lim_{n\to\infty} \widetilde{g}_n = \lim_{n\to\infty} g_n + \lim_{n\to\infty} g_n' = \delta + \delta'$. Thus, our original conditions (6.8–9) are *not* in fact sufficient to characterize whether a sequence of functions has the delta function as a limit. To be absolutely sure, one must, in fact, verify the more comprehensive limiting formula (6.21).

Exercises

6.1.1. Evaluate the following integrals: (a) $\int_{-\pi}^{\pi} \delta(x) \cos x \, dx$, (b) $\int_{1}^{2} \delta(x)\,(x-2)\, dx$, (c) $\int_{0}^{3} \delta_1(x)\, e^x \, dx$, (d) $\int_{1}^{e} \delta(x-2) \log x \, dx$, (e) $\int_{0}^{1} \delta\bigl(x-\tfrac{1}{3}\bigr)\, x^2 \, dx$, (f) $\int_{-1}^{1} \dfrac{\delta(x+2)\, dx}{1+x^2}$.

6.1.2. Simplify the following generalized functions; then write out how they act on a suitable test function $u(x)$: (a) $e^x\,\delta(x)$, (b) $x\,\delta(x-1)$, (c) $3\,\delta_1(x) - 3x\,\delta_{-1}(x)$, (d) $\dfrac{\delta(x-1)}{x+1}$, (e) $(\cos x)\bigl[\,\delta(x) + \delta(x-\pi) + \delta(x+\pi)\bigr]$, (f) $\dfrac{\delta_1(x) - \delta_2(x)}{x^2+1}$.

6.1.3. Define the generalized function $\varphi(x) = \delta(x+1) - \delta(x-1)$: (a) as a limit of ordinary functions; (b) using duality.

6.1.4. Find and sketch a graph of the derivative (in the context of generalized functions) of the following functions:

(a) $f(x) = \begin{cases} x^2, & 0 < x < 3, \\ x, & -1 < x < 0, \\ 0, & \text{otherwise}, \end{cases}$ (b) $g(x) = \begin{cases} \sin|\,x\,|, & |\,x\,| < \tfrac{1}{2}\pi, \\ 0, & \text{otherwise}, \end{cases}$

(c) $h(x) = \begin{cases} \sin \pi x, & x > 1, \\ 1 - x^2, & -1 < x < 1, \\ e^x, & x < -1, \end{cases}$ (d) $k(x) = \begin{cases} \sin x, & x < -\pi, \\ x^2 - \pi^2, & -\pi < x < 0, \\ e^{-x}, & x > 0. \end{cases}$

6.1.5. Find the first and second derivatives of the functions (a) $f(x) = \begin{cases} x+1, & -1 < x < 0, \\ 1 - x, & 0 < x < 1, \\ 0, & \text{otherwise}, \end{cases}$

(b) $k(x) = \begin{cases} |\,x\,|, & -2 < x < 2, \\ 0, & \text{otherwise}, \end{cases}$ (c) $s(x) = \begin{cases} 1 + \cos \pi x, & -1 < x < 1, \\ 0, & \text{otherwise}. \end{cases}$

6.1.6. Find the first and second derivatives of $f(x) =$ (a) $e^{-|\,x\,|}$, (b) $2|\,x\,| - |\,x-1\,|$, (c) $|\,x^2 + x\,|$, (d) $x \,\text{sign}(x^2 - 4)$, (e) $\sin|\,x\,|$, (f) $|\sin x\,|$, (g) $\text{sign}(\sin x)$.

♢ 6.1.7. Explain why the Gaussian functions $g_n(x) = \dfrac{n}{\sqrt{\pi}}\, e^{-n^2 x^2}$ have the delta function $\delta(x)$ as their limit as $n \to \infty$.

♢ 6.1.8. In this exercise, we realize the delta function $\delta_\xi(x)$ as a limit of functions on a finite interval $[a,b]$. Let $a < \xi < b$.

(a) Prove that the functions $\widetilde{g}_n(x) = \dfrac{g_n(x-\xi)}{M_n}$, where $g_n(x)$ is given by (6.10) and $M_n = \int_a^b g_n(x-\xi)\, dx$, satisfy (6.8–9), and hence $\lim_{n\to\infty} \widetilde{g}_n(x) = \delta_\xi(x)$.

(b) One can, alternatively, relax the second condition (6.9) to $\lim_{n\to\infty} \int_a^b g_n(x-\xi)\, dx = 1$. Show that, under this relaxed definition, $\lim_{n\to\infty} g_n(x-\xi) = \delta_\xi(x)$.

♡ 6.1.9. For each positive integer n, let $g_n(x) = \begin{cases} \tfrac{1}{2} n, & |\,x\,| < 1/n, \\ 0, & \text{otherwise}. \end{cases}$ (a) Sketch a graph of $g_n(x)$. (b) Show that $\lim_{n\to\infty} g_n(x) = \delta(x)$. (c) Evaluate $f_n(x) = \int_{-\infty}^{x} g_n(y)\, dy$ and sketch a graph. Does the sequence $f_n(x)$ converge to the step function $\sigma(x)$ as $n \to \infty$? (d) Find the derivative $h_n(x) = g_n'(x)$. (e) Does the sequence $h_n(x)$ converge to $\delta'(x)$ as $n \to \infty$?

♡ 6.1.10. Answer Exercise 6.1.9 for the *hat functions* $g_n(x) = \begin{cases} n - n^2\,|\,x\,|, & |\,x\,| < 1/n, \\ 0, & \text{otherwise}. \end{cases}$

6.1.11. Justify the formula $x\,\delta(x) = 0$ using (*a*) limits, (*b*) duality.

◇ 6.1.12. (*a*) Justify the formula $\delta(2x) = \frac{1}{2}\,\delta(x)$ by (*i*) limits, (*ii*) duality. (*b*) Find a similar formula for $\delta(a\,x)$ when $a > 0$. (*c*) What about when $a < 0$?

6.1.13. (*a*) Prove that $\sigma(\lambda x) = \sigma(x)$ for any $\lambda > 0$. (*b*) What about if $\lambda < 0$? (*c*) Use parts (a,b) to deduce that $\delta(\lambda x) = \dfrac{1}{|\,\lambda\,|}\,\delta(x)$ for any $\lambda \neq 0$.

6.1.14. Let $g(x)$ be a continuously differentiable function with $g'(x) \neq 0$ for all $x \in \mathbb{R}$. Does the composition $\delta(g(x))$ make sense as a distribution? If so, can you identify it?

6.1.15. Let $\xi < a$. Sketch the graphs of (*a*) $s(x) = \displaystyle\int_a^x \delta_\xi(z)\,dz$, (*b*) $r(x) = \displaystyle\int_a^x \sigma_\xi(z)\,dz$.

6.1.16. Justify the formula $\displaystyle\lim_{n\to\infty} n\left[\,\delta\big(x - \tfrac{1}{n}\big) - \delta\big(x + \tfrac{1}{n}\big)\,\right] = -\,2\,\delta'(x)$.

6.1.17. Define the generalized function $\delta''(x)$:
(*a*) as a limit of ordinary functions; (*b*) using duality.

6.1.18. Let $\delta_\xi^{(k)}(x)$ denote the k^{th} derivative of the delta function $\delta_\xi(x)$. Justify the formula $\langle\,\delta_\xi^{(k)}\,, u\,\rangle = (-1)^k\,u^{(k)}(\xi)$ whenever $u \in \mathrm{C}^k$ is k–times continuously differentiable.

6.1.19. According to (6.22), $x\,\delta(x) = 0$. On the other hand, by Leibniz' rule, $(x\,\delta(x))' = \delta(x) + x\,\delta'(x)$ is apparently not zero. Can you explain this paradox?

6.1.20. If $f \in \mathrm{C}^1$, should $(f\,\delta)' = f\,\delta'$ or $f'\,\delta + f\,\delta'$?

◇ 6.1.21. (*a*) Use duality to justify the formula $f(x)\,\delta'(x) = f(0)\,\delta'(x) - f'(0)\,\delta(x)$ when $f \in \mathrm{C}^1$. (*b*) Find a similar formula for $f(x)\,\delta^{(n)}(x)$ as the product of a sufficiently smooth function and the n^{th} derivative of the delta function.

6.1.22. Use Exercise 6.1.21 to simplify the following generalized functions; then write out how they act on a suitable test function $u(x)$:
(*a*) $\varphi(x) = (x-2)\,\delta'(x)$, (*b*) $\psi(x) = (1+\sin x)\big[\,\delta(x) + \delta'(x)\,\big]$,
(*c*) $\chi(x) = x^2\big[\,\delta(x-1) - \delta'(x-2)\,\big]$, (*d*) $\omega(x) = e^x\,\delta''(x+1)$.

◇ 6.1.23. Prove that if $f(x)$ is a continuous function, and $\displaystyle\int_a^b f(x)\,dx = 0$ for every interval $[a,b]$, then $f(x) \equiv 0$ everywhere.

◇ 6.1.24. Write out a rigorous proof that there is no continuous function $\delta_\xi(x)$ such that the inner product identity (6.20) holds for every continuous function $u(x)$.

◇ 6.1.25. *True or false*: The sequence (6.24) converges uniformly.

6.1.26. *True or false*: $\|\,\delta\,\| = 1$.

The Fourier Series of the Delta Function

Let us next investigate the capability of Fourier series to represent generalized functions. We begin with the delta function $\delta(x)$, based at the origin. Using the characterizing properties (6.16), its real Fourier coefficients are

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\cos k\,x\,dx = \frac{1}{\pi}\cos k\,0 = \frac{1}{\pi}\,, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\sin k\,x\,dx = \frac{1}{\pi}\sin k\,0 = 0. \tag{6.35}$$

Therefore, at least on a formal level, its Fourier series is

$$\delta(x) \sim \frac{1}{2\pi} + \frac{1}{\pi}\left(\cos x + \cos 2x + \cos 3x + \cdots\right). \tag{6.36}$$

Since $\delta(x) = \delta(-x)$ is an even function (why?), it should come as no surprise that it has a cosine series. Alternatively, we can rewrite the series in complex form

$$\delta(x) \sim \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{\,i\,k\,x} = \frac{1}{2\pi}\left(\cdots + e^{-2\,i\,x} + e^{-\,i\,x} + 1 + e^{\,i\,x} + e^{2\,i\,x} + \cdots\right), \tag{6.37}$$

where the complex Fourier coefficients are computed[†] as

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(x)\,e^{-\,i\,k\,x}\,dx = \frac{1}{2\pi}\,.$$

Remark: Although we stated that the Fourier series (6.36) represents the delta function, this is not entirely correct. Remember that a Fourier series converges to the 2π–periodic extension of the original function. Therefore, (6.37) actually represents the periodic extension of the delta function, sometimes called the *Dirac comb*,

$$\widetilde{\delta}(x) = \cdots + \delta(x+4\pi) + \delta(x+2\pi) + \delta(x) + \delta(x-2\pi) + \delta(x-4\pi) + \delta(x-6\pi) + \cdots\,, \tag{6.38}$$

consisting of a periodic array of unit impulses concentrated at all integer multiples of 2π.

Let us investigate in what sense (if any) the Fourier series (6.36) or, equivalently, (6.37), represents the delta function. The first observation is that, because its summands do not tend to zero, the series certainly doesn't converge in the usual, calculus, sense. Nevertheless, in a "weak" sense, the series can be regarded as converging to the (periodic extension of the) delta function.

To understand the convergence mechanism, we recall that we already established a formula (3.129) for the partial sums:

$$s_n(x) = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{\,i\,k\,x} = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{n}\cos k\,x = \frac{1}{2\pi}\,\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{1}{2}x}\,. \tag{6.39}$$

Graphs of some of the partial sums on the interval $[-\pi,\pi]$ are displayed in Figure 6.7. Note that, as n increases, the spike at $x = 0$ becomes progressively taller and thinner, converging to an infinitely tall delta spike. (We had to truncate the last two graphs; the spike extends beyond the top.) Indeed, by l'Hôpital's Rule,

$$\lim_{x\to 0}\frac{1}{2\pi}\,\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{1}{2}x} = \lim_{x\to 0}\frac{1}{2\pi}\,\frac{\left(n+\frac{1}{2}\right)\cos\left(n+\frac{1}{2}\right)x}{\frac{1}{2}\cos\frac{1}{2}x} = \frac{n+\frac{1}{2}}{\pi} \longrightarrow \infty \quad \text{as} \quad n\to\infty.$$

(An elementary proof of this formula is to note that, at $x = 0$, every term in the original sum (6.36) is equal to 1.) Furthermore, the integrals remain fixed,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} s_n(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{1}{2}x}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{n} e^{\,i\,k\,x}\,dx = 1, \tag{6.40}$$

[†] Or we could use (3.66).

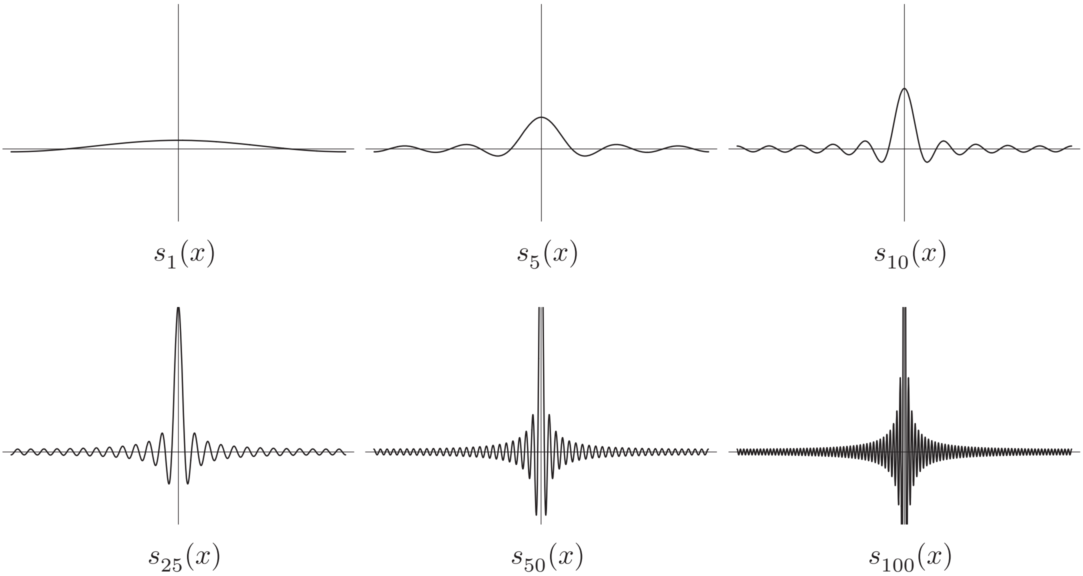


Figure 6.7. Partial Fourier sums approximating the delta function.

as required for convergence to the delta function. However, away from the spike, the partial sums do *not* go to zero! Rather, they oscillate ever more rapidly, while maintaining a fixed overall amplitude of

$$\frac{1}{2\pi}\csc\tfrac{1}{2}x = \frac{1}{2\pi\sin\tfrac{1}{2}x}. \tag{6.41}$$

As n increases, the amplitude function (6.41) can be seen, as in Figure 6.7, as the envelope of the increasingly rapid oscillations. So, roughly speaking, the convergence $s_n(x) \to \delta(x)$ means that the "infinitely fast" oscillations are somehow canceling each other out, and the net effect is zero away from the spike at $x = 0$. So the convergence of the Fourier sums to $\delta(x)$ is much more subtle than in the original limiting definition (6.10).

The technical term is *weak convergence*, which plays a very important role in advanced mathematical analysis, signal processing, composite materials, and elsewhere.

Definition 6.5. A sequence of functions $f_n(x)$ is said to *converge weakly* to $f_\star(x)$ on an interval $[a,b]$ if their L^2 inner products with every continuous *test function* $u(x) \in C^0[a,b]$ converge:

$$\int_a^b f_n(x)\,u(x)\,dx \longrightarrow \int_a^b f_\star(x)\,u(x)\,dx \qquad \text{as} \qquad n \longrightarrow \infty. \tag{6.42}$$

Weak convergence is often indicated by a half-pointed arrow: $f_n \rightharpoonup g$.

Remark: On unbounded intervals, one usually restricts the test functions to have *compact support*, meaning that $u(x) = 0$ for all sufficiently large $|\,x\,| \gg 0$. One can also restrict to smooth test functions only, e.g., require that $u \in C^\infty[a,b]$.

Example 6.6. Let us show that the trigonometric functions $f_n(x) = \cos n x$ converge weakly to the zero function:

$$\cos n x \rightharpoonup 0 \qquad \text{as} \qquad n \longrightarrow \infty \qquad \text{on the interval} \quad [-\pi, \pi].$$

(Actually, this holds on any interval; see Exercise 6.1.38.) According to the definition, we need to prove that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} u(x) \cos n x \, dx = 0$$

for any continuous function $u \in \mathrm{C}^0[-\pi,\pi]$. But this is just a restatement of the Riemann–Lebesgue Lemma 3.40, which says that the high-frequency Fourier coefficients of a continuous (indeed, even square-integrable) function $u(x)$ go to zero. The same remark establishes the weak convergence $\sin n x \rightharpoonup 0$.

Observe that the functions $\cos n x$ fail to converge pointwise to 0 at *any* value of x. Indeed, if x is an integer multiple of 2π, then $\cos n x = 1$ for all n. If x is any other rational multiple of π, the values of $\cos n x$ periodically cycle through a finite number of different values, and never go to 0, while if x is an irrational multiple of π, they oscillate aperiodically between -1 and $+1$. The functions also fail to converge in norm to 0, since their (unscaled) L^2 norms remain fixed at

$$\| \cos n x \| = \sqrt{\int_{-\pi}^{\pi} \cos^2 n x \, dx} = \sqrt{\pi} \qquad \text{for all} \qquad n > 0.$$

The cancellation of oscillations in the high-frequency limit is a characteristic feature of weak convergence.

Let us now explain why, although the Fourier series (6.36) does not converge to the delta function either pointwise or in norm (indeed, $\| \delta \|$ is not even defined!), it does converge weakly on $[-\pi,\pi]$. More specifically, we need to prove that the partial sums $s_n \rightharpoonup \delta$, meaning that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} s_n(x)\, u(x)\, dx = \int_{-\pi}^{\pi} \delta(x)\, u(x)\, dx = u(0) \tag{6.43}$$

for every sufficiently nice function u, or, equivalently,

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) \, \frac{\sin\left(n+\frac{1}{2}\right) x}{\sin \frac{1}{2} x} \, dx = u(0). \tag{6.44}$$

But this is a restatement of a special case of the identities (3.130) used in the proof of the Pointwise Convergence Theorem 3.8 for the Fourier series of a (piecewise) C^1 function. Indeed, summing the two identities in (3.130) and then setting $x = 0$ reproduces (6.44), since, by continuity, $u(0) = \frac{1}{2}\left[\, u(0^+) + u(0^-) \,\right]$. In other words, the pointwise convergence of the Fourier series of a C^1 function is equivalent to the weak convergence† of the Fourier series of the delta function!

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† Definition 6.5 only requires continuity of the test functions, whereas in (6.44) they need to be C^1, so the notion of weak convergence here is slightly slightly more refined. One often restricts further to allow only C^∞ test functions.

Example 6.7. If we differentiate the Fourier series

$$x \sim 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sin kx = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right),$$

we obtain an apparent contradiction:

$$1 \sim 2 \sum_{k=1}^{\infty} (-1)^{k+1} \cos kx = 2\cos x - 2\cos 2x + 2\cos 3x - 2\cos 4x + \cdots . \tag{6.45}$$

But the Fourier series for 1 consists of just a single constant term! (Why?)

The resolution of this paradox is not difficult. The Fourier series (3.37) does *not* converge to x, but rather to its 2π–periodic extension $\widetilde{f}(x)$, which has jump discontinuities of magnitude 2π at odd multiples of π; see Figure 3.1. Thus, Theorem 3.22 is *not* directly applicable. Nevertheless, we can assign a consistent interpretation to the differentiated series. The derivative $\widetilde{f}'(x)$ of the periodic extension is *not* equal to the constant function 1, but rather has an additional delta function concentrated at each jump discontinuity:

$$\widetilde{f}'(x) = 1 - 2\pi \sum_{j=-\infty}^{\infty} \delta\big(\, x - (2j+1)\pi \,\big) = 1 - 2\pi\, \widetilde{\delta}(x-\pi),$$

where $\widetilde{\delta}$ denotes the 2π–periodic extension of the delta function, cf. (6.38). The differentiated Fourier series (6.45) does, in fact, represent $\widetilde{f}'(x)$. Indeed, the Fourier coefficients of $\widetilde{\delta}(x-\pi)$ are

$$a_k = \frac{1}{\pi}\int_0^{2\pi} \delta(x-\pi)\cos kx\,dx = \frac{1}{\pi}\cos k\pi = \frac{(-1)^k}{\pi},$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} \delta(x-\pi)\sin kx\,dx = \frac{1}{\pi}\sin k\pi = 0.$$

Observe that we changed the interval of integration to $[0, 2\pi]$ to avoid placing the delta function singularities at the endpoints. Thus,

$$\delta(x-\pi) \sim \frac{1}{2\pi} + \frac{1}{\pi}\big(-\cos x + \cos 2x - \cos 3x + \cdots\big), \tag{6.46}$$

which serves to resolve the contradiction.

Example 6.8. Let us differentiate the Fourier series

$$\sigma(x) \sim \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right)$$

for the unit step function we found in Example 3.9 and see whether we end up with the Fourier series (6.36) for the delta function. We compute

$$\frac{d\sigma}{dx} \sim \frac{2}{\pi}\big(\cos x + \cos 3x + \cos 5x + \cos 7x + \cdots\big), \tag{6.47}$$

which does *not* agree with (6.36) — half the terms are missing! The explanation is similar to the preceding example: the 2π–periodic extension $\widetilde{\sigma}(x)$ of the step function has two jump discontinuities, of magnitudes +1 at even multiples of π and -1 at odd multiples; see Figure 3.6. Therefore, its derivative

$$\frac{d\widetilde{\sigma}}{dx} = \widetilde{\delta}(x) - \widetilde{\delta}(x - \pi)$$

is the difference of the 2π–periodic extension of the delta function at 0, with Fourier series (6.36), minus the 2π–periodic extension of the delta function at π, with Fourier series (6.46), which produces (6.47).

It is a remarkable, profound fact that Fourier analysis is entirely compatible with the calculus of generalized functions, [**68**]. For instance, term-wise differentiation of the Fourier series for a piecewise C^1 function leads to the Fourier series for the differentiated function that incorporates delta functions of the appropriate magnitude at each jump discontinuity. This fact further reassures us that the rather mysterious construction of delta functions and their generalizations is indeed the right way to extend calculus to functions that do not possess derivatives in the ordinary sense.

Exercises

6.1.27. Determine the real and complex Fourier series for $\delta(x - \xi)$, where $-\pi < \xi < \pi$. What periodic generalized function(s) do they represent?

6.1.28. Determine the Fourier sine series and the Fourier cosine series for $\delta(x - \xi)$, where $0 < \xi < \pi$. Which periodic generalized functions do they represent?

♡ 6.1.29. Let $n > 0$ be a positive integer. (*a*) For integers $0 \leq j < n$, find the complex Fourier series of the 2π–periodically extended delta functions $\widetilde{\delta}_j(x) = \widetilde{\delta}(x - 2j\pi/n)$. (*b*) Prove that their Fourier coefficients satisfy the periodicity condition $c_k = c_l$ whenever $k \equiv l \mod n$. (*c*) Conversely, given complex Fourier coefficients that satisfy the periodicity condition $c_k = c_l$ whenever $k \equiv l \mod n$, prove that the corresponding Fourier series represents a linear combination of the preceding periodically extended delta functions $\widetilde{\delta}_0(x), \ldots, \widetilde{\delta}_{n-1}(x)$. *Hint*: Use Example B.22. (*d*) Prove that a complex Fourier series represents a 2π–periodic function that is constant on the subintervals $2\pi j/n < x < 2\pi(j+1)/n$, for $j \in \mathbb{Z}$, if and only if its Fourier coefficients satisfy the conditions

$$k\,c_k = l\,c_l, \qquad k \equiv l \not\equiv 0 \mod n, \qquad\qquad c_k = 0, \qquad 0 \neq k \equiv 0 \mod n.$$

♣ 6.1.30. (*a*) Find the complex Fourier series for the derivative of the delta function $\delta'(x)$ by direct evaluation of the coefficient formulas. (*b*) Verify that your series can be obtained by term-by-term differentiation of the series for $\delta(x)$. (*c*) Write a formula for the n^{th} partial sum of your series. (*d*) Use a computer graphics package to investigate the convergence of the series.

6.1.31. What is the Fourier series for the generalized function $g(x) = x\,\delta(x)$? Can you obtain this result through multiplication of the individual Fourier series (3.37), (6.37)?

6.1.32. Apply the method of Exercise 3.2.59 to find the complex Fourier series for the function $f(x) = \delta(x)\,e^{\,i\,x}$. Which Fourier series do you get? Can you explain what is going on?

6.1.33. In Exercise 6.1.12 we established the identity $\delta(x) = 2\,\delta(2x)$. Does this hold on the level of Fourier series? Can you explain why or why not?

6.1.34. How should one interpret the formula (6.38) for the periodic extension of the delta function (*a*) as a limit? (*b*) as a linear functional?

6.1.35. Write down the complex Fourier series for e^x. Differentiate term by term. Do you get the same series? Explain your answer.

6.1.36. *True or false*: If you integrate the Fourier series for the delta function $\delta(x)$ term by term, you obtain the Fourier series for the step function $\sigma(x)$.

6.1.37. Find the Fourier series for the function $\delta(x)$ on the interval $-1 \le x \le 1$. Which (generalized) function does the Fourier series represent?

♢ 6.1.38.. Prove that $\cos n x \rightharpoonup 0$ (weakly) as $n \to \infty$ on any bounded interval $[a, b]$.

♢ 6.1.39. Prove that if $u_n \to u$ in norm, then $u_n \rightharpoonup u$ weakly.

6.1.40. *True or false*: (a) If $u_n \to u$ uniformly on $[a, b]$, then $u_n \rightharpoonup u$ weakly.
(b) If $u_n(x) \to u(x)$ pointwise, then $u_n \rightharpoonup u$ weakly.

6.1.41. Prove that the sequence $f_n(x) = \cos^2 n x$ converges weakly on $[-\pi, \pi]$. What is the limiting function?

6.1.42. Answer Exercise 6.1.41 when $f_n(x) = \cos^3 n x$.

6.1.43. Discuss the weak convergence of the Fourier series for the derivative $\delta'(x)$ of the delta function.

6.2 Green's Functions for One–Dimensional Boundary Value Problems

We will now put the delta function to work by developing a general method for solving inhomogeneous linear boundary value problems. The key idea, motivated by the linear algebra technique outlined at the beginning of the previous section, is to first solve the system when subject to a unit delta function impulse, which produces the Green's function. We then apply linear superposition to write down the solution for a general forcing inhomogeneity. The Green's function approach has wide applicability, but will be developed here in the context of a few basic examples.

Example 6.9. The boundary value problem

$$-c u'' = f(x), \qquad u(0) = 0 = u(1), \tag{6.48}$$

models the longitudinal deformation $u(x)$ of a homogeneous elastic bar of unit length and constant stiffness c that is fixed at both ends while subject to an external force $f(x)$. The associated *Green's function* refers to the family of solutions

$$u(x) = G_\xi(x) = G(x; \xi)$$

induced by unit-impulse forces concentrated at a single point $0 < \xi < 1$:

$$-c u'' = \delta(x - \xi), \qquad u(0) = 0 = u(1). \tag{6.49}$$

The solution to the differential equation can be straightforwardly obtained by direct integration. First, by (6.23),

$$u'(x) = -\frac{\sigma(x-\xi)}{c} + a,$$

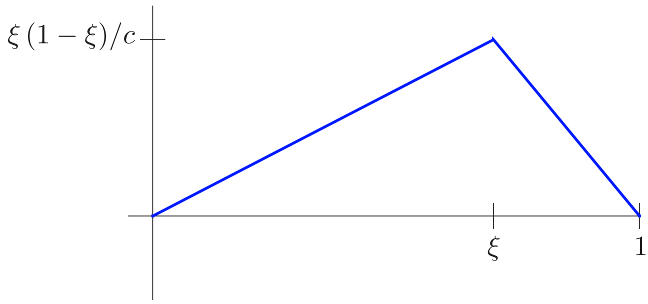

Figure 6.8. Green's function for a bar with fixed ends.

where a is a constant of integration. A second integration leads to

$$u(x) = -\frac{\rho(x-\xi)}{c} + a\,x + b, \qquad (6.50)$$

where ρ is the ramp function (6.25). The integration constants a, b are fixed by the boundary conditions; since $0 < \xi < 1$, we have

$$u(0) = b = 0, \qquad u(1) = -\frac{1-\xi}{c} + a + b = 0, \qquad \text{and so} \qquad a = \frac{1-\xi}{c}, \qquad b = 0.$$

We deduce that the Green's function for the problem is

$$G(x;\xi) = \frac{(1-\xi)\,x - \rho(x-\xi)}{c} = \begin{cases} (1-\xi)\,x/c, & x \le \xi, \\ \xi\,(1-x)/c, & x \ge \xi. \end{cases} \qquad (6.51)$$

As sketched in Figure 6.8, for each fixed ξ, the function $G_\xi(x) = G(x;\xi)$ depends continuously on x; its graph consists of two connected straight line segments, with a corner at the point of application of the unit impulse force.

Once we have determined the Green's function, we are able to solve the general inhomogeneous boundary value problem (6.48) by linear superposition. We first express the forcing function $f(x)$ as a linear combination of impulses concentrated at various points along the bar. Since there is a continuum of possible positions $0 < \xi < 1$ at which impulse forces may be applied, we will use an integral to sum them, thereby writing the external force as

$$f(x) = \int_0^1 \delta(x-\xi)\, f(\xi)\, d\xi. \qquad (6.52)$$

We can interpret (6.52) as the (continuous) superposition of an infinite collection of impulses, namely $f(\xi)\,\delta(x-\xi)$, of magnitude $f(\xi)$ and concentrated at position ξ.

The Superposition Principle states that linear combinations of inhomogeneities produce the selfsame linear combinations of solutions. Again, we adapt this principle to the continuum by replacing the sums by integrals. Thus, the solution to the boundary value problem will be the linear superposition

$$u(x) = \int_0^1 G(x;\xi)\, f(\xi)\, d\xi \qquad (6.53)$$

of the Green's function solutions to the individual unit-impulse problems.

For the particular boundary value problem (6.48), we use the formula (6.51) for the Green's function. Breaking the resulting integral (6.53) into two parts, over the subintervals $0 \le \xi \le x$ and $x \le \xi \le 1$, we arrive at the explicit solution formula

$$u(x) = \frac{1}{c} \int_0^x (1-x)\,\xi\, f(\xi)\, d\xi \;+\; \frac{1}{c} \int_x^1 x\,(1-\xi)\, f(\xi)\, d\xi. \tag{6.54}$$

For example, under a constant unit force f, (6.54) yields the solution

$$u(x) = \frac{f}{c} \int_0^x (1-x)\,\xi\, d\xi \;+\; \frac{f}{c} \int_x^1 x\,(1-\xi)\, d\xi = \frac{f}{2\,c}\,(1-x)\,x^2 + \frac{f}{2\,c}\,x\,(1-x)^2 = \frac{f}{2\,c}\,(x - x^2).$$

Let us, finally, convince ourselves that the superposition formula (6.54) indeed gives the correct answer. First,

$$\begin{aligned} c\,\frac{du}{dx} &= (1-x)\,x\,f(x) + \int_0^x \big(-\xi\, f(\xi)\big)\, d\xi - x\,(1-x)\,f(x) + \int_x^1 (1-\xi)\, f(\xi)\, d\xi \\ &= -\int_0^1 \xi\, f(\xi)\, d\xi + \int_x^1 f(\xi)\, d\xi. \end{aligned}$$

Differentiating again with respect to x, we see that the first term is constant, and so $-\,c\,\dfrac{d^2u}{dx^2} = f(x)$, as claimed.

Remark: In computing the derivatives of u, we made use of the calculus formula

$$\frac{d}{dx} \int_{\alpha(x)}^{\beta(x)} F(x,\xi)\, d\xi = F(x,\beta(x))\,\frac{d\beta}{dx} - F(x,\alpha(x))\,\frac{d\alpha}{dx} + \int_{\alpha(x)}^{\beta(x)} \frac{\partial F}{\partial x}(x,\xi)\, d\xi \tag{6.55}$$

for the derivative of an integral with variable limits — which is a straightforward consequence of the Fundamental Theorem of Calculus and the chain rule, [**8**, **108**]. As always, one must exercise due care when interchanging differentiation and integration.

We note the following basic properties, which serve to uniquely characterize the Green's function. First, since the delta forcing vanishes except at the point $x = \xi$, the Green's function satisfies the homogeneous differential equation†

$$-\,c\,\frac{\partial^2 G}{\partial x^2}(x;\xi) = 0 \qquad \text{for all} \qquad x \ne \xi. \tag{6.56}$$

Second, by construction, it must satisfy the boundary conditions

$$G(0;\xi) = 0 = G(1;\xi).$$

Third, for each fixed ξ, $G(x;\xi)$ is a continuous function of x, but its derivative $\partial G/\partial x$ has a jump discontinuity of magnitude $-1/c$ at the impulse point $x = \xi$. As a result, the second derivative $\partial^2 G/\partial x^2$ has a delta function discontinuity there, and hence solves the original impulse boundary value problem (6.49).

Finally, we cannot help but notice that the Green's function (6.51) is a symmetric function of its two arguments: $G(x;\xi) = G(\xi;x)$. Symmetry has the interesting physical consequence that the displacement of the bar at position x due to an impulse force

† Since $G(x;\xi)$ is a function of two variables, we switch to partial derivative notation to indicate its derivatives.

concentrated at position ξ is exactly the same as the displacement of the bar at ξ due to an impulse of the same magnitude being applied at x. This turns out to be a rather general, although perhaps unanticipated, phenomenon. Symmetry of the Green's function is a consequence of the underlying symmetry, or, more accurately, "self-adjointness", of the boundary value problem, a topic that will be developed in detail in Section 9.2.

Example 6.10. Let $\omega^2 > 0$ be a fixed positive constant. Let us solve the inhomogeneous boundary value problem

$$-u'' + \omega^2 u = f(x), \qquad u(0) = u(1) = 0, \tag{6.57}$$

by constructing its Green's function. To this end, we first analyze the effect of a delta function inhomogeneity

$$-u'' + \omega^2 u = \delta(x - \xi), \qquad u(0) = u(1) = 0. \tag{6.58}$$

Rather than try to integrate this differential equation directly, let us appeal to the defining properties of the Green's function. The general solution to the homogeneous equation is a linear combination of the two basic exponentials $e^{\omega x}$ and $e^{-\omega x}$, or better, the hyperbolic functions

$$\cosh \omega x = \frac{e^{\omega x} + e^{-\omega x}}{2}, \qquad \sinh \omega x = \frac{e^{\omega x} - e^{-\omega x}}{2}. \tag{6.59}$$

The solutions satisfying the first boundary condition are multiples of $\sinh \omega x$, while those satisfying the second boundary condition are multiples of $\sinh \omega (1 - x)$. Therefore, the solution to (6.58) has the form

$$G(x;\xi) = \begin{cases} a \sinh \omega x, & x \leq \xi, \\ b \sinh \omega (1 - x), & x \geq \xi. \end{cases}$$

Continuity of $G(x;\xi)$ at $x = \xi$ requires

$$a \sinh \omega \xi = b \sinh \omega (1 - \xi). \tag{6.60}$$

At $x = \xi$, the derivative $\partial G/\partial x$ must have a jump discontinuity of magnitude -1 in order that the second derivative term in (6.58) match the delta function. (The $\omega^2 u$ term clearly cannot produce the required singularity.) Since

$$\frac{\partial G}{\partial x}(x;\xi) = \begin{cases} a\,\omega \cosh \omega x, & x < \xi, \\ -b\,\omega \cosh \omega (1 - x), & x > \xi, \end{cases}$$

the jump condition requires

$$a\,\omega \cosh \omega \xi - 1 = -b\,\omega \cosh \omega (1 - \xi). \tag{6.61}$$

Multiplying (6.60) by $\omega \cosh \omega (1 - \xi)$ and (6.61) by $\sinh \omega (1 - \xi)$, and then adding the results together, we obtain

$$\sinh \omega (1 - \xi) = a\,\omega \big[\sinh \omega \xi \, \cosh \omega (1 - \xi) + \cosh \omega \xi \, \sinh \omega (1 - \xi) \big] = a\,\omega \sinh \omega, \tag{6.62}$$

where we made use of the addition formula for the hyperbolic sine:

$$\sinh(\alpha + \beta) = \sinh \alpha \, \cosh \beta + \cosh \alpha \, \sinh \beta, \tag{6.63}$$

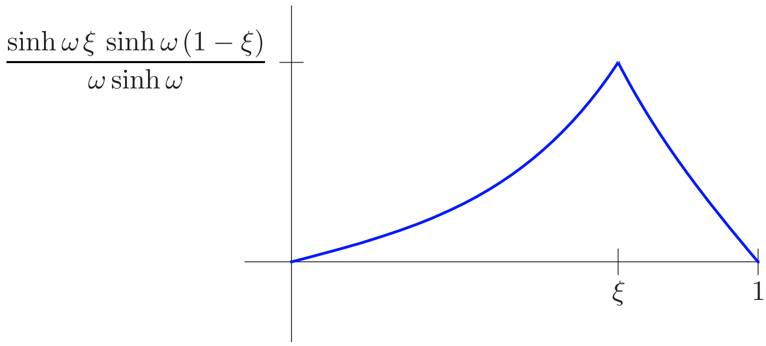

Figure 6.9. Green's function for the boundary value problem (6.57).

which you are asked to prove in Exercise 6.2.13. Therefore, solving (6.61–62) for

$$a = \frac{\sinh \omega (1-\xi)}{\omega \sinh \omega}, \qquad b = \frac{\sinh \omega \xi}{\omega \sinh \omega},$$

produces the explicit formula

$$G(x;\xi) = \begin{cases} \dfrac{\sinh \omega x \, \sinh \omega (1-\xi)}{\omega \sinh \omega}, & x \le \xi, \\ \dfrac{\sinh \omega (1-x) \, \sinh \omega \xi}{\omega \sinh \omega}, & x \ge \xi. \end{cases} \tag{6.64}$$

A representative graph appears in Figure 6.9. As before, a corner, indicating a discontinuity in the first derivative, appears at the point $x = \xi$ where the impulse force is applied. Moreover, as in the previous example, $G(x;\xi) = G(\xi;x)$ is a symmetric function.

The general solution to the inhomogeneous boundary value problem (6.57) is then given by the superposition formula (6.53); explicitly,

$$\begin{aligned} u(x) &= \int_0^1 G(x;\xi) f(\xi)\, d\xi \\ &= \int_0^x \frac{\sinh \omega (1-x) \sinh \omega \xi}{\omega \sinh \omega} f(\xi)\, d\xi \; + \int_x^1 \frac{\sinh \omega x \, \sinh \omega (1-\xi)}{\omega \sinh \omega} f(\xi)\, d\xi. \end{aligned} \tag{6.65}$$

For example, under a constant unit force $f(x) \equiv 1$, the solution is

$$\begin{aligned} u(x) &= \int_0^x \frac{\sinh \omega (1-x) \sinh \omega y}{\omega \sinh \omega}\, d\xi \; + \int_x^1 \frac{\sinh \omega x \, \sinh \omega (1-\xi)}{\omega \sinh \omega}\, d\xi \\ &= \frac{\sinh \omega (1-x) \big(\cosh \omega x - 1\big)}{\omega^2 \sinh \omega} + \frac{\sinh \omega x \big(\cosh \omega (1-x) - 1\big)}{\omega^2 \sinh \omega} \\ &= \frac{1}{\omega^2} - \frac{\sinh \omega x + \sinh \omega (1-x)}{\omega^2 \sinh \omega}. \end{aligned}$$

For comparative purposes, the reader may wish to rederive this particular solution by a direct calculation, without appealing to the Green's function.

Example 6.11. Finally, consider the Neumann boundary value problem

$$-\,c\,u'' = f(x), \qquad u'(0) = 0 = u'(1), \tag{6.66}$$

modeling the equilibrium deformation of a homogeneous bar with two free ends when subject to an external force $f(x)$. The *Green's function* should satisfy the particular case

$$-c\,u'' = \delta(x-\xi), \qquad\qquad u'(0) = 0 = u'(1),$$

when the forcing function is a concentrated impulse. As in Example 6.9, the general solution to the latter differential equation is

$$u(x) = -\,\frac{\rho(x-\xi)}{c} + a\,x + b,$$

where a, b are integration constants, and ρ is the ramp function (6.25). However, the Neumann boundary conditions require that

$$u'(0) = a = 0, \qquad\qquad u'(1) = -\,\frac{1}{c} + a = 0,$$

which cannot both be satisfied. We conclude that there is no Green's function in this case.

The difficulty is that the Neumann boundary value problem (6.66) does not have a unique solution, and hence cannot admit a Green's function solution formula (6.53). Indeed, integrating twice, we find that the general solution to the differential equation is

$$u(x) = a\,x + b - \frac{1}{c}\int_0^x \int_0^y f(z)\,dz\,dy,$$

where a, b are integration constants. Since

$$u'(x) = a - \frac{1}{c}\int_0^x f(z)\,dz,$$

the boundary conditions require that

$$u'(0) = a = 0, \qquad\qquad u'(1) = a - \frac{1}{c}\int_0^1 f(z)\,dz = 0.$$

These equations are compatible if and only if

$$\int_0^1 f(z)\,dz = 0. \tag{6.67}$$

Thus, the Neumann boundary value problem admits a solution if and only if there is no net force on the bar. Indeed, physically, if (6.67) does not hold, then, because its ends are not attached to any support, the bar cannot stay in equilibrium, but will move off in the direction of the net force. On the other hand, if (6.67) holds, then the solution

$$u(x) = b \;-\; \frac{1}{c}\int_0^x \int_0^y f(z)\,dz\,dy$$

is *not unique*, since b is not constrained by the boundary conditions, and so can assume any constant value. Physically, this means that any equilibrium configuration of the bar can be freely translated to assume another valid equilibrium.

Remark: The constraint (6.67) is a manifestation of the *Fredholm Alternative*, to be developed in detail in Section 9.1.

Let us summarize the fundamental properties that serve to completely characterize the Green's function of boundary value problems governed by second-order linear ordinary differential equations

$$p(x)\,\frac{d^2u}{dx^2}+q(x)\,\frac{du}{dx}+r(x)\,u(x)=f(x), \tag{6.68}$$

combined with a pair of homogeneous boundary conditions at the ends of the interval $[a,b]$. We assume that the coefficient functions are continuous, $p,q,r,f\in \mathrm{C}^0[a,b]$, and that $p(x)\neq 0$ for all $a\leq x\leq b$.

Basic Properties of the Green's Function $G(x;\xi)$

(*i*) Solves the homogeneous differential equation at all points $x\neq\xi$.
(*ii*) Satisfies the homogeneous boundary conditions.
(*iii*) Is a continuous function of its arguments.
(*iv*) For each fixed ξ, its derivative $\partial G/\partial x$ is piecewise C^1, with a single jump discontinuity of magnitude $1/p(\xi)$ at the impulse point $x=\xi$.

With the Green's function in hand, we deduce that the solution to the general boundary value problem (6.68) subject to the appropriate homogeneous boundary conditions is expressed by the *Green's Function Superposition Formula*

$$u(x)=\int_a^b G(x;\xi)\,f(\xi)\,d\xi. \tag{6.69}$$

The symmetry of the Green's function is more subtle, for it relies on the self-adjointness of the boundary value problem, an issue to be addressed in detail in Chapter 9. In the present situation, self-adjointness requires that $q(x)=p'(x)$, in which case $G(\xi;x)=G(x;\xi)$ will be symmetric in its arguments.

Finally, as we saw in Example 6.11, not every such boundary value problem admits a solution, and one expects to find a Green's function only in cases in which the solution exists and is unique.

Theorem 6.12. *The following are equivalent*:

- *The only solution to the homogeneous boundary value problem is the zero function.*
- *The inhomogeneous boundary value problem has a unique solution for every choice of forcing function.*
- *The boundary value problem admits a Green's function.*

Exercises

6.2.1. Let $c>0$. Find the Green's function for the boundary value problem $-c\,u''=f(x)$, $u(0)=0$, $u'(1)=0$, which models the displacement of a uniform bar of unit length with one fixed and one free end under an external force. Then use superposition to write down a formula for the solution. Verify that your integral formula is correct by direct differentiation and substitution into the differential equation and boundary conditions.

6.2.2. A uniform bar of length $\ell = 4$ has constant stiffness $c = 2$. Find the Green's function for the case that (*a*) both ends are fixed; (*b*) one end is fixed and the other is free. (*c*) Why is there no Green's function when both ends are free?

6.2.3. A point 2 cm along a 10 cm bar experiences a displacement of 1 mm under a concentrated force of 2 newtons applied at the midpoint of the bar. How far does the midpoint deflect when a concentrated force of 1 newton is applied at the point 2 cm along the bar?

♡ 6.2.4. The boundary value problem $-\dfrac{d}{dx}\left(c(x)\,\dfrac{du}{dx}\right) = f(x),\ u(0) = u(1) = 0$, models the displacement $u(x)$ of a nonuniform elastic bar with stiffness $c(x) = \dfrac{1}{1+x^2}$ for $0 \le x \le 1$. (*a*) Find the displacement when the bar is subjected to a constant external force, $f \equiv 1$. (*b*) Find the Green's function for the boundary value problem. (*c*) Use the resulting superposition formula to check your solution to part (*a*). (*d*) Which point $0 < \xi < 1$ on the bar is the "weakest", i.e., the bar experiences the largest displacement under a unit impulse concentrated at that point?

6.2.5. Answer Exercise 6.2.4 when $c(x) = 1 + x$.

♡ 6.2.6. Consider the boundary value problem $-u'' = f(x),\ u(0) = 0,\ u(1) = 2\,u'(1)$. (*a*) Find the Green's function. (*b*) Which of the fundamental properties does your Green's function satisfy? (*c*) Write down an explicit integral formula for the solution to the boundary value problem, and prove its validity by a direct computation. (*d*) Explain why the related boundary value problem $-u'' = f,\ u(0) = 0,\ u(1) = u'(1)$, does not have a Green's function.

♡ 6.2.7. For n a positive integer, set $f_n(x) = \begin{cases} \frac{1}{2}n, & |\,x-\xi\,| < \frac{1}{n}, \\ 0, & \text{otherwise.} \end{cases}$

(*a*) Find the solution $u_n(x)$ to the boundary value problem $-u'' = f_n(x),\ u(0) = u(1) = 0$, assuming $0 < \xi - \frac{1}{n} < \xi + \frac{1}{n} < 1$. (*b*) Prove that $\lim\limits_{n\to\infty} u_n(x) = G(x;\xi)$ converges to the Green's function (6.51). Why should this be the case? (*c*) Reconfirm the result in part (*b*) by graphing $u_5(x), u_{15}(x), u_{25}(x)$, along with $G(x;\xi)$ when $\xi = .3$.

6.2.8. Solve the boundary value problem $-4u'' + 9u = 0,\ u(0) = 0,\ u(2) = 1$. Is your solution unique?

6.2.9. *True or false*: The Neumann boundary value problem $-u'' + u = 1,\ u'(0) = u'(1) = 0$, has a unique solution.

6.2.10. Use the Green's function (6.64) to solve the boundary value problem (6.57) when the forcing function is $f(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} < x \le 1. \end{cases}$

6.2.11. Let $\omega > 0$. (*a*) Find the Green's function for the mixed boundary value problem

$$-u'' + \omega^2 u = f(x), \quad u(0) = 0, \quad u'(1) = 0.$$

(*b*) Use your Green's function to find the solution when $f(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} < x \le 1. \end{cases}$

6.2.12. Suppose $\omega > 0$. Does the Neumann boundary value problem $-u'' + \omega^2 u = f(x)$, $u'(0) = u'(1) = 0$ admit a Green's function? If not, explain why not. If so, find it, and then write down an integral formula for the solution of the boundary value problem.

◇ 6.2.13. (*a*) Prove the addition formula (6.63) for the hyperbolic sine function. (*b*) Find the corresponding addition formula for the hyperbolic cosine.

◇ 6.2.14. Prove the differentiation formula (6.55).

6.3 Green's Functions for the Planar Poisson Equation

Now we develop the Green's function approach to solving boundary value problems involving the two-dimensional Poisson equation (4.84). As before, the Green's function is characterized as the solution to the homogeneous boundary value problem in which the inhomogeneity is a concentrated unit impulse — a delta function. The solution to the general forced boundary value problem is then obtained via linear superposition, that is, as a convolution integral with the Green's function.

However, before proceeding, we need to quickly review some basic facts concerning vector calculus in the plane. The student may wish to consult a standard multivariable calculus text, e.g., [**8**, **108**], for additional details.

Calculus in the Plane

Let $\mathbf{x} = (x, y)$ denote the usual Cartesian coordinates on $\mathbb{R}^2$. The term *scalar field* is synonymous with a real-valued function $u(x, y)$, defined on a domain $\Omega \subset \mathbb{R}^2$. A vector-valued function

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}(x, y) = \begin{pmatrix} v_1(x, y) \\ v_2(x, y) \end{pmatrix} \tag{6.70}$$

is known as a (planar) *vector field*. A vector field assigns a vector $\mathbf{v}(x, y) \in \mathbb{R}^2$ to each point $(x, y) \in \Omega$ in its domain of definition, and hence defines a function $\mathbf{v}\colon \Omega \to \mathbb{R}^2$. Physical examples include velocity vector fields of fluid flows, heat flux fields in thermodynamics, and gravitational and electrostatic force fields.

The *gradient* operator ∇ maps a scalar field $u(x, y)$ to the vector field

$$\nabla u = \begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \end{pmatrix}. \tag{6.71}$$

The scalar field u is often referred to as a *potential function* for its gradient vector field $\mathbf{v} = \nabla u$. On a connected domain Ω, the potential, when it exists, is uniquely determined up to addition of a constant.

The *divergence* of the planar vector field $\mathbf{v} = (\, v_1, v_2 \,)^T$ is the scalar field

$$\nabla \cdot \mathbf{v} = \operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\,. \tag{6.72}$$

Its *curl* is defined as

$$\nabla \times \mathbf{v} = \operatorname{curl} \mathbf{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\,. \tag{6.73}$$

Notice that the curl of a planar vector field is a scalar field. (In contrast, in three dimensions, the curl of a vector field is another vector field.) Given a smooth potential $u \in \mathrm{C}^2$, the curl of its gradient vector field automatically vanishes:

$$\nabla \times \nabla u = \frac{\partial}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\frac{\partial u}{\partial x} \equiv 0,$$

by the equality of mixed partials. Thus, a necessary condition for a vector field $\mathbf{v}$ to admit a potential is that it be *irrotational*, meaning $\nabla \times \mathbf{v} = 0$; this condition is sufficient if

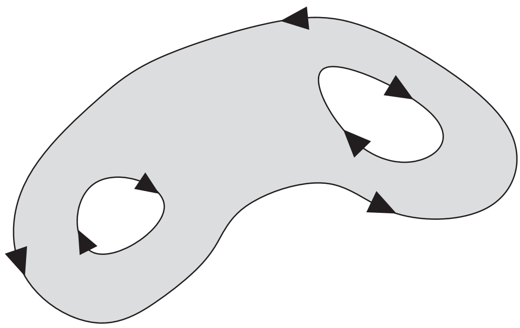

Figure 6.10. Orientation of the boundary of a planar domain.

the underlying domain Ω is *simply connected*, i.e., has no holes. On the other hand, the divergence of a gradient vector field coincides with the Laplacian of the potential function:

$$\nabla \cdot \nabla u = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\,. \tag{6.74}$$

A vector field is *incompressible* if it has zero divergence: $\nabla \cdot \mathbf{v} = 0$; for the velocity vector field of a steady-state fluid flow, incompressibility means that the fluid does not change volume. (Water is, for all practical purposes, an incompressible fluid.) Therefore, an irrotational vector field with potential u is also incompressible if and only if the potential solves the Laplace equation $\Delta u = 0$.

Remark: Because of formula (6.74), the Laplacian operator is also sometimes written as $\Delta = \nabla^2$. The factorization of the Laplacian into the product of the divergence and the gradient operators is, in fact, of great importance, and underlies its "self-adjointness", a fundamental property whose ramifications will be explored in depth in Chapter 9.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain whose boundary $\partial\Omega$ consists of one or more piecewise smooth closed curves. We orient the boundary so that the domain is always on one's left as one goes around the boundary curve(s). Figure 6.10 sketches a domain with two holes; its three boundary curves are oriented according to the directions of the arrows. Note that the outer boundary curve is traversed in a counterclockwise direction, while the two inner boundary curves are oriented clockwise.

Green's Theorem, first formulated by George Green to use in his seminal study of partial differential equations and potential theory, relates certain double integrals over a domain to line integrals around its boundary. It should be viewed as the extension of the Fundamental Theorem of Calculus to double integrals.

Theorem 6.13. *Let* $\mathbf{v}(\mathbf{x})$ *be a smooth*† *vector field defined on a bounded domain* $\Omega \subset \mathbb{R}^2$. *Then the line integral of* $\mathbf{v}$ *around the boundary* $\partial\Omega$ *equals the double integral of its curl over the domain*:

$$\iint_\Omega \nabla \times \mathbf{v}\; dx\, dy = \oint_{\partial\Omega} \mathbf{v} \cdot d\mathbf{x}, \tag{6.75}$$

† To be precise, we require $\mathbf{v}$ to be continuously differentiable within the domain, and continuous up to the boundary, so $\mathbf{v} \in \mathrm{C}^0(\overline{\Omega}) \cap \mathrm{C}^1(\Omega)$, where $\overline{\Omega} = \Omega \cup \partial\Omega$ denotes the closure of the domain Ω.

or, in full detail,

$$\iint_\Omega \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) dx\, dy = \oint_{\partial\Omega} v_1\, dx + v_2\, dy\,. \tag{6.76}$$

Example 6.14. Let us apply Green's Theorem 6.13 to the particular vector field $\mathbf{v} = (\, y, 0\,)^T$. Since $\nabla \times \mathbf{v} \equiv -1$, we obtain

$$\oint_{\partial\Omega} y\, dx = \iint_\Omega (-1)\, dx\, dy = -\operatorname{area}\Omega. \tag{6.77}$$

This means that we can determine the area of a planar domain by computing the negative of the indicated line integral around its boundary.

For later purposes, we rewrite the basic Green identity (6.75) in an equivalent "divergence form". Given a planar vector field $\mathbf{v} = (\, v_1, v_2\,)^T$, let

$$\mathbf{v}^\perp = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} \tag{6.78}$$

denote the "perpendicular" vector field. We note that its curl

$$\nabla \times \mathbf{v}^\perp = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \nabla \cdot \mathbf{v} \tag{6.79}$$

coincides with the divergence of the original vector field.

When we replace $\mathbf{v}$ in Green's identity (6.75) by $\mathbf{v}^\perp$, the result is

$$\iint_\Omega \nabla \cdot \mathbf{v}\; dx\, dy = \iint_\Omega \nabla \times \mathbf{v}^\perp\, dx\, dy = \oint_{\partial\Omega} \mathbf{v}^\perp \cdot\, d\mathbf{x} = \oint_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\; ds,$$

where $\mathbf{n}$ denotes the *unit outwards normal* to the boundary of our domain, while ds denotes the arc-length element along the boundary curve. This yields the *divergence form* of Green's Theorem:

$$\iint_\Omega \nabla \cdot \mathbf{v}\; dx\, dy = \oint_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\; ds. \tag{6.80}$$

Physically, if $\mathbf{v}$ represents the velocity vector field of a steady-state fluid flow, then the line integral in (6.80) represents the net fluid flux out of the region Ω. As a result, the divergence $\nabla \cdot \mathbf{v}$ represents the local change in area of the fluid at each point, which serves to justify our earlier statement on incompressibility.

Consider next the product vector field $u\,\mathbf{v}$ obtained by multiplying a vector field $\mathbf{v}$ by a scalar field u. An elementary computation proves that its divergence is

$$\nabla \cdot (u\,\mathbf{v}) = u\,\nabla \cdot \mathbf{v} + \nabla u \cdot \mathbf{v}. \tag{6.81}$$

Replacing $\mathbf{v}$ by $u\,\mathbf{v}$ in the divergence formula (6.80), we deduce what is usually referred to as *Green's formula*

$$\iint_\Omega \big(\, u\,\nabla \cdot \mathbf{v} + \nabla u \cdot \mathbf{v}\,\big)\, dx\, dy = \oint_{\partial\Omega} u\,(\mathbf{v} \cdot \mathbf{n})\, ds, \tag{6.82}$$

which is valid for arbitrary bounded domains Ω, and arbitrary C^1 scalar and vector fields defined thereon. Rearranging the terms produces

$$\iint_\Omega \nabla u \cdot \mathbf{v}\; dx\, dy = \oint_{\partial\Omega} u\,(\mathbf{v} \cdot \mathbf{n})\, ds - \iint_\Omega u\,\nabla \cdot \mathbf{v}\, dx\, dy. \tag{6.83}$$

We will view this identity as an *integration by parts* formula for double integrals. Indeed, comparing with the one-dimensional integration by parts formula

$$\int_a^b u'(x)\, v(x)\, dx = u(x)\, v(x) \Big|_{x=a}^{b} - \int_a^b u(x)\, v'(x)\, dx, \tag{6.84}$$

we observe that the single integrals have become double integrals; the derivatives are vector derivatives (gradient and divergence), while the boundary contributions at the endpoints of the interval are replaced by a line integral around the entire boundary of the two-dimensional domain.

A useful special case of (6.82) is that in which $\mathbf{v} = \nabla v$ is the gradient of a scalar field v. Then, in view of (6.74), Green's formula (6.82) becomes

$$\iint_\Omega \left(u\, \Delta v + \nabla u \cdot \nabla v \right) dx\, dy = \oint_{\partial\Omega} u\, \frac{\partial v}{\partial \mathbf{n}}\, ds, \tag{6.85}$$

where $\partial v / \partial \mathbf{n} = \nabla v \cdot \mathbf{n}$ is the *normal derivative* of the scalar field v on the boundary of the domain. In particular, setting $v = u$, we deduce

$$\iint_\Omega \left(u\, \Delta u + \|\, \nabla u \,\|^2 \right) dx\, dy = \oint_{\partial\Omega} u\, \frac{\partial u}{\partial \mathbf{n}}\, ds. \tag{6.86}$$

As an application, we establish a basic uniqueness theorem for solutions to the boundary value problems for the Poisson equation:

Theorem 6.15. *Suppose $\widetilde{u}$ and u both satisfy the same inhomogeneous Dirichlet or mixed boundary value problem for the Poisson equation on a connected, bounded domain Ω. Then $\widetilde{u} = u$. On the other hand, if $\widetilde{u}$ and u satisfy the same Neumann boundary value problem, then $\widetilde{u} = u + c$ for some constant c.*

Proof: Since, by assumption, $-\Delta\widetilde{u} = f = -\Delta u$, the difference $v = \widetilde{u} - u$ satisfies the Laplace equation $\Delta v = 0$ in Ω, and satisfies the homogeneous boundary conditions. Therefore, applying (6.86) to v, we find

$$\iint_\Omega \|\, \nabla v \,\|^2\, dx\, dy = \oint_{\partial\Omega} v\, \frac{\partial v}{\partial \mathbf{n}}\, ds = 0,$$

since, at every point on the boundary, either $v = 0$ or $\partial v / \partial \mathbf{n} = 0$. Since the integrand is continuous and everywhere nonnegative, we immediately conclude that $\|\, \nabla v \,\|^2 = 0$, and hence $\nabla v = \mathbf{0}$ throughout Ω. On a connected domain, the only functions annihilated by the gradient operator are the constants:

Lemma 6.16. *If $v(x, y)$ is a C^1 function defined on a connected domain $\Omega \subset \mathbb{R}^2$, then $\nabla v \equiv 0$ if and only if $v(x, y) \equiv c$ is a constant.*

Proof: Let $\mathbf{a}, \mathbf{b}$ be any two points in Ω. Then, by connectivity, we can find a curve C connecting them. The Fundamental Theorem for line integrals, [**8**, **108**], states that

$$\int_C \nabla v \cdot d\mathbf{x} = v(\mathbf{b}) - v(\mathbf{a}).$$

Thus, if $\nabla v \equiv 0$, then $v(\mathbf{b}) = v(\mathbf{a})$ for all $\mathbf{a}, \mathbf{b} \in \Omega$, which implies that v must be constant. *Q.E.D.*

Returning to our proof, we conclude that $\widetilde{u} = u + v = u + c$, which proves the result in the Neumann case. In the Dirichlet or mixed problems, there is at least one point on the boundary where $v = 0$, and hence the only possible constant is $v = c = 0$, proving that $\widetilde{u} = u$. *Q.E.D.*

Thus, the Dirichlet and mixed boundary value problems admit at most one solution, while the Neumann boundary value problem has either no solutions or infinitely many solutions. Proof of existence of solutions is more challenging, and will be left to a more advanced text, e.g., [**35**, **44**, **61**, **70**].

If we subtract from formula (6.85) the formula

$$\iint_\Omega \big(v\,\Delta u + \nabla u \cdot \nabla v \big)\, dx\, dy = \oint_{\partial\Omega} v\, \frac{\partial u}{\partial \mathbf{n}}\, ds, \tag{6.87}$$

obtained by interchanging u and v, we obtain the identity

$$\iint_\Omega \big(u\,\Delta v - v\,\Delta u \big)\, dx\, dy = \oint_{\partial\Omega} \left(u\, \frac{\partial v}{\partial \mathbf{n}} - v\, \frac{\partial u}{\partial \mathbf{n}} \right) ds, \tag{6.88}$$

which will play a major role in our analysis of the Poisson equation. Setting $v = 1$ in (6.87) yields

$$\iint_\Omega \Delta u\, dx\, dy = \oint_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}\, ds. \tag{6.89}$$

Suppose u solves the Neumann boundary value problem

$$-\,\Delta u = f, \quad \text{in} \quad \Omega \qquad\qquad \frac{\partial u}{\partial \mathbf{n}} = h \quad \text{on} \quad \partial\Omega.$$

Then (6.89) requires that

$$\iint_\Omega f\, dx\, dy \;+\; \oint_{\partial\Omega} h\, ds = 0, \tag{6.90}$$

which thus forms a necessary condition for the existence of a solution u to the inhomogeneous Neumann boundary value problem. Physically, if u represents the equilibrium temperature of a plate, then the integrals in (6.89) measure the net gain or loss in heat energy due to, respectively, the external heat source and the heat flux through the boundary. Equation (6.90) is telling us that, for the plate to remain in thermal equilibrium, there can be no net change in its total heat energy.

The Two–Dimensional Delta Function

Now let us return to the business at hand — solving the Poisson equation on a bounded domain $\Omega \subset \mathbb{R}^2$. We will subject the solution to either homogeneous Dirichlet boundary conditions or homogeneous mixed boundary conditions. (As we just noted, the Neumann boundary value problem does not admit a unique solution, and hence does not possess a Green's function.) The Green's function for the boundary value problem arises when the forcing function is a unit impulse concentrated at a single point in the domain.

Thus, our first task is to establish the proper form for a unit impulse in our two-dimensional context. The *delta function* concentrated at a point $\boldsymbol{\xi} = (\xi, \eta) \in \mathbb{R}^2$ is denoted by

$$\delta_{(\xi,\eta)}(x, y) = \delta_{\boldsymbol{\xi}}(\mathbf{x}) = \delta(\mathbf{x} - \boldsymbol{\xi}) = \delta(x - \xi, y - \eta), \tag{6.91}$$

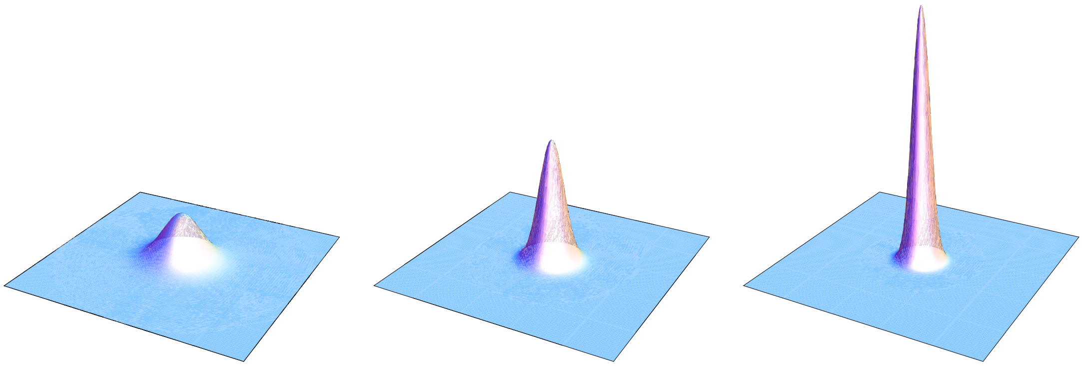

Figure 6.11. Gaussian functions converging to the delta function.

and is designed so that

$$\delta_{\boldsymbol{\xi}}(\mathbf{x}) = 0, \qquad \mathbf{x} \neq \boldsymbol{\xi}, \qquad \iint_{\Omega} \delta_{(\xi,\eta)}(x,y)\, dx\, dy = 1, \qquad \boldsymbol{\xi} \in \Omega. \tag{6.92}$$

In particular, $\delta(x,y) = \delta_{\mathbf{0}}(x,y)$ represents the delta function at the origin. As in the one-dimensional version, there is no ordinary function that satisfies both criteria; rather, $\delta(x,y)$ is to be viewed as the limit of a sequence of more and more highly concentrated functions $g_n(x,y)$, with

$$\lim_{n\to\infty} g_n(x,y) = 0, \quad \text{for} \quad (x,y) \neq (0,0), \qquad \text{while} \qquad \iint_{\mathbb{R}^2} g_n(x,y)\, dx\, dy = 1.$$

A good example of a suitable sequence is provided by the *radial Gaussian functions*

$$g_n(x,y) = \frac{n}{\pi}\, e^{-n\,(x^2+y^2)}\,. \tag{6.93}$$

As plotted in Figure 6.11, as $n \to \infty$, the Gaussian profiles become more and more concentrated near the origin, while maintaining a unit volume underneath their graphs. The fact that their integral over $\mathbb{R}^2$ equals 1 is a consequence of (2.99).

Alternatively, one can assign the delta function a dual interpretation as the linear functional

$$L_{(\xi,\eta)}[\,u\,] = L_{\boldsymbol{\xi}}[\,u\,] = u(\boldsymbol{\xi}) = u(\xi,\eta), \tag{6.94}$$

which assigns to each continuous function $u \in \mathrm{C}^0(\overline{\Omega})$ its value at the point $\boldsymbol{\xi} = (\xi,\eta) \in \Omega$. Then, using the L^2 inner product

$$\langle\, u\,,v\,\rangle = \iint_{\Omega} u(x,y)\, v(x,y)\, dx\, dy \tag{6.95}$$

between scalar fields $u, v \in \mathrm{C}^0(\overline{\Omega})$, we formally identify the linear functional $L_{(\xi,\eta)}$ with the delta "function" by the integral formula

$$\langle\, \delta_{(\xi,\eta)}\,, u\,\rangle = \iint_{\Omega} \delta_{(\xi,\eta)}(x,y)\, u(x,y)\, dx\, dy = \begin{cases} u(\xi,\eta), & (\xi,\eta) \in \Omega, \\ 0, & (\xi,\eta) \in \mathbb{R}^2 \setminus \overline{\Omega}, \end{cases} \tag{6.96}$$

for any $u \in \mathrm{C}^0(\overline{\Omega})$. As in the one-dimensional version, we will avoid defining the integral when the delta function is concentrated at a boundary point of the domain.

Since double integrals can be evaluated as repeated one-dimensional integrals, we can conveniently view

$$\delta_{(\xi,\eta)}(x,y) = \delta_\xi(x)\,\delta_\eta(y) = \delta(x-\xi)\,\delta(y-\eta) \tag{6.97}$$

as the product† of a pair of one-dimensional delta functions. Indeed, if the impulse point

$$(\xi,\eta) \in R = \big\{\, a < x < b,\ c < y < d \,\big\} \subset \Omega$$

is contained in a rectangle that lies within the domain, then

$$\begin{aligned}\iint_\Omega \delta_{(\xi,\eta)}(x,y)\,u(x,y)\,dx\,dy &= \iint_R \delta_{(\xi,\eta)}(x,y)\,u(x,y)\,dx\,dy \\ = \int_a^b \left(\int_c^d \delta(x-\xi)\,\delta(y-\eta)\,u(x,y)\,dy \right) dx &= \int_a^b \delta(x-\xi)\,u(x,\eta)\,dx = u(\xi,\eta).\end{aligned}$$

The Green's Function

As in the one-dimensional context, the Green's function is defined as the solution to the inhomogeneous differential equation when subject to a concentrated unit delta impulse at a prescribed point $\boldsymbol{\xi} = (\xi,\eta) \in \Omega$ inside the domain. In the current situation, the Poisson equation takes the form

$$-\Delta u = \delta_{\boldsymbol{\xi}}, \qquad \text{or, explicitly,} \qquad -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \delta(x-\xi)\,\delta(y-\eta). \tag{6.98}$$

The function $u(x,y)$ is also subject to some homogeneous boundary conditions, e.g., the Dirichlet conditions $u = 0$ on $\partial\Omega$. The resulting solution is called the *Green's function* for the boundary value problem, and written

$$G_{\boldsymbol{\xi}}(\mathbf{x}) = G(\mathbf{x};\boldsymbol{\xi}) = G(x,y;\xi,\eta). \tag{6.99}$$

Once we know the Green's function, the solution to the general Poisson boundary value problem

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega \tag{6.100}$$

is reconstructed as follows. We regard the forcing function

$$f(x,y) = \iint_\Omega \delta(x-\xi)\,\delta(y-\eta) f(\xi,\eta)\,d\xi\,d\eta$$

as a superposition of delta impulses, whose strength equals the value of f at the impulse point. Linearity implies that the solution to the boundary value problem is the corresponding superposition of Green's function responses to each of the constituent impulses. The net result is the fundamental *superposition formula*

$$u(x,y) = \iint_\Omega G(x,y;\xi,\eta)\,f(\xi,\eta)\,d\xi\,d\eta \tag{6.101}$$

† This is an exception to our earlier injunction not to multiply delta functions. Multiplication is allowed when they depend on *different* variables.

for the solution to the boundary value problem. Indeed,

$$
\begin{aligned}
-\Delta u(x,y) &= \iint_\Omega -\Delta G(x,y;\xi,\eta)\, f(\xi,\eta)\, d\xi\, d\eta \\
&= \iint_\Omega \delta(x-\xi, y-\eta)\, f(\xi,\eta)\, d\xi\, d\eta = f(x,y),
\end{aligned}
$$

while the fact that $G(x,y;\xi,\eta) = 0$ for all $(x,y) \in \partial\Omega$ implies that $u(x,y) = 0$ on the boundary.

The Green's function inevitably turns out to be symmetric under interchange of its arguments:

$$G(\xi,\eta;x,y) = G(x,y;\xi,\eta). \tag{6.102}$$

As in the one-dimensional case, symmetry is a consequence of the self-adjointness of the boundary value problem, and will be explained in full in Chapter 9. Symmetry has the following intriguing physical interpretation: Let $\mathbf{x}, \boldsymbol{\xi} \in \Omega$ be any two points in the domain. We apply a concentrated unit force to the membrane at the first point and measure its deflection at the second; the result is exactly the same as if we applied the impulse at the second point and measured the deflection at the first. (Deflections at other points in the domain will typically have no obvious relation with one another.) Similarly, in electrostatics, the solution $u(x,y)$ is interpreted as the electrostatic potential for a system of charges in equilibrium. A delta function corresponds to a point charge, e.g., an electron. The symmetry property says that the electrostatic potential at $\mathbf{x}$ due to a point charge placed at position $\boldsymbol{\xi}$ is exactly the same as the potential at $\boldsymbol{\xi}$ due to a point charge at $\mathbf{x}$. The reader may wish to meditate on the physical plausibility of these striking facts.

Unfortunately, most Green's functions cannot be written down in closed form. One important exception occurs when the domain is the entire plane: $\Omega = \mathbb{R}^2$. The solution to the Poisson equation (6.98) is the *free-space Green's function* $G_0(x,y;\xi,\eta) = G_0(\mathbf{x};\boldsymbol{\xi})$, which measures the effect of a unit impulse, concentrated at $\boldsymbol{\xi}$, throughout two-dimensional space, e.g., the gravitational potential due to a point mass or the electrostatic potential due to a point charge. To motivate the construction, let us appeal to physical intuition. First, since the concentrated impulse is zero when $\mathbf{x} \neq \boldsymbol{\xi}$, the function must solve the homogeneous Laplace equation

$$-\Delta G_0 = 0 \qquad \text{for all} \qquad \mathbf{x} \neq \boldsymbol{\xi}. \tag{6.103}$$

Second, since the Poisson equation is modeling a homogeneous, uniform medium, in the absence of boundary conditions the effect of a unit impulse should depend only on the distance from its source. Therefore, we expect G_0 to be a function of the radial variable alone:

$$G_0(x,y;\xi,\eta) = v(r), \qquad \text{where} \qquad r = \|\,\mathbf{x} - \boldsymbol{\xi}\,\| = \sqrt{(x-\xi)^2 + (y-\eta)^2}\,.$$

According to (4.113), the only radially symmetric solutions to the Laplace equation are

$$v(r) = a + b \log r, \tag{6.104}$$

where a and b are constants. The constant term a has zero derivative, and so cannot contribute to the delta function singularity. Therefore, we expect the required solution to be a multiple of the logarithmic term. To determine the multiple, consider a closed disk of radius $\varepsilon > 0$ centered at $\boldsymbol{\xi}$,

$$D_\varepsilon = \{\, 0 \leq r \leq \varepsilon \,\} = \{\, \|\,\mathbf{x} - \boldsymbol{\xi}\,\| \leq \varepsilon \,\},$$

with circular boundary

$$C_\varepsilon = \partial D_\varepsilon = \{\, r = \|\,\mathbf{x} - \boldsymbol{\xi}\,\| = \varepsilon \,\} = \{\, (\,\xi + \varepsilon \cos\theta, \eta + \varepsilon \sin\theta\,) \;\mid\; -\pi \le \theta \le \pi \,\}.$$

Then, by (6.74) and the divergence form (6.80) of Green's Theorem,

$$\begin{aligned} 1 &= \iint_{D_\varepsilon} \delta(x,y)\,dx\,dy = -\,b \iint_{D_\varepsilon} \Delta(\log r)\,dx\,dy = -\,b \iint_{D_\varepsilon} \nabla\cdot\nabla(\log r)\,dx\,dy \\ &= -\,b \oint_{C_\varepsilon} \frac{\partial(\log r)}{\partial \mathbf{n}}\,ds = -\,b \oint_{C_\varepsilon} \frac{\partial(\log r)}{\partial r}\,ds = -\,b \oint_{C_\varepsilon} \frac{1}{r}\,ds = -\,b \int_{-\pi}^{\pi} d\theta = -2\,\pi\, b, \end{aligned} \tag{6.105}$$

and hence $b = -1/(2\,\pi)$. We conclude that the free-space Green's function should have the logarithmic form

$$G_0(x,y;\xi,\eta) = -\,\frac{1}{2\pi}\,\log r = -\,\frac{1}{2\pi}\log\|\,\mathbf{x}-\boldsymbol{\xi}\,\| = -\,\frac{1}{4\pi}\log\big[\,(x-\xi)^2 + (y-\eta)^2\,\big]. \tag{6.106}$$

A fully rigorous, albeit more difficult, justification of (6.106) comes from the following important result, known as *Green's representation formula*.

Theorem 6.17. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, with piecewise C^1 boundary $\partial\Omega$. Suppose $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^1(\overline{\Omega})$. Then, for any $(x,y) \in \Omega$,*

$$\begin{aligned} u(x,y) = &-\iint_\Omega G_0(x,y;\xi,\eta)\,\Delta u(\xi,\eta)\,d\xi\,d\eta \\ &+ \oint_{\partial\Omega} \left(G_0(x,y;\xi,\eta)\,\frac{\partial u}{\partial \mathbf{n}}\,(\xi,\eta) - \frac{\partial G_0}{\partial \mathbf{n}}\,(x,y;\xi,\eta)\,u(\xi,\eta) \right) ds, \end{aligned} \tag{6.107}$$

where the Laplacian and the normal derivatives on the boundary are all taken with respect to the integration variables $\boldsymbol{\xi} = (\xi,\eta)$.

In particular, if both u and $\partial u/\partial \mathbf{n}$ vanish on $\partial\Omega$, then (6.107) reduces to

$$u(x,y) = -\iint_{\mathbb{R}^2} G_0(x,y;\xi,\eta)\,\Delta u(\xi,\eta)\,d\xi\,d\eta.$$

Invoking the definition of the delta function on the left-hand side and formally applying the Green identity (6.88) to the right-hand side produces

$$\iint_{\mathbb{R}^2} \delta(x-\xi)\,\delta(y-\eta)\,u(\xi,\eta)\,d\xi\,d\eta = \iint_{\mathbb{R}^2} -\,\Delta G_0(x,y;\xi,\eta)\,u(\xi,\eta)\,d\xi\,d\eta. \tag{6.108}$$

It is in this dual sense that we justify the desired formula

$$-\,\Delta G_0(\mathbf{x};\boldsymbol{\xi}) = \frac{1}{2\pi}\,\Delta\big(\log\|\,\mathbf{x}-\boldsymbol{\xi}\,\|\big) = \delta(\mathbf{x}-\boldsymbol{\xi}). \tag{6.109}$$

Proof of Theorem 6.17: We first note that, even though $G_0(\mathbf{x},\boldsymbol{\xi})$ has a logarithmic singularity at $\mathbf{x} = \boldsymbol{\xi}$, the double integral in (6.107) is finite. Indeed, after introducing polar coordinates $\xi = x + r\cos\theta,\ \eta = y + r\sin\theta$, and recalling $d\xi\,d\eta = r\,dr\,d\theta$, we see that it equals

$$\frac{1}{2\pi}\iint (r\log r)\,\Delta u\,dr\,d\theta.$$

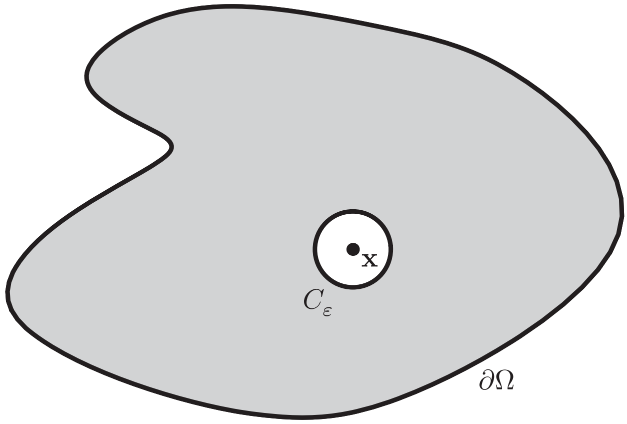

Figure 6.12. Domain $\Omega_\varepsilon = \Omega \setminus D_\varepsilon(\mathbf{x})$.

The product $r \log r$ is everywhere continuous — even at $r = 0$ — and so, provided Δu is well behaved, e.g., continuous, the integral is finite. There is, of course, no problem with the line integral in (6.107), since the contour does not go through the singularity.

Let us now avoid dealing directly with the singularity by working on a subdomain

$$\Omega_\varepsilon = \Omega \setminus D_\varepsilon(\mathbf{x}) = \{\, \boldsymbol{\xi} \in \Omega \mid \ \| \mathbf{x} - \boldsymbol{\xi} \| > \varepsilon \,\}$$

obtained by cutting out a small disk

$$D_\varepsilon(\mathbf{x}) = \{\, \boldsymbol{\xi} \mid \ \| \mathbf{x} - \boldsymbol{\xi} \| \leq \varepsilon \,\}$$

of radius $\varepsilon > 0$ centered at $\mathbf{x}$. We choose ε sufficiently small in order that $D_\varepsilon(\mathbf{x}) \subset \Omega$, and hence

$$\partial\Omega_\varepsilon = \partial\Omega \cup C_\varepsilon, \qquad \text{where} \qquad C_\varepsilon = \big\{\, \| \mathbf{x} - \boldsymbol{\xi} \| = \varepsilon \,\big\}$$

is the circular boundary of the disk. The subdomain Ω_ε is represented by the shaded region in Figure 6.12. Since the double integral is well defined, we can approximate it by integrating over Ω_ε:

$$\iint_\Omega G_0(x,y;\xi,\eta)\, \Delta u(\xi,\eta)\, d\xi\, d\eta = \lim_{\varepsilon \to 0} \iint_{\Omega_\varepsilon} G_0(x,y;\xi,\eta)\, \Delta u(\xi,\eta)\, d\xi\, d\eta. \tag{6.110}$$

Since G_0 has no singularities in Ω_ε, we are able to apply the Green formula (6.85) and then (6.103) to evaluate

$$\begin{aligned} &\iint_{\Omega_\varepsilon} G_0(x,y;\xi,\eta)\, \Delta u(\xi,\eta)\, d\xi\, d\eta \\ &\qquad = \oint_{\partial\Omega} \left(G_0(x,y;\xi,\eta)\, \frac{\partial u}{\partial \mathbf{n}}(\xi,\eta) - \frac{\partial G_0}{\partial \mathbf{n}}(x,y;\xi,\eta)\, u(\xi,\eta) \right) ds \\ &\qquad\qquad - \oint_{C_\varepsilon} \left(G_0(x,y;\xi,\eta)\, \frac{\partial u}{\partial \mathbf{n}}(\xi,\eta) - \frac{\partial G_0}{\partial \mathbf{n}}(x,y;\xi,\eta)\, u(\xi,\eta) \right) ds, \end{aligned} \tag{6.111}$$

where the line integral around C_ε is taken in the usual counterclockwise direction — the opposite orientation to that induced by its status as part of the boundary of Ω_ε. Now, on the circle C_ε,

$$G_0(x,y;\xi,\eta) = -\left.\frac{\log r}{2\pi}\right|_{r=\varepsilon} = -\frac{\log\varepsilon}{2\pi}, \tag{6.112}$$

while, in view of Exercise 6.3.1,

$$\frac{\partial G_0}{\partial \mathbf{n}}(x,y;\xi,\eta) = -\left.\frac{1}{2\pi}\frac{\partial(\log r)}{\partial r}\right|_{r=\varepsilon} = -\frac{1}{2\pi\varepsilon}. \tag{6.113}$$

Therefore,

$$\oint_{C_\varepsilon}\frac{\partial G_0}{\partial \mathbf{n}}(x,y;\xi,\eta)\,u(\xi,\eta)\,ds = -\frac{1}{2\pi\varepsilon}\oint_{C_\varepsilon}u(\xi,\eta)\,ds,$$

which we recognize as minus the *average* of u on the circle of radius ε. As $\varepsilon\to 0$, the circles shrink down to their common center, and so, by continuity, the averages tend to the value $u(x,y)$ at the center; thus,

$$\lim_{\varepsilon\to 0}\oint_{C_\varepsilon}\frac{\partial G_0}{\partial \mathbf{n}}(x,y;\xi,\eta)\,u(\xi,\eta)\,ds = -u(x,y). \tag{6.114}$$

On the other hand, using (6.112), and then (6.89) on the disk D_ε, we have

$$\begin{aligned}\oint_{C_\varepsilon}G_0(x,y;\xi,\eta)\,\frac{\partial u}{\partial \mathbf{n}}(\xi,\eta)\,ds &= -\frac{\log\varepsilon}{2\pi}\oint_{C_\varepsilon}\frac{\partial u}{\partial \mathbf{n}}(\xi,\eta)\,ds \\ &= -\frac{\log\varepsilon}{2\pi}\iint_{D_\varepsilon}\Delta u(\xi,\eta)\,d\xi\,d\eta = -\,(\varepsilon^2\log\varepsilon)\,\overline{\Delta u}_\varepsilon,\end{aligned}$$

where

$$\overline{\Delta u}_\varepsilon = \frac{1}{2\pi\varepsilon^2}\iint_{D_\varepsilon}\Delta u(\xi,\eta)\,d\xi\,d\eta$$

is the *average* of Δu over the disk D_ε. As above, as $\varepsilon\to 0$, the averages over the disks converge to the value at their common center, $\overline{\Delta u}_\varepsilon\to\Delta u(x,y)$, and hence

$$\lim_{\varepsilon\to 0}\oint_{C_\varepsilon}G_0(x,y;\xi,\eta)\,\frac{\partial u}{\partial \mathbf{n}}(\xi,\eta)\,ds = \lim_{\varepsilon\to 0}(-\,\varepsilon^2\log\varepsilon)\,\overline{\Delta u}_\varepsilon = 0. \tag{6.115}$$

In view of (6.110, 114, 115), the $\varepsilon\to 0$ limit of (6.111) is exactly the Green representation formula (6.107). *Q.E.D.*

As noted above, the free space Green's function (6.106) represents the gravitational potential in empty two-dimensional space due to a unit point mass, or, equivalently, the two-dimensional electrostatic potential due to a unit point charge sitting at position $\boldsymbol{\xi}$. The corresponding gravitational or electrostatic force field is obtained by taking its gradient:

$$\mathbf{F} = \nabla G_0 = -\frac{\mathbf{x}-\boldsymbol{\xi}}{2\pi\,\|\,\mathbf{x}-\boldsymbol{\xi}\,\|^2}.$$

Its magnitude

$$\|\,\mathbf{F}\,\| = \frac{1}{2\pi\,\|\,\mathbf{x}-\boldsymbol{\xi}\,\|}$$

is inversely proportional to the distance from the mass or charge, which is the two-dimensional form of Newton's and Coulomb's three-dimensional inverse square laws.

The gravitational potential due to a two-dimensional mass, e.g., a flat plate, in the shape of a domain $\Omega \subset \mathbb{R}^2$ is obtained by superimposing delta function sources with strengths equal to the density of the material at each point. The result is the potential function

$$u(x,y) = -\frac{1}{4\pi} \iint_\Omega \rho(\xi,\eta) \log\left[\,(x-\xi)^2 + (y-\eta)^2\,\right] d\xi\, d\eta, \tag{6.116}$$

in which $\rho(\xi,\eta)$ denotes the density at position $(\xi,\eta) \in \Omega$.

Example 6.18. The gravitational potential due to a circular disk $D = \{\, x^2 + y^2 \le 1 \,\}$ of unit radius and unit density $\rho \equiv 1$ is

$$u(x,y) = -\frac{1}{4\pi} \iint_D \log\left[\,(x-\xi)^2 + (y-\eta)^2\,\right] d\xi\, d\eta. \tag{6.117}$$

A direct evaluation of this double integral is not so easy. However, we can write down the potential in closed form by recalling that it solves the Poisson equation

$$-\Delta u = \begin{cases} 1, & \|\,\mathbf{x}\,\| < 1, \\ 0, & \|\,\mathbf{x}\,\| > 1. \end{cases} \tag{6.118}$$

Moreover, u is clearly radially symmetric, and hence a function of r alone. Thus, in the polar coordinate expression (4.105) for the Laplacian, the θ derivative terms vanish, and so (6.118) reduces to

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = \begin{cases} -1, & r < 1, \\ 0, & r > 1, \end{cases}$$

which is effectively a first-order linear ordinary differential equation for du/dr. Solving separately on the two subintervals produces

$$u(r) = \begin{cases} a + b \log r - \frac{1}{4} r^2, & r < 1, \\ c + d \log r, & r > 1, \end{cases}$$

where a, b, c, d are constants. Continuity of $u(r)$ and $u'(r)$ at $r = 1$ implies $c = a - \frac{1}{4}$, $d = b - \frac{1}{2}$. Moreover, the potential for a non-concentrated mass cannot have a singularity at the origin, and so $b = 0$. Direct evaluation of (6.117) at $x = y = 0$, using polar coordinates, proves that $a = \frac{1}{4}$. We conclude that the gravitational potential (6.117) due to a uniform disk of unit radius, and hence total mass (area) π, is, explicitly,

$$u(x,y) = \begin{cases} \frac{1}{4}(1 - r^2) = \frac{1}{4}(1 - x^2 - y^2), & x^2 + y^2 \le 1, \\ -\frac{1}{2}\log r = -\frac{1}{4}\log(x^2 + y^2), & x^2 + y^2 \ge 1. \end{cases} \tag{6.119}$$

Observe that, outside the disk, the potential is exactly the same as the logarithmic potential due to a point mass of magnitude π located at the origin. Consequently, the gravitational force field outside a uniform disk is the same as if all its mass were concentrated at the origin.

With the free-space logarithmic potential in hand, let us return to the question of finding the Green's function for a boundary value problem on a bounded domain $\Omega \subset \mathbb{R}^2$. Since the logarithmic potential (6.106) is a particular solution to the Poisson equation (6.98), the general solution, according to Theorem 1.6, is given by $u = G_0 + z$, where z is an arbitrary solution to the homogeneous equation $\Delta z = 0$, i.e., an arbitrary harmonic function. Thus, constructing the Green's function has been reduced to the problem of finding the harmonic function z such that $G = G_0 + z$ satisfies the desired homogeneous boundary conditions. Let us explicitly formulate this result for the (inhomogeneous) Dirichlet problem.

Theorem 6.19. *The Green's function for the Dirichlet boundary value problem for the Poisson equation on a bounded domain* $\Omega \subset \mathbb{R}^2$ *has the form*

$$G(x, y; \xi, \eta) = G_0(x, y; \xi, \eta) + z(x, y; \xi, \eta), \tag{6.120}$$

where the first term is the logarithmic potential (6.106), *while, for each* $(\xi, \eta) \in \Omega$, *the second term is the harmonic function that solves the boundary value problem*

$$\begin{aligned} \Delta z &= 0 \qquad \text{on} \qquad \Omega, \\ z(x, y; \xi, \eta) &= \frac{1}{4\pi} \log\big[\, (x-\xi)^2 + (y-\eta)^2 \,\big] \qquad \text{for} \qquad (x, y) \in \partial\Omega. \end{aligned} \tag{6.121}$$

If $u(x, y)$ *is a solution to the inhomogeneous Dirichlet problem*

$$-\Delta u = f, \qquad \mathbf{x} \in \Omega, \qquad\qquad u = h, \qquad \mathbf{x} \in \partial\Omega, \tag{6.122}$$

then

$$u(x, y) = \iint_\Omega G(x, y; \xi, \eta)\, f(\xi, \eta)\, d\xi\, d\eta - \oint_{\partial\Omega} \frac{\partial G}{\partial \mathbf{n}}\,(x, y; \xi, \eta)\, h(\xi, \eta)\, ds, \tag{6.123}$$

where the normal derivative of G *is taken with respect to* $(\xi, \eta) \in \partial\Omega$.

Proof: To show that (6.120) is the Green's function, we note that

$$-\Delta G = -\Delta G_0 - \Delta z = \delta_{(\xi,\eta)} \qquad \text{in} \qquad \Omega, \tag{6.124}$$

while

$$G(x, y; \xi, \eta) = G_0(x, y; \xi, \eta) + z(x, y; \xi, \eta) = 0 \qquad \text{on} \qquad \partial\Omega. \tag{6.125}$$

Next, to establish the solution formula (6.123), since both z and u are C^2, we can use (6.88) (with $v = z$, keeping in mind that $\Delta z = 0$) to establish

$$\begin{aligned} 0 = -&\iint_\Omega z(x, y; \xi, \eta)\, \Delta u(\xi, \eta)\, d\xi\, d\eta \\ &+ \oint_{\partial\Omega} \left(z(x, y; \xi, \eta)\, \frac{\partial u}{\partial \mathbf{n}}\,(\xi, \eta) - \frac{\partial z}{\partial \mathbf{n}}\,(x, y; \xi, \eta)\, u(\xi, \eta) \right) ds. \end{aligned}$$

Adding this to Green's representation formula (6.107), and using (6.125), we deduce that

$$u(x, y) = -\iint_\Omega G(x, y; \xi, \eta)\, \Delta u(\xi, \eta)\, d\xi\, d\eta - \oint_{\partial\Omega} \frac{\partial G(x, y; \xi, \eta)}{\partial \mathbf{n}}\, u(\xi, \eta)\, ds,$$

which, given (6.122), produces (6.123). *Q.E.D.*

The one subtle issue left unresolved is the existence of the solution. Read properly, Theorem 6.19 states that *if* a classical solution exists, *then* it is necessarily given by the Green's function formula (6.123). Proving existence of the solution — and also the existence of the Green's function, or equivalently, the solution z to (6.121) — requires further in-depth analysis, lying beyond the scope of this text. In particular, to guarantee existence, the underlying domain must have a reasonably nice boundary, e.g., a piecewise smooth curve without sharp cusps. Interestingly, lack of regularity at sharp cusps in the boundary underlies the electromagnetic phenomenon known as St. Elmo's fire, cf. [**121**]. Extensions to irregular domains, e.g., those with fractal boundaries, is an active area of contemporary research. Moreover, unlike one-dimensional boundary value problems, mere continuity of the forcing function f is not quite sufficient to ensure the existence of a classical solution to the Poisson boundary value problem; differentiability does suffice, although this assumption can be weakened. We refer to [**61**, **70**], for a development of the Perron method based on approximating the solution by a sequence of *subsolutions*, which, by definition, solve the differential inequality $-\Delta u \leq f$. An alternative proof, using the direct method of the calculus of variations, can be found in [**35**]. The latter proof relies on the characterization of the solution by a minimization principle, which we discuss in some detail in Chapter 9.

Exercises

♢ 6.3.1. Let C_R be a circle of radius R centered at the origin and $\mathbf{n}$ its unit outward normal. Let $f(r,\theta)$ be a function expressed in polar coordinates. Prove that $\partial f/\partial \mathbf{n} = \partial f/\partial r$ on C_R.

6.3.2. Let $f(x) > 0$ be a continuous, positive function on the interval $a \leq x \leq b$. Let Ω be the domain lying between the graph of $f(x)$ on the interval $[a,b]$ and the x–axis. Explain why (6.77) reduces to the usual calculus formula for the area under the graph of f.

6.3.3. Explain what happens to the conclusion of Lemma 6.16 if Ω is not a connected domain.

6.3.4. Can you find constants c_n such that the functions $g_n(x,y) = c_n[1 + n^2(x^2+y^2)]^{-1}$ converge to the two-dimensional delta function: $g_n(x,y) \to \delta(x,y)$ as $n \to \infty$?

♢ 6.3.5. Explain why the two-dimensional delta function satisfies the scaling law

$$\delta(\beta\, x, \beta\, y) = \frac{1}{\beta^2}\,\delta(x,y), \qquad \text{for} \qquad \beta \neq 0.$$

♢ 6.3.6. Write out a polar coordinate formula, in terms of $\delta(r - r_0)$ and $\delta(\theta - \theta_0)$, for the two-dimensional delta function $\delta(x - x_0, y - y_0) = \delta(x - x_0)\,\delta(y - y_0)$.

6.3.7. *True or false*: $\delta(\mathbf{x}) = \delta(\|\,\mathbf{x}\,\|)$.

♢ 6.3.8. Suppose that $\xi = f(x,y)$, $\eta = g(x,y)$ defines a one-to-one C^1 map from a domain $D \subset \mathbb{R}^2$ to the domain $\Omega = \{\,(\xi,\eta) = (f(x,y), g(x,y)) \,|\, (x,y) \in D\,\} \subset \mathbb{R}^2$, and has nonzero Jacobian determinant: $J(x,y) = f_x g_y - f_y g_x \neq 0$ for all $(x,y) \in D$. Suppose further that $(0,0) = (f(x_0,y_0), g(x_0,y_0)) \in \Omega$ for $(x_0,y_0) \in D$. Prove the following formula governing the effect of the map on the two-dimensional delta function:

$$\delta(f(x,y), g(x,y)) = \frac{\delta(x - x_0, y - y_0)}{|\,J(x_0,y_0)\,|}\,. \tag{6.126}$$

6.3.9. Suppose $f(x,y) = \begin{cases} 1, & 3x - 2y > 1, \\ 0, & 3x - 2y < 1. \end{cases}$ Compute its partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ in the sense of generalized functions.

6.3.10. Find a series solution to the rectangular boundary value problem (4.91–92) when the boundary data $f(x) = \delta(x - \xi)$ is a delta function at a point $0 < \xi < a$. Is your solution infinitely differentiable inside the rectangle?

6.3.11. Answer Exercise 6.3.10 when $f(x) = \delta'(x - \xi)$ is the derivative of the delta function.

6.3.12. A 1 meter square plate is subject to the Neumann boundary conditions $\partial u/\partial \mathbf{n} = 1$ on its entire boundary. What is the equilibrium temperature? Explain.

♢ 6.3.13. A *conservation law* for an equilibrium system in two dimensions is, by definition, a divergence expression

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0 \tag{6.127}$$

that vanishes for all solutions.

(a) Given a conservation law prescribed by $\mathbf{v} = (X, Y)$ defined on a simply connected domain D, show that the line integral $\int_C \mathbf{v} \cdot \mathbf{n}\, ds = \int_C X\, dy - Y\, dx$ is path-independent, meaning that its value depends only on the endpoints of the curve C.

(b) Show that the Laplace equation can be written as a conservation law, and write down the corresponding path-independent line integral.

Note: Path-independent integrals are of importance in the study of cracks, dislocations, and other material singularities, [**49**].

♢ 6.3.14. In two-dimensional dynamics, a *conservation law* is an equation of the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0, \tag{6.128}$$

in which T is the *conserved density*, while $\mathbf{v} = (X, Y)$ represents the associated *flux*.

(a) Prove that, on a bounded domain $\Omega \subset \mathbb{R}^2$, the rate of change of the integral $\iint_\Omega T\, dx\, dy$ of the conserved density depends only on the flux through the boundary $\partial\Omega$.

(b) Write the partial differential equation $u_t + u\,u_x + u\,u_y = 0$ as a conservation law. What is the integrated version?

The Method of Images

The preceding analysis exposes the underlying form of the Green's function, but we are still left with the determination of the harmonic component $z(x, y)$ required to match the logarithmic potential boundary values, cf. (6.121). We will discuss two principal analytic techniques employed to produce explicit formulas. The first is an adaptation of the method of separation of variables, which leads to infinite series expressions. We will not dwell on this approach here, although a couple of the exercises ask the reader to work through some of the details; see also the discussion leading up to (9.110). The second is the *Method of Images*, which will be developed in this section. Another approach is based on the theory of *conformal mapping*; it can be found in books on complex analysis, including [**53**, **98**]. While the first two methods are limited to a fairly small class of domains, they extend to higher-dimensional problems, as well as to certain other types of elliptic boundary value problems, whereas conformal mapping is, unfortunately, restricted to two-dimensional problems involving the Laplace and Poisson equations.

We already know that the singular part of the Green's function for the two-dimensional Poisson equation is provided by a logarithmic potential. The problem, then, is to construct the harmonic part, called $z(x, y)$ in (6.120), so that the sum has the correct homogeneous boundary values, or, equivalently, so that $z(x, y)$ has the same boundary values as the logarithmic potential. In certain cases, $z(x, y)$ can be thought of as the potential induced by one or more hypothetical electric charges (or, equivalently, gravitational point masses) that are located *outside* the domain Ω, arranged in such a manner that their combined electrostatic potential happens to coincide with the logarithmic potential on the boundary of the domain. The goal, then, is to place image charges of suitable strengths in the appropriate positions.

Here, we will only consider the case of a single image charge, located at a position $\boldsymbol{\eta} \notin \Omega$. We scale the logarithmic potential (6.106) by the charge strength, and, for added

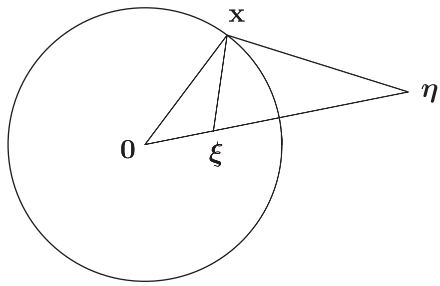

Figure 6.13. Method of Images for the unit disk.

flexibility, include an additional constant — the charge's potential baseline:

$$z(x,y) = a \log \| \mathbf{x} - \boldsymbol{\eta} \| + b, \qquad \boldsymbol{\eta} \in \mathbb{R}^2 \setminus \overline{\Omega}.$$

The function $z(x,y)$ is harmonic inside Ω, since the logarithmic potential is harmonic everywhere except at the external singularity $\boldsymbol{\eta}$. For the Dirichlet boundary value problem, then, for each point $\boldsymbol{\xi} \in \Omega$, we must find a corresponding image point $\boldsymbol{\eta} \in \mathbb{R}^2 \setminus \overline{\Omega}$ and constants $a, b \in \mathbb{R}$ such that†

$$\log \| \mathbf{x} - \boldsymbol{\xi} \| = a \log \| \mathbf{x} - \boldsymbol{\eta} \| + b \qquad \text{for all} \qquad \mathbf{x} \in \partial\Omega,$$

or, equivalently,

$$\| \mathbf{x} - \boldsymbol{\xi} \| = \lambda \| \mathbf{x} - \boldsymbol{\eta} \|^a \qquad \text{for all} \qquad \mathbf{x} \in \partial\Omega, \tag{6.129}$$

where $\lambda = e^b$. For each fixed $\boldsymbol{\xi}, \boldsymbol{\eta}, \lambda, a$, the equation in (6.129) will, typically, implicitly prescribe a plane curve, but it is not clear that one can always arrange that these curves all coincide with the boundary of our domain.

To make further progress, we appeal to a geometric construction based on similar triangles. Let us select $\boldsymbol{\eta} = c\,\boldsymbol{\xi}$ to be a point lying on the ray through $\boldsymbol{\xi}$. Its location is chosen so that the triangle with vertices $\mathbf{0}, \mathbf{x}, \boldsymbol{\eta}$ is similar to the triangle with vertices $\mathbf{0}, \boldsymbol{\xi}, \mathbf{x}$, noting that they have the same angle at the common vertex $\mathbf{0}$ — see Figure 6.13. Similarity requires that the triangles' corresponding sides have a common ratio, and so

$$\frac{\| \boldsymbol{\xi} \|}{\| \mathbf{x} \|} = \frac{\| \mathbf{x} \|}{\| \boldsymbol{\eta} \|} = \frac{\| \mathbf{x} - \boldsymbol{\xi} \|}{\| \mathbf{x} - \boldsymbol{\eta} \|} = \lambda. \tag{6.130}$$

The last equality implies that (6.129) holds with $a = 1$. Consequently, if we choose

$$\| \boldsymbol{\eta} \| = \frac{1}{\| \boldsymbol{\xi} \|}, \qquad \text{so that} \qquad \boldsymbol{\eta} = \frac{\boldsymbol{\xi}}{\| \boldsymbol{\xi} \|^2}, \tag{6.131}$$

then

$$\| \mathbf{x} \|^2 = \| \boldsymbol{\xi} \| \, \| \boldsymbol{\eta} \| = 1.$$

† To simplify the formulas, we have omitted the $1/(2\pi)$ factor, which can easily be reinstated at the end of the analysis.

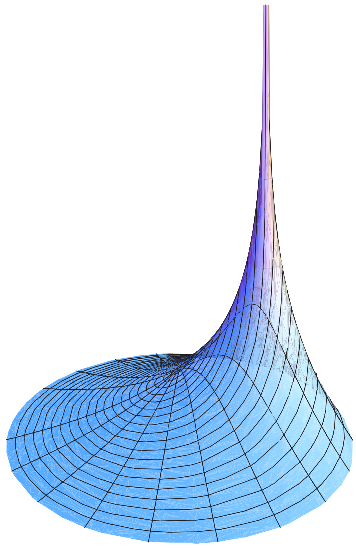

Figure 6.14. Green's function for the unit disk.

Thus $\mathbf{x}$ lies on the unit circle, and, as a result, $\lambda = \| \boldsymbol{\xi} \| = 1/\| \boldsymbol{\eta} \|$. The map taking a point $\boldsymbol{\xi}$ inside the disk to its image point $\boldsymbol{\eta}$ defined by (6.131) is known as *inversion* with respect to the unit circle.

We have now demonstrated that the potentials

$$\frac{1}{2\pi} \log \| \mathbf{x} - \boldsymbol{\xi} \| = \frac{1}{2\pi} \log\big(\| \boldsymbol{\xi} \| \, \| \mathbf{x} - \boldsymbol{\eta} \| \big) = \frac{1}{2\pi} \log \frac{\| \, \| \boldsymbol{\xi} \|^2 \, \mathbf{x} - \boldsymbol{\xi} \, \|}{\| \boldsymbol{\xi} \|}, \qquad \| \mathbf{x} \| = 1, \tag{6.132}$$

have the same boundary values on the unit circle. Consequently, their difference

$$G(\mathbf{x}; \boldsymbol{\xi}) = -\frac{1}{2\pi} \log \| \mathbf{x} - \boldsymbol{\xi} \| + \frac{1}{2\pi} \log \frac{\| \, \| \boldsymbol{\xi} \|^2 \, \mathbf{x} - \boldsymbol{\xi} \, \|}{\| \boldsymbol{\xi} \|} = \frac{1}{2\pi} \log \frac{\| \, \| \boldsymbol{\xi} \|^2 \, \mathbf{x} - \boldsymbol{\xi} \, \|}{\| \boldsymbol{\xi} \| \, \| \mathbf{x} - \boldsymbol{\xi} \|} \tag{6.133}$$

has the required properties for the Green's function for the Dirichlet problem on the unit disk. Writing this in terms of polar coordinates

$$\mathbf{x} = (r \cos \theta, r \sin \theta), \qquad \boldsymbol{\xi} = (\rho \cos \phi, \rho \sin \phi),$$

and applying the Law of Cosines to the triangles in Figure 6.13 produces the explicit formula

$$G(r, \theta; \rho, \phi) = \frac{1}{4\pi} \log \left(\frac{1 + r^2 \rho^2 - 2\, r \rho \cos(\theta - \phi)}{r^2 + \rho^2 - 2\, r \rho \cos(\theta - \phi)} \right). \tag{6.134}$$

In Figure 6.14 we sketch the Green's function for the Dirichlet boundary value problem corresponding to a unit impulse being applied at a point halfway between the center and the edge of the disk. We also require its radial derivative

$$\frac{\partial G}{\partial r}(r, \theta; \rho, \phi) = -\frac{1}{2\pi} \, \frac{1 - r^2}{1 + r^2 - 2\, r \cos(\theta - \phi)}, \tag{6.135}$$

which coincides with its normal derivative on the unit circle. Thus, specializing (6.123), we arrive at a solution to the general Dirichlet boundary value problem for the Poisson equation in the unit disk.

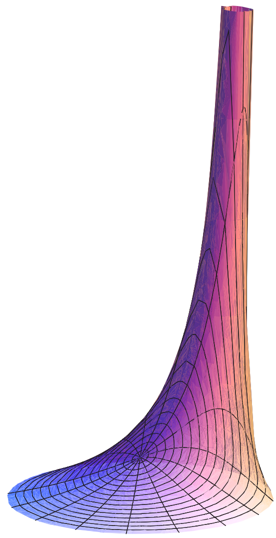

Figure 6.15. The Poisson kernel.

Theorem 6.20. *The solution to the inhomogeneous Dirichlet boundary value problem*

$$-\Delta u = f, \quad \textit{for} \quad r = \|\mathbf{x}\| < 1, \qquad u = h, \quad \textit{for} \quad r = 1,$$

is, when expressed in polar coordinates,

$$u(r,\theta) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_0^1 f(\rho,\phi) \, \log\left(\frac{1 + r^2\rho^2 - 2\,r\rho\cos(\theta-\phi)}{r^2+\rho^2 - 2\,r\rho\cos(\theta-\phi)} \right) \rho\, d\rho\, d\phi \\ + \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\phi) \, \frac{1-r^2}{1+r^2 - 2\,r\cos(\theta-\phi)} \, d\phi. \tag{6.136}$$

When $f \equiv 0$, formula (6.136) recovers the Poisson integral formula (4.126) for the solution to the Dirichlet boundary value problem for the Laplace equation. In particular, the boundary data $h(\theta) = \delta(\theta - \phi)$, corresponding to a concentrated unit heat source applied to a single point on the boundary, produces the *Poisson kernel*

$$u(r,\theta) = \frac{1-r^2}{2\pi\big(1 + r^2 - 2\,r\cos(\theta-\phi)\big)}\,. \tag{6.137}$$

The reader may enjoy verifying that this function indeed solves the Laplace equation and has the correct boundary values in the limit as $r \to 1$.

Exercises

6.3.15. A circular disk of radius 1 is subject to a heat source of unit magnitude on the subdisk $r \le \frac{1}{2}$. Its boundary is kept at 0°.

(*a*) Write down an integral formula for the equilibrium temperature.
(*b*) Use radial symmetry to find an explicit formula for the equilibrium temperature.

6.3.16. A circular disk of radius 1 meter is subject to a unit concentrated heat source at its center and has completely insulated boundary. What is the equilibrium temperature?

♡ 6.3.17. (*a*) For $n > 0$, find the solution to the boundary value problem

$$-\Delta u = \frac{n}{\pi}\, e^{-n\,(x^2+y^2)}, \qquad x^2+y^2<1, \qquad u(x,y)=0, \qquad x^2+y^2=1.$$

(*b*) Discuss what happens in the limit as $n \to \infty$.

♡ 6.3.18. (*a*) Use the Method of Images to construct the Green's function for a half-plane $\{\, y > 0 \,\}$ that is subject to homogeneous Dirichlet boundary conditions. *Hint*: The image point is obtained by reflection. (*b*) Use your Green's function to solve the boundary value problem

$$-\Delta u = \frac{1}{1+y}, \qquad y > 0, \qquad u(x,0)=0.$$

6.3.19. Construct the Green's function for the half-disk $\Omega = \{\, x^2+y^2<1,\ y>0 \,\}$ when subject to homogeneous Dirichlet boundary conditions. *Hint*: Use three image points.

6.3.20. Prove directly that the Poisson kernel (6.137) solves the Laplace equation for all $r < 1$.

♡ 6.3.21. Provide the details for the following alternative method for solving the homogeneous Dirichlet boundary value problem for the Poisson equation on the unit square:

$$-u_{xx}-u_{yy}=f(x,y), \quad u(x,0)=0, \quad u(x,1)=0, \quad u(0,y)=0, \quad u(1,y)=0, \quad 0<x,\ y<1.$$

(*a*) Write both $u(x,y)$ and $f(x,y)$ as Fourier sine series in y whose coefficients depend on x. (*b*) Substitute these series into the differential equation, and equate Fourier coefficients to obtain an infinite system of ordinary boundary value problems for the x-dependent Fourier coefficients of u. (*c*) Use the Green's functions for each boundary value problem to write out the solution and hence a series for the solution to the original boundary value problem. (*d*) Implement this method for the following forcing functions:
(*i*) $f(x,y)=\sin \pi y$, (*ii*) $f(x,y)=\sin \pi x\, \sin 2\pi y$, (*iii*) $f(x,y)=1$.

♢ 6.3.22. Use the method of Exercise 6.3.21 to find a series representation for the Green's function of a unit square subject to Dirichlet boundary conditions.

6.3.23. Write out the details of how to derive (6.134) from (6.133).

6.3.24. *True or false*: If the gravitational potential at a point **a** is greater than its value at the point **b**, then the magnitude of the gravitational force at **a** is greater than its value at **b**.

♠ 6.3.25. (*a*) Write down integral formulas for the gravitational potential and force due to a square plate $S=\{\,-1\le x,y\le 1\,\}$ of unit density $\rho=1$. (*b*) Use numerical integration to calculate the gravitational force at the points $(2,0)$ and $(\sqrt{2},\sqrt{2})$. Before starting, try to predict which point experiences the stronger force, and then check your prediction.

♠ 6.3.26. An equilateral triangular plate with unit area exerts a gravitational force on an observer sitting a unit distance away from its center. Is the force greater if the observer is located opposite a vertex of the triangle or opposite a side? Is the force greater than or less than that exerted by a circular plate of the same area? Use numerical integration to evaluate the double integrals.

6.3.27. Consider the wave equation $u_{tt}=c^2u_{xx}$ on the line $-\infty<x<\infty$. Use the d'Alembert formula (2.82) to solve the initial value problem $u(0,x)=\delta(x-a)$, $u_t(0,x)=0$. Can you realize your solution as the limit of classical solutions?

♢ 6.3.28. Consider the wave equation $u_{tt}=c^2u_{xx}$ on the line $-\infty<x<\infty$. Use the d'Alembert formula (2.82) to solve the initial value problem $u(0,x)=0$, $u_t(0,x)=\delta(x-a)$, modeling the effect of striking the string with a highly concentrated blow at the point $x=a$. Graph the solution at several times. Discuss the behavior of any discontinuities in the solution. In particular, show that $u(t,x)\ne 0$ on the domain of influence of the point $(0,a)$.

6.3.29. (*a*) Write down the solution $u(t,x)$ to the wave equation $u_{tt} = 4u_{xx}$ on the real line with initial data $u(0,x) = \begin{cases} 1 - |\,x\,|, & |\,x\,| \le 1, \\ 0, & \text{otherwise}, \end{cases}$ $\dfrac{\partial u}{\partial t}(0,x) = 0$. (*b*) Explain why $u(t,x)$ is not a classical solution to the wave equation. (*c*) Determine the derivatives $\partial^2 u/\partial t^2$ and $\partial^2 u/\partial x^2$ in the sense of distributions (generalized functions) and use this to justify the fact that $u(t,x)$ solves the wave equation in a distributional sense.

♡ 6.3.30. A piano string of length $\ell = 3$ and wave speed $c = 2$ with both ends fixed is hit by a hammer $\frac{1}{3}$ of the way along. The initial-boundary value problem that governs the resulting vibrations of the string is

$$\frac{\partial^2 u}{\partial t^2} = 4\,\frac{\partial^2 u}{\partial x^2}\,, \qquad u(t,0) = 0 = u(t,3), \qquad u(0,x) = 0, \qquad \frac{\partial u}{\partial t}(0,x) = \delta(x-1).$$

(*a*) What are the fundamental frequencies of vibration?
(*b*) Write down the solution to the initial-boundary value problem in Fourier series form.
(*c*) Write down the Fourier series for the velocity $\partial u/\partial t$ of your solution.
(*d*) Write down the d'Alembert formula for the solution, and sketch a picture of the string at four or five representative times.
(*e*) *True or false*: The solution is periodic in time. If true, what is the period? If false, explain what happens as t increases.

6.3.31. (*a*) Write down a Fourier series for the solution to the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\,, \qquad u(t,-1) = 0 = u(t,1), \qquad u(0,x) = \delta(x), \qquad \frac{\partial u}{\partial t}(0,x) = 0.$$

(*b*) Write down an analytic formula for the solution, i.e., sum your series.
(*c*) In what sense does the series solution in part (a) converge to the true solution? Do the partial sums provide a good approximation to the actual solution?

6.3.32. Answer Exercise 6.3.31 for

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\,, \qquad u(t,-1) = 0 = u(t,1), \qquad u(0,x) = 0, \qquad \frac{\partial u}{\partial t}(0,x) = \delta(x).$$