Chapter 4
Separation of Variables

Three cardinal linear second-order partial differential equations have collectively driven the development of the entire subject. The first two we have already encountered: The *wave equation* describes vibrations and waves in continuous media, including sound waves, water waves, elastic waves, electromagnetic waves, and so on. The *heat equation* models diffusion processes, including thermal energy in solids, solutes in liquids, and biological populations. Third, and in many ways the most important of all, is the *Laplace equation* and its inhomogeneous counterpart, the *Poisson equation*, which govern equilibrium mechanics. The latter two equations arise in an astonishing variety of mathematical and physical contexts, ranging through elasticity and solid mechanics, fluid mechanics, electromagnetism, potential theory, thermomechanics, geometry, probability, number theory, and many other fields. The solutions to the Laplace equation are known as harmonic functions, and the discovery of their many remarkable properties forms one of the most celebrated chapters in the history of mathematics. All three equations, along with their multi-dimensional kin, will appear repeatedly throughout this text.

The aim of the current chapter is to develop the method of separation of variables for solving these key partial differential equations in their two-independent-variable incarnations. For the wave and heat equations, the variables are time, t, and a single space coordinate, x, leading to initial-boundary value problems modeling the dynamical behavior of a one-dimensional medium. For the Laplace and Poisson equations, both variables represent space coordinates, x and y, and the associated boundary value problems model the equilibrium configuration of a planar body, e.g., the deformations of a membrane. Separation of variables seeks special solutions that can be written as the product of functions of the individual variables, thereby reducing the partial differential equation to a pair of ordinary differential equations. More-general solutions can then be expressed as infinite series in the appropriate separable solutions. For the two-variable equations considered here, this results in a Fourier series representation of the solution. In the case of the wave equation, separation of variables serves to focus attention on the vibrational character of the solution, whereas the earlier d'Alembert approach emphasizes its particle-like aspects. Unfortunately, for the Laplace equation, separation of variables applies only to boundary value problems in very special geometries, e.g., rectangles and disks. Further development of the separation of variables method for solving partial differential equations in three or more variables can be found in Chapters 11 and 12.

In the final section, we take the opportunity to summarize the fundamental tripartite classification of planar second-order partial differential equations. Each of the three

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paradigmatic equations epitomizes one of the classes: *hyperbolic*, such as the wave equation; *parabolic*, such as the heat equation; and *elliptic*, such as the Laplace and Poisson equations. Each category enjoys its own distinctive properties and features, both analytic and numeric, and, in effect, forms a separate mathematical subdiscipline.

4.1 The Diffusion and Heat Equations

Let us begin with a brief physical derivation of the heat equation from first principles. We consider a *bar* — meaning a thin, heat-conducting body. "Thin" means that we can regard the bar as a one-dimensional continuum with no significant transverse temperature variation. We will assume that the bar is fully insulated along its length, and so heat can enter (or leave) only through its uninsulated endpoints. We use t to represent time, and $a \leq x \leq b$ to denote spatial position along the bar, which occupies the interval $[a, b]$. Our goal is to find the temperature $u(t, x)$ of the bar at position x and time t.

The dynamical equations governing the temperature are based on three fundamental physical principles. First is the Law of Conservation of Heat Energy. Recalling the general Definition 2.7, this particular conservation law takes the form

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial w}{\partial x} = 0, \tag{4.1}$$

in which $\varepsilon(t, x)$ represents the thermal *energy density* at time t and position x, while $w(t, x)$ denotes the *heat flux*, i.e., the rate of flow of thermal energy along the bar. Our sign convention is that $w(t, x) > 0$ at points where the energy flows in the direction of increasing x (left to right). The integrated form (2.49) of the conservation law, namely

$$\frac{d}{dt} \int_a^b \varepsilon(t, x)\, dx = w(t, a) - w(t, b), \tag{4.2}$$

states that the rate of change in the thermal energy within the bar is equal to the total heat flux passing through its uninsulated ends. The signs of the boundary terms confirm that heat flux *into* the bar results in an increase in temperature.

The second ingredient is a *constitutive assumption* concerning the bar's material properties. It has been observed that, under reasonable conditions, thermal energy is proportional to temperature:

$$\varepsilon(t, x) = \sigma(x)\, u(t, x). \tag{4.3}$$

The factor

$$\sigma(x) = \rho(x)\, \chi(x) > 0 \tag{4.4}$$

is the product of the *density* ρ of the material and its *specific heat capacity* χ, which is the amount of heat energy required to raise the temperature of a unit mass of the material by one degree. Note that we are assuming that the medium is not changing in time, and so physical quantities such as density and specific heat depend only on position x. We also assume, perhaps with less physical justification, that its material properties do not depend upon the temperature; otherwise, we would be forced to deal with a much thornier nonlinear diffusion equation, [**70**, **99**].

The third physical principle relates heat flux and temperature. Physical experiments show that the thermal energy moves from hot to cold at a rate that is in direct proportion to the temperature gradient, which, in the one-dimensional case, means its derivative $\partial u/\partial x$. The resulting relation

$$w(t,x) = -\,\kappa(x)\,\frac{\partial u}{\partial x} \tag{4.5}$$

is known as *Fourier's Law of Cooling*. The proportionality factor $\kappa(x) > 0$ is the *thermal conductivity* of the bar at position x, and the minus sign reflects the everyday observation that heat energy moves from hot to cold. A good heat conductor, e.g., silver, will have high conductivity, while a poor conductor, e.g., glass, will have low conductivity.

Combining the three laws (4.1, 3, 5) produces the *linear diffusion equation*

$$\frac{\partial}{\partial t}\big(\,\sigma(x)\,u\,\big) = \frac{\partial}{\partial x}\left(\kappa(x)\,\frac{\partial u}{\partial x}\right), \qquad a < x < b, \tag{4.6}$$

governing the thermodynamics of a one-dimensional medium. It is also used to model a wide variety of diffusive processes, including chemical diffusion, diffusion of contaminants in liquids and gases, population dispersion, and the spread of infectious diseases. If there is an external heat source along the length of the bar, then the diffusion equation acquires an additional prescribed inhomogeneous term:

$$\frac{\partial}{\partial t}\big(\,\sigma(x)\,u\,\big) = \frac{\partial}{\partial x}\left(\kappa(x)\,\frac{\partial u}{\partial x}\right) + h(t,x), \qquad a < x < b. \tag{4.7}$$

In order to uniquely prescribe the solution $u(t,x)$, we need to specify an initial temperature distribution

$$u(t_0,x) = f(x), \qquad a \le x \le b. \tag{4.8}$$

In addition, we must impose a suitable boundary condition at each end of the bar. There are three common types. The first is a *Dirichlet boundary condition*, where the end is held at a prescribed temperature. For example,

$$u(t,a) = \alpha(t) \tag{4.9}$$

fixes the temperature (possibly time-varying) at the left end. Alternatively, the *Neumann boundary condition*

$$\frac{\partial u}{\partial x}\,(t,a) = \mu(t) \tag{4.10}$$

prescribes the heat flux $w(t,a) = -\,\kappa(a)u_x(t,a)$ there. In particular, a homogeneous Neumann condition, $u_x(t,a) \equiv 0$, models an insulated end that prevents thermal energy flowing in or out. The *Robin*† *boundary condition*,

$$\frac{\partial u}{\partial x}\,(t,a) + \beta(t)\,u(t,a) = \tau(t), \tag{4.11}$$

models the heat exchange resulting from the end of the bar being placed in a heat bath (thermal reservoir) at temperature $\tau(t)$.

Each end of the bar is required to satisfy one of these boundary conditions. For example, a bar with both ends having prescribed temperatures is governed by the pair of Dirichlet boundary conditions

$$u(t,a) = \alpha(t), \qquad u(t,b) = \beta(t), \tag{4.12}$$

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† Since it is named after the nineteenth-century French analyst Victor Gustave Robin, the pronunciation should be with a French accent.

whereas a bar with two insulated ends requires two homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial x}(t,a) = 0, \qquad\qquad \frac{\partial u}{\partial x}(t,b) = 0. \tag{4.13}$$

Mixed boundary conditions, with one end at a fixed temperature and the other insulated, are similarly formulated, e.g.,

$$u(t,a) = \alpha(t), \qquad\qquad \frac{\partial u}{\partial x}(t,b) = 0. \tag{4.14}$$

Finally, the *periodic boundary conditions*

$$u(t,a) = u(t,b), \qquad\qquad \frac{\partial u}{\partial x}(t,a) = \frac{\partial u}{\partial x}(t,b), \tag{4.15}$$

correspond to a circular *ring* obtained by joining the two ends of the bar. As before, we are assuming that the heat is allowed to flow only around the ring — insulation prevents the radiation of heat from one side of the ring affecting the other side.

The Heat Equation

In this book, we will retain the term "heat equation" to refer to the case in which the bar is composed of a uniform material, and so its density ρ, conductivity κ, and specific heat χ are all positive constants. We also exclude external heat sources (other than at the endpoints), meaning that the bar remains insulated along its entire length. Under these assumptions, the general diffusion equation (4.6) reduces to the homogeneous *heat equation*

$$\frac{\partial u}{\partial t} = \gamma\, \frac{\partial^2 u}{\partial x^2} \tag{4.16}$$

for the temperature $u(t,x)$ at time t and position x. The constant

$$\gamma = \frac{\kappa}{\sigma} = \frac{\kappa}{\rho\,\chi} \tag{4.17}$$

is called the *thermal diffusivity*; it incorporates all of the bar's relevant physical properties. The solution $u(t,x)$ will be uniquely prescribed once we specify initial conditions (4.8) and a suitable boundary condition at both of its endpoints.

As we learned in Section 3.1, the separable solutions to the heat equation are based on the exponential ansatz[†]

$$u(t,x) = e^{-\lambda t}\, v(x), \tag{4.18}$$

where $v(x)$ depends only on the spatial variable. Functions of this form, which "separate" into a product of a function of t times a function of x, are known as *separable solutions*. Substituting (4.18) into (4.16) and canceling the common exponential factors, we find that $v(x)$ must solve the second-order linear ordinary differential equation

$$-\gamma\, \frac{d^2 v}{dx^2} = \lambda\, v.$$

[†] Anticipating the eventual signs of the eigenvalues, and to facilitate later discussions, we now include a minus sign in the exponential term.

Each nontrivial solution $v(x) \not\equiv 0$ is an *eigenfunction*, with associated *eigenvalue* λ, for the linear differential operator $L[v] = -\gamma v''(x)$. With the separable eigensolutions (4.18) in hand, we will then be able to reconstruct the desired solution $u(t,x)$ as a linear combination, or rather infinite series, thereof.

Let us concentrate on the simplest case: a uniform, insulated bar of length ℓ that is held at zero temperature at both ends. We specify its initial temperature $f(x)$ at time $t_0 = 0$, and so the relevant initial and boundary conditions are

$$\begin{aligned} u(t,0) = 0, \qquad u(t,\ell) = 0, \qquad & t \geq 0, \\ u(0,x) = f(x), \qquad & 0 \leq x \leq \ell. \end{aligned} \tag{4.19}$$

The eigensolutions (4.18) are found by solving the Dirichlet boundary value problem

$$\gamma \frac{d^2 v}{dx^2} + \lambda v = 0, \qquad v(0) = 0, \qquad v(\ell) = 0. \tag{4.20}$$

By direct calculation (as you are asked to do in Exercises 4.1.19–20), one finds that if λ is either complex, or real and nonpositive, then the only solution to the boundary value problem (4.20) is the trivial solution $v(x) \equiv 0$. This means that all the eigenvalues must necessarily be real and positive. In fact, the reality and positivity of the eigenvalues need not be explicitly checked. Rather, they follow from very general properties of positive definite boundary value problems, of which (4.20) is a particular case. See Section 9.5 for the underlying theory and Theorem 9.34 for the relevant result.

When $\lambda > 0$, the general solution to the differential equation is a trigonometric function

$$v(x) = a \cos \omega x + b \sin \omega x, \qquad \text{where} \qquad \omega = \sqrt{\lambda/\gamma},$$

and a and b are arbitrary constants. The first boundary condition requires $v(0) = a = 0$. This serves to eliminate the cosine term, and then the second boundary condition requires

$$v(\ell) = b \sin \omega \ell = 0.$$

Therefore, since we require $b \neq 0$ — otherwise, the solution is trivial and does not qualify as an eigenfunction — $\omega \ell$ must be an integer multiple of π, and so

$$\omega = \frac{\pi}{\ell}, \qquad \frac{2\pi}{\ell}, \qquad \frac{3\pi}{\ell}, \qquad \ldots\,.$$

We conclude that the eigenvalues and eigenfunctions of the boundary value problem (4.20) are

$$\lambda_n = \gamma \left(\frac{n\pi}{\ell}\right)^2, \qquad v_n(x) = \sin \frac{n\pi x}{\ell}, \qquad n = 1, 2, 3, \ldots\,. \tag{4.21}$$

The corresponding eigensolutions (4.18) are

$$u_n(t,x) = \exp\left(-\frac{\gamma n^2 \pi^2 t}{\ell^2}\right) \sin \frac{n\pi x}{\ell}, \qquad n = 1, 2, 3, \ldots\,. \tag{4.22}$$

Each represents a trigonometrically oscillating temperature profile that maintains its form while decaying to zero at an exponentially fast rate.

To solve the general initial value problem, we assemble the eigensolutions into an infinite series,

$$u(t,x) = \sum_{n=1}^{\infty} b_n u_n(t,x) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{\gamma n^2 \pi^2 t}{\ell^2}\right) \sin \frac{n\pi x}{\ell}, \tag{4.23}$$

whose coefficients b_n are to be fixed by the initial conditions. Indeed, assuming that the series converges, the initial temperature profile is

$$u(0,x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell} = f(x). \tag{4.24}$$

This has the form of a Fourier sine series (3.52) on the interval $[0,\ell]$. Thus, the coefficients are determined by the Fourier formulae (3.53), and so

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell}\, dx, \qquad n = 1,2,3,\ldots\ . \tag{4.25}$$

The resulting formula (4.23) describes the Fourier sine series for the temperature $u(t,x)$ of the bar at each later time $t \geq 0$.

Example 4.1. Consider the initial temperature profile

$$u(0,x) = f(x) = \begin{cases} -x, & 0 \leq x \leq \frac{1}{5}, \\ x - \frac{2}{5}, & \frac{1}{5} \leq x \leq \frac{7}{10}, \\ 1 - x, & \frac{7}{10} \leq x \leq 1, \end{cases} \tag{4.26}$$

on a bar of length 1, plotted in the first graph in Figure 4.1. Using (4.25), the first few Fourier coefficients of $f(x)$ are computed (by either exact or numerical integration) to be

$$\begin{array}{lllll} b_1 \approx .0897, & b_2 \approx -.1927, & b_3 \approx -.0289, & b_4 = 0, & \\ b_5 \approx -.0162, & b_6 \approx .0132, & b_7 \approx .0104, & b_8 = 0, & \ldots\ . \end{array}$$

The resulting Fourier series solution to the heat equation is

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty} b_n\, u_n(t,x) = \sum_{n=1}^{\infty} b_n\, e^{-\gamma n^2 \pi^2 t} \sin n\pi x \\ &\approx .0897\, e^{-\gamma\pi^2 t} \sin \pi x - .1927\, e^{-4\gamma\pi^2 t} \sin 2\pi x - .0289\, e^{-9\gamma\pi^2 t} \sin 3\pi x - \cdots\ . \end{aligned}$$

In Figure 4.1, the solution, for $\gamma = 1$, is plotted at some representative times. Observe that the corners in the initial profile are immediately smoothed out. As time progresses, the solution decays, at a fast exponential rate of $e^{-\pi^2 t} \approx e^{-9.87\, t}$, to a uniform, zero temperature, which is the equilibrium temperature distribution for the homogeneous Dirichlet boundary conditions. As the solution decays to thermal equilibrium, the higher Fourier modes rapidly disappear, and the solution assumes the progressively more symmetric shape of a single sine arc, of rapidly decreasing amplitude.

Smoothing and Long–Time Behavior

The fact that we can write the solution to an initial-boundary value problem in the form of an infinite series (4.23) is progress of a sort. However, because we are unable to sum the series in closed form, this "solution" is much less satisfying than a direct, explicit formula. Nevertheless, there are important qualitative and quantitative features of the solution that can be easily gleaned from such series expansions.

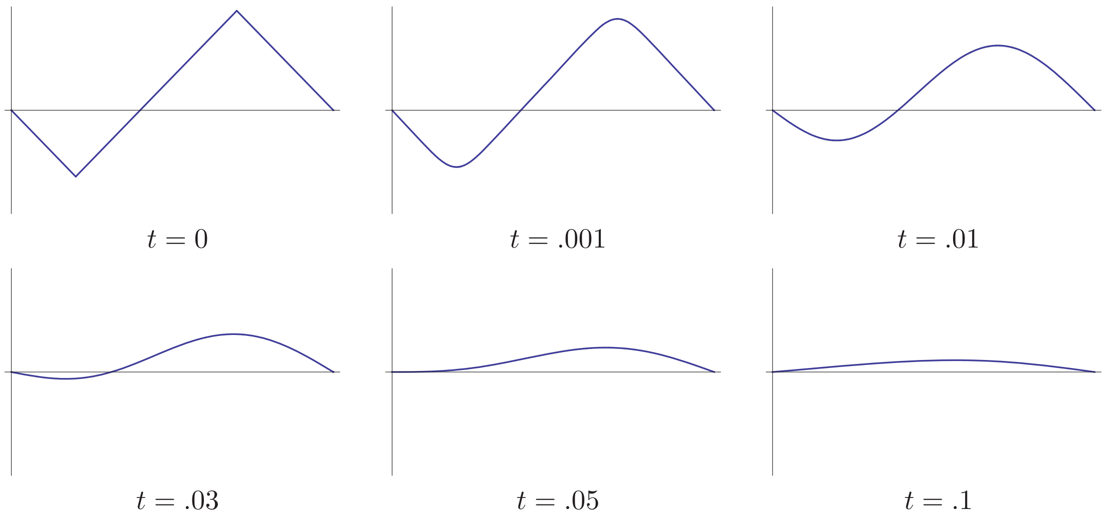

Figure 4.1. A solution to the heat equation.

If the initial data $f(x)$ is integrable (e.g., piecewise continuous), then its Fourier coefficients are uniformly bounded; indeed, for any $n \geq 1$,

$$|\, b_n \,| \;\leq\; \frac{2}{\ell} \int_0^\ell \left|\, f(x) \sin \frac{n \pi x}{\ell} \,\right| dx \;\leq\; \frac{2}{\ell} \int_0^\ell |\, f(x) \,|\, dx \equiv M. \tag{4.27}$$

This property holds even for quite irregular data. Under these conditions, each term in the series solution (4.23) is bounded by an exponentially decaying function

$$\left|\; b_n \exp\left(-\, \frac{\gamma\, n^2 \pi^2}{\ell^2}\, t \right) \sin \frac{n \pi x}{\ell} \;\right| \;\leq\; M \exp\left(-\, \frac{\gamma\, n^2 \pi^2}{\ell^2}\, t \right).$$

This means that, as soon as $t > 0$, most of the high-frequency terms, $n \gg 0$, will be extremely small. Only the first few terms will be at all noticeable, and so the solution essentially degenerates into a finite sum over the first few Fourier modes. As time increases, more and more of the Fourier modes will become negligible, and the sum further degenerates into fewer and fewer significant terms. Eventually, as $t \to \infty$, *all* of the Fourier modes will decay to zero. Therefore, the solution will converge exponentially fast to a zero temperature profile: $u(t,x) \to 0$ as $t \to \infty$, representing the bar in its final uniform thermal equilibrium. The fact that its equilibrium temperature is zero is the result of holding both ends of the bar fixed at zero temperature, whereby any initial thermal energy is eventually dissipated away through the ends. The small-scale temperature fluctuations tend to rapidly cancel out through diffusion of thermal energy, and the last term to disappear is the one with the slowest decay, namely

$$u(t,x) \;\approx\; b_1 \exp\left(-\, \frac{\gamma \pi^2}{\ell^2}\, t \right) \sin \frac{\pi x}{\ell}, \qquad \text{where} \qquad b_1 = \frac{2}{\ell} \int_0^\ell f(x) \sin \frac{\pi x}{\ell}\, dx. \tag{4.28}$$

For generic initial data, the coefficient $b_1 \neq 0$, and the solution approaches thermal equilibrium at an exponential rate prescribed by the smallest eigenvalue, $\lambda_1 = \gamma \pi^2 / \ell^2$, which is proportional to the thermal diffusivity divided by the square of the length of the bar. The

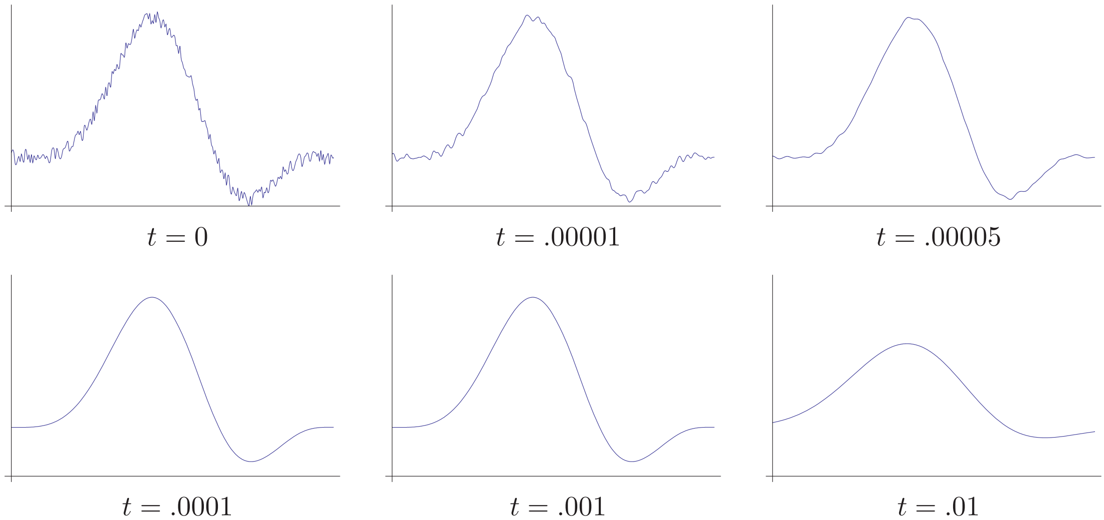

Figure 4.2. Denoising a signal with the heat equation.

longer the bar, or the smaller the diffusivity, the longer it takes for the effect of holding the ends at zero temperature to propagate along its entire length. Also, again provided $b_1 \neq 0$, the asymptotic shape of the temperature profile is a small, exponentially decaying sine arc, just as we observed in Example 4.1. In exceptional situations, namely when $b_1 = 0$, the solution decays even faster, at a rate equal to the eigenvalue $\lambda_k = \gamma\, k^2 \pi^2/\ell^2$ corresponding to the first nonzero term, $b_k \neq 0$, in the Fourier series; its asymptotic shape now oscillates k times over the interval.

Another, closely related, observation is that, for any fixed time $t > 0$ after the initial moment, the coefficients in the Fourier sine series (4.23) decay exponentially fast as $n \to \infty$. According to Corollary 3.32, this implies that the Fourier series converges to an infinitely differentiable function of x at each positive time t, *no matter how unsmooth the initial temperature profile*. We have discovered the basic smoothing property of heat flow, which we state for a general initial time t_0.

Theorem 4.2. *If $u(t,x)$ is a solution to the heat equation with piecewise continuous initial data $f(x) = u(t_0,x)$, or, more generally, initial data satisfying* (4.27), *then, for any $t > t_0$, the solution $u(t,x)$ is an infinitely differentiable function of x.*

In other words, the heat equation *instantaneously* smoothes out any discontinuities and corners in the initial temperature profile by fast damping of the high-frequency modes. The heat equation's effect on irregular initial data underlies its effectiveness for smoothing and denoising signals. We take the initial data $u(0,x) = f(x)$ to be a noisy signal, and then evolve the heat equation forward to a prescribed time $t^\star > 0$. The resulting function $g(x) = u(t^\star, x)$ will be a smoothed version of the original signal $f(x)$ in which most of the high-frequency noise has been eliminated. Of course, if we run the heat flow for too long, all of the low-frequency features will also be smoothed out and the result will be a uniform, constant signal. Thus, the choice of *stopping time* $t^\star$ is crucial to the success of this method. Figure 4.2 shows the effect of running the heat equation,† with $\gamma = 1$, on a signal that has been contaminated by random noise. Observe how quickly the noise is removed. By the final time, the overall smoothing effect of the heat flow has caused significant degradation (blurring) of the original signal. The heat equation approach to denoising has the advantage that no Fourier coefficients need be explicitly computed, nor does one need to reconstruct the smoothed signal. Basic numerical solution schemes for the heat equation are to be discussed in Chapter 5.

An important theoretical consequence of the smoothing property is that diffusion is a one-way process — one cannot run time backwards and accurately infer what a temperature distribution looked like in the past. In particular, if the initial data $u(0,x) = f(x)$ is not smooth, then the value of $u(t,x)$ for any $t < 0$ cannot be defined, because if $u(t_0,x)$ were defined and integrable at some $t_0 < 0$ then, by Theorem 4.2, $u(t,x)$ would be smooth at all subsequent times $t > t_0$, including $t = 0$, in contradiction to our assumption. Moreover, for most initial data, the Fourier coefficients in the solution formula (4.23) are, at any $t < 0$, exponentially *growing* as $n \to \infty$, indicating that high-frequency noise has completely overwhelmed the solution, thereby precluding any kind of convergence of the Fourier series.

Mathematically, we can reverse future and past by changing t to $-t$. In the differential equation, this merely reverses the sign of the time-derivative term; the x derivatives are unaffected. Thus, by the above reasoning, the *backwards heat equation*

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^2 u}{\partial x^2}, \qquad \text{with a negative diffusion coefficient} \qquad -\gamma < 0, \tag{4.29}$$

is an *ill-posed problem* in the sense that small changes in the initial data — e.g., a small perturbation of a high-frequency mode — can produce arbitrarily large changes in the solution arbitrarily close to the initial time. In other words, the solution does not depend continuously on the initial data. Even worse, for nonsmooth initial data, the solution is not even well defined in forwards time $t > 0$ (although it is well-posed if we run t backwards). The same holds for more general diffusion processes, e.g., (4.6). If, as in all physically relevant cases, the coefficient of u_{xx} is everywhere positive, then the initial value problem is well-posed for $t > 0$, but ill-posed for $t < 0$. On the other hand, if the coefficient is everywhere negative, the reverse holds. A coefficient that changes signs would cause the differential equation to be ill-posed in both directions.

While theoretically undesirable, the unsmoothing effect of the backwards heat equation has potential benefits in certain contexts. For example, in image processing, diffusion will gradually blur an image by damping out the high-frequency modes. Image enhancement is the reverse process, and can be based on running the heat flow backwards in some stable manner. In forensics, determining the time of death based on the current temperature of a corpse also requires running the equations governing the dissipation of body heat backwards in time. One option would be to restrict the backwards evolution to the first few Fourier modes, which prevents the small-scale fluctuations from overwhelming the computation. Ill-posed problems also arise in the reconstruction of subterranean profiles from seismic data, a central problem of the oil and gas industry. These and other applications are driving contemporary research into how to cleverly circumvent the ill-posedness of backwards diffusion processes.

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† To avoid artifacts at the ends of the interval, we are, in fact, using periodic boundary conditions in the plots. Away from the ends, running the equation with Dirichlet boundary conditions leads to almost identical results.

Remark: The irreversibility of the heat equation, along with the irreversibility of nonlinear transport in the presence of shock waves discussed in Section 2.3, highlight a crucial distinction between partial differential equations and ordinary differential equations. Ordinary differential equations are always reversible — the existence, uniqueness, and continuous dependence properties of solutions are all equally valid in reverse time (although their detailed qualitative and quantitative properties will, of course, depend upon whether time is running forwards or backwards). The irreversibility and ill-posedness of partial differential equations modeling thermodynamical, biological, and other diffusive processes in our universe may explain why Time's Arrow points exclusively to the future.

The Heated Ring Redux

Let us next consider the periodic boundary value problem modeling heat flow in an insulated circular ring. We fix the length of the ring to be $\ell = 2\pi$, with $-\pi \le x \le \pi$ representing the "angular" coordinate around the ring. For simplicity, we also choose units in which the thermal diffusivity is $\gamma = 1$. Thus, we seek to solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad -\pi < x < \pi, \qquad t > 0, \tag{4.30}$$

subject to periodic boundary conditions

$$u(t,-\pi) = u(t,\pi), \qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi), \qquad t \ge 0, \tag{4.31}$$

that ensure continuity of the solution when the angular coordinate switches from $-\pi$ to π. The initial temperature distribution is

$$u(0,x) = f(x), \qquad -\pi < x \le \pi. \tag{4.32}$$

The resulting temperature $u(t,x)$ will be a periodic function in x of period 2π.

Substituting the separable solution ansatz (3.15) into the heat equation and the boundary conditions results in the periodic eigenvalue problem

$$\frac{d^2 v}{dx^2} + \lambda\, v = 0, \qquad v(-\pi) = v(\pi), \qquad v'(-\pi) = v'(\pi). \tag{4.33}$$

As we already noted in Section 3.1, the eigenvalues of this particular boundary value problem are $\lambda_n = n^2$, where $n = 0, 1, 2, \ldots$ is a nonnegative integer; the corresponding eigenfunctions are the trigonometric functions

$$v_n(x) = \cos n x, \qquad \widetilde{v}_n(x) = \sin n x, \qquad n = 0, 1, 2, \ldots .$$

Note that $\lambda_0 = 0$ is a simple eigenvalue, with constant eigenfunction $\cos 0 x = 1$ — the sine solution $\sin 0 x \equiv 0$ is trivial — while the positive eigenvalues are, in fact, double, each possessing two linearly independent eigenfunctions. The corresponding eigensolutions to the heated ring equation (4.30–31) are

$$u_n(t,x) = e^{-n^2 t} \cos n x, \qquad \widetilde{u}_n(t,x) = e^{-n^2 t} \sin n x, \qquad n = 0, 1, 2, 3, \ldots .$$

The resulting infinite series solution is

$$u(t,x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n e^{-n^2 t} \cos n x + b_n e^{-n^2 t} \sin n x \right), \tag{4.34}$$

with as yet unspecified coefficients a_n, b_n. The initial conditions require

$$u(0,x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = f(x), \tag{4.35}$$

which is precisely the complete Fourier series (3.34) of the initial temperature profile $f(x)$. Consequently,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \tag{4.36}$$

are its usual Fourier coefficients (3.35).

As in the Dirichlet problem, after the initial instant, the high-frequency terms in the series (4.34) become extremely small, since $e^{-n^2 t} \ll 1$ for $n \gg 0$. Therefore, as soon as $t > 0$, the solution instantaneously becomes smooth, and quickly degenerates into what is in essence a finite sum over the first few Fourier modes. Moreover, as $t \to \infty$, *all* of the Fourier modes will decay to zero with the exception of the constant mode, associated with the null eigenvalue $\lambda_0 = 0$. Consequently, the solution will converge, at an exponential rate, to a constant-temperature profile,

$$u(t,x) \longrightarrow \tfrac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx,$$

which equals the *average* of the initial temperature profile. In physical terms, since the insulation prevents any thermal energy from escaping the ring, it rapidly redistributes itself so that the ring achieves a uniform constant temperature — its eventual equilibrium state.

Prior to attaining equilibrium, only the very lowest frequency Fourier modes will still be noticeable, and so the solution will asymptotically look like

$$u(t,x) \approx \tfrac{1}{2}a_0 + e^{-t}(a_1\cos x + b_1 \sin x) = \tfrac{1}{2}a_0 + r_1 e^{-t}\cos(x+\delta_1), \tag{4.37}$$

where

$$a_1 = r_1\cos\delta_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx, \qquad b_1 = r_1\sin\delta_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx.$$

Thus, for most initial data, the solution approaches thermal equilibrium at an exponential rate of e^{-t}. The exceptions are when $a_1 = b_1 = 0$, for which the rate of convergence is even faster, namely at a rate $e^{-k^2 t}$, where k is the smallest integer such that at least one of the k^{th} order Fourier coefficients a_k, b_k is nonzero.

In fact, once we are convinced that the bar must tend to thermal equilibrium as $t \to \infty$, we can predict the final temperature without knowing the explicit solution formula. Our derivation in Section 4.1 implies that the heat equation has the form of a conservation law (4.1), with the conserved density being the temperature $u(t,x)$. As in (4.2), the integrated form of the conservation law reads

$$\begin{aligned}\frac{d}{dt}\int_{-\pi}^{\pi} u(t,x)\,dx = \int_{-\pi}^{\pi}\frac{\partial u}{\partial t}(t,x)\,dx &= \gamma\int_{-\pi}^{\pi}\frac{\partial^2 u}{\partial x^2}(t,x)\,dx \\ &= \gamma\left[\frac{\partial u}{\partial x}(t,\pi) - \frac{\partial u}{\partial x}(t,-\pi)\right] = 0,\end{aligned}$$

where the flux terms cancel thanks to the periodic boundary conditions (4.31). Physically, any flux out of one end of the circular bar is immediately fed into the other, abutting end, and so there is no net loss of thermal energy. We conclude that, for the periodic boundary value problem, the total *thermal energy*

$$E(t) = \int_{-\pi}^{\pi} u(t,x)\,dx = \text{constant} \tag{4.38}$$

remains constant for all time. (In contrast, the thermal energy does *not* remain constant for the Dirichlet boundary value problem, decaying steadily to 0 due to the out-flux of heat through the ends of the bar; see Exercise 4.1.13 for further details.)

Remark: More correctly, according to (4.3), the thermal energy is obtained by multiplying the temperature by the product, $\sigma = \rho\,\chi$, of the density and the specific heat of the body. For the heat equation, both are constant, and so the physical thermal energy equals $\sigma\,E(t)$. Mathematically, we can safely ignore this extra constant factor, or, equivalently, work in physical units in which $\sigma = 1$. This does not extend to nonuniform bodies, whose *thermal energy* is given by $E(t) = \displaystyle\int_{-\pi}^{\pi} \sigma(x)\,u(t,x)\,dx$, and whose constancy, under suitable boundary conditions, follows from the conservation-law form (4.6) of the linear diffusion equation.

In general, a system is in (static) *equilibrium* if it remains unaltered as time progresses. Thus, any equilibrium configuration has the form $u = u^\star(x)$, and hence satisfies $\partial u^\star/\partial t = 0$. If, in addition, $u^\star(x)$ is an equilibrium solution to the periodic heat equation (4.30–33), then it must satisfy

$$\frac{\partial u^\star}{\partial t} = 0 = \frac{\partial^2 u^\star}{\partial x^2}\,, \qquad u^\star(-\pi) = u^\star(\pi), \qquad \frac{\partial u^\star}{\partial x}(-\pi) = \frac{\partial u^\star}{\partial x}(\pi). \tag{4.39}$$

In other words, $u^\star$ is a solution to the periodic boundary value problem (4.33) for the null eigenvalue $\lambda = 0$. Thus, *the null eigenfunctions* (*including the zero solution*) *are all the possible equilibrium solutions*. In particular, for the periodic boundary value problem, the null eigenfunctions are constant, and therefore solutions to the periodic heat equation will tend to a constant equilibrium temperature.

Now, once we know that the solution tends to a constant, $u(t,x) \to a$ as $t \to \infty$, then its thermal energy tends to

$$E(t) = \int_{-\pi}^{\pi} u(t,x)\,dx \;\longrightarrow\; \int_{-\pi}^{\pi} a\,dx = 2\pi a \qquad \text{as} \qquad t \longrightarrow \infty.$$

On the other hand, as we just demonstrated, the thermal energy is constant, so

$$E(t) = E(0) = \int_{-\pi}^{\pi} u(0,x)\,dx = \int_{-\pi}^{\pi} f(x)\,dx.$$

Combining these two, we conclude that

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a, \quad \text{and so the equilibrium temperature} \quad a = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

equals the average initial temperature. This reconfirms our earlier result, but avoids having to know an explicit series solution formula. As a result, the latter method can be applied to a much wider range of situations.

Inhomogeneous Boundary Conditions

So far, we have concentrated our attention on homogeneous boundary conditions. There is a simple trick that will convert a boundary value problem with inhomogeneous but constant Dirichlet boundary conditions,

$$\frac{\partial u}{\partial t} = \gamma \, \frac{\partial^2 u}{\partial x^2} \,, \qquad u(t,0) = \alpha, \qquad u(t,\ell) = \beta, \qquad t \geq 0, \tag{4.40}$$

into a homogeneous Dirichlet problem. We begin by solving for the equilibrium temperature profile. As in (4.39), the equilibrium does not depend on t and hence satisfies the boundary value problem

$$\frac{\partial u^\star}{\partial t} = 0 = \gamma \, \frac{\partial^2 u^\star}{\partial x^2} \,, \qquad u^\star(0) = \alpha, \qquad u^\star(\ell) = \beta.$$

Solving the ordinary differential equation yields $u^\star(x) = a + b\,x$, where the constants a, b are fixed by the boundary conditions. We conclude that the equilibrium solution is a straight line connecting the boundary values:

$$u^\star(x) = \alpha + \frac{\beta - \alpha}{\ell}\, x. \tag{4.41}$$

The difference

$$\widetilde{u}(t,x) = u(t,x) - u^\star(x) = u(t,x) - \alpha - \frac{\beta - \alpha}{\ell}\, x \tag{4.42}$$

measures the deviation of the solution from equilibrium. It clearly satisfies the homogeneous boundary conditions at both ends:

$$\widetilde{u}(t,0) = 0 = \widetilde{u}(t,\ell).$$

Moreover, by linearity, since both $u(t,x)$ and $u^\star(x)$ are solutions to the heat equation, so is $\widetilde{u}(t,x)$. The initial data must be similarly adapted:

$$\widetilde{u}(0,x) = u(t,x) - u^\star(x) = f(x) - \alpha - \frac{\beta - \alpha}{\ell}\, x \equiv \widetilde{f}(x). \tag{4.43}$$

Solving the resulting homogeneous initial-boundary value problem, we write $\widetilde{u}(t,x)$ in Fourier series form (4.23), where the Fourier coefficients are specified by the modified initial data $\widetilde{f}(x)$ in (4.43). The solution to the inhomogeneous boundary value problem thus has the series form

$$u(t,x) = \alpha + \frac{\beta - \alpha}{\ell}\, x \; + \; \sum_{n=1}^{\infty} \widetilde{b}_n \, \exp\left(-\, \frac{\gamma\, n^2 \pi^2}{\ell^2}\, t \right) \, \sin \frac{n \pi x}{\ell} \,, \tag{4.44}$$

where

$$\widetilde{b}_n = \frac{2}{\ell} \int_0^\ell \widetilde{f}(x)\, \sin \frac{n \pi x}{\ell}\, dx, \qquad n = 1, 2, 3, \ldots \;. \tag{4.45}$$

Since $\widetilde{u}(t,0)$ decays to zero at an exponential rate as $t \to \infty$, the actual temperature profile (4.44) will asymptotically decay to the equilibrium profile,

$$u(t,x) \quad \longrightarrow \quad u^\star(x) = \alpha \; + \; \frac{\beta - \alpha}{\ell}\, x,$$

at the same exponentially fast rate, governed by the first eigenvalue $\lambda_1 = \pi^2/\ell^2$ — unless $\widetilde{b}_1 = 0$, in which case the decay rate is even faster.

This method does not work as well when the boundary conditions are time-dependent:

$$u(t,0) = \alpha(t), \qquad u(t,\ell) = \beta(t).$$

Attempting to mimic the preceding technique, we discover that the deviation[†]

$$\widetilde{u}(t,x) = u(t,x) - u^\star(t,x), \qquad \text{where} \qquad u^\star(t,x) = \alpha(t) + \frac{\beta(t)-\alpha(t)}{\ell}\, x, \tag{4.46}$$

satisfies the homogeneous boundary conditions, but now solves an inhomogeneous or forced version of the heat equation:

$$\frac{\partial \widetilde{u}}{\partial t} = \frac{\partial^2 \widetilde{u}}{\partial x^2} + h(t,x), \quad \text{where} \quad h(t,x) = -\,\frac{\partial u^\star}{\partial t}\,(t,x) = -\,\alpha'(t) - \frac{\beta'(t)-\alpha'(t)}{\ell}\, x. \tag{4.47}$$

Solution techniques for the latter partial differential equation will be discussed in Section 8.1 below.

Robin Boundary Conditions

Consider a bar of unit length and unit thermal diffusivity, insulated along its length, which has one of its ends held at 0° and the other put in a heat bath. The resulting thermodynamics are modeled by the heat equation subject to Dirichlet boundary conditions at $x = 0$ and Robin boundary conditions at $x = 1$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \qquad u(t,0) = 0, \qquad \frac{\partial u}{\partial x}\,(t,1) + \beta\, u(t,1) = 0, \tag{4.48}$$

where $\beta \neq 0$ is a constant[‡] that measures the rate of transfer of thermal energy, with $\beta > 0$ when the bath is cold and so the energy is being extracted from the bar. As before, the general solution to the resulting initial-boundary value problem can be assembled from the separable eigensolutions based on our usual exponential ansatz $u(t,x) = e^{-\lambda t}\, v(x)$. Substituting this expression into (4.48), we find that the eigenfunction $v(x)$ must satisfy the boundary value problem

$$-\,\frac{d^2 v}{dx^2} = \lambda\, v, \qquad v(0) = 0, \qquad v'(1) + \beta\, v(1) = 0. \tag{4.49}$$

In order to find nontrivial solutions $v(x) \not\equiv 0$ to (4.49), let us first assume $\lambda = \omega^2 > 0$, where, without loss of generality, $\omega > 0$. The solution to the ordinary differential equation that satisfies the Dirichlet boundary condition at $x = 0$ is a constant multiple of $v(x) = \sin \omega\, x$. Substituting this function into the Robin boundary condition at $x = 1$, we find

$$\omega \cos \omega + \beta \sin \omega = 0, \qquad \text{or, equivalently,} \qquad \omega = -\,\beta \tan \omega. \tag{4.50}$$

It is not hard to see that there is an infinite number of real, positive solutions $0 < \omega_1 < \omega_2 < \omega_3 < \cdots \to \infty$ to the latter transcendental equation. Indeed, they can be characterized as the abscissas $\omega_n > 0$ of the intersection points of the graphs of the two functions $f(\omega) = \omega$

[†] In this case, $u^\star(t,x)$ is not an equilibrium solution. Indeed, we do not expect the bar to go to equilibrium if the temperature of its endpoints is constantly changing.

[‡] The case $\beta = 0$ reduces to the mixed boundary value problem, whose analysis is left to the reader.

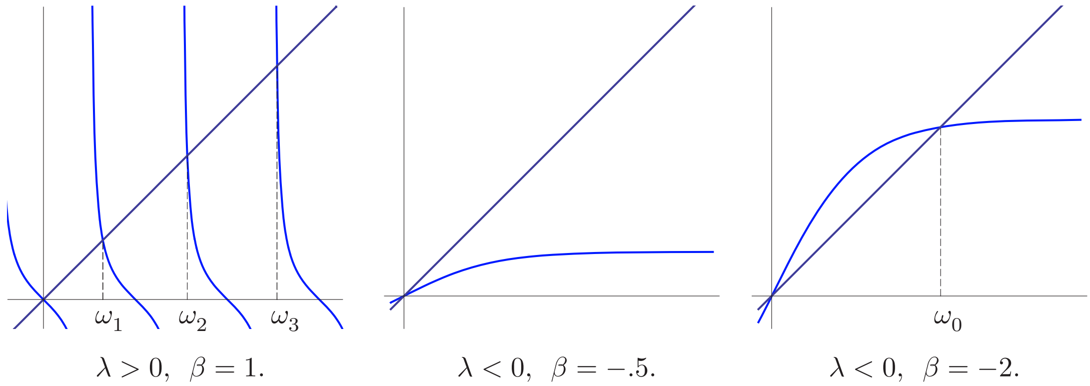


Figure 4.3. Eigenvalue equation for Robin boundary conditions.

and $g(\omega) = -\beta \tan \omega$, as shown in the first plot in Figure 4.3. Each root ω_n defines a positive eigenvalue $\lambda_n = \omega_n^2 > 0$ to the boundary value problem (4.49) and hence an exponentially decaying eigensolution

$$u_n(t,x) = e^{-\lambda_n t} \sin \omega_n x \tag{4.51}$$

to the Robin boundary value problem (4.48). While there is no explicit formula, numerical approximations to the eigenvalues are easily found via a numerical root finder, e.g., Newton's Method, [**24, 94**]. In particular, for $\beta = 1$, the first three eigenvalues are $\lambda_1 = \omega_1^2 \approx 4.1159$, $\lambda_2 = \omega_2^2 \approx 24.1393$, $\lambda_3 = \omega_3^2 \approx 63.6591$.

What about a zero eigenvalue? If $\lambda = 0$ in (4.49), then the solution to the ordinary differential equation that satisfies the Dirichlet boundary condition is a constant multiple of $v(x) = x$. This function satisfies the Robin boundary condition $v'(1) + \beta\, v(1) = 0$ if and only if $\beta = -1$. In this special configuration, the heat equation admits a time-independent eigensolution $u_0(t,x) = x$ with eigenvalue $\lambda_0 = 0$. Physically, the rate of transfer of thermal energy into the bar through its end in the heat bath is exactly enough to cancel the heat loss through the Dirichlet end, resulting in a steady-state solution. All other eigenmodes correspond to positive eigenvalues, and hence are exponentially decaying. The general solution decays to the steady state, which is a constant multiple of the null eigensolution: $u(t,x) \to c\,x$ as $t \to \infty$, at an exponential rate prescribed, generically, by the first positive eigenvalue $\lambda_1 > 0$.

However, in contrast to the more common types of boundary conditions (Dirichlet, Neumann, mixed, periodic), we cannot automatically rule out the existence of negative eigenvalues in the Robin case. Suppose $\lambda = -\omega^2 < 0$ with $\omega > 0$. Now the solution to (4.49) that satisfies the Dirichlet boundary condition at $x = 0$ is a constant multiple of the hyperbolic sine function $v(x) = \sinh \omega\, x$. Substituting this expression into the Robin boundary condition at $x = 1$ produces

$$\omega \cosh \omega + \beta \sinh \omega = 0, \qquad \text{or, equivalently,} \qquad \omega = -\,\beta \tanh \omega, \tag{4.52}$$

where

$$\tanh \omega = \frac{\sinh \omega}{\cosh \omega} = \frac{e^{\omega} - e^{-\omega}}{e^{\omega} + e^{-\omega}} \tag{4.53}$$

is the hyperbolic tangent. If $\beta > -1$, there are no solutions $\omega > 0$ to this transcendental equation, and in this case all the eigenvalues are strictly positive and all solutions to the heat equation are exponentially decaying. On the other hand, if $\beta < -1$, there is a single solution $\omega_0 > 0$, which produces a single negative eigenvalue $\lambda_0 = -\omega_0^2$. Representative graphs illustrating the two possibilities appear in Figure 4.3; in the first, the graph of $f(\omega) = \omega$ does not intersect the graph of $g(\omega) = \frac{1}{2}\tanh\omega$ when $\omega > 0$, whereas it intersects the graph of $\widehat{g}(\omega) = 2\tanh\omega$ at a single point, with abscissa $\omega_0 \approx 1.9150$, producing the negative eigenvalue $\lambda_0 \approx -\omega_0^2 \approx -3.6673$. Thus, when $\beta < -1$, there is, in addition to all the exponentially decaying eigenmodes associated with the positive eigenvalues, a single unstable exponentially *growing* eigenmode

$$u_0(t,x) = e^{\lambda_0 t}\,\sinh\omega_0\,x. \tag{4.54}$$

Physically, $\beta < -1$ implies that thermal energy is entering the Robin end of the bar at a faster rate than can be removed through the Dirichlet end, and hence the bar experiences an exponential increase in its overall temperature.

Remark: Even though some Robin boundary conditions admit exponentially growing solutions, and hence lead to *unstable* dynamics, the initial-boundary value problem remains *well-posed* because the solution exists and is uniquely determined by the initial data, and, moreover, small changes in the initial conditions induce relatively small changes in the resulting solution on bounded time intervals.

The Root Cellar Problem

As a final example, we discuss a problem that involves analysis of the heat equation on a semi-infinite interval. The question is this: how deep should you dig a root cellar? In the prerefrigeration era, a root cellar was used to keep food cool in the summer, but not freeze in the winter. We assume that the temperature inside the Earth depends only on the depth and the time of year. Let $u(t,x)$ denote the deviation in the temperature from its annual mean at depth $x > 0$ and time t. We shall assume that the temperature at the Earth's surface, $x = 0$, fluctuates in a periodic manner; specifically, we set

$$u(t,0) = a\,\cos\omega\,t, \tag{4.55}$$

where the oscillatory frequency

$$\omega = \frac{2\pi}{365.25\text{ days}} = 2.0\times 10^{-7}\text{sec}^{-1} \tag{4.56}$$

refers to yearly temperature variations. In this model, we shall ignore daily temperature fluctuations, since their effect is not significant below a very thin surface layer. At large depths the temperature is assumed to be unvarying:

$$u(t,x) \longrightarrow 0 \qquad \text{as} \qquad x \longrightarrow \infty, \tag{4.57}$$

where 0 refers to the mean temperature.

Thus, we must solve the heat equation on a semi-infinite bar $0 < x < \infty$, with time-dependent boundary conditions $(4.55, 57)$ at the ends. The analysis will be simplified a little if we replace the cosine by a complex exponential, and so we look for a complex solution with boundary conditions

$$u(t,0) = a\,e^{\,\mathrm{i}\,\omega\,t}, \qquad\qquad \lim_{x\to\infty} u(t,x) = 0. \tag{4.58}$$

Let us try a separable solution of the form

$$u(t,x) = v(x)\, e^{\,\mathrm{i}\,\omega\, t}. \tag{4.59}$$

Substituting this expression into the heat equation $u_t = \gamma\, u_{xx}$ leads to

$$\mathrm{i}\,\omega\, v(x)\, e^{\,\mathrm{i}\,\omega\, t} = \gamma\, v''(x)\, e^{\,\mathrm{i}\,\omega\, t}.$$

Canceling the common exponential factors, we conclude that $v(x)$ should solve the boundary value problem

$$\gamma\, v''(x) = \mathrm{i}\,\omega\, v, \qquad v(0) = a, \qquad \lim_{x\to\infty} v(x) = 0.$$

The solutions to the ordinary differential equation are

$$v_1(x) = e^{\sqrt{\mathrm{i}\,\omega/\gamma}\, x} = e^{\sqrt{\omega/(2\gamma)}\,(1+\mathrm{i})\, x}, \qquad v_2(x) = e^{-\sqrt{\mathrm{i}\,\omega/\gamma}\, x} = e^{-\sqrt{\omega/(2\gamma)}\,(1+\mathrm{i})\, x}.$$

The first solution is exponentially growing as $x \to \infty$, and so not germane to our problem. The solution to the boundary value problem must therefore be a multiple of the exponentially decaying solution:

$$v(x) = a\, e^{-\sqrt{\omega/(2\gamma)}\,(1+\mathrm{i})\, x}.$$

Substituting back into (4.59), we find the (complex) solution to the root cellar problem to be

$$u(t,x) = a\, e^{-x\sqrt{\omega/(2\gamma)}}\; e^{\,\mathrm{i}\,(\omega\, t - \sqrt{\omega/(2\gamma)}\, x)}. \tag{4.60}$$

The corresponding real solution is obtained by taking the real part,

$$u(t,x) = a\, e^{-x\sqrt{\omega/(2\gamma)}}\, \cos\left(\omega\, t - \sqrt{\frac{\omega}{2\gamma}}\, x\right). \tag{4.61}$$

The first factor in (4.61) is exponentially decaying as a function of the depth. Thus, the further underground one is, the less noticeable is the effect of the surface temperature fluctuations. The second factor is periodic in time, with the same annual frequency ω. The interesting feature is that the temperature variations (4.61) are typically out of phase with respect to the surface temperature fluctuations, having an overall *phase lag* of

$$\delta = \sqrt{\frac{\omega}{2\gamma}}\, x$$

that depends linearly on the depth x. In particular, a cellar built at a depth where δ is an odd multiple of π will be completely out of phase, being hottest in the winter, and coldest in the summer. Thus, the (shallowest) ideal depth at which to build a root cellar would take $\delta = \pi$, corresponding to a depth of

$$x = \pi\sqrt{\frac{2\gamma}{\omega}}\,. \tag{4.62}$$

For typical soils in the Earth, $\gamma \approx 10^{-6}$ meters^2 sec^{-1}, and so, with ω given by (4.56), $x \approx 9.9$ meters. However, at this depth, the relative amplitude of the oscillations is

$$e^{-x\sqrt{\omega/2\gamma}} = e^{-\pi} = .04,$$

and hence there is only a 4% temperature fluctuation. In Minneapolis, the temperature varies, roughly, from $-40°$C to $+40°$C, and hence our 10-meter-deep root cellar would experience only a 3.2°C annual temperature deviation from the winter, when it is the warmest, to the summer, when it is the coldest. Building the cellar twice as deep would lead to a temperature fluctuation of .2%, now in phase with the surface variations, which means that the cellar would be, for all practical purposes, at constant temperature year round.

Exercises

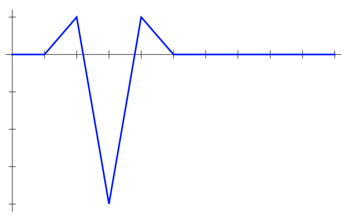
4.1.1. Suppose the ends of a bar of length 1 and thermal diffusivity $\gamma = 1$ are held fixed at respective temperatures 0° and 10°. (a) Determine the equilibrium temperature profile. (b) Determine the rate at which the equilibrium temperature profile is approached. (c) What does the temperature profile look like as it nears equilibrium?

4.1.2. A uniform insulated bar 1 meter long is stored at room temperature of 20° Celsius. An experimenter places one end of the bar in boiling water and the other end in ice water. (a) Set up an initial-boundary value problem that models the temperature in the bar. (b) Find the equilibrium temperature distribution. (c) Discuss how your answer depends on the material properties of the bar.

4.1.3. Consider the initial-boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad \begin{array}{ll} u(t,0) = 0 = u(t,10), & t > 0, \\ u(0,x) = f(x), & 0 < x < 10, \end{array}$$

for the heat equation where the initial data has the following form:



$$f(x) = \begin{cases} x - 1, & 1 \le x \le 2, \\ 11 - 5x, & 2 \le x \le 3, \\ 5x - 19, & 3 \le x \le 4, \\ 5 - x, & 4 \le x \le 5, \\ 0, & \text{otherwise.} \end{cases}$$

Discuss what happens to the solution as t increases. You do *not* need to write down an explicit formula, but for full credit you must explain (sketches can help) at least three or four interesting things that happen to the solution as time progresses.

4.1.4. Find a series solution to the initial-boundary value problem for the heat equation $u_t = u_{xx}$ for $0 < x < 1$ when one the end of the bar is held at 0° and the other is insulated. Discuss the asymptotic behavior of the solution as $t \to \infty$.

4.1.5. Answer Exercise 4.1.4 when both ends of the bar are insulated.

4.1.6. A metal bar, of length $\ell = 1$ meter and thermal diffusivity $\gamma = 2$, is taken out of a 100° oven and then fully insulated except for *one* end, which is fixed to a large ice cube at 0°. (a) Write down an initial-boundary value problem that describes the temperature $u(t,x)$ of the bar at all subsequent times. (b) Write a series formula for the temperature distribution $u(t,x)$ at time $t > 0$. (c) What is the equilibrium temperature distribution in the bar, i.e., for $t \gg 0$? How fast does the solution go to equilibrium? (d) Just before the temperature distribution reaches equilibrium, what does it look like? Sketch a picture and discuss.

4.1.7. A metal bar of length $\ell = 1$ and thermal diffusivity $\gamma = 1$ is fully insulated, including its ends. Suppose the initial temperature distribution is $u(0,x) = \begin{cases} x, & 0 \le x \le \frac{1}{2}, \\ 1 - x, & \frac{1}{2} \le x \le 1. \end{cases}$
(a) Use Fourier series to write down the temperature distribution at time $t > 0$.

(b) What is the equilibrium temperature distribution in the bar, i.e., for $t \gg 0$?
(c) How fast does the solution go to equilibrium? (d) Just before the temperature distribution reaches equilibrium, what does it look like? Sketch a picture and discuss.

4.1.8. (a) Find the series solution to the heat equation $u_t = u_{xx}$ on $-2 < x < 2,\ t > 0$, when subject to the Dirichlet boundary conditions $u(t,-2) = u(t,2) = 0$ and the initial condition $u(0,x) = \begin{cases} x, & |x| < 1, \\ 0, & \text{otherwise.} \end{cases}$ (b) Sketch a graph of the solution at some representative times. (c) At what rate does the temperature approach thermal equilibrium?

4.1.9. Solve the heat equation when the right-hand end of a bar of unit length is held at a fixed constant temperature α while the left-hand end is insulated. Discuss the asymptotic behavior of the solution.

4.1.10. For each of the following initial temperature distributions, (i) write out the Fourier series solution to the heated ring (4.30–32), and (ii) find the resulting equilibrium temperature as $t \to \infty$: (a) $\cos x$, (b) $\sin^3 x$, (c) $|x|$, (d) $\begin{cases} 1, & -\pi < x < 0, \\ 0, & 0 < x < \pi. \end{cases}$

♢ 4.1.11. Suppose that the temperature $u(t,x)$ of a homogeneous bar satisfies the heat equation. Show that the associated heat flux $w(t,x)$ is also a solution to the same heat equation.

♢ 4.1.12. Show that the time derivative $v = u_t$ of any solution to the heat equation is also a solution. If $u(t,x)$ satisfies the initial condition $u(0,x) = f(x)$, what initial condition does $v(t,x)$ inherit?

♢ 4.1.13. Explain why the thermal energy $E(t) = \displaystyle\int_0^\ell u(t,x)\,dx$ is not constant for the Dirichlet initial-boundary value problem for the heat equation on the interval $[0,\ell]$.

♢ 4.1.14. (a) Show that the thermal energy $E(t) = \displaystyle\int_0^\ell u(t,x)\,dx$ is constant for the Neumann boundary value problem on the interval $[0,\ell]$. (b) Use part (a) to prove that the constant equilibrium solution for the homogeneous Neumann boundary value problem is equal to the mean initial temperature $u(0,x)$.

4.1.15. Let $u(t,x)$ be any nonconstant solution to the periodic heat equation (4.30–31). Prove that the squared L^2 norm of the solution, $N(t) = \displaystyle\int_{-\pi}^{\pi} u(t,x)^2\,dx$, is a strictly decreasing function of t. *Remark*: Interestingly, comparing this result with formula (4.38), we find that, for the periodic boundary value problem, the integral of u is constant, but the integral of u^2 is strictly decreasing. How is this possible?

♡ 4.1.16. The *cable equation* $v_t = \gamma v_{xx} - \alpha v$, with $\gamma, \alpha > 0$, also known as the *lossy heat equation*,was derived by the nineteenth-century Scottish physicist William Thomson to model propagation of signals in a transatlantic cable. Later, in honor of his work on thermodynamics, including determining the value of absolute zero temperature, he was named Lord Kelvin by Queen Victoria. The cable equation was later used to model the electrical activity of neurons. (a) Show that the general solution to the cable equation is given by $v(t,x) = e^{-\alpha t}\,u(t,x)$, where $u(t,x)$ solves the heat equation $u_t = \gamma u_{xx}$.
(b) Find a Fourier series solution to the Dirichlet initial-boundary value problem
$$v_t = \gamma v_{xx} - \alpha v, \qquad v(0,x) = f(x), \qquad v(t,0) = 0 = v(t,1), \qquad 0 \le x \le 1, \qquad t > 0.$$
Does your solution approach an equilibrium value? If so, how fast?
(c) Answer part (b) for the Neumann problem
$$v_t = \gamma v_{xx} - \alpha v, \qquad v(0,x) = f(x), \qquad v_x(t,0) = 0 = v_x(t,1), \qquad 0 \le x \le 1, \qquad t > 0.$$

♢ 4.1.17. The *convection-diffusion equation* $u_t + c u_x = \gamma u_{xx}$ is a simple model for the diffusion of a pollutant in a fluid flow moving with constant speed c. Show that $v(t,x) = u(t, x + ct)$ solves the heat equation. What is the physical interpretation of this change of variables?

4.1.18. Combine Exercises 4.1.16–17 to solve the *lossy convection-diffusion equation*
$$u_t = \gamma u_{xx} + c u_x - \alpha u.$$

♢ 4.1.19. Let $\gamma > 0$ and $\lambda \leq 0$. (*a*) Find all solutions to the differential equation $\gamma\, v'' + \lambda\, v = 0$. (*b*) Prove that the only solution that satisfies the boundary conditions $v(0) = 0$, $v(\ell) = 0$, is the zero solution $v(x) \equiv 0$.

♢ 4.1.20. Answer Exercise 4.1.19 when λ is a non-real complex number.

4.2 The Wave Equation

Let us return to the one-dimensional *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2\, \frac{\partial^2 u}{\partial x^2}\,, \tag{4.63}$$

with constant wave speed $c > 0$, used to model the vibrations of bars and strings. In Chapter 2, we learned how to explicitly solve the wave equation by the method of d'Alembert. Unfortunately, d'Alembert's approach does not extend to other equations of interest to us, and so alternative solution techniques, particularly those based on Fourier methods, are worth developing. Indeed, the resulting series solutions provide valuable insight into wave dynamics on bounded intervals.

Separation of Variables and Fourier Series Solutions

One of the oldest — and still one of the most widely used — techniques for constructing explicit analytic solutions to a wide range of linear partial differential equations is the method of *separation of variables*. We have, in fact, already employed a simplified version of the method when constructing each eigensolution to the heat equation as an exponential function of t times a function of x. In general, the separation of variables method seeks solutions to the partial differential equation that can be written as the product of functions of the individual independent variables. For the wave equation, we seek solutions

$$u(t,x) = w(t)\, v(x) \tag{4.64}$$

that can be written as the product of a function of t alone and a function of x alone. When the method succeeds (which is not guaranteed in advance), both factors are found as solutions to certain ordinary differential equations.

Let us see whether such an expression can possibly solve the wave equation. First of all,

$$\frac{\partial^2 u}{\partial t^2} = w''(t)\, v(x), \qquad\qquad \frac{\partial^2 u}{\partial x^2} = w(t)\, v''(x),$$

where the primes indicate ordinary derivatives. Substituting these expressions into the wave equation (4.63), we obtain

$$w''(t)\, v(x) = c^2\, w(t)\, v''(x).$$

Dividing both sides by $w(t)\, v(x)$ (which we assume is not identically zero, since otherwise, the solution would be trivial) yields

$$\frac{w''(t)}{w(t)} = c^2\, \frac{v''(x)}{v(x)}\,,$$

which effectively "separates" the t and x variables on each side of the equation, whence the name "separation of variables".

Now, how could a function of t alone be equal to a function of x alone? A moment's reflection should convince the reader that this can happen if and only if the two functions are constant,† so

$$\frac{w''(t)}{w(t)} = c^2\,\frac{v''(x)}{v(x)} = \lambda, \tag{4.65}$$

where we use λ to indicate the common *separation constant*. Thus, the individual factors $w(t)$ and $v(x)$ must satisfy ordinary differential equations

$$\frac{d^2w}{dt^2} - \lambda\,w = 0, \qquad\qquad \frac{d^2v}{dx^2} - \frac{\lambda}{c^2}\,v = 0,$$

as promised. We already know how to solve both of these ordinary differential equations by elementary techniques. There are three different cases, depending on the sign of the separation constant λ. As a result, each value of λ leads to four independent separable solutions to the wave equation, as listed in the accompanying table.

Separable Solutions to the Wave Equation

λ	$w(t)$	$v(x)$	$u(t,x) = w(t)\,v(x)$
$\lambda = -\omega^2 < 0$	$\cos\omega t,\ \sin\omega t$	$\cos\frac{\omega x}{c},\ \sin\frac{\omega x}{c}$	$\cos\omega t\ \cos\frac{\omega x}{c},\ \cos\omega t\ \sin\frac{\omega x}{c},$ $\sin\omega t\ \cos\frac{\omega x}{c},\ \sin\omega t\ \sin\frac{\omega x}{c}$
$\lambda = 0$	$1,\ t$	$1,\ x$	$1,\ x,\ t,\ tx$
$\lambda = \omega^2 > 0$	$e^{-\omega t},\ e^{\omega t}$	$e^{-\omega x/c},\ e^{\omega x/c}$	$e^{-\omega(t+x/c)},\ e^{\omega(t-x/c)},$ $e^{-\omega(t-x/c)},\ e^{\omega(t+x/c)}$

So far, we have not taken the boundary conditions into account. Consider first the case of a string of length ℓ with two fixed ends, and thus subject to homogeneous Dirichlet boundary conditions

$$u(t,0) = 0 = u(t,\ell).$$

Substituting the separable ansatz (4.65), we find that $v(x)$ must satisfy

$$\frac{d^2v}{dx^2} - \frac{\lambda}{c^2}\,v = 0, \qquad\qquad v(0) = 0 = v(\ell). \tag{4.66}$$

The complete system of (nontrivial) solutions to this boundary value problem were found in (4.21):

$$v_n(x) = \sin\frac{n\pi x}{\ell}, \qquad\quad \lambda_n = -\left(\frac{n\pi c}{\ell}\right)^2, \qquad\quad n = 1, 2, 3, \ldots\,.$$

† *Technical detail*: one should assume that the underlying domain is connected for this to be valid as stated. In practice, this technicality can be safely ignored.

Hence, according to the table, the corresponding separable solutions are

$$u_n(t,x) = \cos\frac{n\pi c\,t}{\ell}\ \sin\frac{n\pi x}{\ell}, \qquad \widetilde{u}_n(t,x) = \sin\frac{n\pi c\,t}{\ell}\ \sin\frac{n\pi x}{\ell}. \tag{4.67}$$

We will now employ these solutions to construct a candidate series solution to the wave equation subject to the prescribed boundary conditions:

$$u(t,x) = \sum_{n=1}^{\infty}\left[b_n \cos\frac{n\pi c\,t}{\ell}\ \sin\frac{n\pi x}{\ell} + d_n \sin\frac{n\pi c\,t}{\ell}\ \sin\frac{n\pi x}{\ell}\right]. \tag{4.68}$$

The solution is thus a linear combination of the natural Fourier modes vibrating with frequencies

$$\omega_n = \frac{n\pi c}{\ell} = \frac{n\pi}{\ell}\sqrt{\frac{\kappa}{\rho}}, \qquad n = 1,2,3,\ldots, \tag{4.69}$$

where the second expression follows from (2.66). Observe that, the longer the length ℓ of the string, or the higher its density ρ, the slower the vibrations, whereas increasing its stiffness or tension κ speeds them up — in exact accordance with our physical intuition.

The Fourier coefficients b_n and d_n in (4.68) will be uniquely determined by the initial conditions

$$u(0,x) = f(x), \qquad \frac{\partial u}{\partial t}(0,x) = g(x), \qquad 0 < x < \ell.$$

Differentiating the series term by term, we discover that we must represent the initial displacement and velocity as Fourier sine series

$$u(0,x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} = f(x), \qquad \frac{\partial u}{\partial t}(0,x) = \sum_{n=1}^{\infty} d_n \frac{n\pi c}{\ell}\sin\frac{n\pi x}{\ell} = g(x).$$

Therefore,

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx, \qquad n = 1,2,3,\ldots, \tag{4.70}$$

are the Fourier sine coefficients (3.85) of the initial displacement $f(x)$, while

$$d_n = \frac{2}{n\pi c}\int_0^{\ell} g(x)\sin\frac{n\pi x}{\ell}\,dx, \qquad n = 1,2,3,\ldots. \tag{4.71}$$

are rescaled versions of the Fourier sine coefficients of the initial velocity $g(x)$.

Example 4.3. A string of unit length fixed at both ends is held taut at its center and then released. Our task is to describe the ensuing vibrations. Let us assume that the physical units are chosen so that $c^2 = 1$, and so we are asked to solve the initial-boundary value problem

$$u_{tt} = u_{xx}, \qquad u(0,x) = f(x), \qquad u_t(0,x) = 0, \qquad u(t,0) = u(t,1) = 0. \tag{4.72}$$

To be specific, we assume that the center of the string has been moved by half a unit, and so the initial displacement is

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2}, \\ 1-x, & \frac{1}{2} \le x \le 1. \end{cases}$$

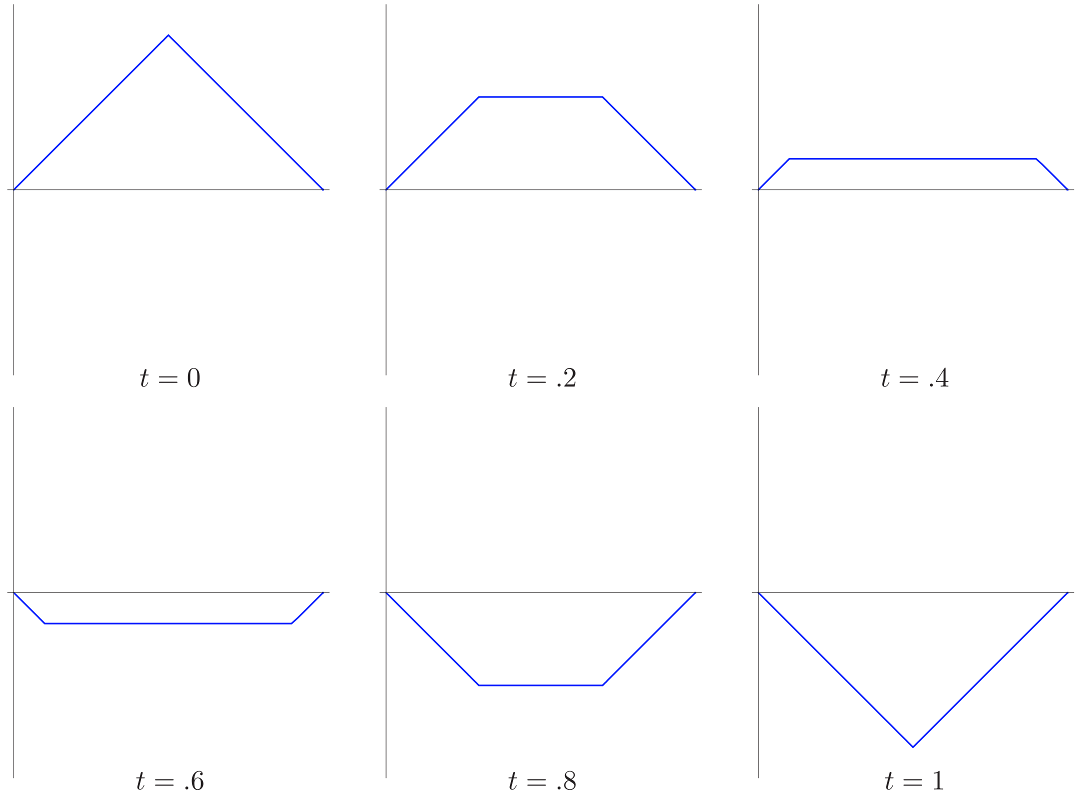


Figure 4.4. Plucked string solution of the wave equation.

The vibrational frequencies $\omega_n = n\pi$ are the integral multiples of π, and so the natural modes of vibration are

$$\cos n\pi t \ \sin n\pi x \qquad \text{and} \qquad \sin n\pi t \ \sin n\pi x \qquad \text{for} \qquad n = 1, 2, \ldots .$$

Consequently, the general solution to the boundary value problem is

$$u(t,x) = \sum_{n=1}^{\infty} \big(b_n \cos n\pi t \ \sin n\pi x + d_n \sin n\pi t \ \sin n\pi x \big),$$

where

$$b_n = 2 \int_0^1 f(x) \sin n\pi x \, dx = \begin{cases} 4 \displaystyle\int_0^{1/2} x \sin n\pi x \, dx = \dfrac{4\,(-1)^k}{(2k+1)^2\pi^2}\,, & n = 2k+1, \\ 0, & n = 2k, \end{cases}$$

while $d_n = 0$. Therefore, the solution is the Fourier sine series

$$u(t,x) = \frac{4}{\pi^2} \sum_{k=0}^{\infty} (-1)^k \, \frac{\cos(2k+1)\pi t \ \sin(2k+1)\pi x}{(2k+1)^2}\,, \tag{4.73}$$

whose profile is depicted in Figure 4.4. At time $t = 1$, the original displacement is reproduced exactly, but upside down. The subsequent dynamics proceeds as before, but in mirror-image form. The original displacement reappears at time $t = 2$, after which time the motion is periodically repeated. Interestingly, at times $t_k = .5, 1.5, 2.5, \ldots$, the displacement is identically zero, $u(t_k, x) \equiv 0$, although the velocity is not, $u_t(t_k, x) \not\equiv 0$. The solution appears to be piecewise affine, i.e., its graph is a collection of straight lines. This can, in fact, be proved as a consequence of the d'Alembert formula; see Exercise 4.2.13. Observe that, unlike the heat equation, the wave equation does *not* smooth out discontinuities and corners in the initial data. And, although we will loosely refer to such piecewise C^2 functions as "solutions", they are not, in fact, classical solutions. (Their status as weak solutions, though, can be established using the methods of Section 10.4.)

While the series form (4.68) of the solution is perhaps less satisfying than a d'Alembert-style formula, we can still use it to deduce important qualitative properties. First of all, since each term is periodic in t with period $2\,\ell/c$, the entire solution is time periodic with that period: $u(t + 2\,\ell/c, x) = u(t, x)$. In fact, after half a period, the solution reduces to

$$u\left(\frac{\ell}{c}, x\right) = \sum_{n=1}^{\infty} (-1)^n\, b_n \sin\frac{n\pi x}{\ell} = -\sum_{n=1}^{\infty} b_n \sin\frac{n\pi(\ell - x)}{\ell} = -\,u(0, \ell - x) = -\,f(\ell - x).$$

In general,

$$u\left(t + \frac{\ell}{c},\, x\right) = -\,u(t, \ell - x), \qquad\qquad u\left(t + \frac{2\,\ell}{c},\, x\right) = u(t, x). \tag{4.74}$$

Therefore, the initial wave form is reproduced, first as an upside down mirror image of itself at time $t = \ell/c$, and then in its original form at time $t = 2\,\ell/c$. This has the important consequence that vibrations of (homogeneous) one-dimensional media are inherently periodic, because the fundamental frequencies (4.69) are all integer multiples of the lowest one: $\omega_n = n\,\omega_1$.

Remark: The immediately preceding remark has important musical consequences. To the human ear, sonic vibrations that are integral multiples of a single frequency, and thus periodic in time, sound harmonious, whereas those with irrationally related frequencies, and hence experiencing aperiodic vibrations, sound dissonant. This is why most tonal instruments rely on vibrations in one dimension, be it a violin or piano string, a column of air in a wind instrument (flute, clarinet, trumpet, or saxophone), a xylophone bar, or a triangle. On the other hand, most percussion instruments rely on the vibrations of two-dimensional media, e.g., drums and cymbals, or three-dimensional solid bodies, e.g., blocks. As we shall see in Chapters 11 and 12, the frequency ratios of the latter are irrationally related, and hence their motion is only quasiperiodic, as in Example 2.20. For some reason, our appreciation of music is psychologically attuned to the differences between rationally related/periodic and irrationally related/quasiperiodic vibrations, [**105**].

Consider next a string with both ends left free, and so subject to the Neumann boundary conditions

$$\frac{\partial u}{\partial x}(t, 0) = 0 = \frac{\partial u}{\partial x}(t, \ell). \tag{4.75}$$

The solutions of (4.66) satisfying $v'(0) = 0 = v'(\ell)$ are now

$$v_n(x) = \cos\frac{n\pi x}{\ell} \qquad \text{with} \qquad \omega_n = \frac{n\pi c}{\ell}, \qquad n = 0, 1, 2, 3, \ldots\,.$$

The resulting solution takes the form of a Fourier cosine series

$$u(t, x) = a_0 + c_0\, t + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi c\,t}{\ell}\,\cos\frac{n\pi x}{\ell} + c_n \sin\frac{n\pi c\,t}{\ell}\,\cos\frac{n\pi x}{\ell}\right). \tag{4.76}$$

The first two terms come from the null eigenfunction $v_0(x) = 1$ with $\omega_0 = 0$. The string vibrates with the same fundamental frequencies (4.69) as in the fixed-end case, but there is now an additional *unstable mode* $c_0 t$ that is no longer periodic, but grows linearly in time. In general, the presence of null eigenfunctions implies that the wave equation admits unstable modes.

Substituting (4.76) into the initial conditions

$$u(0,x) = f(x), \qquad \frac{\partial u}{\partial t}(0,x) = g(x), \qquad 0 < x < \ell,$$

we find that the Fourier coefficients are prescribed, as before, by the initial displacement and velocity:

$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\,\cos\frac{n\pi x}{\ell}\,dx, \qquad c_n = \frac{2}{n\pi c}\int_0^\ell g(x)\,\cos\frac{n\pi x}{\ell}\,dx, \qquad n = 1,2,3,\ldots\,.$$

The order-zero coefficients[†]

$$a_0 = \frac{1}{\ell}\int_0^\ell f(x)\,dx, \qquad c_0 = \frac{1}{\ell}\int_0^\ell g(x)\,dx,$$

are equal to the average initial displacement and average initial velocity of the string. In particular, when $c_0 = 0$, there is no net initial velocity, and the unstable mode is not excited. In this case, the solution is time-periodic, oscillating around the position given by the average initial displacement. On the other hand, if $c_0 \neq 0$, the string will move off with constant average speed c_0, all the while vibrating at the same fundamental frequencies.

Similar considerations apply to the periodic boundary value problem for the wave equation on a circular ring. The details are left as Exercise 4.2.6 for the reader.

Exercises

4.2.1. In music, an octave corresponds to doubling the frequency of the sound waves. On my piano, the middle C string has length .7 meter, while the string for the C an octave higher has length .6 meter. Assuming that they have the same density, how much tighter does the shorter string need to be tuned?

4.2.2. How much longer would a piano string have to be to make the same sound when it is pulled twice as tight?

4.2.3. Write down the solutions to the following initial-boundary value problems for the wave equation in the form of a Fourier series:
(a) $u_{tt} = u_{xx}$, $u(t,0) = u(t,\pi) = 0$, $u(0,x) = 1$, $u_t(0,x) = 0$;
(b) $u_{tt} = 2u_{xx}$, $u(t,0) = u(t,\pi) = 0$, $u(0,x) = 0$, $u_t(0,x) = 1$;
(c) $u_{tt} = 3u_{xx}$, $u(t,0) = u(t,\pi) = 0$, $u(0,x) = \sin^3 x$, $u_t(0,x) = 0$;
(d) $u_{tt} = 4u_{xx}$, $u(t,0) = u(t,1) = 0$, $u(0,x) = x$, $u_t(0,x) = -x$;
(e) $u_{tt} = u_{xx}$, $u(t,0) = u_x(t,1) = 0$, $u(0,x) = 1$, $u_t(0,x) = 0$;
(f) $u_{tt} = 2u_{xx}$, $u_x(t,0) = u_x(t,2\pi) = 0$, $u(0,x) = -1$, $u_t(0,x) = 1$;
(g) $u_{tt} = u_{xx}$, $u_x(t,0) = u_x(t,1) = 0$, $u(0,x) = x(1-x)$, $u_t(0,x) = 0$.

[†] Note that we have not included the usual $\frac{1}{2}$ factor in the constant terms in the Fourier series (4.76).

4.2.4. Find all separable solutions to the wave equation $u_{tt} = u_{xx}$ on the interval $0 \le x \le \pi$ subject to (a) mixed boundary conditions $u(t,0) = 0$, $u_x(t,\pi) = 0$;
(b) Neumann boundary conditions $u_x(t,0) = 0$, $u_x(t,\pi) = 0$.

4.2.5. (a) Under what conditions is the solution to the Neumann boundary value problem (4.75) a periodic function of t? What is the period? (b) Establish explicit periodicity formulas of the form (4.74). (c) Under what conditions is the velocity $\partial u/\partial t$ periodic in t?

♡ 4.2.6. (a) Formulate the periodic initial-boundary value problem for the wave equation on the interval $-\pi \le x \le \pi$, modeling the vibrations of a circular ring. (b) Write out a formula for the solution to your problem in the form of a Fourier series. (c) Is the solution a periodic function of t? If so, what is the period? (d) Suppose the initial displacement coincides with that in Figure 4.6, while the initial velocity is zero. Describe what happens to the solution as time evolves.

4.2.7. Show that the time derivative, $v = \partial u/\partial t$, of any solution to the wave equation is also a solution. If you know the initial conditions of u, what initial conditions does v satisfy?

4.2.8. Find all the separable real solutions to the wave equation subject to a restoring force: $u_{tt} = u_{xx} - u$. Discuss their long-term behavior.

♡ 4.2.9. Let $a, c > 0$ be positive constants. The *telegrapher's equation* $u_{tt} + a\,u_t = c^2 u_{xx}$ represents a damped version of the wave equation. Consider the Dirichlet boundary value problem $u(t,0) = u(t,1) = 0$, on the interval $0 \le x \le 1$, with initial conditions $u(0,x) = f(x)$, $u_t(0,x) = 0$. (a) Find all separable solutions to the telegrapher's equation that satisfy the boundary conditions. (b) Write down a series solution for the initial boundary value problem. (c) Discuss the long term behavior of your solution. (d) State a criterion that distinguishes overdamped from underdamped versions of the equation.

4.2.10. The fourth-order partial differential equation $u_{tt} = -\,u_{xxxx}$ is a simple model for a vibrating elastic beam. (a) Find all separable real solutions to the beam equation.
(b) Show that any C^4 (complex) solution to the Schrödinger equation $\mathrm{i}\,u_t = u_{xx}$ solves the beam equation.

4.2.11. The initial-boundary value problem

$$u_{tt} = -\,u_{xxxx}, \qquad \begin{array}{c} u(t,0) = u_{xx}(t,0) = u(t,1) = u_{xx}(t,1) = 0, \\ u(0,x) = f(x), \qquad u_t(0,x) = 0, \end{array} \qquad \begin{array}{c} 0 < x < 1, \\ t > 0, \end{array}$$

models the vibrations of an elastic beam of unit length with simply supported ends, subject to a nonzero initial displacement $f(x)$ and zero initial velocity. (a) What are the vibrational frequencies for the beam? (b) Write down the solution to the initial-boundary value problem as a Fourier series. (c) Does the beam vibrate periodically
(i) for all initial conditions? (ii) for some initial conditions? (iii) for no initial conditions?

4.2.12. *Multiple choice*: The initial-boundary value problem

$$u_{tt} = u_{xxxx}, \qquad \begin{array}{c} u(t,0) = u_{xx}(t,0) = u(t,1) = u_{xx}(t,1) = 0, \\ u(0,x) = f(x), \qquad u_t(0,x) = g(x), \end{array} \qquad \begin{array}{c} 0 < x < 1, \\ t > 0, \end{array}$$

is well-posed for (a) $t > 0$; (b) $t < 0$; (c) all t; (d) no t. Explain your answer.

The d'Alembert Formula for Bounded Intervals

In Theorem 2.15, we derived the explicit d'Alembert formula

$$u(t,x) = \frac{f(x - c\,t) + f(x + c\,t)}{2} + \frac{1}{2\,c}\int_{x-c\,t}^{x+c\,t} g(z)\,dz, \tag{4.77}$$

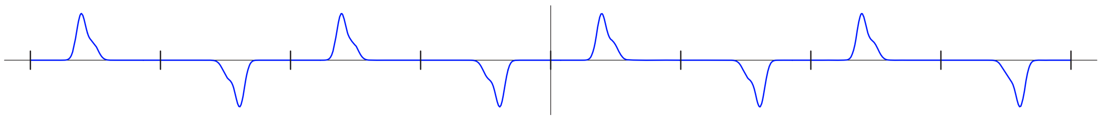


Figure 4.5. Odd periodic extension of a concentrated pulse.

for solving the basic initial value problem for the wave equation on an infinite interval:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,x) = f(x), \qquad \frac{\partial u}{\partial t}(0,x) = g(x), \qquad -\infty < x < \infty.$$

In this section we explain how to adapt the formula in order to solve initial-boundary value problems on bounded intervals, thereby effectively summing the Fourier series solution.

The easiest case to deal with is the periodic problem on $0 \le x \le \ell$, with boundary conditions

$$u(t,0) = u(t,\ell), \qquad u_x(t,0) = u_x(t,\ell). \tag{4.78}$$

If we extend the initial displacement $f(x)$ and velocity $g(x)$ to be periodic functions of period ℓ, so $f(x+\ell) = f(x)$ and $g(x+\ell) = g(x)$ for all $x \in \mathbb{R}$, then the resulting d'Alembert solution (4.77) will also be periodic in x, so $u(t, x+\ell) = u(t,x)$. In particular, it satisfies the boundary conditions (4.78) and so coincides with the desired solution. Details are to be supplied in Exercises 4.2.27–28.

Next, suppose we have fixed (Dirichlet) boundary conditions

$$u(t,0) = 0, \qquad u(t,\ell) = 0. \tag{4.79}$$

The resulting solution can be written as a Fourier sine series (4.68), and hence is both odd and 2ℓ–periodic in x. Therefore, to write the solution in d'Alembert form (4.77), we extend the initial displacement $f(x)$ and velocity $g(x)$ to be odd, periodic functions of period 2ℓ:

$$f(-x) = -f(x), \qquad f(x+2\ell) = f(x), \qquad g(-x) = -g(x), \qquad g(x+2\ell) = g(x).$$

This will ensure that the d'Alembert solution also remains odd and periodic. As a result, it satisfies the homogeneous Dirichlet boundary conditions (4.79) for all t, cf. Exercise 4.2.31. Keep in mind that, while the solution $u(t,x)$ is defined for all x, the only physically relevant values occur on the interval $0 \le x \le \ell$. Nevertheless, the effects of displacements in the unphysical regime will eventually be felt as the propagating waves pass through the physical interval.

For example, consider an initial displacement that is concentrated near $x = \xi$ for some $0 < \xi < \ell$. Its odd 2ℓ–periodic extension consists of two sets of replicas: those of the same form occurring at positions $\xi \pm 2\ell,\ \xi \pm 4\ell,\ \ldots$, and their upside-down mirror images at the intermediate positions $-\xi,\ -\xi \pm 2\ell,\ -\xi \pm 4\ell,\ \ldots$; Figure 4.5 shows a representative example. The resulting solution begins with each of the pulses, both positive and negative, splitting into two half-size replicas that propagate with speed c in opposite directions. When a left and right moving pulse meet, they emerge from the interaction unaltered. The process repeats periodically, with an infinite row of half-size pulses moving to the right kaleidoscopically interacting with an infinite row moving to the left.

However, only the part of this solution that lies on $0 \le x \le \ell$ is actually observed on the physical string. The effect is as if one were watching the full solution as it passes by a window of length ℓ. Such observers will interpret what they see a bit differently. To

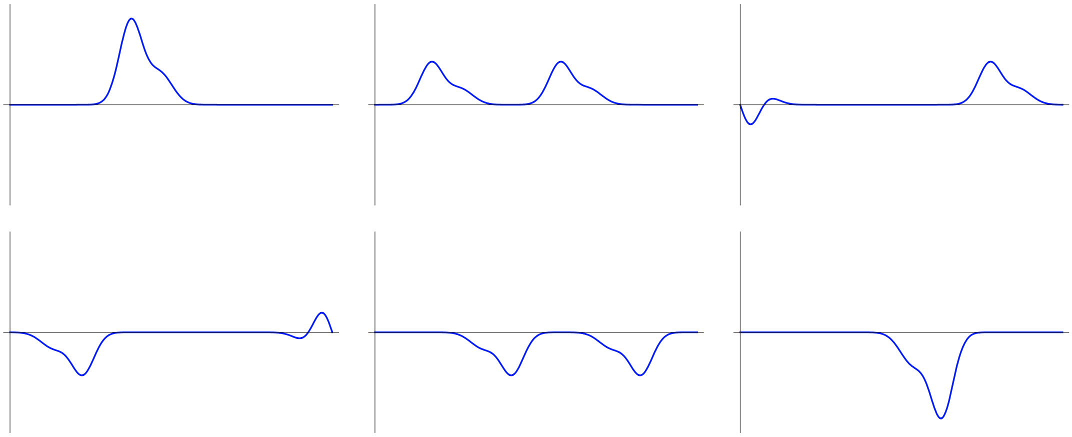


Figure 4.6. Solution to wave equation with fixed ends.

wit, the original pulse starting at position $0 < \xi < \ell$ splits up into two half-size replicas that move off in opposite directions. As each half-size pulse reaches an end of the string, it meets a mirror-image pulse that has been propagating in the opposite direction from the nonphysical regime. The pulse is reflected at the end of the interval and becomes an upside-down mirror image moving in the opposite direction. The original positive pulse has moved off the end of the string just as its mirror image has moved into the physical regime. (A common physical realization is a pulse propagating down a jump rope that is held fixed at its end; the reflected pulse returns upside down.) A similar reflection occurs as the other half-size pulse hits the other end of the physical interval, after which the solution consists of two upside-down half-size pulses moving back towards each other. At time $t = \ell/c$ they recombine at the point $\ell - \xi$ to instantaneously form a full-sized, but upside-down mirror image of the original disturbance — in accordance with (4.74). The recombined pulse in turn splits apart into two upside-down half-size pulses that, when each collides with the end, reflect and return to their original upright form. At time $t = 2\ell/c$, the pulses recombine to exactly reproduce the original displacement. The process then repeats, and the solution is periodic in time with period $2\ell/c$.

In Figure 4.6, the first picture displays the initial displacement. In the second, it has split into left- and right-moving half-size clones. In the third picture, the left-moving bump is in the process of colliding with the left end of the string. In the fourth picture, it has emerged from the collision, and is now upside down, reflected, and moving to the right. Meanwhile, the right-moving pulse is starting to collide with the right end. In the fifth picture, both pulses have completed their collisions and are now moving back towards each other, where, in the last picture, they recombine into an upside-down mirror image of the original pulse. The process then repeats itself, in mirror image, finally recombining to the original pulse, at which point the entire process starts over.

The Neumann (free) boundary value problem

$$\frac{\partial u}{\partial x}(t,0) = 0, \qquad \frac{\partial u}{\partial x}(t,\ell) = 0, \tag{4.80}$$

is handled similarly. Since the solution has the form of a Fourier cosine series in x, we extend the initial conditions to be *even* 2ℓ–periodic functions

$$f(-x) = f(x), \qquad f(x+2\ell) = f(x), \qquad g(-x) = g(x), \qquad g(x+2\ell) = g(x).$$

The resulting d'Alembert solution (4.77) is also even and 2ℓ–periodic in x, and hence satisfies the boundary conditions, cf. Exercise 4.2.31(b). In this case, when a pulse hits one of the ends, its reflection remains upright, but becomes a mirror image of the original; a familiar physical illustration is a water wave that reflects off a solid wall. Further details are left to the reader in Exercise 4.2.22

In summary, we have now studied two very different ways to solve the one-dimensional wave equation. The first, based on the d'Alembert formula, emphasizes their particle-like aspects, where individual wave packets collide with each other, or reflect at the boundary, all the while maintaining their overall form, while the second, based on Fourier analysis, emphasizes the vibrational or wave-like character of the solutions. Some solutions look like vibrating waves, while others appear much more like interacting particles. But, like the proverbial blind men describing an elephant, these are merely two facets of the *same* solution. The Fourier series formula shows how every particle-like solution can be decomposed into its constituent vibrational modes, while the d'Alembert formula demonstrates how vibrating solutions combine into moving wave packets.

The coexistence of particle and wave features is reminiscent of the long-running historical debate over the nature of light. Newton and his disciples proposed a particle-based theory, anticipating the modern concept of photons. However, until the beginning of the twentieth century, most physicists advocated a wave-like or vibrational viewpoint. Einstein's explanation of the photoelectric effect served to resurrect the particle interpretation. Only with the establishment of quantum mechanics was the debate resolved — light, and, indeed, all subatomic particles manifest *both* particle and wave features, depending upon the experiment and the physical situation. But a theoretical basis for the perplexing wave-particle duality could have been found already in Fourier's and d'Alembert's competing solution formulae for the classical wave equation!

Exercises

♢ 4.2.13. (a) Solve the initial-boundary value problem from Example 4.3 using the d'Alembert method. (b) Verify that your solution coincides with the Fourier series solution derived above. (c) Justify our earlier observation that, at each time t, the solution $u(t,x)$ is a piecewise affine function of x.

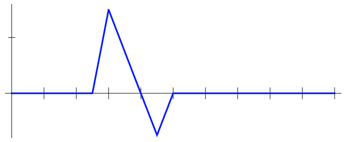
4.2.14. Sketch the solution of the wave equation $u_{tt} = u_{xx}$ and describe its behavior when the initial displacement is the box function $u(0,x) = \begin{cases} 1, & 1 < x < 2, \\ 0, & \text{otherwise,} \end{cases}$ while the initial velocity is 0 in each of the following scenarios: (a) on the entire line $-\infty < x < \infty$; (b) on the half-line $0 \le x < \infty$, with homogeneous Dirichlet boundary condition at the end; (c) on the half-line $0 \le x < \infty$, with homogeneous Neumann boundary condition at the end; (d) on the bounded interval $0 \le x \le 5$ with homogeneous Dirichlet boundary conditions; (e) on the bounded interval $0 \le x \le 5$ with homogeneous Neumann boundary conditions.

4.2.15. Answer Exercise 4.2.14 when the initial velocity is the box function, while the initial displacement is zero.

4.2.16. Consider the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad \begin{array}{ll} u(t,0) = 0 = u(t,10), & t > 0, \\ u(0,x) = f(x), \quad u_t(0,x) = 0, & 0 < x < 10, \end{array}$$

for the wave equation, where the initial data has the following form:

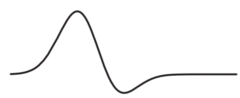


$$f(x) = \begin{cases} 3x - 7.5, & 2.5 \le x \le 3, \\ 6 - 1.5x, & 3 \le x \le 4.5, \\ 1.5x - 7.5, & 4.5 \le x \le 5, \\ 0, & \text{otherwise}. \end{cases}$$

Discuss what happens to the solution. You do *not* need to write down an explicit formula for the solution, but for full credit you must explain (sketches can help) at least three or four interesting things that happen to the solution as time progresses.

4.2.17. Repeat Exercise 4.2.16 for the Neumann boundary conditions.

4.2.18. Suppose the initial displacement of a string of length ℓ looks like the graph to the right. Assuming that the ends of the string are held fixed, graph the string's profile at times $t = \ell/c$ and $2\ell/c$.



♣ 4.2.19. Consider the wave equation $u_{tt} = u_{xx}$ on the interval $0 \le x \le 1$, with homogeneous Dirichlet boundary conditions at both ends. (*a*) Use the d'Alembert formula to explicitly solve the initial value problem $u(0,x) = x - x^2$, $u_t(0,x) = 0$. (*b*) Graph the solution profile at some representative times, and discuss what you observe. (*c*) Find the Fourier series at each t of your solution and compare the two. (*d*) How many terms do you need to sum to obtain a reasonable approximation to the exact solution?

♣ 4.2.20. Solve Exercise 4.2.19 for the initial conditions $u(0,x) = 0$, $u_t(0,x) = x^2 - x$.

♣ 4.2.21. Solve (*i*) Exercise 4.2.19, (*ii*) Exercise 4.2.20, when the solution is subject to homogeneous Neumann boundary conditions.

♢ 4.2.22. Under what conditions is the solution to the Neumann boundary value problem for the wave equation on a bounded interval $[0,\ell]$ periodic in time? What is the period?

4.2.23. Discuss and sketch the behavior of the solution to the Neumann boundary value problem $u_{tt} = 4u_{xx}$, $0 < x < 1$, $u_x(t,0) = 0 = u_x(t,1)$, $u(0,x) = f(x)$, $u_t(0,x) = g(x)$, for

(*a*) a localized initial displacement: $f(x) = \begin{cases} 1, & .2 < x < .3, \\ 0, & \text{otherwise}. \end{cases} \quad g(x) = 0;$

(*b*) a localized initial velocity: $f(x) = 0$, $g(x) = \begin{cases} 1, & .2 < x < .3, \\ 0, & \text{otherwise}. \end{cases}$

4.2.24. (*a*) Explain how to solve the Neumann initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial u}{\partial x}(t,0) = 0 = \frac{\partial u}{\partial x}(t,1), \qquad u(0,x) = f(x), \qquad \frac{\partial u}{\partial t}(0,x) = g(x),$$

on the interval $0 \le x \le 1$.

(*b*) Let $f(x) = \begin{cases} x - \frac14, & \frac14 \le x \le \frac12, \\ \frac34 - x, & \frac12 \le x \le \frac34, \\ 0, & \text{otherwise}, \end{cases}$ and $g(x) = 0$. Sketch the graph of the solution at a few representative times, and discuss what is happening. Is the solution periodic in time? If so, what is the period?

(*c*) Do the same when $f(x) = 0$ and $g(x) = x$.

4.2.25. (*a*) Write down a formula for the solution $u(t,x)$ to the initial-boundary value problem
$$\frac{\partial^2 u}{\partial t^2} - 4\,\frac{\partial^2 u}{\partial x^2} = 0, \quad u(0,x) = \sin x, \quad \frac{\partial u}{\partial t}(0,x) = \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,\pi) = 0, \quad 0 < x < \pi, \quad t > 0.$$
(*b*) Find $u\left(\frac{\pi}{2},\frac{\pi}{2}\right)$. (*c*) Prove that $h(t) = u\left(t,\frac{\pi}{2}\right)$ is a periodic function of t and find its period. (*d*) Does $\dfrac{\partial u}{\partial x}$ have any discontinuities? If so, discuss their behavior.

4.2.26. Answer Exercise 4.2.25 for the mixed boundary conditions $u(t,0) = 0 = u_x(t,\pi)$.

♡ 4.2.27. (*a*) Explain how to use d'Alembert's formula (4.77) to solve the periodic initial-boundary value problem for the wave equation given in Exercise 4.2.6.
(*b*) Do the d'Alembert and Fourier series formulae represent the same solution? If so, can you justify it? If not, explain why they are different.

♢ 4.2.28. Show that the solution $u(t,x)$ to the wave equation on an interval $[0,\ell]$, subject to periodic boundary conditions $u(t,0) = u(t,\ell)$, $u_x(t,0) = u_x(t,\ell)$, is a periodic function of t if and only if there is no net initial velocity: $\displaystyle\int_0^\ell g(x)\,dx = 0$.

4.2.29. (*a*) Explain how to solve the wave equation on a half-line $x > 0$ when subject to Dirichlet boundary conditions $u(t,0) = 0$. (*b*) Assuming $c = 1$, find the solution satisfying $u(0,x) = (x-2)\,e^{-5\,(x-2.2)^2}$, $u_t(0,x) = 0$. (*c*) Sketch a picture of your solution at some representative times, and discuss what is happening.

4.2.30. Solve Exercise 4.2.29 for homogeneous Neumann boundary conditions at $x = 0$.

♢ 4.2.31. (*a*) Given that $f(x)$ is odd and 2ℓ–periodic, explain why $f(0) = 0 = f(\ell)$.
(*b*) Given that $f(x)$ is even and 2ℓ–periodic, explain why $f'(0) = 0 = f'(\ell)$.

♢ 4.2.32. (*a*) Prove that if $f(-x) = -f(x)$, $f(x+2\ell) = f(x)$, for all x, then $u(t,x) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]$ satisfies the Dirichlet boundary conditions (4.79).
(*b*) Prove that if $g(-x) = -g(x)$, $g(x+2\ell) = g(x)$ for all x, then
$\displaystyle u(t,x) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$ also satisfies the Dirichlet boundary conditions.

4.2.33. If both $u(0,x) = f(x)$ and $u_t(0,x) = g(x)$ are even functions, show that the solution $u(t,x)$ of the wave equation is even in x for all t.

4.2.34. (*a*) Prove that the solution $u(t,x)$ to the wave equation for $x \in \mathbb{R}$ is an even function of t if and only if its initial velocity, at $t = 0$, is zero.
(*b*) Under what conditions is $u(t,x)$ an odd function of t?

♢ 4.2.35. Let $u(t,x)$ be a classical solution to the wave equation $u_{tt} = c^2 u_{xx}$ on the interval $0 < x < \ell$, satisfying homogeneous Dirichlet boundary conditions. The *total energy* of u at time t is
$$E(t) = \int_0^\ell \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x}\right)^2\right] dx. \tag{4.81}$$
Establish the *Law of Conservation of Energy* by showing that $E(t) = E(0)$ is a constant function.

♢ 4.2.36. (*a*) Use Exercise 4.2.35 to prove that the only C^2 solution to the initial-boundary value problem $v_{tt} = c^2 v_{xx}$, $v(t,0) = v(t,\ell) = 0$, $v(0,x) = 0$, $v_t(0,x) = 0$, is the trivial solution $v(t,x) \equiv 0$. (*b*) Establish the following *Uniqueness Theorem* for the wave equation: given $f(x), g(x) \in \mathrm{C}^2$, there is at most one C^2 solution $u(t,x)$ to the initial-boundary value problem $u_{tt} = c^2 u_{xx}$, $u(t,0) = u(t,\ell) = 0$, $u(0,x) = f(x)$, $u_t(0,x) = g(x)$.

4.2.37. Referring back to Exercises 4.2.35 and 4.2.36: (*a*) Does conservation of energy hold for solutions to the homogeneous Neumann initial-boundary value problem?
(*b*) Can you establish a uniqueness theorem for the Neumann problem?

4.2.38. Explain how to solve the Dirichlet initial-boundary value problem

$$u_{tt} = c^2 u_{xx} + F(t,x), \qquad u(0,x) = f(x), \qquad u_t(0,x) = g(x), \qquad u(t,0) = u(t,\ell) = 0,$$

for the wave equation subject to an external forcing on the interval $[0,\ell]$.

4.3 The Planar Laplace and Poisson Equations

The two-dimensional *Laplace equation* is the second-order linear partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{4.82}$$

named in honor of the influential eighteenth-century French mathematician Pierre–Simon Laplace. It, along with its higher-dimensional versions, is arguably the most important differential equation in all of mathematics. A real-valued solution $u(x,y)$ to the Laplace equation is known as a *harmonic function*. The space of harmonic functions can thus be identified as the kernel of the second-order linear partial differential operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,, \tag{4.83}$$

known as the *Laplace operator*, or *Laplacian* for short. The inhomogeneous or forced version, namely

$$-\Delta[u] = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x,y), \tag{4.84}$$

is known as *Poisson's equation*, named after Siméon–Denis Poisson, who was taught by Laplace. The mathematical and physical reasons for including the minus sign will gradually become clear.

Besides their theoretical importance, the Laplace and Poisson equations arise as the basic equilibrium equations in a remarkable variety of physical systems. For example, we may interpret $u(x,y)$ as the displacement of a *membrane*, e.g., a drum skin; the inhomogeneity $f(x,y)$ in the Poisson equation represents an external forcing over the surface of the membrane. Another example is in the thermal equilibrium of flat plates; here $u(x,y)$ represents the temperature and $f(x,y)$ an external heat source. In fluid mechanics, $u(x,y)$ represents the potential function whose gradient $\mathbf{v} = \nabla u$ is the velocity vector field of a steady planar fluid flow. Similar considerations apply to two-dimensional electrostatic and gravitational potentials. The dynamical counterparts to the Laplace equation are the two-dimensional versions of the heat and wave equations, to be analyzed in Chapter 11.

Since both the Laplace and Poisson equations describe equilibrium configurations, they almost always appear the context of boundary value problems. We seek a solution $u(x,y)$ to the partial differential equation defined at points (x,y) belonging to a bounded, open domain $\Omega \subset \mathbb{R}^2$. The solution is required to satisfy suitable conditions on the boundary of the domain, denoted by $\partial\Omega$, which will consist of one or more simple closed curves, as illustrated in Figure 4.7. As in one-dimensional boundary value problems, there are several especially important types of boundary conditions.

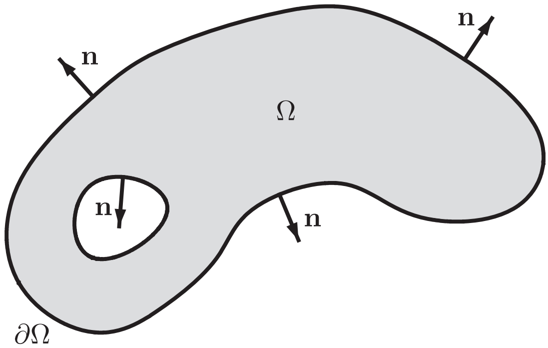


Figure 4.7. A planar domain with outward unit normals on its boundary.

The first are the *fixed* or *Dirichlet boundary conditions*, which specify the value of the function u on the boundary:

$$u(x,y) = h(x,y) \qquad \text{for} \qquad (x,y) \in \partial\Omega. \tag{4.85}$$

Under mild regularity conditions on the domain Ω, the boundary values h, and the forcing function f, the Dirichlet conditions (4.85) serve to uniquely specify the solution $u(x,y)$ to the Laplace or the Poisson equation. Physically, in the case of a free or forced membrane, the Dirichlet boundary conditions correspond to gluing the edge of the membrane to a wire at height $h(x,y)$ over each boundary point $(x,y) \in \partial\Omega$, as illustrated in Figure 4.8. A physical realization can be easily obtained by dipping the wire in a soap solution; the resulting soap film spanning the wire forms a *minimal surface*, which, if the wire is reasonably close to planar shape,† is the solution to the Dirichlet problem prescribed by the wire. Similarly, in the modeling of thermal equilibrium, a Dirichlet boundary condition represents the imposition of a prescribed temperature distribution, represented by the function h, along the boundary of the plate.

The second important class consists of the *Neumann boundary conditions*

$$\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n} = k(x,y) \qquad \text{on} \qquad \partial\Omega, \tag{4.86}$$

in which the *normal derivative* of the solution u on the boundary is prescribed. In general, $\mathbf{n}$ denotes the *unit outwards normal* to the boundary $\partial\Omega$, i.e., the vector of unit length, $\| \mathbf{n} \| = 1$, that is orthogonal to the tangent to the boundary and points *away* from the domain; see Figure 4.7. For example, in thermomechanics, a Neumann boundary condition specifies the heat flux out of a plate through its boundary. The "no-flux" or homogeneous Neumann boundary conditions, where $k(x,y) \equiv 0$, correspond to a fully insulated boundary. In the case of a membrane, homogeneous Neumann boundary conditions correspond to a free, unattached edge of a drum. In fluid mechanics, the Neumann conditions prescribe the fluid flux through the boundary; in particular, homogeneous Neumann boundary conditions

† More generally, the minimal surface formed by the soap film solves the vastly more complicated nonlinear *minimal surface equation* $(1+u_x^2)u_{xx} - 2\,u_x u_y u_{xy} + (1+u_y^2)u_{yy} = 0$, which, for surfaces with small variation, i.e., with $\| \nabla u \| \ll 1$, can be approximated by the Laplace equation.

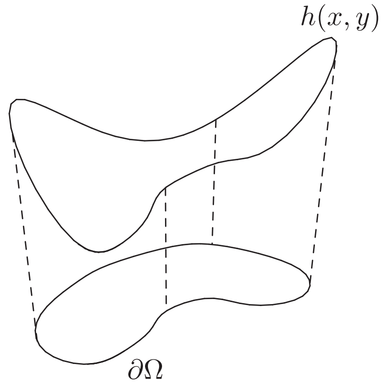


Figure 4.8. Dirichlet boundary conditions.

correspond to a solid boundary that the fluid cannot penetrate. More generally, the *Robin boundary conditions*

$$\frac{\partial u}{\partial \mathbf{n}} + \beta(x,y)\,u = k(x,y) \qquad \text{on} \qquad \partial\Omega,$$

also known as *impedance boundary conditions* due to their applications in electromagnetism, are used to model insulated plates in heat baths, or membranes attached to springs.

Finally, one can mix the previous kinds of boundary conditions, imposing, say, Dirichlet conditions on part of the boundary and Neumann conditions on the complementary part. A typical *mixed boundary value problem* has the form

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = h \quad \text{on} \quad D, \qquad \frac{\partial u}{\partial \mathbf{n}} = k \quad \text{on} \quad N, \tag{4.87}$$

with the boundary $\partial\Omega = D \cup N$ being the disjoint union of a "Dirichlet segment", denoted by D, and a "Neumann segment" N. For example, if u represents the equilibrium temperature in a plate, then the Dirichlet segment of the boundary is where the temperature is fixed, while the Neumann segment is insulated, or, more generally, has prescribed heat flux. Similarly, when modeling the displacement of a membrane, the Dirichlet segment is where the edge of the drum is attached to a support, while the homogeneous Neumann segment is left hanging free.

Exercises

4.3.1. (*a*) Solve the boundary value problem $\Delta u = 1$ for $x^2 + y^2 < 1$ and $u(x, y) = 0$ for $x^2 + y^2 = 1$ directly. *Hint*: The solution is a simple polynomial.
(*b*) Graph your solution, interpreting it as the equilibrium displacement of a circular drum under a constant gravitational force.

4.3.2. Set up the boundary value problem corresponding to the equilibrium of a circular membrane subject to a constant downwards gravitational force, half of whose boundary is glued to a flat semicircular wire, while the other half is unattached.

4.3.3. Set up the boundary value problem corresponding to the thermal equilibrium of a rectangular plate that is insulated on two of its sides, has 0° at its top edge and 100° at the bottom edge. Where do you expect the maximum temperature to be located? What is its value? Can you find a formula for the temperature inside the plate? *Hint*: The solution is constant along horizontal lines.

4.3.4. Set up the boundary value problem corresponding to the thermal equilibrium of an insulated semi-circular plate with unit diameter, whose curved edge is kept at 0° and whose straight edge is at 50°.

4.3.5. Explain why the solution to the homogeneous Neumann boundary value problem for the Laplace equation is *not* unique.

4.3.6. Write down the Dirichlet boundary value problem for the Laplace equation on the unit square $0 \le x, y \le 1$ that is satisfied by $u(x,y) = 1 + x\,y$.

4.3.7. Write down the Neumann boundary value problem for the Poisson equation on the unit disk $x^2 + y^2 \le 1$ that is satisfied by $u(x,y) = x^3 + x\,y^2$.

♢ 4.3.8. Suppose $u(x,y)$ is a solution to the Laplace equation.
(*a*) Show that any translate $U(x,y) = u(x-a, y-b)$, where $a, b \in \mathbb{R}$, is also a solution.
(*b*) Show that the rotated function $U(x,y) = u(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$, where $-\pi < \theta \le \pi$, is also a solution.

♢ 4.3.9. (*a*) Show that if $u(x,y)$ solves the Laplace equation, then so does the rescaled function $U(x,y) = c\,u(\alpha\,x, \alpha\,y)$ for any constants c, α.
(*b*) Discuss the effect of scaling on the Dirichlet boundary value problem.
(*c*) What happens if we use different scaling factors in x and y?

Separation of Variables

Our first approach to solving the Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{4.88}$$

will be based on the method of *separation of variables*. As in (4.64), we seek solutions that can be written as a product

$$u(x,y) = v(x)\,w(y) \tag{4.89}$$

of a function of x alone times a function of y alone. We compute

$$\frac{\partial^2 u}{\partial x^2} = v''(x)\,w(y), \qquad \frac{\partial^2 u}{\partial y^2} = v(x)\,w''(y),$$

and so

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = v''(x)\,w(y) + v(x)\,w''(y) = 0.$$

We then *separate the variables* by placing all the terms involving x on one side of the equation and all the terms involving y on the other; this is accomplished by dividing by $v(x)\,w(y)$ and then writing the resulting equation in the separated form

$$\frac{v''(x)}{v(x)} = -\,\frac{w''(y)}{w(y)} = \lambda. \tag{4.90}$$

As we argued in (4.65), the only way a function of x alone can be equal to a function of y alone is if both functions are equal to a common *separation constant* λ. Thus, the factors $v(x)$ and $w(y)$ must satisfy the elementary ordinary differential equations

$$v'' - \lambda\, v = 0, \qquad\qquad w'' + \lambda\, w = 0.$$

As before, the solution formulas depend on the sign of the separation constant λ. We list the resulting collection of separable harmonic functions in the following table:

Separable Solutions to Laplace's Equation

λ	$v(x)$	$w(y)$	$u(x,y) = v(x)\, w(y)$
$\lambda = -\omega^2 < 0$	$\cos\omega\, x,\ \sin\omega\, x$	$e^{-\omega\, y},\ e^{\omega\, y},$	$e^{\omega\, y}\cos\omega\, x,\quad e^{\omega\, y}\sin\omega\, x,$ $e^{-\omega\, y}\cos\omega\, x,\ \ e^{-\omega\, y}\sin\omega\, x$
$\lambda = 0$	$1,\ x$	$1,\ y$	$1,\ x,\ y,\ x\, y$
$\lambda = \omega^2 > 0$	$e^{-\omega\, x},\ e^{\omega\, x}$	$\cos\omega\, y,\ \sin\omega\, y$	$e^{\omega\, x}\cos\omega\, y,\quad e^{\omega\, x}\sin\omega\, y,$ $e^{-\omega\, x}\cos\omega\, y,\ \ e^{-\omega\, x}\sin\omega\, y$

Since Laplace's equation is a homogeneous linear system, any linear combination of solutions is also a solution. So, we can build more general solutions as finite linear combinations, or, provided we pay proper attention to convergence issues, infinite series in the separable solutions. Our goal is to solve boundary value problems, and so we must ensure that the resulting combination satisfies the boundary conditions. But this is not such an easy task, unless the underlying domain has a rather special geometry.

In fact, the only bounded domains on which we can explicitly solve boundary value problems using the preceding separable solutions are rectangles. So, we will concentrate on boundary value problems for Laplace's equation

$$\Delta u = 0 \qquad \text{on a rectangle} \qquad R = \{\, 0 < x < a,\ \ 0 < y < b \,\}. \tag{4.91}$$

To make progress, we will allow nonzero boundary values on only one of the four sides of the rectangle. To illustrate, we will focus on the following Dirichlet boundary conditions:

$$u(x,0) = f(x), \qquad u(x,b) = 0, \qquad u(0,y) = 0, \qquad u(a,y) = 0. \tag{4.92}$$

Once we know how to solve this type of problem, we can employ linear superposition to solve the general Dirichlet boundary value problem on a rectangle; see Exercise 4.3.12 for details. Other boundary conditions can be treated in a similar fashion — with the proviso that the condition on each side of the rectangle is either entirely Dirichlet or entirely Neumann or, more generally, entirely Robin with constant transfer coefficient.

To solve the boundary value problem (4.91–92), the first step is to narrow down the separable solutions to only those that respect the three homogeneous boundary conditions. The separable function $u(x,y) = v(x)\, w(y)$ will vanish on the top, right, and left sides of the rectangle, provided

$$v(0) = v(a) = 0 \qquad \text{and} \qquad w(b) = 0.$$

Referring to the preceding table, the first condition $v(0) = 0$ requires

$$v(x) = \begin{cases} \sin \omega x, & \lambda = -\omega^2 < 0, \\ x, & \lambda = 0, \\ \sinh \omega x, & \lambda = \omega^2 > 0, \end{cases}$$

where $\sinh z = \frac{1}{2}(e^z - e^{-z})$ is the usual hyperbolic sine function. However, the second and third cases cannot satisfy the second boundary condition $v(a) = 0$, and so we discard them. The first case leads to the condition

$$v(a) = \sin \omega a = 0, \qquad \text{and hence} \qquad \omega a = \pi,\ 2\pi,\ 3\pi,\ \ldots .$$

The corresponding separation constants and solutions (up to constant multiple) are

$$\lambda_n = -\omega^2 = -\frac{n^2\pi^2}{a^2}, \qquad v_n(x) = \sin \frac{n\pi x}{a}, \qquad n = 1, 2, 3, \ldots . \tag{4.93}$$

Note: So far, we have merely recomputed the known eigenvalues and eigenfunctions of the familiar boundary value problem $v'' - \lambda v = 0,\ v(0) = v(a) = 0$.

Next, since $\lambda = -\omega^2 < 0$, we have $w(y) = c_1 e^{\omega y} + c_2 e^{-\omega y}$ for constants c_1, c_2. The third boundary condition $w(b) = 0$ then requires that, up to constant multiple,

$$w_n(y) = \sinh \omega\,(b - y) = \sinh \frac{n\pi(b-y)}{a}. \tag{4.94}$$

We conclude that the harmonic functions

$$u_n(x,y) = \sin \frac{n\pi x}{a}\ \sinh \frac{n\pi(b-y)}{a}, \qquad n = 1, 2, 3, \ldots , \tag{4.95}$$

provide a complete list of separable solutions that satisfy the three homogeneous boundary conditions. It remains to analyze the inhomogeneous boundary condition along the bottom edge of the rectangle. To this end, let us try a linear superposition of the relevant separable solutions in the form of an infinite series

$$u(x,y) = \sum_{n=1}^{\infty} c_n u_n(x,y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{a}\ \sinh \frac{n\pi\,(b-y)}{a},$$

whose coefficients $c_1, c_2, \ldots$ are to be prescribed by the remaining boundary condition. At the bottom edge, $y = 0$, we find

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi b}{a}\ \sin \frac{n\pi x}{a} = f(x), \qquad 0 \le x \le a, \tag{4.96}$$

which takes the form of a Fourier sine series for the function $f(x)$. Let

$$b_n = \frac{2}{a}\int_0^a f(x) \sin \frac{n\pi x}{a}\, dx \tag{4.97}$$

be its Fourier sine coefficients, whence $c_n = b_n / \sinh(n\pi b/a)$. We thus anticipate that the solution to the boundary value problem can be expressed as the infinite series

$$u(x,y) = \sum_{n=1}^{\infty} \frac{b_n \sin \dfrac{n\pi x}{a}\ \sinh \dfrac{n\pi(b-y)}{a}}{\sinh \dfrac{n\pi b}{a}}. \tag{4.98}$$

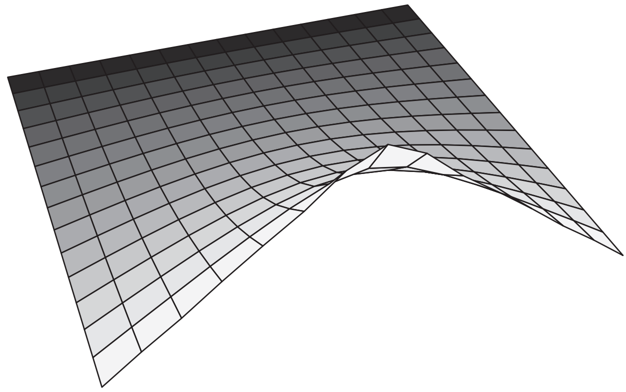


Figure 4.9. Square membrane on a wire.

Does this series actually converge to the solution to the boundary value problem? Fourier analysis says that, under very mild conditions on the boundary function $f(x)$, the answer is yes. Suppose that its Fourier coefficients are uniformly bounded,

$$|\, b_n \,| \leq M \qquad \text{for all} \qquad n \geq 1, \tag{4.99}$$

which, according to (4.27), is true whenever $f(x)$ is piecewise continuous or, more generally, integrable: $\displaystyle\int_0^a |\, f(x) \,|\, dx < \infty$. In this case, as you are asked to prove in Exercise 4.3.20, the coefficients of the Fourier sine series (4.98) go to zero exponentially fast:

$$\frac{b_n \sinh \dfrac{n\pi(b-y)}{a}}{\sinh \dfrac{n\pi b}{a}} \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty \qquad \text{for all} \quad 0 < y \leq b, \tag{4.100}$$

and so, at each point inside the rectangle, the series can be well approximated by partial summation. Theorem 3.31 tells us that, for each $0 < y \leq b$, the solution $u(x,y)$ is an infinitely differentiable function of x. Moreover, by term-wise differentiation of the series with respect to y and use of Proposition 3.28, we also establish that the solution is infinitely differentiable with respect to y; see Exercise 4.3.21. (In fact, as we shall see, solutions to the Laplace equation are *always analytic* functions inside their domain of definition — even when their boundary values are rather rough.) Since the individual terms all satisfy the Laplace equation, we conclude that the series (4.98) is indeed a classical solution to the boundary value problem.

Example 4.4. A membrane is stretched over a wire in the shape of a unit square with one side bent in half, as graphed in Figure 4.9. The precise boundary conditions are

$$u(x,y) = \begin{cases} x, & 0 \leq x \leq \frac{1}{2}, \quad y = 0, \\ 1-x, & \frac{1}{2} \leq x \leq 1, \quad y = 0, \\ 0, & 0 \leq x \leq 1, \quad y = 1, \\ 0, & x = 0, \quad 0 \leq y \leq 1, \\ 0, & x = 1, \quad 0 \leq y \leq 1. \end{cases}$$

The Fourier sine series of the inhomogeneous boundary function is readily computed:

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2}, \\ 1-x, & \frac{1}{2} \le x \le 1, \end{cases}$$
$$= \frac{4}{\pi^2}\left(\sin \pi x - \frac{\sin 3\pi x}{9} + \frac{\sin 5\pi x}{25} - \cdots \right) = \frac{4}{\pi^2} \sum_{j=0}^{\infty} (-1)^j \, \frac{\sin (2j+1)\pi x}{(2j+1)^2} \,.$$

Specializing (4.98) to $a = b = 1$, we conclude that the solution to the boundary value problem can be expressed as a Fourier series

$$u(x,y) = \frac{4}{\pi^2} \sum_{j=0}^{\infty} (-1)^j \, \frac{\sin(2j+1)\pi x \ \sinh(2j+1)\pi(1-y)}{(2j+1)^2 \ \sinh(2j+1)\pi} \,.$$

In Figure 4.9 we plot the sum of the first 10 terms in the series. This gives a reasonably good approximation to the actual solution, except when we are very close to the raised corner of the boundary wire — which is the point of maximal displacement of the membrane.

Exercises

4.3.10. Solve the following boundary value problems for Laplace's equation on the square $\Omega = \{ 0 \le x \le \pi, \quad 0 \le y \le \pi \}$.
(a) $u(x,0) = \sin^3 x, \quad u(x,\pi) = 0, \quad u(0,y) = 0, \quad u(\pi,y) = 0.$
(b) $u(x,0) = 0, \quad u(x,\pi) = 0, \quad u(0,y) = \sin y, \quad u(\pi,y) = 0.$
(c) $u(x,0) = 0, \quad u(x,\pi) = 1, \quad u(0,y) = 0, \quad u(\pi,y) = 0.$
(d) $u(x,0) = 0, \quad u(x,\pi) = 0, \quad u(0,y) = 0, \quad u(\pi,y) = y(\pi - y).$

♢ 4.3.11. (a) Explain how to use linear superposition to solve the boundary value problem

$$\Delta u = 0, \qquad u(x,0) = f(x), \qquad u(x,b) = g(x), \qquad u(0,y) = h(y), \qquad u(a,y) = k(y),$$

on the rectangle $R = \{ 0 < x < a,\ 0 < y < b \}$, by splitting it into four separate boundary value problems for which each of the solutions vanishes on three sides of the rectangle.
(b) Write down a series formula for the resulting solution.

4.3.12. Solve the following Dirichlet problems for Laplace's equation on the unit square $S = \{ 0 < x, y < 1 \}$. *Hint*: Use superposition as in Exercise 4.3.11.
(a) $u(x,0) = \sin \pi x, \quad u(x,1) = 0, \quad u(0,y) = \sin \pi y, \quad u(1,y) = 0;$
(b) $u(x,0) = 1, \quad u(x,1) = 0, \quad u(0,y) = 1, \quad u(1,y) = 0;$
(c) $u(x,0) = 1, \quad u(x,1) = 1, \quad u(0,y) = 0, \quad u(1,y) = 0;$
(d) $u(x,0) = x, \quad u(x,1) = 1 - x, \quad u(0,y) = y, \quad u(1,y) = 1 - y.$

4.3.13. Solve the following mixed boundary value problems for Laplace's equation $\Delta u = 0$ on the square $S = \{ 0 < x, y < \pi \}$.
(a) $u(x,0) = \sin \frac{1}{2} x, \quad u_y(x,\pi) = 0, \quad u(0,y) = 0, \quad u_x(\pi,y) = 0;$
(b) $u(x,0) = \sin \frac{1}{2} x, \quad u_y(x,\pi) = 0, \quad u_x(0,y) = 0, \quad u_x(\pi,y) = 0;$
(c) $u(x,0) = x, \quad u(x,\pi) = 0, \quad u_x(0,y) = 0, \quad u_x(\pi,y) = 0;$
(d) $u(x,0) = x, \quad u(x,\pi) = 0, \quad u(0,y) = 0, \quad u_x(\pi,y) = 0.$

4.3.14. Find the solution to the boundary value problem

$$\Delta u = 0, \qquad \begin{array}{cc} u_y(x,0) = u_y(x,2) = 0, & 0 < x < 1, \\ u(0,y) = 2\cos \pi y - 1, \quad u(1,y) = 0, & 0 < y < 2. \end{array}$$

4.3.15. Find the solution to the boundary value problem
$$\Delta u = 0, \qquad \begin{array}{c} u(x,0) = 2\cos 7\pi x - 4, \quad u(x,1) = 5\cos 3\pi x, \\ u_x(0,y) = u_x(1,y) = 0, \end{array} \qquad 0 < x, y < 1.$$

4.3.16. Let $u(x,y)$ be the solution to the boundary value problem
$$\Delta u = 0,\ u(x,-1) = f(x),\ u(x,1) = 0,\ u(-1,y) = 0,\ u(1,y) = 0,\ -1 < x < 1,\ -1 < y < 1.$$
(a) *True or false*: If $f(-x) = -f(x)$ is odd, then $u(0,y) = 0$ for all $-1 \le y \le 1$.
(b) *True or false*: If $f(0) = 0$, then $u(0,y) = 0$ for all $-1 \le y \le 1$.
(c) Under what conditions on $f(x)$ is $u(x,0) = 0$ for all $-1 \le x \le 1$?

4.3.17. Use separation of variables to solve the following boundary value problem:
$$u_{xx} + 2u_y + u_{yy} = 0, \quad u(x,0) = 0, \quad u(x,1) = f(x), \quad u(0,y) = 0, \quad u(1,y) = 0.$$

4.3.18. Use separation of variables to solve the Helmholtz boundary value problem $\Delta u = u$, $u(x,0) = 0$, $u(x,1) = f(x)$, $u(0,y) = 0$, $u(1,y) = 0$, on the unit square $0 < x, y < 1$.

♢ 4.3.19. Provide the details for the derivation of (4.94).

♢ 4.3.20. Justify the statement that if $|\,b_n\,| \le M$ are uniformly bounded, then the coefficients given in (4.100) go to zero exponentially fast as $n \to \infty$ for any $0 < y \le b$.

♢ 4.3.21. Let $u(x,y)$ denote the solution to the boundary value problem (4.91–92). (a) Write down the Fourier sine series for $\partial u/\partial y$. (b) Prove that $\partial u/\partial y$ is an infinitely differentiable function of x. (c) Justify the same result for the functions $\partial^k u/\partial y^k$ for each $k \ge 0$. *Hint*: Don't forget that $u(x,y)$ solves the Laplace equation.

Polar Coordinates

The method of separation of variables can be successfully exploited in certain other very special geometries. One particularly important case is a circular disk. To be specific, let us take the disk to have radius 1 and be centered at the origin. Consider the Dirichlet boundary value problem

$$\Delta u = 0, \qquad x^2 + y^2 < 1, \qquad \text{and} \qquad u = h, \qquad x^2 + y^2 = 1, \tag{4.101}$$

so that the function $u(x,y)$ satisfies the Laplace equation on the unit disk and the specified Dirichlet boundary conditions on the unit circle. For example, $u(x,y)$ might represent the displacement of a circular drum that is attached to a wire of height

$$h(x,y) = h(\cos\theta, \sin\theta) \equiv h(\theta), \qquad -\pi < \theta \le \pi, \tag{4.102}$$

at each point $(x,y) = (\cos\theta, \sin\theta)$ on its edge.

The rectangular separable solutions are not particularly helpful in this situation, and so we look for solutions that are better adapted to a circular geometry. This inspires us to adopt *polar coordinates*

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad \text{or} \qquad r = \sqrt{x^2+y^2}\,, \qquad \theta = \tan^{-1}\frac{y}{x}\,, \tag{4.103}$$

and write the solution $u(r,\theta)$ as a function thereof.

Warning: We will often retain the same symbol, e.g., u, when rewriting a function in a different coordinate system. This is the convention of tensor analysis, physics, and differential geometry, [**3**], that treats the function (scalar field) as an intrinsic object, which is concretely realized through its formula in any chosen coordinate system. For instance, if $u(x,y) = x^2 + 2y$ in rectangular coordinates, then its expression in polar coordinates is $u(r,\theta) = (r\cos\theta)^2 + 2r\sin\theta$, *not* $r^2 + 2\theta$. This convention avoids the inconvenience of having to devise new symbols when changing coordinates.

We need to relate derivatives with respect to x and y to those with respect to r and θ. Performing a standard multivariate chain rule computation based on (4.103), we obtain

$$\begin{aligned} \frac{\partial}{\partial r} &= \cos\theta\,\frac{\partial}{\partial x} + \sin\theta\,\frac{\partial}{\partial y}, \\ \frac{\partial}{\partial \theta} &= -r\sin\theta\,\frac{\partial}{\partial x} + r\cos\theta\,\frac{\partial}{\partial y}, \end{aligned} \qquad \text{so} \qquad \begin{aligned} \frac{\partial}{\partial x} &= \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}, \\ \frac{\partial}{\partial y} &= \sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial \theta}. \end{aligned} \tag{4.104}$$

Applying the squares of the latter differential operators to $u(r,\theta)$, we find, after a calculation in which many of the terms cancel, the *polar coordinate form of the Laplace equation*:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{4.105}$$

The boundary conditions are imposed on the unit circle $r = 1$, and so, by (4.102), take the form

$$u(1,\theta) = h(\theta). \tag{4.106}$$

Keep in mind that, in order to be single-valued functions of x, y, the solution $u(r,\theta)$ and its boundary values $h(\theta)$ must both be 2π–periodic functions of the angular coordinate:

$$u(r,\theta + 2\pi) = u(r,\theta), \qquad h(\theta + 2\pi) = h(\theta). \tag{4.107}$$

Polar *separation of variables* is based on the ansatz

$$u(r,\theta) = v(r)\,w(\theta), \tag{4.108}$$

which assumes that the solution is a product of functions of the individual variables. Substituting (4.108) into the polar form (4.105) of Laplace's equation yields

$$v''(r)\,w(\theta) + \frac{1}{r}\,v'(r)\,w(\theta) + \frac{1}{r^2}\,v(r)\,w''(\theta) = 0.$$

We now separate variables by moving all the terms involving r onto one side of the equation and all the terms involving θ onto the other. This is accomplished by first multiplying the equation by $r^2/\big(\,v(r)\,w(\theta)\,\big)$ and then moving the final term to the right-hand side:

$$\frac{r^2\,v''(r) + r\,v'(r)}{v(r)} = -\,\frac{w''(\theta)}{w(\theta)} = \lambda.$$

As in the rectangular case, a function of r can equal a function of θ if and only if both are equal to a common separation constant, which we call λ. The partial differential equation thus splits into a pair of ordinary differential equations

$$r^2\,v'' + r\,v' - \lambda\,v = 0, \qquad w'' + \lambda\,w = 0, \tag{4.109}$$

that will prescribe the separable solution (4.108). Observe that both have the form of an eigenfunction equation in which the separation constant λ plays the role of the eigenvalue. We are, as always, interested only in nonzero solutions.

We have already solved the eigenvalue problem for $w(\theta)$. According to (4.107), $w(\theta + 2\pi) = w(\theta)$ must be a 2π–periodic function. Therefore, by our earlier discussion, this periodic boundary value problem has the nonzero eigenfunctions

$$1, \qquad \sin n\theta, \qquad \cos n\theta, \qquad n = 1, 2, \ldots\,, \tag{4.110}$$

corresponding to the eigenvalues (separation constants)

$$\lambda = n^2, \qquad n = 0, 1, 2, \ldots .$$

With the value of λ fixed, the linear ordinary differential equation for the radial component,

$$r^2 v'' + r\, v' - n^2 v = 0, \tag{4.111}$$

does not have constant coefficients. But, fortunately, it has the form of a second-order *Euler ordinary differential equation*, [**23**, **89**], and hence can be readily solved by substituting the power ansatz $v(r) = r^k$. (See also Exercise 4.3.23.) Note that

$$v'(r) = k\, r^{k-1}, \qquad v''(r) = k\,(k-1)\, r^{k-2},$$

and hence, by substituting into the differential equation,

$$r^2 v'' + r\, v' - n^2 v = \left[\, k\,(k-1) + k - n^2 \,\right] r^k = (k^2 - n^2) r^k .$$

Thus, r^k is a solution if and only if

$$k^2 - n^2 = 0, \qquad \text{and hence} \qquad k = \pm\, n.$$

For $n \neq 0$, we have found the two linearly independent solutions:

$$v_1(r) = r^n, \qquad v_2(r) = r^{-n}, \qquad n = 1, 2, \ldots . \tag{4.112}$$

When $n = 0$, the power ansatz yields only the constant solution. But in this case, the equation $r^2 v'' + r\, v' = 0$ is effectively of first order and linear in v', and hence readily integrated. This provides the two independent solutions

$$v_1(r) = 1, \qquad v_2(r) = \log r, \qquad n = 0. \tag{4.113}$$

Combining (4.110) and (4.112–113), we produce the complete list of separable polar coordinate solutions to the Laplace equation:

$$\begin{matrix} 1, & r^n \cos n\theta, & r^n \sin n\theta, \\ \log r, & r^{-n} \cos n\theta, & r^{-n} \sin n\theta, \end{matrix} \qquad n = 1, 2, 3, \ldots . \tag{4.114}$$

Now, the solutions in the top row of (4.114) are continuous (in fact analytic) at the origin, where $r = 0$, whereas the solutions in the bottom row have singularities as $r \to 0$. The latter are not of use in the present situation, since we require that the solution remain bounded and smooth — even at the center of the disk. Thus, we should use only the nonsingular solutions to concoct a candidate series solution

$$u(r, \theta) \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n r^n \cos n\theta + b_n r^n \sin n\theta \right). \tag{4.115}$$

The coefficients a_n, b_n will be prescribed by the boundary conditions (4.106). Substituting $r = 1$, we obtain

$$u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \bigl(a_n \cos n\theta + b_n \sin n\theta \bigr) = h(\theta).$$

We recognize this as a standard Fourier series (3.29) (with θ replacing x) for the 2π periodic function $h(\theta)$. Therefore,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos n\theta \, d\theta, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\sin n\theta \, d\theta, \tag{4.116}$$

are precisely its Fourier coefficients, cf. (3.35). In this manner, we have produced a series solution (4.115) to the boundary value problem (4.105–106).

Remark: Introducing the complex variable

$$z = x + \mathrm{i}\, y = r\, e^{\mathrm{i}\,\theta} = r\cos\theta + \mathrm{i}\, r \sin\theta \tag{4.117}$$

allows us to write

$$z^n = r^n\, e^{\mathrm{i}\, n\theta} = r^n \cos n\theta + \mathrm{i}\, r^n \sin n\theta. \tag{4.118}$$

Therefore, the nonsingular separable solutions are the *harmonic polynomials*

$$r^n \cos n\theta = \operatorname{Re} z^n, \qquad r^n \sin n\theta = \operatorname{Im} z^n. \tag{4.119}$$

The first few are listed in the following table:

n	$\operatorname{Re} z^n$	$\operatorname{Im} z^n$
0	1	0
1	x	y
2	$x^2 - y^2$	$2xy$
3	$x^3 - 3xy^2$	$3x^2y - y^3$
4	$x^4 - 4x^2y^2 + y^4$	$4x^3y - 4xy^3$

Their general expression is obtained using the Binomial Formula:

$$\begin{aligned} z^n &= (x + \mathrm{i}\, y)^n \\ &= x^n + n x^{n-1}(\mathrm{i}\, y) + \binom{n}{2} x^{n-2}(\mathrm{i}\, y)^2 + \binom{n}{3} x^{n-3}(\mathrm{i}\, y)^3 + \cdots + (\mathrm{i}\, y)^n \\ &= x^n + \mathrm{i}\, n x^{n-1} y - \binom{n}{2} x^{n-2} y^2 - \mathrm{i}\binom{n}{3} x^{n-3} y^3 + \cdots, \end{aligned}$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \tag{4.120}$$

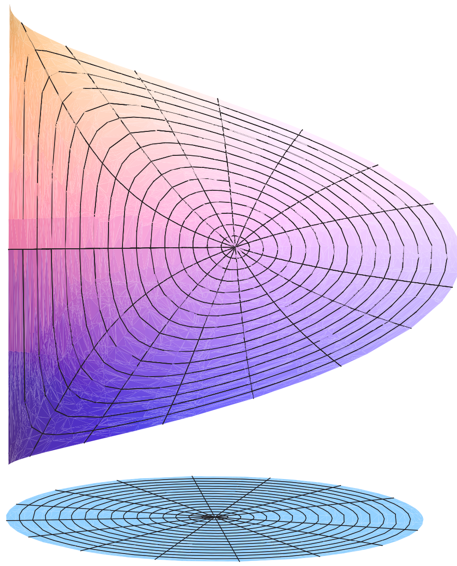


Figure 4.10. Membrane attached to a helical wire.

are the usual *binomial coefficients*. Separating the real and imaginary terms, we produce the explicit formulae

$$\begin{aligned} r^n \cos n\theta &= \operatorname{Re} z^n = x^n - \binom{n}{2} x^{n-2} y^2 + \binom{n}{4} x^{n-4} y^4 + \cdots , \\ r^n \sin n\theta &= \operatorname{Im} z^n = n x^{n-1} y - \binom{n}{3} x^{n-3} y^3 + \binom{n}{5} x^{n-5} y^5 + \cdots , \end{aligned} \tag{4.121}$$

for the two independent harmonic polynomials of degree n.

Example 4.5. Consider the Dirichlet boundary value problem on the unit disk with

$$u(1,\theta) = \theta \qquad \text{for} \qquad -\pi < \theta < \pi. \tag{4.122}$$

The boundary data can be interpreted as a wire in the shape of a single turn of a spiral helix sitting over the unit circle. The wire has a single jump discontinuity, of magnitude 2π, at the boundary point $(-1,0)$. The required Fourier series

$$h(\theta) = \theta \ \sim \ 2\left(\sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \frac{\sin 4\theta}{4} + \cdots\right)$$

was already computed in Example 3.3. Therefore, invoking our solution formula (4.115–116), we have

$$u(r,\theta) = 2\left(r\sin\theta - \frac{r^2\sin 2\theta}{2} + \frac{r^3\sin 3\theta}{3} - \frac{r^4\sin 4\theta}{4} + \cdots\right) \tag{4.123}$$

is the desired solution, which is plotted in Figure 4.10. In fact, this series can be explicitly summed. In view of (4.119) and the usual formula (A.13) for the complex logarithm, we have

$$u = 2\operatorname{Im}\left(z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots\right) = 2\operatorname{Im}\log(1+z) = 2\psi, \tag{4.124}$$

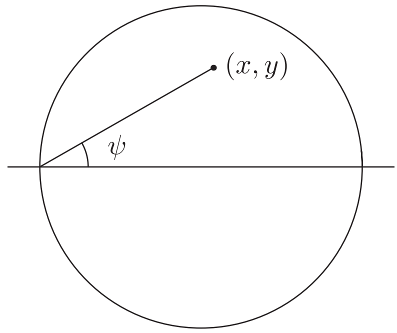


Figure 4.11. Geometric construction of the solution.

where

$$\psi = \tan^{-1} \frac{y}{1+x}$$

is the angle that the line passing through the two points (x, y) and $(-1, 0)$ makes with the x-axis, as sketched in Figure 4.11. You should try to convince yourself that, on the unit circle, $2\,\psi = \theta$ has the correct boundary values. Observe that, even though the boundary values are discontinuous, the solution is an analytic function inside the disk.

In fact, unlike the rectangular series (4.98), the general polar series solution formula (4.115) can, in fact, be summed in closed form! If we substitute the explicit Fourier formulae (4.116) into (4.115) — remembering to change the integration variable to, say, ϕ to avoid a notational conflict — we obtain

$$\begin{aligned} u(r,\theta) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n r^n \cos n\,\theta + b_n r^n \sin n\,\theta \big) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\phi)\, d\phi + \sum_{n=1}^{\infty} \left[\frac{r^n \cos n\,\theta}{\pi} \int_{-\pi}^{\pi} h(\phi) \cos n\,\phi\, d\phi + \frac{r^n \sin n\,\theta}{\pi} \int_{-\pi}^{\pi} h(\phi) \sin n\,\phi\, d\phi \right] \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} h(\phi) \left[\frac{1}{2} + \sum_{n=1}^{\infty} r^n \big(\cos n\,\theta \cos n\,\phi + \sin n\,\theta \sin n\,\phi \big) \right] d\phi \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} h(\phi) \left[\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n\,(\theta - \phi) \right] d\phi. \end{aligned} \tag{4.125}$$

We next show how to sum the final series. Using (4.118), we can write it as the real part of a geometric series:

$$\begin{aligned} \frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n\,\theta &= \operatorname{Re} \left(\frac{1}{2} + \sum_{n=1}^{\infty} z^n \right) = \operatorname{Re} \left(\frac{1}{2} + \frac{z}{1-z} \right) = \operatorname{Re} \left(\frac{1+z}{2\,(1-z)} \right) \\ &= \operatorname{Re} \left(\frac{(1+z)(1-\overline{z})}{2\,|\,1-z\,|^2} \right) = \frac{\operatorname{Re}\,(1 + z - \overline{z} - |\,z\,|^2)}{2\,|\,1-z\,|^2} = \frac{1-|\,z\,|^2}{2\,|\,1-z\,|^2} = \frac{1-r^2}{2\,(1+r^2-2\,r\cos\theta)}\,, \end{aligned}$$

which is known as the *Poisson kernel*. Substituting back into (4.125) establishes the important *Poisson Integral Formula* for the solution to the boundary value problem.

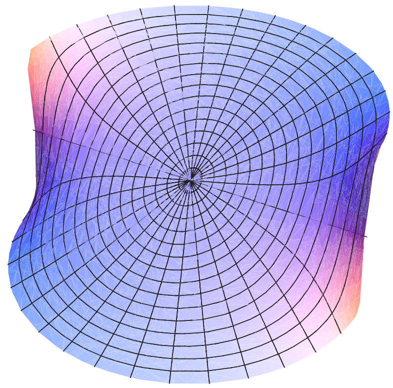


Figure 4.12. Equilibrium temperature of a disk.

Theorem 4.6. *The solution to the Laplace equation in the unit disk subject to Dirichlet boundary conditions* $u(1,\theta) = h(\theta)$ *is*

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\, \frac{1-r^2}{1+r^2-2\,r\cos(\theta-\phi)}\, d\phi. \tag{4.126}$$

Example 4.7. A uniform metal disk of unit radius has half of its circular boundary held at 1°, while the other half is held at 0°. Our task is to find the equilibrium temperature $u(x,y)$. In other words, we seek the solution to the Dirichlet boundary value problem

$$\Delta u = 0, \qquad x^2+y^2<1, \qquad u(x,y) = \begin{cases} 1, & x^2+y^2=1, \quad y>0, \\ 0, & x^2+y^2=1, \quad y<0. \end{cases} \tag{4.127}$$

In polar coordinates, the boundary data is a (periodic) step function

$$h(\theta) = \begin{cases} 1, & 0<\theta<\pi, \\ 0, & -\pi<\theta<0. \end{cases}$$

Therefore, according to the Poisson formula (4.126), the solution is given by†

$$u(r,\theta) = \frac{1}{2\pi}\int_{0}^{\pi} \frac{1-r^2}{1+r^2-2\,r\cos(\theta-\phi)}\, d\phi = \begin{cases} 1 - \dfrac{1}{\pi}\tan^{-1}\left(\dfrac{1-r^2}{2\,r\sin\theta}\right), & 0<\theta<\pi, \\[2ex] \dfrac{1}{2}, & \theta = 0, \pm\pi, \\[2ex] -\dfrac{1}{\pi}\tan^{-1}\left(\dfrac{1-r^2}{2\,r\sin\theta}\right), & -\pi<\theta<0, \end{cases} \tag{4.128}$$

† The detailed derivation of the final expressions is left to the reader as Exercise 4.3.40.

where we use the principal branch $-\frac{1}{2}\pi < \tan^{-1} t < \frac{1}{2}\pi$ of the inverse tangent. Reverting to rectangular coordinates, we find that the equilibrium temperature has the explicit formula

$$u(x,y) = \begin{cases} 1 - \dfrac{1}{\pi} \tan^{-1}\left(\dfrac{1-x^2-y^2}{2\,y}\right), & x^2+y^2<1, \quad y>0, \\ \dfrac{1}{2}, & x^2+y^2<1, \quad y=0, \\ -\dfrac{1}{\pi} \tan^{-1}\left(\dfrac{1-x^2-y^2}{2\,y}\right), & x^2+y^2<1, \quad y<0. \end{cases} \tag{4.129}$$

The result is depicted in Figure 4.12.

Averaging, the Maximum Principle, and Analyticity

Let us investigate some important consequences of the Poisson integral formula (4.126). First, setting $r=0$ yields

$$u(0,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\phi)\, d\phi. \tag{4.130}$$

The left-hand side is the value of u at the origin — the center of the disk — and so independent of θ; the right-hand side is the *average* of its boundary values around the unit circle. This formula is a particular instance of an important general fact.

Theorem 4.8. *Let $u(x,y)$ be harmonic inside a disk of radius a centered at a point (x_0,y_0) with piecewise continuous (or, more generally, integrable) boundary values on the circle $C = \{\,(x-x_0)^2+(y-y_0)^2 = a^2\,\}$. Then its value at the center of the disk is equal to the average of its values on the boundary circle*:

$$u(x_0,y_0) = \frac{1}{2\pi a} \oint_C u\, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_0 + a\cos\theta, y_0 + a\sin\theta)\, d\theta. \tag{4.131}$$

Proof: One approach is to use the scaling and translation symmetries of the Laplace equation, cf. Exercises 4.3.8–9, to map the disk of radius a centered at (x_0,y_0) to the unit disk centered at the origin, and then invoke (4.130). Specifically, we set

$$U(x,y) = u(x_0 + a\,x, y_0 + a\,y). \tag{4.132}$$

An easy chain rule computation proves that $U(x,y)$ also satisfies the Laplace equation on the unit disk $x^2+y^2<1$, with boundary values

$$h(\theta) = U(\cos\theta, \sin\theta) = u(x_0 + a\cos\theta, y_0 + a\sin\theta).$$

Therefore, by (4.130),

$$U(0,0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\theta)\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} U(\cos\theta, \sin\theta)\, d\theta.$$

Replacing U by its formula (4.132) produces the desired result for solutions defined by the Poisson integral formula. However, it is not a priori clear that all solutions, i.e., all harmonic functions, are necessarily of this form. This will follow eventually from the Uniqueness Theorem 4.10; however, its proof relies on formula (4.131), leading to a circular argument.

A better proof, which does not rely on the solution formula (4.115, 116), is the following. Given the harmonic function $u(x,y)$, consider the scalar function

$$g(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_0 + a\cos\theta, y_0 + a\sin\theta)\, d\theta,$$

which is well defined for $a > 0$ sufficiently small. Since $u \in \mathrm{C}^2$, we can calculate the derivative of g as follows:

$$\begin{aligned} g'(a) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\cos\theta \, \frac{\partial u}{\partial x}\, (x_0 + a\cos\theta, y_0 + a\sin\theta) + \sin\theta\, \frac{\partial u}{\partial y}\, (x_0 + a\cos\theta, y_0 + a\sin\theta) \right] d\theta \\ &= \frac{1}{2\pi a} \oint_C \frac{\partial u}{\partial \mathbf{n}}\, ds, \end{aligned}$$

where $\mathbf{n} = (\cos\theta, \sin\theta)$ defines the unit normal to C at the point $(x_0 + a\cos\theta, y_0 + a\sin\theta)$ and $ds = a\,d\theta$ is the arc length element. Letting $D = \{\,(x - x_0)^2 + (y - y_0)^2 \le a^2\,\}$ denote the disk of radius a, so $C = \partial D$ is its boundary, the divergence identity (6.89), which is an easy consequence of Green's Theorem, implies that the latter integral equals

$$\oint_C \frac{\partial u}{\partial \mathbf{n}}\, ds = \iint_D \Delta u \, dx\, dy = 0$$

because u is harmonic. Thus, $g'(a) = 0$ for all $a > 0$ sufficiently small, which implies $g(a) = c$ is constant. But $g(a)$ represents the average of $u(x,y)$ on the circle C of radius a and hence, as $a \to 0$, the average $g(a) \to u(x_0, y_0)$. We conclude that $g(a) = u(x_0, y_0)$ for all $a > 0$ such that $u(x,y)$ is harmonic in the disk of radius a, which establishes (4.131) for all such harmonic functions. *Q.E.D.*

An important consequence of the integral formula (4.131) is the *Strong Maximum Principle* for harmonic functions.

Theorem 4.9. *Let u be a* nonconstant *harmonic function defined on a bounded domain Ω and continuous on $\partial\Omega$. Then u achieves its maximum and minimum values only at boundary points of the domain. In other words, if*

$$m = \min\{\, u(x,y) \mid (x,y) \in \partial\Omega \,\}, \qquad M = \max\{\, u(x,y) \mid (x,y) \in \partial\Omega \,\},$$

are, respectively, its maximum and minimum values on the boundary, then

$$m < u(x,y) < M \qquad \text{at all interior points} \qquad (x,y) \in \Omega.$$

Proof: Let $M^\star \ge M$ be the maximum value of u on all of $\overline{\Omega} = \Omega \cup \partial\Omega$, and assume $u(x_0, y_0) = M^\star$ at some interior point $(x_0, y_0) \in \Omega$. Theorem 4.8 implies that $u(x_0, y_0)$ equals its average over any circle C centered at (x_0, y_0) that bounds a closed disk contained in Ω. Since u is continuous and $\le M^\star$ on C, its average must be strictly less than $M^\star$ — except in the trivial case in which it is constant and equal to $M^\star$ on all of C. Thus, our assumption implies that $u(x,y) = M^\star = u(x_0, y_0)$ for all (x,y) belonging to any circle $C \subset \Omega$ centered at (x_0, y_0). Since Ω is connected, this allows us to conclude† that $u(x,y) = M^\star$ is constant throughout Ω, in contradiction to our original assumption.

A similar argument works for the minimum; alternatively, one can interchange maximum and minimum by replacing u by $-u$. *Q.E.D.*

† You are asked to supply the details in Exercise 4.3.42.

Physically, if we interpret $u(x,y)$ as the vertical displacement of a membrane stretched over a wire, then Theorem 4.9 says that, in the absence of external forcing, the membrane cannot have any internal bumps — its highest and lowest points are necessarily on the boundary of the domain. This reconfirms our physical intuition: the restoring force exerted by the stretched membrane will serve to flatten any bump, and hence a membrane with a local maximum or minimum cannot be in equilibrium. A similar interpretation holds for heat conduction. A body in thermal equilibrium will achieve its maximum and minimum temperature only at boundary points. Indeed, thermal energy would flow away from any internal maximum, or towards any local minimum, and so if the body contained a local maximum or minimum in its interior, it could not remain in thermal equilibrium.

The Maximum Principle immediately implies the uniqueness of solutions to the Dirichlet boundary value problem for both the Laplace and Poisson equations:

Theorem 4.10. *If u and $\widetilde{u}$ both satisfy the same Poisson equation $-\Delta u = f = -\Delta \widetilde{u}$ within a bounded domain Ω, and $u = \widetilde{u}$ on $\partial\Omega$, then $u \equiv \widetilde{u}$ throughout Ω.*

Proof: By linearity, the difference $v = u - \widetilde{u}$ satisfies the homogeneous boundary value problem $\Delta v = 0$ in Ω and $v = 0$ on $\partial\Omega$. Our assumption implies that the maximum and minimum boundary values of v are both $0 = m = M$. Theorem 4.9 implies that $v(x,y) \equiv 0$ at all $(x,y) \in \Omega$, and hence $u \equiv \widetilde{u}$ everywhere in Ω. *Q.E.D.*

Finally, let us discuss the analyticity of harmonic functions. In view of (4.119), the n^{th} order term in the polar series solution (4.115), namely,

$$a_n r^n \cos n\theta + b_n r^n \sin n\theta = a_n \operatorname{Re} z^n + b_n \operatorname{Im} z^n = \operatorname{Re}\left[\,(a_n - \mathrm{i}\, b_n)\, z^n\,\right],$$

is, in fact, a homogeneous polynomial in (x,y) of degree n. This means that, when written in rectangular coordinates x and y, (4.115) is, in fact, a *power series* for the harmonic function $u(x,y)$. It is well known, [**8**, **23**, **97**], that any convergent power series converges to an analytic function — in this case $u(x,y)$. Moreover, the power series must, in fact, be the *Taylor series* for $u(x,y)$ based at the origin, and so its coefficients are multiples of the derivatives of u at $x = y = 0$. Details are worked out in Exercise 4.3.49.

We can adapt this argument to prove analyticity of *all* solutions to the Laplace equation. Note especially the contrast with the wave equation, which has many non-analytic solutions.

Theorem 4.11. *A harmonic function is analytic at every point in the interior of its domain of definition.*

Proof: Let $u(x,y)$ be a solution to the Laplace equation on the open domain $\Omega \subset \mathbb{R}^2$. Let $\mathbf{x}_0 = (x_0, y_0) \in \Omega$, and choose $a > 0$ such that the closed disk of radius a centered at $\mathbf{x}_0$ is entirely contained within Ω:

$$D_a(\mathbf{x}_0) = \{\, \|\, \mathbf{x} - \mathbf{x}_0 \,\| \le a \,\} \subset \Omega,$$

where $\|\cdot\|$ is the usual Euclidean norm. Then the function $U(x,y)$ defined by (4.132) is harmonic on the unit disk, with well-defined boundary values. Thus, by the preceding remarks, $U(x,y)$ is analytic at every point inside the unit disk, and hence so is

$$u(x,y) = U\left(\frac{x - x_0}{a}, \frac{y - y_0}{a}\right)$$

at every point (x,y) in the interior of the disk $D_a(\mathbf{x}_0)$. Since $\mathbf{x}_0 \in \Omega$ was arbitrary, this establishes the analyticity of u throughout the domain. *Q.E.D.*

This concludes our discussion of the method of separation of variables for the planar Laplace equation and some of its important consequences. The method can be used in a few other special coordinate systems. See [**78**, **79**] for a complete account, including the fascinating connections with the underlying symmetry properties of the equation.

Exercises

4.3.22. Solve the following Euler differential equations by use of the power ansatz:
(a) $x^2 u'' + 5 x u' - 5 u = 0$, (b) $2 x^2 u'' - x u' - 2 u = 0$, (c) $x^2 u'' - u = 0$,
(d) $x^2 u'' + x u' - 3 u = 0$, (e) $3 x^2 u'' - 5 x u' - 3 u = 0$, (f) $\dfrac{d^2u}{dx^2} + \dfrac{2}{x}\dfrac{du}{dx} = 0$.

♢ 4.3.23. (*i*) Show that if $u(x)$ solves the *Euler differential equation*

$$a x^2 \frac{d^2u}{dx^2} + b x \frac{du}{dx} + c u = 0, \tag{4.133}$$

then $v(y) = u(e^y)$ solves a linear constant-coefficient differential equation.
(*ii*) Use this technique to solve the Euler differential equations in Exercise 4.3.22.

4.3.24. (a) Use the method in Exercise 4.3.23 to solve an Euler equation whose characteristic equation has a double root $r_1 = r_2 = r$. (b) Solve the specific equations

$$(i)\ \ x^2 u'' - x u' + u = 0, \qquad (ii)\ \ \frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} = 0.$$

4.3.25. Solve the following boundary value problems:
(a) $\Delta u = 0,\ x^2 + y^2 < 1,\quad u = x^3,\ x^2 + y^2 = 1$;
(b) $\Delta u = 0,\ x^2 + y^2 < 2,\quad u = \log(x^2 + y^2),\ x^2 + y^2 = 1$;
(c) $\Delta u = 0,\ x^2 + y^2 < 4,\quad u = x^4,\ x^2 + y^2 = 4$;
(d) $\Delta u = 0,\ x^2 + y^2 < 1,\quad \dfrac{\partial u}{\partial \mathbf{n}} = x,\ x^2 + y^2 = 1$.

4.3.26. Let $u(x,y)$ be the solution to the boundary value problem $u_{xx} + u_{yy} = 0,\ x^2 + y^2 < 1$, $u(x,y) = x^2,\ x^2 + y^2 = 1$. Find $u(0,0)$.

♡ 4.3.27. (a) Find the equilibrium temperature on a disk of radius 1 when half the boundary is held at 1° and the other half is held at -1°. (b) Find the equilibrium temperature on a half-disk of radius 1 when the temperature is held to 1° on the curved edge and 0° on the straight edge. (c) Find the equilibrium temperature on a half disk of radius 1 when the temperature is held to 0° on the curved edge and 1° on the straight edge.

4.3.28. Find the solution to Laplace's equation $u_{xx} + u_{yy} = 0$ on the semi-disk $x^2 + y^2 < 1$, $y > 0$, that satisfies the boundary conditions $u(x,0) = 0$ for $-1 < x < 1$ and $u(x,y) = y^3$ for $x^2 + y^2 = 1,\ y > 0$.

4.3.29. Find the equilibrium temperature on a half-disk of radius 1 when the temperature is held to 1° on the curved edge, while the straight edge is insulated.

4.3.30. Solve the Dirichlet boundary value problem for the Laplace equation on the pie wedge $W = \{0 < \theta < \frac{1}{4}\pi,\ \ 0 < r < 1\}$, when the nonzero boundary data $u(1,\theta) = h(\theta)$ appears only on the curved portion of its boundary.

4.3.31. Find a harmonic function $u(x,y)$ defined on the annulus $\frac{1}{2} < r < 1$ subject to the constant Dirichlet boundary conditions $u = a$ on $r = \frac{1}{2}$ and $u = b$ on $r = 1$.

4.3.32. Boiling water flows continually through a long circular metal pipe of inner radius 1 cm and outer radius 1.2 cm placed in an ice water bath. *True or false*: The temperature at the midpoint, at radius 1.1 cm, is 50°. If false, what is the temperature at this point?

4.3.33. Write out the series solution to the boundary value problem $u(1,\theta) = 0$, $u(2,\theta) = h(\theta)$, for the Laplace equation on an annulus $1 < r < 2$. *Hint*: Use all of the separable solutions listed in (4.114).

4.3.34. Solve the following boundary value problems for the Laplace equation on the annulus $1 < r < 2$: (*a*) $u(1,\theta) = 0$, $u(2,\theta) = 1$, (*b*) $u(1,\theta) = 0$, $u(2,\theta) = \cos\theta$, (*c*) $u(1,\theta) = \sin 2\theta$, $u(2,\theta) = \cos 2\theta$, (*d*) $u_r(1,\theta) = 0$, $u(2,\theta) = 1$, (*e*) $u_r(1,\theta) = 0$, $u(2,\theta) = \sin 2\theta$, (*f*) $u_r(1,\theta) = 0$, $u_r(2,\theta) = 1$, (*g*) $u_r(1,\theta) = 2$, $u_r(2,\theta) = 1$.

4.3.35. Solve the following boundary value problems for the Laplace equation on the semi-annular domain $D = \{1 < x^2 + y^2 < 2,\ y > 0\}$:
(*a*) $u(x,y) = 0$, $x^2 + y^2 = 1$, $u(x,y) = 1$, $x^2 + y^2 = 2$, $u(x,0) = 0$;
(*b*) $u(x,y) = 0$, $x^2 + y^2 = 1$ or 2, $u(x,0) = 0$, $x > 0$, $u(x,0) = 1$, $x < 0$.

4.3.36. Solve the following boundary value problem:

$$(x^2 + y^2)(u_{xx} + u_{yy}) + 2\,x\,u_x + 2\,y\,u_y = 0,\quad x^2 + y^2 < 1,\quad u(x,y) = 1 + 3\,x,\quad x^2 + y^2 = 1.$$

♢ 4.3.37. Justify the chain rule computation (4.104). Then justify formula (4.105) for the Laplacian in polar coordinates.

4.3.38. Suppose $\displaystyle\int_{-\pi}^{\pi} |\,h(\theta)\,|\,d\theta < \infty$. Prove that (4.115) converges uniformly to the solution to the boundary value problem (4.101) on any smaller disk $D_{r_\star} = \{\, r \le r_\star < 1 \,\} \subsetneq D_1$.

4.3.39. Prove directly that (4.124) satisfies the boundary conditions (4.122).

♢ 4.3.40. Justify the integration formula in (4.128).

4.3.41. Provide a complete proof that (4.129) is indeed the solution to the boundary value problem (4.127).

♢ 4.3.42. Complete the proof of Theorem 4.9 by showing that $u(x,y) = M^\star$ for all $(x,y) \in \Omega$. *Hint*: Join (x_0,y_0) to (x,y) by a curve $C \subset \Omega$ of finite length, and use the preceding part of the proof to inductively deduce the existence of a finite sequence of points $(x_i,y_i) \in C$, $i = 0,\ldots,n$, with $(x_n,y_n) = (x,y)$, and such that $u(x_i,y_i) = M^\star$.

♢ 4.3.43. Derive the analogue of the Poisson integral formula for the solution to the Neumann boundary value problem $\Delta u = 0$, $x^2 + y^2 < 1$, $\partial u/\partial \mathbf{n} = h$, $x^2 + y^2 = 1$, on the unit disk. Pay careful attention to the existence and uniqueness of solutions in your formulation.

4.3.44. Give an example of a solution to Poisson's equation on the unit disk that achieves its maximum at an interior point. Interpret your construction physically.

4.3.45. Let $p(x,y)$ be a polynomial (not necessarily harmonic). Suppose $u(x,y)$ is harmonic and equals $p(x,y)$ on the unit circle $x^2 + y^2 = 1$. Prove that $u(x,y)$ is a harmonic polynomial.

4.3.46. Write down an integral formula for the solution to the Dirichlet boundary value problem on a disk of radius $R > 0$, namely, $\Delta u = 0$, $x^2 + y^2 < R^2$, $u = h$, $x^2 + y^2 = R^2$.

4.3.47. State and prove a one-dimensional version of Theorem 4.8. Does the analogue of Theorem 4.9 hold?

4.3.48. A unit area square plate has 100° temperature on its top edge and 0° on its three other edges. *True or false*: The temperature at the center equals the average edge temperature.

♢ 4.3.49. Let $u(x,y)$ be a harmonic function on the unit disk with boundary values $h(\theta)$ when $r = 1$. Using the fact that (4.115) is the Taylor series for $u(x,y)$ at the origin: (*a*) Find integral formulas for its partial derivatives $u_x(0,0)$, $u_y(0,0)$, involving the boundary values $h(\theta)$. (*b*) Generalize part (a) to the second-order derivatives $u_{xx}(0,0)$, $u_{xy}(0,0)$, $u_{yy}(0,0)$.

4.3.50. Prove that if $u(x,y)$ is a bounded harmonic function defined on all of $\mathbb{R}^2$, then u is constant. *Hint*: First generalize Exercise 4.3.49(a) to find the value of its gradient, $\nabla u(x_0, y_0)$, in terms of the values of u on a circle of radius a centered at (x_0, y_0). Then see what happens when the radius of the circle goes to ∞.

4.4 Classification of Linear Partial Differential Equations

We have, at last, been introduced to the three paradigmatic linear second-order partial differential equations for functions of two variables. The homogeneous versions are

(a)	The wave equation:	$u_{tt} - c^2 u_{xx} = 0,$	*hyperbolic*,
(b)	The heat equation:	$u_t - \gamma u_{xx} = 0,$	*parabolic*,
(c)	Laplace's equation:	$u_{xx} + u_{yy} = 0,$	*elliptic.*

The last column indicates the equation's *type*, in accordance with the standard taxonomy of partial differential equations; an explanation will appear momentarily. The wave, heat, and Laplace equations are the prototypical representatives of these three fundamental genres. Each genre has its own distinctive analytic features, physical manifestations, and even numerical solution schemes. Equations governing vibrations, such as the wave equation, are typically hyperbolic. Equations modeling diffusion, such as the heat equation, are parabolic. Hyperbolic and parabolic equations both typically represent dynamical processes, and so one of the independent variables is identified as time. On the other hand, equations modeling equilibrium phenomena, including the Laplace and Poisson equations, are usually elliptic, and involve only spatial variables. Elliptic partial differential equations are associated with boundary value problems, whereas parabolic and hyperbolic equations require initial and initial-boundary value problems.

The classification theory of real linear second-order partial differential equations for a scalar-valued function $u(t,x)$ depending on two variables† proceeds as follows. The most general such equation has the form

$$L[u] = A u_{tt} + B u_{tx} + C u_{xx} + D u_t + E u_x + F u = G, \tag{4.134}$$

where the coefficients A, B, C, D, E, F are all allowed to be functions of (t,x), as is the inhomogeneity or forcing function $G(t,x)$. The equation is *homogeneous* if and only if $G \equiv 0$. We assume that at least one of the leading coefficients A, B, C is not identically zero, since otherwise, the equation degenerates to a first-order equation.

The key quantity that determines the *type* of such a partial differential equation is its *discriminant*

$$\Delta = B^2 - 4AC. \tag{4.135}$$

This should (and for good reason) remind the reader of the discriminant of the quadratic equation

$$Q(x,y) = A x^2 + B x y + C y^2 + D x + E y + F = 0, \tag{4.136}$$

whose solutions trace out a plane curve — a conic section. In the nondegenerate cases, the discriminant (4.135) fixes its geometric type:

† For equilibrium equations, we identify t with the space variable y.

- a hyperbola when $\Delta > 0$,
- a parabola when $\Delta = 0$,
- an ellipse when $\Delta < 0$.

This motivates the choice of terminology used to classify second-order partial differential equations.

Definition 4.12. At a point (t, x), the linear second-order partial differential equation (4.134) is called

- *hyperbolic* if $\Delta(t, x) > 0$,
- *parabolic* if $\Delta(t, x) = 0$, but $A^2 + B^2 + C^2 \neq 0$,
- *elliptic* if $\Delta(t, x) < 0$,
- *singular* if $A = B = C = 0$.

In particular:

- The wave equation $u_{tt} - u_{xx} = 0$ has discriminant $\Delta = 4$, and is hyperbolic.
- The heat equation $u_{xx} - u_t = 0$ has discriminant $\Delta = 0$, and is parabolic.
- The Poisson equation $u_{tt} + u_{xx} = -f$ has discriminant $\Delta = -4$, and is elliptic.

Example 4.13. When the coefficients A, B, C vary, the type of the partial differential equation may not remain fixed over the entire domain. Equations that change type are less common, as well as being much harder to analyze and solve, both analytically and numerically. One example arising in the theory of supersonic aerodynamics, [**44**], is the *Tricomi equation*

$$x \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{4.137}$$

Comparing with (4.134), we find that

$$A = x, \quad B = 0, \quad C = -1, \qquad \text{while} \qquad D = E = F = G = 0.$$

The discriminant in this particular case is

$$\Delta = B^2 - 4AC = 4x,$$

and hence the equation is hyperbolic when $x > 0$, elliptic when $x < 0$, and parabolic on the transition line $x = 0$. In the physical model, the hyperbolic region corresponds to subsonic flow, while the supersonic regions are of elliptic type. The transitional parabolic boundary represents the shock line between the sub- and super-sonic regions — the familiar sonic boom as an airplane crosses the sound barrier.

While this tripartite classification into hyperbolic, parabolic, and elliptic equations initially appears in the bivariate context, the terminology, underlying properties, and associated physical models carry over to second-order partial differential equations in higher dimensions. Most of the partial differential equations arising in applications fall into one of these three categories, and it is fair to say that the field of partial differential equations splits into three distinct subfields. Or rather four subfields, the last containing all the equations, including higher-order equations, that do not fit into the preceding categorization. (One important example appears in Section 8.5.)

Remark: The classification into hyperbolic, parabolic, elliptic, and singular types carries over as stated to *quasilinear* second-order equations, whose coefficients $A, \ldots, G$ are allowed to depend on u and its first-order derivatives, u_t, u_x. Here the type of the equation can vary with both the point in the domain and the particular solution being considered. Even more generally, for a *fully nonlinear* second-order partial differential equation

$$H(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0, \tag{4.138}$$

one defines its *discriminant* to be

$$\Delta = \left(\frac{\partial H}{\partial u_{tx}}\right)^2 - 4\,\frac{\partial H}{\partial u_{tt}}\,\frac{\partial H}{\partial u_{xx}}\,. \tag{4.139}$$

Its sign determines the type of the equation as above — again depending on the point in the domain and the solution under consideration.

Exercises

4.4.1. Plot the following conic sections and classify their type:
(a) $x^2 + 3y^2 = 1$, (b) $xy + x + y = 4$, (c) $x^2 - xy + y^2 = x - 2y$,
(d) $x^2 + 2xy + y^2 + y = 1$, (e) $x^2 - 2y^2 = 6x + 8y + 1$.

4.4.2. Determine the type of the following partial differential equations:
(a) $u_{tt} + 3u_{xx} = 0$, (b) $u_{tx} + u_t + u_x = u$, (c) $u_{tt} + u_t + u_x = 0$,
(d) $u_{tt} - u_{tx} + u_{xx} = u$, (e) $u_{tt} + 4u_{tx} + 4u_{xx} = u_t$, (f) $u_{tx} + u_{xx} = 0$.

4.4.3. Consider the partial differential equation $x\,u_{tt} + (t + x)\,u_{xx} = 0$. At what points of the plane is the equation elliptic? hyperbolic? parabolic? degenerate?

4.4.4. Answer Exercise 4.4.3 for the equations
(a) $x^2 u_{xx} + x\,u_x + u_{yy} = 0$, (b) $\partial_x(x\,u_x) = \partial_y(y\,u_y)$, (c) $u_t = \partial_x[(x+t)u_x]$,
(d) $\nabla \cdot (c(x,y)\nabla u) = u$, where $c(x,y)$ is a given function.

4.4.5. Steady flow of air past an airplane is modeled by the partial differential equation $(m^2 - 1)u_{xx} + u_{yy} = 0$, in which x is the flight direction, y the transverse direction, and $m \geq 0$ is the *Mach number* — the ratio of the airplane's speed to the speed of sound. Show that the equation is hyperbolic for subsonic flight, but elliptic for supersonic flight.

4.4.6. Show that the second-order partial differential equation

$$-\frac{\partial}{\partial x}\left(p(x,y)\,\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(q(x,y)\,\frac{\partial u}{\partial y}\right) + r(x,y)\,u = f(x,y)$$

is elliptic if and only if $p(x,y)$ and $q(x,y)$ are nonzero and have the same sign.

♢ 4.4.7. *True or false*: The type of a linear second-order partial differential equation is not affected by a change of independent variables: $\tau = \varphi(t,x)$, $\xi = \psi(t,x)$.

4.4.8. Let $v(t,x) = a(t,x)\,u(t,x) + b(t,x)$, where a, b are fixed functions with $a \neq 0$. Suppose u is a solution to a second-order linear partial differential equation. Prove that v also solves a linear partial differential equation of the same type.

♢ 4.4.9. *True or false*: The polar coordinate form (4.105) of the Laplace equation is elliptic.

4.4.10. Rewrite the Laplace equation $u_{xx} + u_{yy} = 0$ in terms of *parabolic coordinates* ξ, η, as defined by the equations $x = \xi^2 - \eta^2$, $y = 2\xi\eta$. Is the resulting equation elliptic?

♢ 4.4.11. Prove that the complex change of variables $x = x$, $t = \mathrm{i}\,y$, maps the Laplace equation $u_{xx} + u_{yy} = 0$ to the wave equation $u_{tt} = u_{xx}$. Explain why the type of a partial differential equation is *not* necessarily preserved under a complex change of variables.

♡ 4.4.12. Suppose, against all advice, we pose the elliptic Laplace equation as an initial value problem, namely

$$u_{tt} = -\,u_{xx} \qquad \text{for} \qquad 0 < x < 1, \qquad t > 0,$$
$$u(0,x) = f(x), \qquad u_t(0,x) = 0, \qquad 0 \le x \le 1, \qquad u(t,0) = 0 = u(t,1), \qquad t \ge 0.$$

(*a*) Prove that for any positive integer $n > 0$, the function $u_n(t,x) = \dfrac{\sin n\pi t\, \cosh n\pi x}{n}$ satisfies the initial value problem. Determine the initial condition $u_n(0,x) = f_n(x)$.

(*b*) Prove that, as $n \to \infty$, the initial condition $f_n(x) \to 0$ becomes vanishingly small, whereas, at any $t > 0$, the solution value $u_n\left(t, \frac{1}{2}\right) \to \infty$.

(*c*) Explain why this represents an ill-posed problem.

4.4.13. The *minimal surface equation* $(1+u_x^2)u_{xx} - 2\,u_x u_y u_{xy} + (1+u_y^2)u_{yy} = 0$ is (*a*) hyperbolic, (*b*) parabolic, (*c*) elliptic, (*d*) singular, (*e*) of variable type depending on the point in the domain, or (*f*) of variable type depending on the solution and the point in the domain.

Characteristics and the Cauchy Problem

In Chapter 2, we discovered that the characteristic curves guide the behavior of solutions to first-order partial differential equations. Characteristics play a similarly fundamental role in the analysis of more general hyperbolic partial differential equations and systems. In particular, they provide a mechanism for distinguishing among the various classes of second-order partial differential equations.

As above, we will focus our attention on partial differential equations involving two independent variables. The starting point is the general initial value problem, also known as the Cauchy problem, in honor of the prolific nineteenth-century French mathematician Augustin–Louis Cauchy, justly famous for his wide-ranging contributions throughout mathematics and its applications, including the Cauchy–Schwarz inequality, many of the fundamental concepts in complex analysis, as well as the foundations of elasticity and materials science. The general *Cauchy problem* specifies appropriate initial data along a smooth curve† $\Gamma \subset \mathbb{R}^2$ and seeks a solution to the partial differential equation that assumes the given initial data on Γ. In all our examples, the curve in question has been a straight line, e.g., the x–axis, but one could easily envisage more general situations. If the partial differential equation has order n, then the *Cauchy data* consists of the values of the dependent variable u along with all its partial differential equations up to order $n-1$ on the curve Γ. For most curves, there is a unique solution $u(t,x)$ to the partial differential equation that achieves the specified values along Γ. More rigorously, if we are in the analytic category, meaning that the partial differential equation, the curve, and the Cauchy data are all specified by analytic functions, then the fundamental *Cauchy–Kovalevskaya Theorem* guarantees the existence of an analytic solution $u(t,x)$ to the Cauchy problem near any point on the initial curve. The statement of proof of this important theorem, due to Cauchy and, in general form, the influential nineteenth-century Russian mathematician Sofia Kovalevskaya, relies on the construction of convergent power series for the desired

† More generally, for partial differential equations in $m > 2$ independent variables, the curve is replaced by a hypersurface $S \subset \mathbb{R}^m$ of dimension $m-1$.

solution and would take us too far afield. We refer the interested reader to [**35**, **44**]. The exceptional curves, for which the Cauchy–Kovalevskaya Existence Theorem does not apply, are called the characteristics of the underlying partial differential equations.

More prosaically, a curve Γ will be called *non-characteristic* for the given partial differential equation if one can determine the values of *all* the derivatives of u along Γ from the specified Cauchy data. Indeed, the determination of the values of the higher-order derivatives along the curve is a necessary preliminary step towards establishing the Cauchy–Kovalevskaya existence result. As we will now show, this requirement serves to distinguish the characteristic and non-characteristic curves for the examples we have already encountered, and hence to lead to their characterization in much more general contexts.

To illustrate the preceding requirement, let us begin with a first-order linear partial differential equation of the form

$$\frac{\partial u}{\partial t} + c(t,x)\,\frac{\partial u}{\partial x} = f(t,x). \tag{4.140}$$

Let $\Gamma \subset \mathbb{R}^2$ be a smooth curve parametrized[‡] by $\mathbf{x}(s) = \big(\,t(s), x(s)\,\big)^T$, where smoothness necessitates that its tangent vector not vanish: $\mathbf{x}'(s) = (\,dt/ds, dx/ds\,)^T \neq \mathbf{0}$. Since the equation is of order $n = 1$, the Cauchy data requires specifying the values of the dependent variable u only along Γ — in other words, the function

$$h(s) = u\big(t(s), x(s)\big). \tag{4.141}$$

The curve will be non-characteristic if we can then determine the values of the derivatives of u along Γ, starting with

$$\frac{\partial u}{\partial t}\big(t(s), x(s)\big), \qquad\qquad \frac{\partial u}{\partial x}\big(t(s), x(s)\big). \tag{4.142}$$

To this end, let us differentiate the Cauchy data (4.141): applying the chain rule, we obtain

$$h'(s) = \frac{d}{ds}\,u\big(t(s), x(s)\big) = \frac{\partial u}{\partial t}\big(t(s), x(s)\big)\,\frac{dt}{ds} + \frac{\partial u}{\partial x}\big(t(s), x(s)\big)\,\frac{dx}{ds}\,. \tag{4.143}$$

On the other hand, we are assuming that $u(t,x)$ solves the partial differential equation (4.140) at all points in its domain of definition. In particular, at points on the curve Γ, the partial differential equation requires

$$\frac{\partial u}{\partial t}\big(t(s), x(s)\big) + c\big(t(s), x(s)\big)\,\frac{\partial u}{\partial x}\big(t(s), x(s)\big) = f\big(t(s), x(s)\big). \tag{4.144}$$

We can regard (4.143–144) as a pair of inhomogeneous linear algebraic equations, which can be uniquely solved for the as yet unknown quantities (4.142), *unless* the determinant of their coefficient matrix vanishes:

$$\det \begin{pmatrix} 1 & c\big(t(s), x(s)\big) \\ dt/ds & dx/ds \end{pmatrix} = \frac{dx}{ds} - c\big(t(s), x(s)\big)\,\frac{dt}{ds} = 0. \tag{4.145}$$

This condition serves to define a *characteristic curve* for the first-order partial differential equation (4.140). In particular, if the curve is parametrized by $s = t$, i.e., can be identified with the graph of a function $x = g(t)$, then the characteristic condition (4.145) reduces to

$$\frac{dx}{dt} = c(t,x), \tag{4.146}$$

[‡] The parameter s could be the arc length, but this is not required. See also Exercise 4.4.20.

thus reproducing our original definition of characteristic curve, as in (2.18) and, more generally, Exercise 2.2.26. On the other hand, if the determinant (4.145) is nonzero, then one can solve (4.143–144) for the values of the first-order derivatives (4.142) along Γ. Further differentiation of these conditions proves that one can, in fact, determine the values of all the higher-order derivatives of the solution u along the curve, which is hence non-characteristic.

Next, consider a nonsingular linear second-order partial differential equation of the form (4.134). Since the equation has order $n = 2$, the Cauchy data along a curve Γ parametrized as above consists of the values of the function and its first derivatives:

$$u\big(t(s), x(s)\big), \qquad \frac{\partial u}{\partial t}\big(t(s), x(s)\big), \qquad \frac{\partial u}{\partial x}\big(t(s), x(s)\big). \tag{4.147}$$

However, the latter cannot be specified independently. Indeed, given the value of the dependent variable, $h(s) = u\big(t(s), x(s)\big)$, along Γ, its derivative

$$h'(s) = \frac{d}{ds}\, u\big(t(s), x(s)\big) = \frac{\partial u}{\partial t}\big(t(s), x(s)\big)\, \frac{dt}{ds} + \frac{\partial u}{\partial x}\big(t(s), x(s)\big)\, \frac{dx}{ds} \tag{4.148}$$

prescribes a particular combination of the two first-order derivatives. Thus, once the value of one derivative of u on Γ is known, the other is automatically fixed by the relation (4.148). For example, if $dx/ds \neq 0$, we can use (4.148) to determine $u_x\big(t(s), x(s)\big)$, knowing $u\big(t(s), x(s)\big)$ and $u_t\big(t(s), x(s)\big)$. Similarly, if we differentiate the values of the first-order derivatives with respect to the curve parameter, we can determine two combinations of second-order derivatives along the curve Γ:

$$\begin{aligned} \frac{d}{ds}\,\frac{\partial u}{\partial t}\big(t(s), x(s)\big) &= \frac{\partial^2 u}{\partial t^2}\big(t(s), x(s)\big)\, \frac{dt}{ds} + \frac{\partial^2 u}{\partial t\, \partial x}\big(t(s), x(s)\big)\, \frac{dx}{ds}\,, \\ \frac{d}{ds}\,\frac{\partial u}{\partial x}\big(t(s), x(s)\big) &= \frac{\partial^2 u}{\partial t\, \partial x}\big(t(s), x(s)\big)\, \frac{dt}{ds} + \frac{\partial^2 u}{\partial x^2}\big(t(s), x(s)\big)\, \frac{dx}{ds}\,. \end{aligned} \tag{4.149}$$

On the other hand, the partial differential equation (4.134) induces yet a third relation among the second-order partial derivatives u_{tt}, u_{tx}, u_{xx}. These three linear equations can be uniquely solved for values of these derivatives on Γ if and only if the determinant of their coefficient matrix is nonzero:

$$\det \begin{pmatrix} A(t,x) & B(t,x) & C(t,x) \\ dt/ds & dx/ds & 0 \\ 0 & dt/ds & dx/ds \end{pmatrix} = A(t,x) \left(\frac{dx}{ds} \right)^2 - B(t,x)\, \frac{dt}{ds}\, \frac{dx}{ds} + C(t,x) \left(\frac{dt}{ds} \right)^2 = 0. \tag{4.150}$$

We conclude that a smooth curve $\mathbf{x}(s) = \big(\, t(s), x(s)\, \big)^T \subset \mathbb{R}^2$ is a *characteristic curve* for the nonsingular linear second-order partial differential equation (4.134) whenever its tangent vector $\mathbf{x}'(s) = (\, dt/ds, dx/ds\,)^T \neq \mathbf{0}$ satisfies the quadratic *characteristic equation* (4.150). Conversely, if the curve is non-characteristic, meaning that its tangent does not satisfy (4.150) anywhere, then one can, with some further work, determine all the higher-order derivatives of the solution $u(t,x)$ along Γ, and then, at least in the analytic category, prove existence of a solution to the Cauchy problem, [**35**].

According to Exercise 4.4.20, the status of a curve as characteristic or not does not depend on the choice of parametrization. In particular, if the curve is given by the graph of the function $x = x(t)$, which we parametrize by $s = t$, then the characteristic equation

(4.150) takes the form of a quadratically nonlinear first-order ordinary differential equation

$$A(t,x)\left(\frac{dx}{dt}\right)^2 - B(t,x)\,\frac{dx}{dt} + C(t,x) = 0, \tag{4.151}$$

whose solutions are characteristic curves of the second-order partial differential equation.

Warning: If $A(t,x) = 0$, then the partial differential equation admits characteristic curves with vertical tangents that cannot be parametrized by $s = t$. For example, if $A(t,x) \equiv 0$, then the vertical lines e.g., $t =$ constant, $x = s$, are characteristic, satisfying (4.150), but do not appear as solutions to (4.151).

For example, consider the hyperbolic wave equation

$$u_{tt} - c^2\, u_{xx} = 0.$$

According to (4.151), any characteristic curve that is given by the graph of $x(t)$ must solve

$$\left(\frac{dx}{dt}\right)^2 - c^2 = 0, \qquad \text{which implies that} \qquad \frac{dx}{dt} = \pm\, c.$$

Thus, in accordance with our previous analysis, the characteristic curves are the straight lines of slope $\pm\, c$, and there are two characteristic curves passing through each point of the (t,x)–plane. On the other hand, the elliptic Laplace equation

$$u_{tt} + u_{xx} = 0$$

has no (real) characteristic curves, since the characteristic equation (4.150) reduces to

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dt}{ds}\right)^2 = 0,$$

and t_s and x_s are not allowed to vanish simultaneously. Finally, for the parabolic heat equation

$$u_{xx} - u_t = 0,$$

the characteristic curve equation (4.150) is simply

$$\left(\frac{dt}{ds}\right)^2 = 0$$

(since the first-derivative term plays no role), and so there is only one characteristic curve passing through each point, namely the vertical line $t =$ constant. Observe that the standard initial value problem $u(0,x) = f(x)$ for the heat equation takes place on a characteristic curve — the x–axis — but does not take the form of a Cauchy problem, which would also require specifying the first-order derivatives $u_t(0,x), u_x(0,x)$ there. And indeed, the standard initial value problem is not well-posed near the characteristic x–axis for negative $t < 0$.

In general, the number of real solutions to the nondegenerate quadratic characteristic curve equation (4.150) depends on its discriminant $\Delta = B^2 - 4AC$: In the hyperbolic case, $\Delta > 0$, and there are two real characteristic curves passing through each point; in the parabolic case, $\Delta = 0$, and there is just one real characteristic curve passing through each point; in the elliptic case, $\Delta < 0$, and there are no real characteristic curves. In this manner, elliptic, parabolic, and hyperbolic partial differential equations are distinguished by the number of (real) characteristic curves passing through a point — namely, zero, one, and two, respectively. First-order partial differential equations are also viewed as *hyperbolic*, since they always admit real characteristic curves.

With further analysis, [**35**, **70**, **122**], it can be shown that, as with the wave equation, signals and disturbances propagate along characteristic curves. Thus, hyperbolic equations share many qualitative properties with the wave equation, with signals moving in two different directions. For example, light rays move along characteristic curves, and are thereby subject to the optical phenomena of refraction and focusing. Similarly, since the characteristic curves for the parabolic heat equation are the vertical lines, this indicates that the effect of a disturbance at a point $(t,x)=(t_0,x_0)$ is simultaneously felt along the entire contemporaneous vertical line $t=t_0$. This has the implication that disturbances in the heat equation propagate at infinite speed — a counterintuitive fact that will be further expounded on in Section 8.1. Elliptic equations have no characteristics, and as a consequence, do not support propagating signals; indeed, the effect of a localized disturbance is immediately felt throughout the domain. For example, even when an external force is concentrated near a single point, it displaces the entire membrane.

Exercises

4.4.14. Find and graph the real characteristic curves for each of the partial differential equations in Exercise 4.4.2.

4.4.15. Graph the characteristic curves for the Tricomi equation (4.137) in its hyperbolic region. What happens to the characteristics as one approaches the parabolic transition boundary?

4.4.16. *True or false*: The characteristic curves of the *Helmholtz equation* $u_{xx}+u_{yy}-u=0$ are circles.

4.4.17. (*a*) At what points of the plane is the partial differential equation $x\,u_{xx}+y\,u_{yy}=0$ elliptic? parabolic? hyperbolic? (*b*) How many characteristics are there through the point $(1,-1)$? (*c*) Find them explicitly.

4.4.18. Consider the partial differential equation $u_{xx}+y\,u_{xy}=y^2$.
(*a*) On which regions of the (x,y)–plane is the equation elliptic? parabolic? hyperbolic?
(*b*) Find the characteristics in the hyperbolic region.
(*c*) Find the general solution in the hyperbolic region. *Hint*: Use characteristic coordinates.

4.4.19. Find a partial differential equation whose characteristic curves are:
(*a*) the lines $x-y=a$, $x+2y=b$, where $a,b\in\mathbb{R}$ are arbitrary constants;
(*b*) the exponential curves $y=c\,e^x$ for $c\in\mathbb{R}$;
(*c*) the concentric circles $x^2+y^2=a$ for $a\geq 0$, and the rays $y=b\,x$.

♢ 4.4.20. Prove that any reparametrization of a characteristic curve for a given second-order linear partial differential equation is also a characteristic curve.

4.4.21. *True or false*: You can uniquely recover a second-order partial differential equation by knowing all its characteristic curves.

♢ 4.4.22. Prove that any invertible change of variables, as in Exercise 4.4.7, maps the characteristic curves of the original linear partial differential equation to the characteristic curves of the transformed equation. Thus, characteristic curves are intrinsic: they do not depend on the parametrization, nor on the coordinates used to represent the partial differential equation.